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THE WORKSHOP MATHEMATICS PROJECT

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Nancy Baxter Hastings • Barbara Reynolds

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# WORKSHOP CALCULUS WITH GRAPHING CALCULATORS


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GUIDED EXPLORATION WITH REVIEW

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VOLUME

1





*Workshop  
Calculus  
with Graphing  
Calculators*

**Guided Exploration with Review**

**Volume 1**

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# *Workshop Calculus*

*with Graphing  
Calculators*

**Guided Exploration with Review**

**Volume 1**

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**Dickinson College**

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*To my husband,  
David,  
and our family,  
Erica and Mark,  
Benjamin,  
Karin and Matthew,  
Mark, Margie, and Morgan,  
John and Laura.*



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# *Preface*

## TO THE INSTRUCTOR



*I hear, I forget.  
I see, I remember.  
I do, I understand.*

Anonymous

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### OBJECTIVES OF WORKSHOP CALCULUS

1. Impel students to be active learners.
2. Help students to develop confidence about their ability to think about and do mathematics.
3. Encourage students to read, write, and discuss mathematical ideas.
4. Enhance students' understanding of the fundamental concepts underlying the calculus.
5. Prepare students to use calculus in other disciplines.
6. Inspire students to continue their study of mathematics.
7. Provide an environment where students enjoy learning and doing mathematics.

## THE WORKSHOP APPROACH

*Workshop Calculus with Graphing Calculators: Guided Exploration with Review* provides students with a gateway into the study of calculus. The two-volume series integrates a review of basic precalculus ideas with the study of concepts traditionally encountered in beginning calculus: functions, limits, derivatives, integrals, and an introduction to integration techniques and differential equations. It seeks to help students develop the confidence, understanding, and skills necessary for using calculus in the natural and social sciences, and for continuing their study of mathematics.

In the workshop environment, students learn by doing and reflecting on what they have done. No formal distinction is made between classroom and laboratory work. Lectures are replaced by an interactive teaching format, with the following components:

- **Summary discussion:** Typically, the beginning of each class is devoted to summarizing what happened in the last class, reviewing important ideas, and presenting additional theoretical material. Although this segment of a class may take only 10 minutes or so, many students claim that it is one of the most important parts of the course, since it helps them make connections and focus on the overall picture. Students understand, and consequently value, the discussion because it relates directly to the work they have done.
- **Introductory remarks:** The summary discussion leads into a brief introduction to what's next. The purpose of this initial presentation is to help guide students' thoughts in appropriate directions without giving anything away. New ideas and concepts are introduced in an intuitive way, without giving any formal definitions, proofs of theorems, or detailed examples.
- **Collaborative activities:** The major portion of the class is devoted to students working collaboratively on the tasks and exercises in their *Workshop Calculus* book, which we refer to as their "activity guide." As students work together, the instructor moves from group to group, guiding discussions, posing questions, and responding to queries.

Workshop Calculus is part of Dickinson College's Workshop Mathematics Program, which also includes Workshop Statistics and Workshop Quantitative Reasoning, developed by my colleague Allan Rossman. Based on our experiences and those of others who have taught workshop courses, Allan developed the following helpful list, which he calls "A Dozen (Plus or Minus Two) Suggestions for Workshop Instructors":

- **Take control of the course.** Perhaps this goes without saying, but it is very important for an instructor to establish that he or she has control of the course. It is a mistake to think of Workshop Mathematics courses as self-paced, where the instructor plays but a minor role.

- **Keep the class roughly together.** We suggest that you take control of the course in part by keeping the students roughly together with the material, not letting some groups get too far ahead or lag behind.
- **Allow students to discover.** We encourage you to resist the temptation to tell students too much. Rather, let them do the work to discover ideas for themselves. Try not to fall into let-me-show-you-how-to-do-this mode.
- **Promote collaborative learning among students.** We suggest that you have students work together on the tasks in pairs or groups of three. We do recommend, however, that students be required to write their responses in their books individually.
- **Encourage students' guessing and development of intuition.** We believe that much can be gained by asking students to think and make predictions about issues before analyzing them in detail.
- **Lecture when appropriate.** By no means do we propose never speaking to the class as a whole. As a general rule, however, we advocate lecturing on an idea only after students have had an opportunity to grapple with it themselves.
- **Have students do some work by hand.** While we strongly believe in using technology to explore mathematical phenomena, we think students have much to gain by first becoming competent at performing computations, doing symbolic manipulations, and sketching graphs by hand.
- **Use technology as a tool.** The counterbalance to the previous suggestion is that students should come to regard technology as an invaluable tool for modeling situations and analyzing functions.
- **Be proactive in approaching students.** As your students work through the tasks, we strongly suggest that you mingle with them. Ask questions. Join their discussions.
- **Give students access to "right" answers.** Some students are fearful of a self-discovery approach because they worry about discovering "wrong" things. We appreciate this objection, for it makes a strong case for providing students with regular and consistent feedback.
- **Provide plenty of feedback.** An instructor can supply much more personalized, in-class feedback with the workshop approach than in a traditional lecture classroom, and the instructor is positioned to continually assess how students are doing. We also encourage you to collect a regular sampling of tasks and homework exercises as another type of feedback.
- **Stress good writing.** We regard writing-to-learn as an important aspect of a workshop course. Many activities call for students to write interpretations and to explain their findings. We insist that students relate these to the context at hand.
- **Implore students to read well.** Students can do themselves a great service by taking their time and reading not only the individual questions carefully, but also the short blurbs between tasks, which summarize what they have done and point the way to what is to come.



- *Have fun!* We enjoy teaching more with the workshop approach than with lecturing, principally because we get to know the students better and we love to see them actively engaged with the material. We genuinely enjoy talking with individual students and small groups of students on a regular basis, as we try to visit each group several times during a class period. We sincerely hope that you and your students will have as much fun as we do in a Workshop Mathematics course.

## INSTRUCTIONAL MATERIALS

A key aspect of the Workshop Calculus materials is their flexibility. The length of class sessions, the balance between lecture and laboratory time, the type of technology that is used, the intended mathematical level, and the specific computer instructions can be varied by the local instructor. Although the two-volume set of materials is intended for a year-long integrated Precalculus/Calculus I course, subsets of the materials can be used in one-semester Precalculus and Calculus I courses, or they can be supplemented for use in an Advanced Placement Calculus course. Moreover, the activity guide may be used as a stand-alone book or in conjunction with other materials.

*Workshop Calculus* is a collection of guided inquiry notes presented in a workbook format. As students begin to use the book, encourage them to tear out the pages for the current section and to place them in a three-ring binder. These pages can then be interspersed with lecture/discussion notes, responses to homework exercises, supplemental activities, and so on. During the course, they will put together their own book.

Each section in the book consists of a sequence of tasks followed by a set of homework exercises. These activities are designed to help students think like mathematicians—to make observations and connections, ask questions, explore, guess, learn from their errors, share ideas, read, write, and discuss mathematics.

The tasks are designed to help students explore new concepts or discover ways to solve problems. The steps in the tasks provide students with a substantial amount of guidance. Students make predictions, do calculations, and enter observations directly in their activity guide. At the conclusion of each task, the main ideas are summarized, and students are given a brief overview of what they will be doing in the next task. The tasks are intended to be completed in a linear fashion.

The homework exercises provide students with an opportunity to utilize new techniques, to think more deeply about the concepts introduced in the section, and occasionally to tackle new ideas. New information that is presented in an exercise usually is not needed in subsequent tasks. However, if a task does rely on a concept introduced in an exercise, students are referred back to the exercise for review. Any subset of the exercises may be assigned, and they may be completed in any order or interspersed with associated tasks.

The homework exercises probably should be called “post-task activities,” since the term “homework” implies that they are to be done outside of class. This is not our intention; both tasks and exercises may be completed either in or out of class.

At the conclusion of each unit, students reflect on what they have learned in their “journal entry” for the unit. They are asked to describe in their own words the concepts they have studied, how they fit together, which ones were easy, and which were hard. They are also asked to reflect on the learning environment for the course. We view this activity as one of the most important in a unit. Not only do the journal entries provide us with feedback and enable us to catch any last misconceptions, but more important, they provide the students with an opportunity to think about what has been going on and to write about their observations.

Technology plays an important role in Workshop Calculus. In Unit 1, students use a motion detector connected to a computer-based laboratory (CBL) interface to create distance versus time functions and to analyze their behavior. In Unit 2, they learn to use their graphing calculators while exploring various ways to represent functions. Then, throughout the remainder of the materials, they use their calculators to do numerical and graphical manipulations and to form mental images associated with abstract mathematical ideas, such as the limiting behavior of a function.

The homework exercises contain optional activities for students who have access to a graphing calculator that does symbolic manipulation, such as a TI-89 or a TI-92, or to a computer that is equipped with a computer algebra system (CAS), such as *Mathematica*<sup>®</sup>, Maple<sup>®</sup>, or Derive. These optional activities are labeled “CAS activity.”

Although the Workshop Calculus materials are graphing calculator dependent, there are no references in the text to a particular type of calculator or to a specific CAS package. However, the book does contain an appendix for a TI-83, which gives an overview of the features that are used in Workshop Calculus. This appendix is also available electronically from our Web site at Dickinson College and can be downloaded and customized for use with other types of calculators.

In addition to the appendix for the TI-83, a set of notes to the instructor is also available electronically. These notes contain topics for discussion and review; suggested timing for each task; solutions to homework exercises; and sample schedules, syllabi, and exams. Visit <http://calc.dickinson.edu>.

## ACKNOWLEDGMENTS

The computer-based versions of *Workshop Calculus*, Volumes 1 and 2, appeared in 1997 and 1998, respectively. Their publication marked the culmination of seven years of testing and development. While we were working on the computer-based version, graphing calculators were becoming more and more powerful and user-friendly. Again and again, colleagues would remark that calculators were a viable alternative to a fully equipped

computer laboratory. Realizing that the Workshop Calculus materials could be implemented on a graphing calculator is one thing; actually doing the work is another. My close friend and colleague, Barbara E. Reynolds at Cardinal Stritch University, agreed to undertake the transformation. She worked carefully through the tasks in Volume 1, replacing the ISETL and CAS activities with equivalent graphing calculator versions. She wrote the initial version of the appendices for using a TI-83 and for using a TI-92 in Workshop Calculus. Without her contributions, the calculator version of *Workshop Calculus* would still be just an idea, not a reality.

The tasks contained in the Workshop Calculus materials were developed in consultation with my physics colleague, Priscilla Laws, and a former student, Christa Fratto.

Priscilla Laws was the impelling force behind the project. She developed many of the applications that appear in the text, and her award-winning Workshop Physics project provided a model for the Workshop Calculus materials and the underlying pedagogical approach.

Christa Fratto graduated from Dickinson College in 1994 and is currently teaching at The Episcopal School in Philadelphia. Christa started working on the workshop materials as a Dana Student Intern, during the summer of 1992. She quickly became an indispensable partner in the project. She tested activities, offered in-depth editorial comments, developed problem sets, helped collect and analyze assessment data, and supervised the student assistants for Workshop Calculus classes. Following graduation, she continued to work on the project, writing the handouts for the software tools used in the original computer-based version and the answer key for the homework exercises in Volume 1, and developing new tasks. After reviewing the appendices that Barbara Reynolds wrote, Christa rewrote the TI-83 appendix, contained in this volume, using the format for the computer-based version of *Workshop Calculus*.

Other major contributors include Kevin Callahan and Mark Botorff, who helped design, write, and test initial versions of the material while on the faculty at Dickinson College. Kevin is now using the materials at California State University at Hayward, and Mark is completing his Ph.D. degree in Mathematical Physics.

A number of other colleagues have tested the materials and provided constructive feedback. These include Peter Martin, Shari Prevost, Judy Roskowski, Barry Tesman, Jack Stodghill, and Blayne Carroll at Dickinson College; Carol Harrison at Susquehanna University; Nancy Johnson at Lake Brantley High School; Michael Kantor at Knox College; Stacy Landry at The Potomac School; Sandy Skidmore and Julia Clark at Emory and Henry College; Sue Suran at Gettysburg High School; Sam Tumolo at Cincinnati Country Day School; and Barbara Wahl at Hanover College.

The Dickinson College students who assisted in Workshop Calculus classes helped make the materials more learner-centered and user-friendly. These students include Jennifer Becker, Jason Cutshall, Amy Demski, Kimberly Kendall, Greta Kramer, Russell LaMantia, Tamara Manahan, Susan Nouse, Alexandria Pefkaros, Benjamin Seward, Melissa Tan, Katharyn Wilber, and Jennifer Wysocki. In addition, Quian Chen, Kathy Clawson, Christa Fratto, Hannah Hazard, Jennifer Hoenstine, Linda Mellott, Marlo

Mewherter, Matthew Parks, and Katherine Reynolds worked on the project as Dana Student Interns, reviewing the materials, analyzing assessment data, developing answer keys, and designing Web pages. Virginia Laws did the initial version of the illustrations. Sarah Buchan and Matthew Weber proofread the final versions of Volumes 1 and 2, respectively.

The development of the Workshop Calculus materials was also influenced by helpful suggestions from Jack Bookman of Duke University, Steve Davis of Davidson College, and Murray Kirch of the Richard Stockton College of New Jersey, who served as outside reviewers for the manuscript; Ed Dubinsky of Georgia State University, who served as the project's mathematics education research consultant; and David Smith of Duke University, who served as the project's outside evaluator.

An important aspect of the development of the Workshop Calculus project is the ongoing assessment activities. With the help of Jack Bookman, who served as the project's outside evaluation expert, we have analyzed student attitudes and learning gains, observed gender differences, collected retention data, and examined performance in subsequent classes. The information has provided the program with documented credibility and has been used to refine the materials for publication.

The Workshop Mathematics Program has received generous support from the U.S. Department of Education's Fund for Improvement of Post Secondary Education (FIPSE #P116B50675 and FIPSE #P116B11132), the National Science Foundation (NSF/USE #9152325, NSF/DUE #9450746, and NSF/DUE #9554684), and the Knight Foundation. For the past six years, Joanne Weissman has served as the project manager for the Workshop Mathematics Program. She has done a superb job, keeping the program running smoothly and keeping us focused and on task.

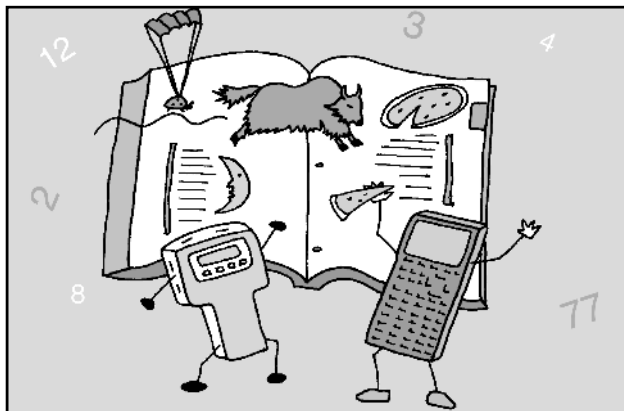
Publication of the calculator version of the Workshop Calculus activity guides marks the culmination of eight years of testing and development. We have enjoyed working with Jerry Lyons, Editorial Director of Physical Sciences at Springer-Verlag. Jerry is a kindred spirit who shares our excitement and understands our vision. We appreciate his support, value his advice, and enjoy his friendship. And, finally, we wish to thank Kim Banister, who did the illustrations for the manuscript. In her drawings, she has caught the essence of the workshop approach: students exploring mathematical ideas, working together, and enjoying the learning experience.

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# Preface

## TO THE STUDENT



*Everyone knows that if you want to do physics or engineering, you had better be good at mathematics. More and more people are finding out that if you want to work in certain areas of economics or biology, you had better brush up on your mathematics. Mathematics has penetrated sociology, psychology, medicine and linguistics . . . it has been infiltrating the field of history. Why is this so? What gives mathematics its power? What makes it work?*

*. . . the universe expresses itself naturally in the language of mathematics. The force of gravity diminishes as the second power of the distance; the planets go around the sun in ellipses, light travels in a straight line. . . . Mathematics in this view, has evolved precisely as a symbolic counterpart of this universe. It is no wonder then, that mathematics works: that is exactly its reason for existence. The universe has imposed mathematics upon humanity. . . .*

Philip J. Davis and Rubin Hersh  
Co-authors of *The Mathematical Experience*  
Birkhäuser, Boston, 1981

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### Task P-1: Why Are You Taking This Course?

Briefly summarize the reasons you decided to enroll in this calculus course. What do you hope you will gain by taking it?



## Why Study Calculus?

Why should you study calculus? When students like yourself are asked their reasons for taking calculus courses, they often give reasons such as, “It’s required for my major.” “My parents want me to take it.” “I like math.” Mathematics teachers would love to have more students give idealistic answers such as, “Calculus is a great intellectual achievement that has made major contributions to the development of philosophy and science. Without an understanding of calculus and an appreciation of its inherent beauty, one cannot be considered an educated person.”

Although most mathematicians and scientists believe that becoming an educated person ought to be the major reason why you should study calculus, we can think of two other equally important reasons for studying this branch of mathematics: (1) mastering calculus can provide you with conceptual tools that will contribute to your understanding of phenomena in many other fields of study, and (2) the process of learning calculus can help you acquire invaluable critical thinking skills that will enrich the rest of your life.

## What Is Calculus?

We have made several claims about calculus and its importance without saying anything about what it is or why it’s so useful. It is difficult to explain this branch of mathematics to someone who has not had direct experience with it over an extended period of time. Hopefully, the overview that follows will help you get started with your study of calculus.

Basically, calculus is a branch of mathematics that has been developed to describe relationships between things that can change continuously. For example, consider the *mathematical relationship* between the diameter of a pizza and its area. You know from geometry that the area of a perfectly round pizza is related to its diameter by the equation

$$A = \frac{1}{4}\pi d^2.$$

You also know that the diameter can be changed *continuously*. Thus, you don’t have to make just 9" pizzas or 12" pizzas. You could decide to make one that is 10.12" or one that is 10.13", or one whose diameter is halfway between these two sizes. A pizza maker could use calculus to figure out how the area of a pizza changes when the diameter changes a little more easily than a person who only knows geometry.

But it is not only pizza makers who could benefit by studying calculus. Someone working for the Federal Reserve might want to figure out how



much metal would be saved if the size of a coin is reduced. A biologist might want to study how the growth rate of a bacterial colony in a circular petri dish changes over time. An astronomer might be curious about the accretion of material in Saturn's famous rings. All these questions can be answered by using calculus to find the relationship between the change in the diameter of a circle and its area.

---

## Calculus and the Study of Motion

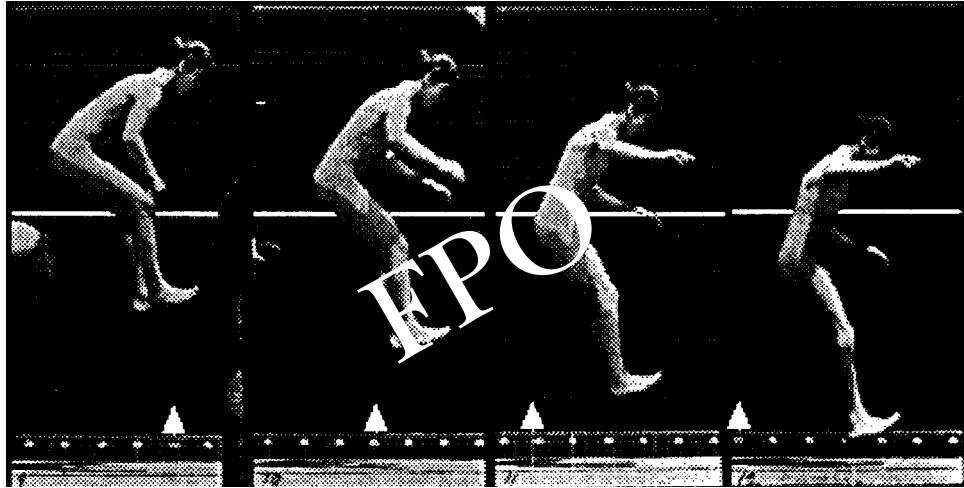
Ever since antiquity, philosophers and scientists have been fascinated with the nature of motion. Since motion seems intuitively to involve continuous change from moment to moment and seems to have regular patterns, it is not surprising that calculus can be used to describe motion in a very elegant manner.

The link between calculus and motion is quite fundamental. Isaac Newton, a seventeenth-century scientist, who made major contributions to the development of calculus, did so because he was primarily interested in describing motion mathematically. If motion is continuous, then it is possible to use calculus to find the speed of a runner or jumper at various times, if the relationship between his position and time is known. *Differential calculus* can be used to find the link between the change in position and velocity. Conversely, *integral calculus* can be used to find the position of a moving person whose velocity is known.



**Figure 1:** Panathenaic Prize amphora depicting the motion of a runner, ca. 530 B.C. Attributed to Euphiletos Painter, Terracotta. The Metropolitan Museum of Art, all rights reserved.

The potential of finding speed from changing positions and position from changing speeds can be illustrated by considering a sequence of photographs of a leaping boy taken by an eccentric artist-photographer, Eadweard Muybridge.<sup>1</sup>



**Figure 2:** Sequence of photos of a young boy leap frogging over the head of a companion. This series was taken by a famous artist-photographer Eadweard Muybridge. The original photos have been enhanced with lines and markers. *Boys Leapfrogging* (Plate 168). The Trout Gallery, Dickinson College. Gift of Samuel Moyerman, 87.4.8.

---

### Task P-2: Continuous Motion or Motion Containing Instantaneous Jumps and Jerks?

1. Suppose the time interval between frames in Figure 2 is  $1/10$ th of a second. What might be happening to the position of the boy during the time between frames?

---

<sup>1</sup>Muybridge used a series of cameras linked electrically to obtain photos equally spaced in time. His mastery of technology was quite advanced for the time. Muybridge engaged in exploits other than photography, as he is reputed to have killed his wife's lover in a fit of rage. For this murder, a sympathetic jury acquitted him of his crime of passion. See G. Hendricks, *Eadweard Muybridge: The Father of the Motion Picture*, New York, Grossman, 1975.

2. Do you think people and animals move from place to place in a series of little jumps and jerks or continuously? Explain the reasons for your answer.
  
  
  
  
  
  
  
  
  
  
3. Describe a way to use modern technology to determine if a leaping boy, like the one depicted in Figure 2, is moving in continuous motion or in instantaneous jumps and jerks.

---

## Summary

Most people believe that objects move *continuously* from position to position no matter how small the time interval between positions is. Thus, if the boy's big toe is at point  $A$  at one time and point  $B$  at another time, at any point between  $A$  and  $B$  there is a time when the boy's toe will be at that position.

As a result of the presumed continuity of motion, calculus is a powerful intellectual tool for exploring the nature of the boy's motion. For example, if you know the mathematical relationship between the boy's toe and time, you can use differential calculus to find its velocity—in other words, the rate at which the position of his toe changes. Conversely, if you know when the boy's toe is at a particular time and the mathematical relationship describing the velocity of the toe at each moment of time, you can use integral calculus to find a relationship for the position of the toe at each moment of time.

Calculus is not only useful in the study of motion. As we mentioned earlier, it is concerned with the study of mathematical relationships among two or more quantities that can vary continuously. The uses of calculus to study continuous changes are widespread and varied. It can be used to help understand many types of relationships, such as population changes of living organisms, the accumulation of the national debt, and the relationship between the concentration of chemicals and their reaction rates.

---

## Using Technology and Collaboration to Study Calculus

As you complete the activities designed for this course, you will learn a lot about the nature of calculus. You will develop a conceptual understanding of the fundamental ideas underlying calculus: function, limit, derivative,

antiderivative, and definite integral. You will discover how to use calculus to solve problems. Along the way, you will review the necessary algebraic and trigonometric concepts needed to study calculus.

The methods used to teach Workshop Calculus may be new to you. In the workshop environment, formal lectures are replaced by an interactive teaching format. You will learn by doing and reflecting on what you have done. Initially, new ideas will be introduced to you in an informal and intuitive way. You will then work collaboratively with your classmates on the activities in this workbook—which we refer to as your “activity guide”—exploring and discovering mathematical concepts on your own. You will be encouraged to share your observations during class discussions.

Although we can take responsibility for designing a good learning environment and attempting to teach you calculus, you must take responsibility for learning it. No one else can learn it for you. You should find the thinking skills and mathematical techniques acquired in this course useful in the future. Most importantly, we hope you enjoy the study of calculus and begin to appreciate its inherent beauty.

A number of the activities in this course will involve using technology to enhance your learning. Using technology will help you develop a conceptual understanding of important mathematical concepts and help you focus on significant ideas, rather than spending a lot of time on extraneous details.

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### Task P-3: Getting Started with Your Graphing Calculator

Familiarize yourself with the calculator that you will be using in this course. (See “Calculator Basics” in the appendix for your calculator.)

1. Determine how to turn your graphing calculator on and off.
2. Note how the keyboard is organized. Locate the graphing keys, the cursor keys, the editing keys, the advanced function keys, and the scientific calculator keys. Observe where the predefined functions and constants are located.
3. Most keys can perform two or more functions. Determine how to access the functions that are represented by the same key.
4. Determine how to:
  - return to the home screen.
  - adjust the contrast on the screen.
  - select an item from a list or menu.
  - access the catalog feature.
  - edit and reexecute code.
  - transfer programs between calculators.
  - execute a program.





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## Some Important Advice Before You Begin

- ***Put together your own book.*** Remove the pages for the current section from your activity guide, and place them in a three-ring binder. Intersperse the pages with lecture and discussion notes, answers to homework problems, and handouts from your instructor.
- ***Read carefully the short blurbs at the beginning of each section and prior to each task.*** These blurbs summarize what you have done and point the way to what is to come. They contain important and useful information.
- ***Work closely with the members of your group.*** Think about the tasks together. Discuss how you might respond to a given question. Share your thoughts and your ideas. Help one another. Talk mathematics.
- ***Answer the questions in your activity guide in your own words.*** Work together, but when it comes time to write down the answer to a question, do not simply copy what one of your partners has written.
- ***Use separate sheets of paper for homework problems.*** Unless otherwise instructed, do not try to squish the answers in between the lines in your activity guide.
- ***Think about what your graphing calculator is doing.*** Whenever you ask your calculator to perform a task, think about how it might be processing the information that you have given it, keeping in mind:
  - What you have commanded your calculator to do.
  - Why you asked it to do whatever it is doing.
  - How it might be doing whatever you have told it to do.
  - What the results mean.
- ***Have fun!***

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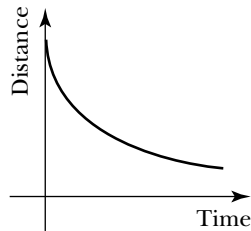
*Volume*

*1*

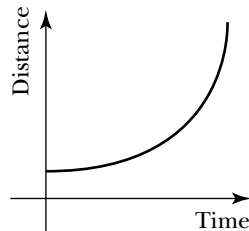
### Task 1-8: Contemplating Concavity

1. Examine the location of the curve with respect to the tangent line. Does the curve lie above the tangent line or below?

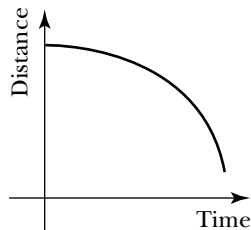
Consider the following graphs:



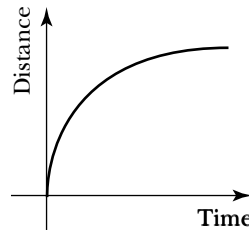
Graph 1



Graph 2



Graph 3



Graph 4

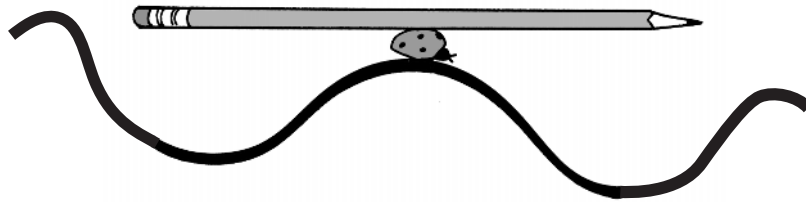
- a. Pick a point on each graph and use a straightedge to draw the tangent line at that point.
  - b. Graphs 1 and 2 are concave up. Describe the location of the curve with respect to the tangent line in both of these cases.
  - c. Graphs 3 and 4 are concave down. Describe the location of the curve with respect to the tangent line in both of these cases.
2. Describe how you must move to produce each of the distance versus time graphs shown above.
    - a. Graph 1:
    - b. Graph 2:

- c. Graph 3:
- d. Graph 4:
3. Use the motion detector to check your responses to part 2. Try to create distance versus time graphs that have the same shape as Graphs 1 through 4. If necessary, modify your descriptions in part 2.

*Hint: use a short time interval, say 2 to 3 seconds.*

4. Remember the pencil-laden bug? Analyze the bug's graph.

*Note: In reality, the bug has zero mass and the tangent line touches the curve. You will have to use your imagination.*



- a. Label the intervals on the bug's graph where the function is concave up and where it is concave down.
- b. Describe the location of the ant (and the pencil) as the bug crawls along a portion of the curve that is concave up.
- c. What happens to the bug (and the pencil) when the curve changes concavity, from up to down or down to up?

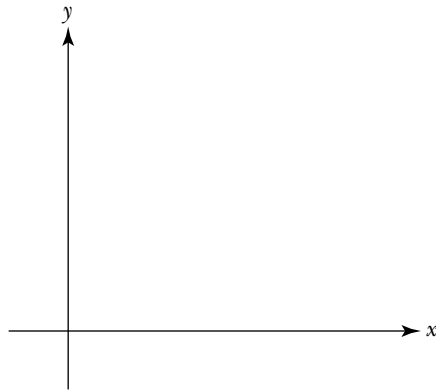
*Note: This is called an inflection point.*

- d. Mark all the local maxima on the bug's curve. Describe the concavity at a local maximum.

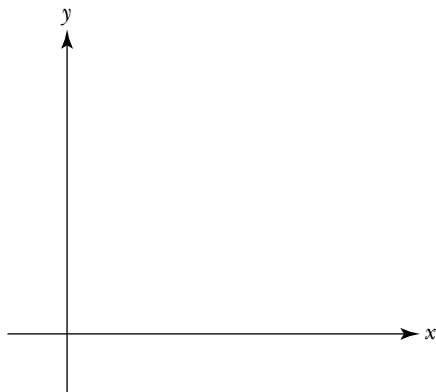
- e. Mark all the local minima on the bug's curve. Describe the concavity at a local minimum.

5. Sketch a graph of some basic situations. For each of the following:

- i. Sketch a graph satisfying the specified conditions.
  - ii. Draw some typical tangent lines.
- a. The function is increasing. The graph is concave up.

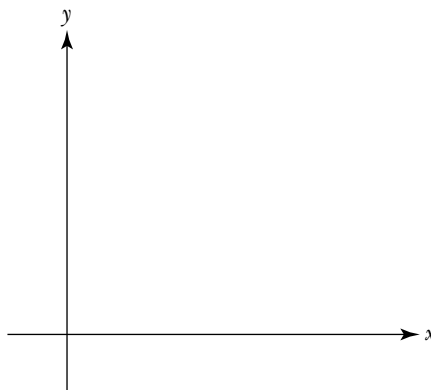


- b. The function is decreasing. The graph is concave up.

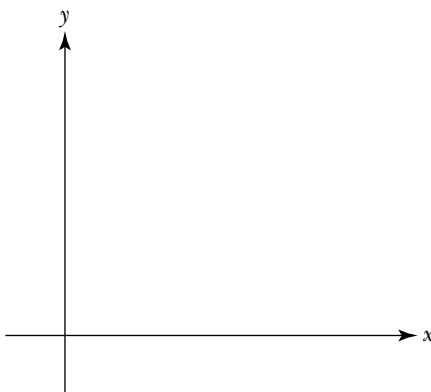




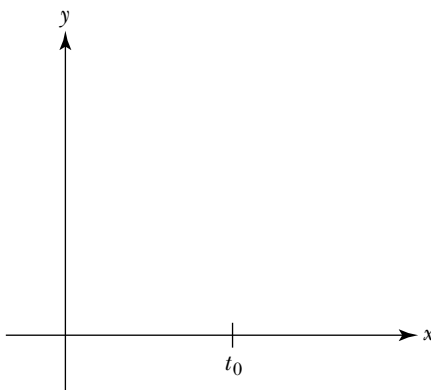
c. The function is increasing. The graph is concave down.



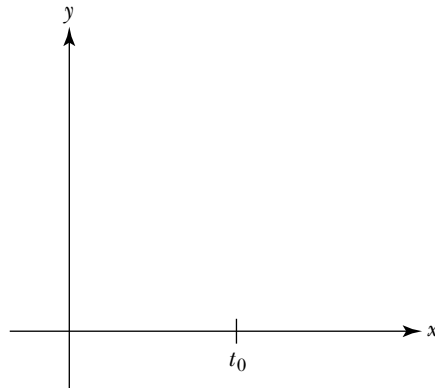
d. The function is decreasing. The graph is concave down.



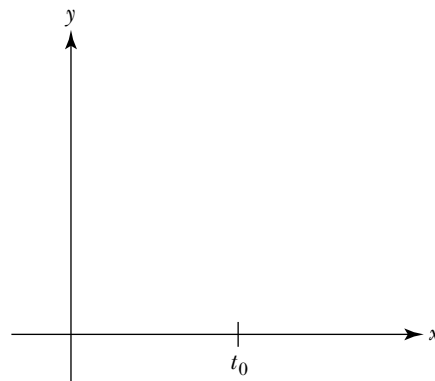
e. The function is increasing. The tangent line at  $t_0$  cuts through the graph of the function. (Note that the point where the tangent line passes through the graph is an inflection point.)



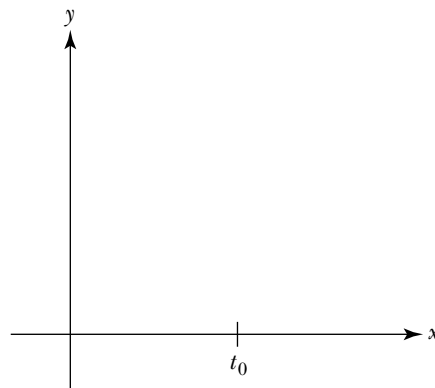
- f. The function is decreasing.  $t_0$  is an inflection point.



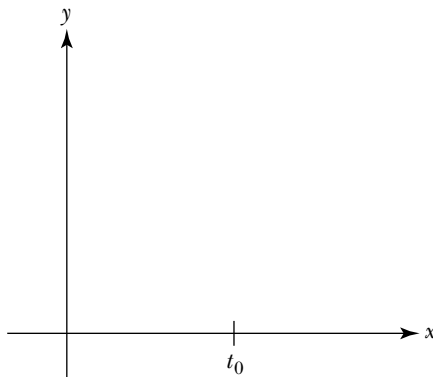
- g. The tangent lines at points to the left of  $t_0$  have negative slopes. The slope of the tangent line at  $t_0$  is 0. The tangent lines at points to the right of  $t_0$  have positive slopes.



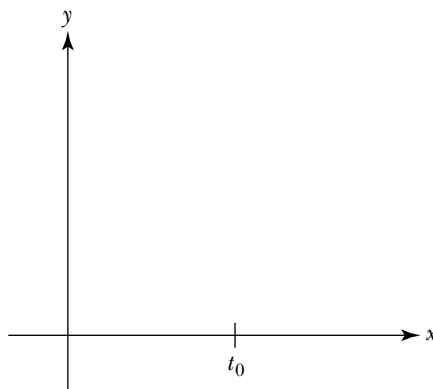
- h. The tangent lines at points to the left of  $t_0$  have positive slopes. The slope of the tangent line at  $t_0$  is 0. The tangent lines at points to the right of  $t_0$  have negative slopes.



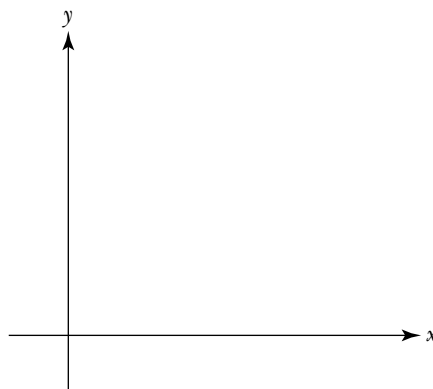
- i. The tangent lines at points to the left and to the right of  $t_0$  have positive slopes. The slope of the tangent line at  $t_0$  is 0.



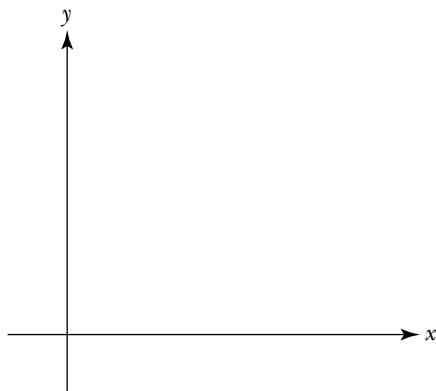
- j. The tangent lines at points to the left and to the right of  $t_0$  have negative slopes. The slope of the tangent line at  $t_0$  is 0.



- k. The function is increasing. The rate of change is increasing—that is, the slope of the tangent line is increasing as the line moves along the curve from left to right.



1. The function is increasing. The rate of change is decreasing—that is, the slope of the tangent line is decreasing as the line moves along the curve from left to right.



Whenever you use the motion detector to create a function, the graph lies in the first quadrant of the  $xy$ -plane, since, in this case, time and distance are positive quantities. This, of course, is not always what happens. For example, if you lived in upstate New York and graphed the temperature (in Celsius) versus time over a one-year period, your graph would dip below the horizontal axis since there would be a number of days when the temperature drops below freezing. One way to indicate when the graph of a function lies below the horizontal axis, when it touches the axis and when it lies above the axis, is to use a *sign chart*. In the next task, you will create a sign chart for a given function. In addition, you will reverse the process and sketch the graph of a function which satisfies a given chart. But first a few words about functional notation.

When you represent a function by an expression, you give the independent variable a name, such as  $x$  or *price*, and then define the dependent variable in terms of the independent variable. There are several ways to do this. One approach is to give the dependent variable a name, such as  $y$  or *tax*, and represent the function by equations such as

$$y = x^2 - 5x - 1 \quad \text{or} \quad \text{tax} = 0.06 \cdot \text{price}$$

Another approach is to give the function a name, such as  $f$  or  $T$ , in which case you would write

$$f(x) = x^2 - 5x - 1 \quad \text{or} \quad T(\text{price}) = 0.06 \cdot \text{price}$$

where  $f(x)$  is read “ $f$  of  $x$ ” and  $T(\text{price})$  is read “ $T$  of *price*.”

To find the output value corresponding to a given input value, you assign the independent variable the desired value and solve for the value of the dependent variable. For example, when  $x = 4$ ,

$$y = 4^2 - 5 \cdot 4 - 1 = -5 \quad \text{or} \quad f(4) = 4^2 - 5 \cdot 4 - 1 = -5$$

Consequently, the point  $P(4, -5)$  lies on the graph of  $f$ . Moreover, since the value of  $f(4)$  is negative,  $P$  lies below the horizontal axis.

The *sign of  $f(x)$*  is the sign of the output value corresponding to  $x$ . For instance, the sign of  $f$  is negative ( $-$ ) when  $x = 4$ , since  $f(4) = -5 < 0$ . Similarly, the sign of  $f$  is positive ( $+$ ) when  $x = 6$ , since  $f(6) = 5 > 0$ . You can record this information on the sign chart for  $f$  as follows:



But  $x = 4$  and  $x = 6$  are only two of the values in the domain of  $f$ . What happens at the other values in the domain? What does the sign chart tell you about the location of the graph of  $f$ ? You will explore answers to these questions in the next task.

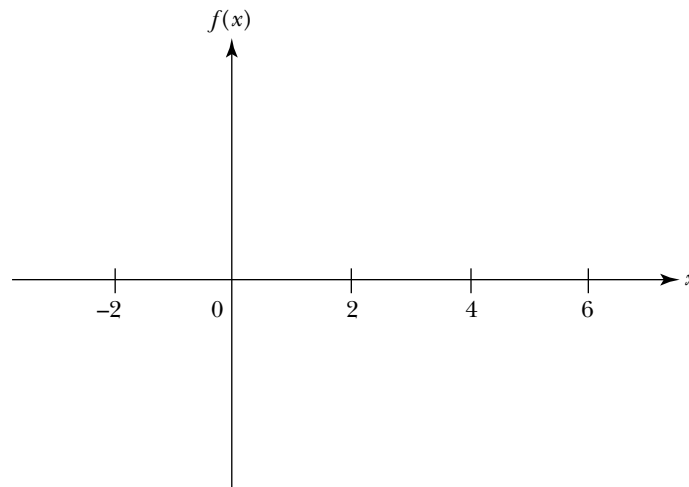
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### Task 1-9: Interpreting Sign Charts

1. Create some sign charts.

a. Consider the function  $f(x) = 2x - 2$ .

(1) Sketch the graph of  $f$  on the pair of axes given below. Label the vertical axis.



- a. You cannot divide a real number by 0 (and get a real number).
- b. You cannot find the square root of a negative number (and get a real number).
- c. The slope of a horizontal line is always 0.
- d. The slope of a line which rises from left to right is always positive.
- e. The slope of a line which falls from left to right is always negative.
- f. The slope of a vertical line is undefined.

## SECTION 2

### Analyzing Linear Functions

You will begin your study of functions by using a motion detector to create your own functions. As you walk, jump, or run in front of the motion detector, it sends out a series of ultrasonic pulses which bounce off an object in its path (in this case, you) and back to the detector. The computer or calculator records the time it takes each sound pulse to make the round-trip between the detector and the object, uses this information to calculate the object's distance from the detector, and instantaneously displays the distance versus time graph generated by the motion on the screen.

In the first task<sup>1</sup> you will use the motion detector to create some *linear* functions—that is, functions whose graphs are straight lines—and then find an expression that models your curve. You will create increasing and decreasing functions and examine how varying the rate at which you move affects the shape of the graph representing your motion. You will examine some ways of representing a function and think about the meaning of domain and range. Before beginning the task, review the calculator basics, how to use the CBL tools, and how to analyze data collected by the CBL System in the appendix for your graphing calculator.

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#### Task 1-4: Creating Linear Functions

*As you go through this task, read each of the questions very carefully before you begin. Try to do exactly what it says. Work as a team. Share the calculations. Compare your results. Talk about what is going on. Help each other out. Make sure each person takes a turn using the motion detector. You may try a number of times. Get the times right. Get the distances right.*

Connect a motion detector to your graphing calculator and open the program MOTION.

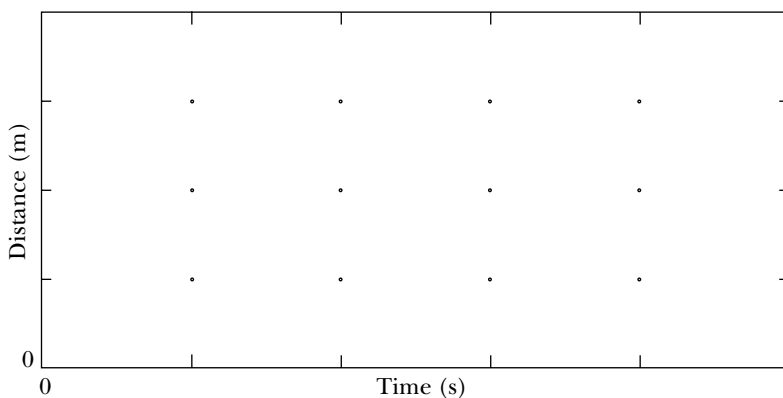
1. Create two linear graphs while increasing your distance from the detector.

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<sup>1</sup>The motion detector investigations are based on activities developed by Ron Thornton of Tufts University, David Sokoloff of the University of Oregon, and Priscilla Laws of Dickinson College.



- a. Create a graph by increasing your distance while walking very slowly. Start at the  $\frac{1}{2}$ -meter mark. Increase your distance from the detector walking at a *very* slow, constant pace, for 5 seconds. Keep trying, until you are satisfied with your graph.
- b. Save the data for your first graph.
- c. Create a second graph by increasing your distance while walking a little faster than you did with the first graph. Start at the  $\frac{1}{2}$ -meter mark. Increase your distance from the detector walking at a slightly faster, constant pace. As you did with the first graph, keep trying, until you are satisfied with the result.
- d. Sketch the graphs of the functions on the axes given below. Label the axes.



- e. Quit program MOTION and return to your home screen.
2. Consider the linear functions which you created by increasing your distance from the detector in part 1.
  - a. View the graphs of the two functions in the graphing window on your calculator.

*Note: Adjust the viewing WINDOW if necessary. Observe that you can use the TRACE feature to move the cursor along a graph and to move from one graph to another graph.*

- b. Analyze the first graph, which you created walking very slowly.
  - (1) Use your graph to estimate your distance when
    - (a)  $t = 1$  second
    - (b)  $t = 3.5$  seconds
  - (2) In one or two sentences, describe the process you used to find your distance at a given time based on your graph.

(3) Although you tried to walk at a steady pace, the graph representing your motion is probably a little bumpy. Try to find a line that approximates or “fits” your graph.

(a) Approximate the slope of the line that best fits your graph.

*Note: In this case, the unit for the slope is meters per second.*

(b) Find an equation for the line that best fits your graph, using the *point-slope equation* for a line and the value for the slope from part (a).

c. Analyze the second graph, which you created by walking a little more rapidly.

(1) Find an equation which represents your new function by fitting a line to your graph.

(2) If  $d$  is the name of your function, then  $d(t)$ —which is read “ $d$  of  $t$ ”—gives the output, or your distance from the detector, at time  $t$ . Use your graph to approximate the values of

(a)  $d(1.0)$

(b)  $d(2.3)$

(3) Represent your new function by a *table* by approximating the values for  $d(t)$  at some specified times.

$t$	1.5	2.1	2.9	3.8
$d(t)$				

(4) Describe how the columns in the table relate to the points on your graph.

d. The slope of the line represents your *rate of change* or your *velocity*. Describe how the rate at which you walk affects the value of the slope of the function modeling your motion.

e. When you increase your distance from the detector, the slope of the associated graph is always positive. Explain why.

f. The *domain* of a function is the set of all acceptable input values. In the case of a motion-detector generated function the domain is the time interval during which the detector is collecting data.

(1) Find a value of  $t$  that is in the domain of the second function, which you created. State the value and mark its location on the horizontal axis of the sketch that you made in part 1.

(2) Give a value of  $t$  that is not in the domain of the second function.

(3) Find the domain of the second function.

(4) Describe in general how you can determine the domain of a function by looking at its graph.

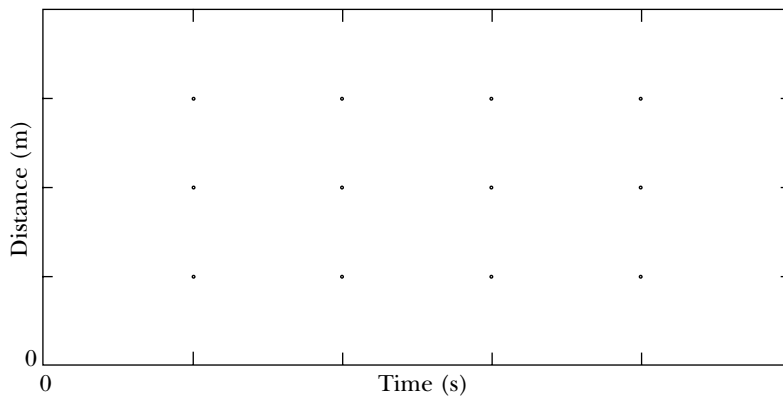
3. Repeat the process, but this time create two linear graphs while decreasing your distance from the detector.

a. Create a graph by decreasing your distance while walking very slowly. Start at the 2.5-meter mark. Decrease your distance from the detector walking at a *very* slow, constant pace, for 5 seconds.

b. Save the data for your first graph.

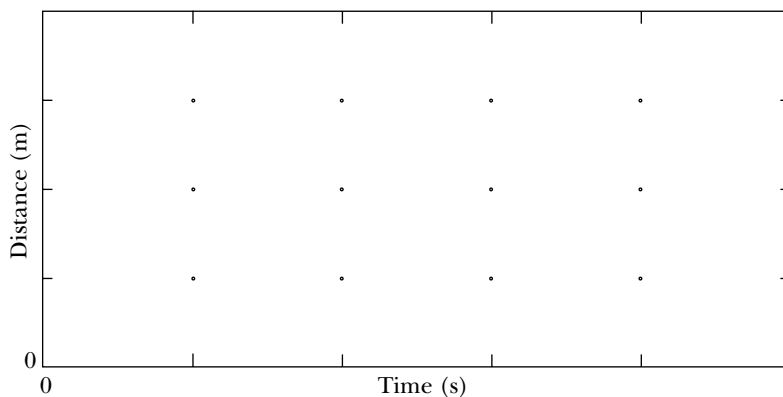
c. Create a graph by decreasing your distance while walking a little faster than you did with the first graph. Start at the 2.5-meter mark. Decrease your distance from the detector walking at a slightly faster, constant pace.

d. Sketch the graphs of the functions on the axes given below. Label the axes.



4. Consider the graphs of the linear functions that you created by decreasing your distance from the detector in part 3.
  - a. Consider the graph of the first function.
    - (1) Find an equation which represents your function by fitting a line to your graph.
  
    - (2) Use *your* equation to approximate your distance from the detector when  $t$  is one second.
  
    - (3) Find the domain of the function.
  - b. Find the slope of the line that best fits the graph of the second function which you created by walking a little more rapidly.
  - c. Describe how the rate at which you walk affects the value of the slope of the function modeling your motion.
  - d. When you decrease your distance from the detector, the slope of the associated graph, and hence your rate of change or velocity, is always negative. Explain why.
  - e. The *range* of a function is the set containing all the possible output values of the function.
    - (1) Find the range of the first function you created.
  
    - (2) Find the range of the second function.
  
    - (3) Describe how you can determine the range of a function by looking at its graph.

5. Investigate what happens when you stand still in front of the detector—that is, when you neither increase nor decrease your distance.
- a. Stand 1.5 meters from the detector for 5 seconds. Sketch the associated distance versus time graph.



- b. What is your rate of change in this case?
- c. Model your motion with an equation.

When you created each of the graphs in the last task, your rate of change was constant. Consequently, the graphs representing your motion could be modeled by a line. What happens if you vary your rate of change? How does this affect the shape of the associated graph? In the next section, you will explore the answers to these questions when your rate of change varies continuously, for example, when you walk faster and faster and faster. First examine the situation where you move at one constant rate of change, followed by another constant rate, followed by another. In this case, the graph representing your motion consists of a bunch of line segments. The graph is said to be *piecewise-linear*, since each piece of the graph is itself a line. In the next task, you will convert among verbal, graphic and symbolic representations of piecewise-linear functions, and use them to model situations.

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### Task 1-5: Examining Piecewise-Linear Functions

1. Convert from a verbal representation to a graphical representation and model the function.
- a. Predict the graph that would be produced if you were to start at the 1-meter mark, increase your distance from the detector walking at a slow constant pace for 4 seconds, stop for 4 seconds, and then decrease your distance walking at a rapid constant pace for 2 seconds. Use a dotted line to draw your prediction on the first pair of axes given on the next page.

**Task 1-2: Describing a Process for Finding the Position at a Given Time**

1. What assumptions do you have to make about the nature of the motion of the horse’s nose in order to make a reasonable estimate of its position (distance from the marker pole) at a time when a photograph was not made?
  
2. Is it possible for the horse’s nose to be at two different positions at a single time?
  
3. Estimate the position of the horse’s nose at 0.12 seconds and at 0.38 seconds.
  - a. Enter your estimates in the table below.

Time (seconds)	Position (centimeters)
0.12	
0.38	

- b. In one or two sentences, describe the process that you used to obtain your estimates in part a.
  
4. Suppose you wanted to estimate the position of the horse’s nose at any time between 0.00 and 0.60 seconds. Describe the process you would use to obtain this estimation.



At any given time between 0 and 0.6 seconds, the horse's nose is at one and only one position (with respect to the marker pole). Moreover, you can describe a process for determining the position corresponding to a particular time. Consequently, you say that there is a *functional relationship* between position and time or that position is a *function of time*. The important observation is that corresponding to each input (time), there is exactly one output (position).

Apply the idea of a function to another situation. Suppose you have collected \$10 to buy pizza for your hungry friends. An 8" pizza costs \$5 and a 12" pizza costs \$10. Would it be better to buy two 8" pizzas for the gang or one 12" pizza? Is area a function of diameter in the mathematical sense? In the following task, you will find a function that *models* this situation and then use the function to determine which combination gives you more for your money.

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### Task 1-3: Using the Concept of Function to Buy Pizza

1. Assuming you have collected \$10 to buy pizza and that an 8" pizza costs \$5 and a 12" pizza costs \$10, offhand what do you think will give you the most pizza to eat? One 12" pizza or two 8" pizzas?
2. Use a formula to represent the functional relationship between the area  $A$  of a circle and its diameter  $d$ .
3. Give a verbal description of a process you can use to find the area of a pizza if you know its diameter.
4. In a functional relationship there is exactly one output corresponding to each input.
  - a. Give the input quantity in the model for the pizza dilemma.

## SECTION 3

## Analyzing Smooth Curves

In the last section, you created linear functions by walking at a constant pace—that is, by keeping your rate of change constant. You observed that your rate of change, or velocity, corresponded to the slope of the line which modeled your movement. When you increased your distance from the detector, your rate of change was positive and you created a line with a positive slope. Conversely, when you decreased your distance, your rate of change was negative and you created a line with a negative slope.

In this section, you will look at functions whose graphs bend and wiggle, but which do not have any sharp corners or jumps. These graphs are called *smooth curves*. As you probably guessed, you can create a smooth curve by varying your rate of change—for instance, by walking faster and faster or by walking back and forth in front of the detector. The questions in which you are interested are: How does varying your rate of change affect the shape of the curve? How do you move to create a curve that opens up, opens down, or turns? How can you measure your rate of change at a given instant?

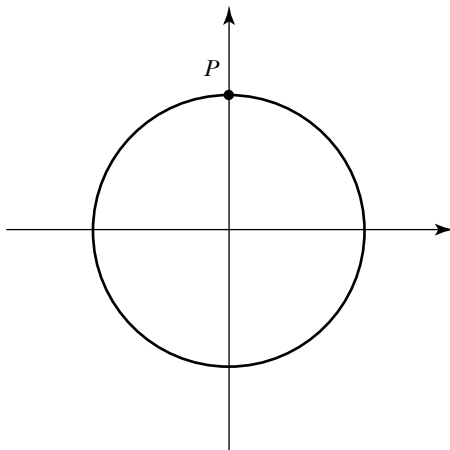
One way to answer these questions is to analyze the behavior of the *tangent line* to a curve. In the next task, you will develop an intuitive understanding of what a tangent line to a curve is. In subsequent tasks, you will use the motion detector to explore the relationship between the tangent line and a smooth curve.

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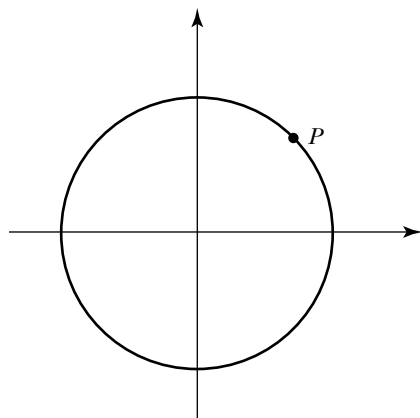
**Task 1-6: Developing an Intuitive Understanding of a Tangent Line to a Curve**

1. Begin by thinking about a familiar situation: a tangent line to a circle. A *tangent line to a point on a circle* is the line that touches the circle at exactly that one point, even though the line extends indefinitely in both directions.
  - a. Use a straightedge to sketch the tangent line to each of the circles given below at the point  $P$ .

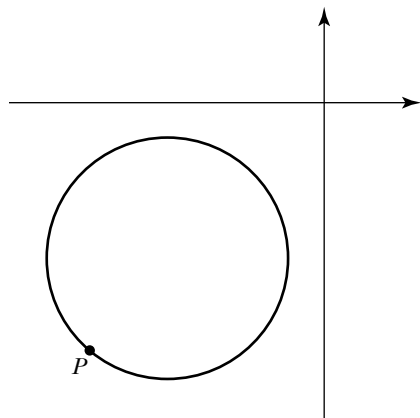
(1)



(2)

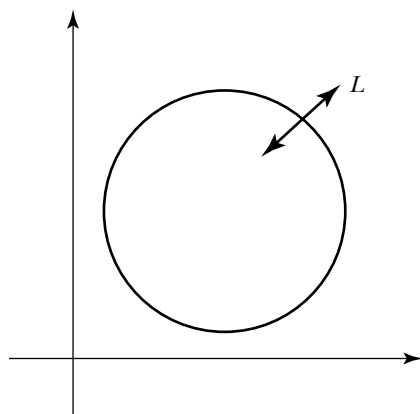


(3)

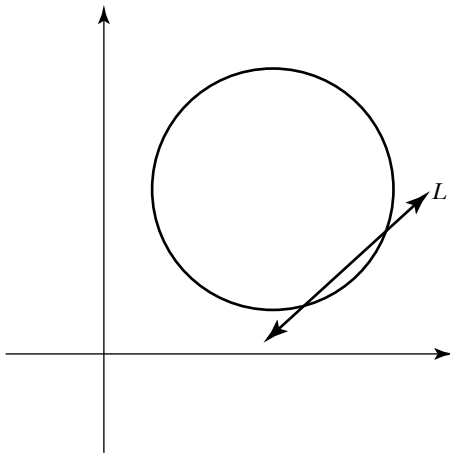


b. In each of the following diagrams, explain why the line  $L$  is *not* a tangent line to the given circle.

(1)



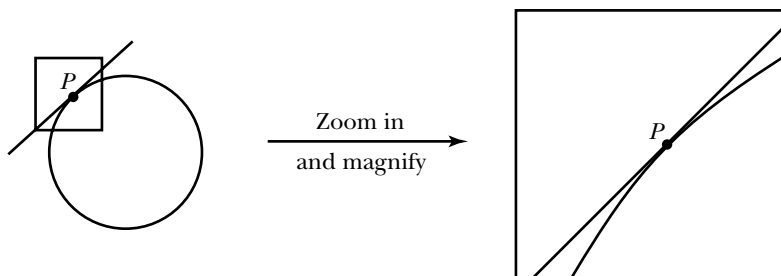
(2)



c. Describe a way of constructing the tangent line to a circle at a given point.

d. Another way to identify the tangent line at a point on a circle is to repeatedly “zoom in” and magnify a small region containing the point. As you get closer and closer, the tangent line and the part of the circle near the point merge.

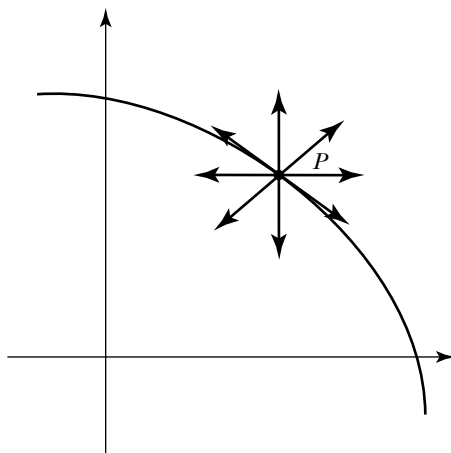
Convince yourself that this is the case by repeating the following zoom-in-and-magnify process: Draw a small box around  $P$ , discard the portion of the graph outside the small box, and then copy the contents of the box into a larger box nearby. The diagram given below illustrates the process for the first time. Repeat this zoom-in-and-magnify process two more times.



2. In part 1 you examined the idea of a tangent line to a circle. Next extend this notion to a smooth curve whose graph bends and wiggles.

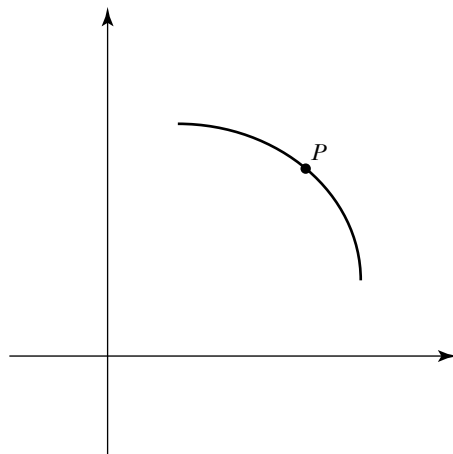
One way to find the tangent line at a given point  $P$  on a smooth curve is to repeatedly zoom in and magnify the portion of the curve containing  $P$ . As you get closer and closer, the graph of the smooth curve will “straighten out” and merge with the graph of the tangent line to the curve at  $P$ . This, of course, is exactly what happens in the case of the tangent line to a circle.

- a. Identify the tangent line to the graph of the function at the point  $P$  in the diagram below. Use a zoom-in-and-magnify argument to justify your choice.



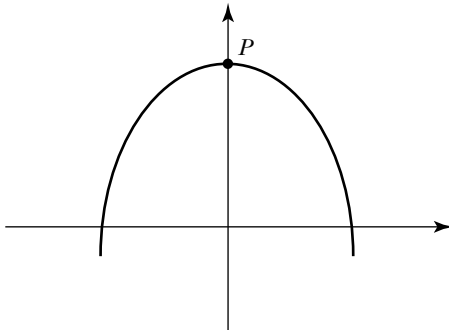
- b. Carefully sketch the tangent line to each of the smooth curves given below at the point  $P$ . In each case indicate the *sign* of the slope of the tangent line—that is, indicate whether the value of the slope is a positive number (+), a negative number (−), zero (0), or undefined.

(1)



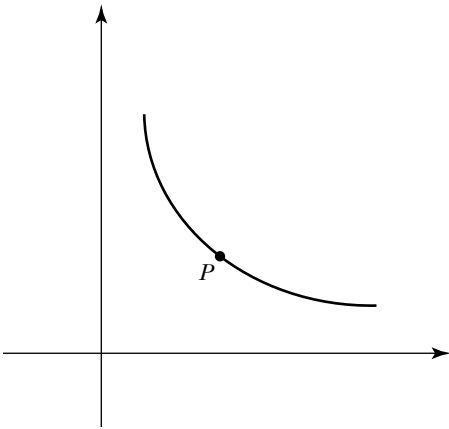
Sign of the slope of the tangent line at  $P$ :

(2)



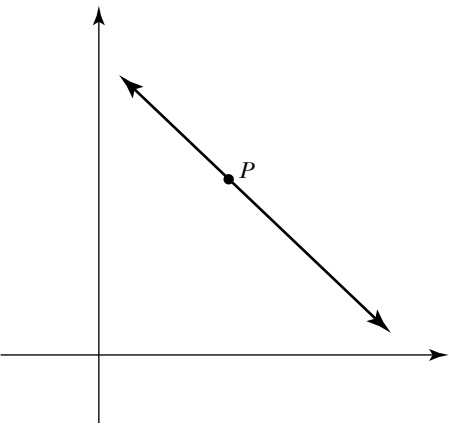
Sign of the slope of the tangent line at  $P$ :

(3)



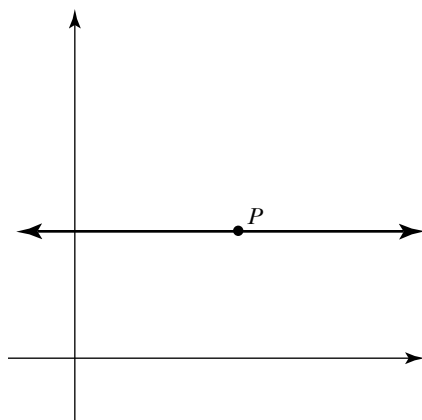
Sign of the slope of the tangent line at  $P$ :

(4)



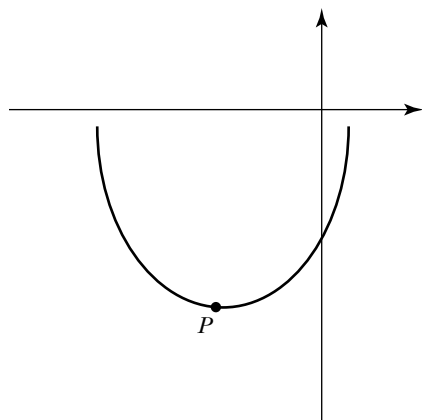
Sign of the slope of the tangent line at  $P$ :

(5)



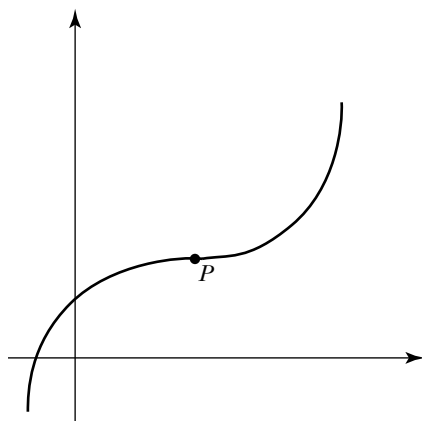
Sign of the slope of the tangent line at  $P$ :

(6)



Sign of the slope of the tangent line at  $P$ :

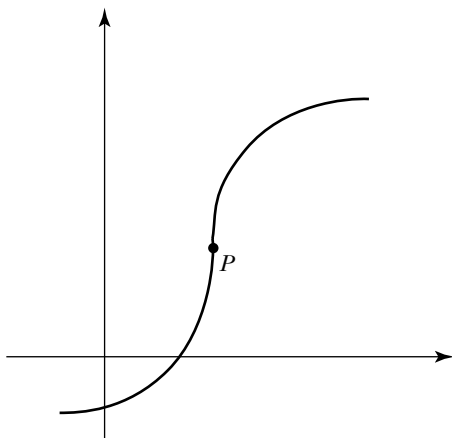
(7)



Sign of the slope of the tangent line at  $P$ :

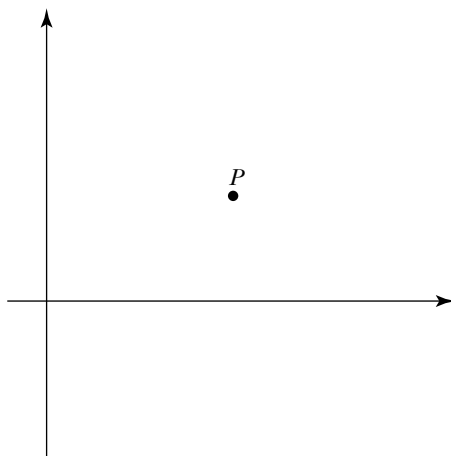


(8)



Sign of the slope of the tangent line at  $P$ :

- c. The tangent line at a point  $P$  on a circle intersects the circle at exactly one point, namely  $P$ . The tangent line to a smooth curve, however, may intersect the curve at more than one point. Illustrate this fact with a sketch; that is, draw a smooth curve that passes through  $P$ , where the tangent line at  $P$  intersects the curve at at least one point other than  $P$ .



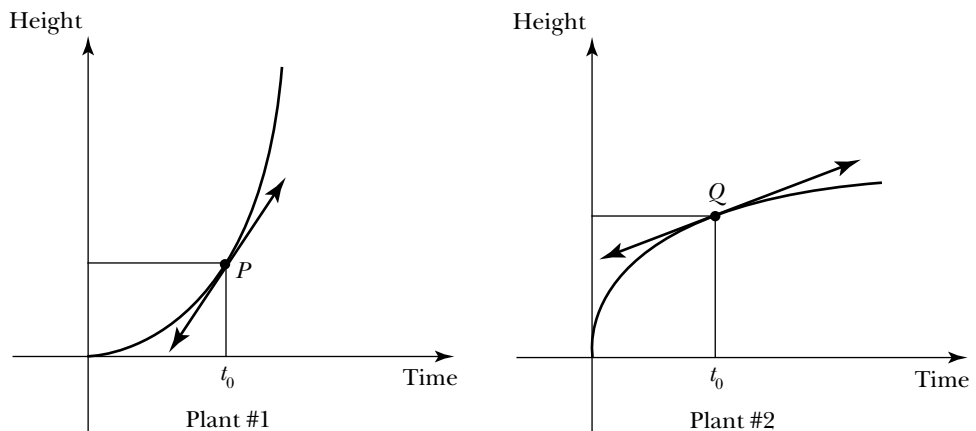
- d. If a tangent line exists at a point then it is unique—that is, there is one and only one. Use a zoom-in-and-magnify argument to explain why this is a reasonable claim.

e. Tangent lines do not exist at sharp peaks and dips.



Zoom in and magnify a tip of the dude's hair. What happens in this case? Based on your observations and on the properties of tangent lines, explain why it is reasonable to conclude that tangent lines do not exist at sharp peaks and dips.

The slope of the tangent line to a smooth curve gives the *instantaneous rate of change* of the function at the point  $P$ . This is a very important value, since it indicates the rate at which the function is changing. For example, suppose you planted two different types of plants, Plant #1 and Plant #2, and graphed their heights over a given time period using the same scales for each pair of axes.



The slope of the tangent line to each point on the curve indicates how fast the plant is growing at that time. Look closely at what happens at time  $t_0$ . Plant #1 is growing more rapidly than Plant #2, since the slope of its tangent line, and hence its rate of change, is greater. On the other hand, it is interesting to note that even though Plant #1 is growing faster, it is shorter than Plant #2 at time  $t_0$ . (You will have a chance to explain what is happening here in the homework exercises.)

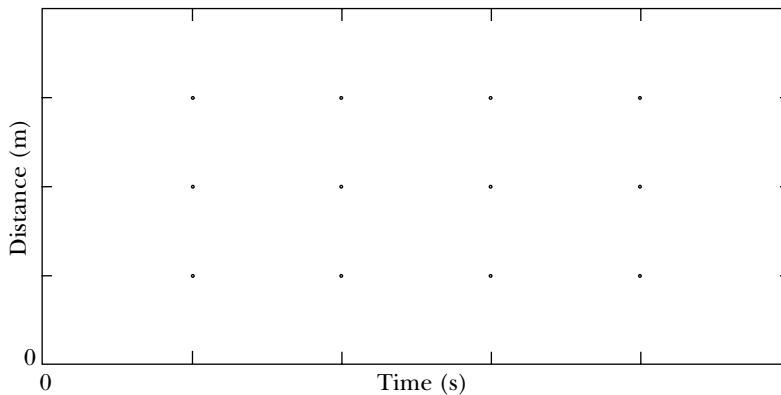
Being able to calculate the rate of change of a function at a given point is very useful. There's one big problem, however. You do not know how to find the actual value of the slope of the tangent line to a smooth curve! In

order to use the slope formula which you learned in algebra, you need two points, and in the case of a tangent line to a curve, you have only one, namely  $P$ . The good news is that calculus will come to your rescue and provide you with a way of doing this. In the meantime, you can use your intuitive understanding of tangent line which you developed in the last task. You know the circumstances under which it exists. You know what it looks like, and given a point, you can sketch a reasonable approximation. The motion detector can help you develop a feel for what rate of change means, since its value depends on how fast you are moving and its sign depends on whether you are increasing or decreasing your distance from the detector. In the following task, you will use the motion detector to discover some of the relationships between the shape of a graph and the behavior of the tangent line as it moves along the curve.

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### Task 1-7: Investigating the Behavior of the Tangent Line Near a Turning Point

1. Create a smooth curve by walking back and forth in front of the motion detector for 10 seconds. Change your direction at least two times. Sketch your graph below. Label the axes.

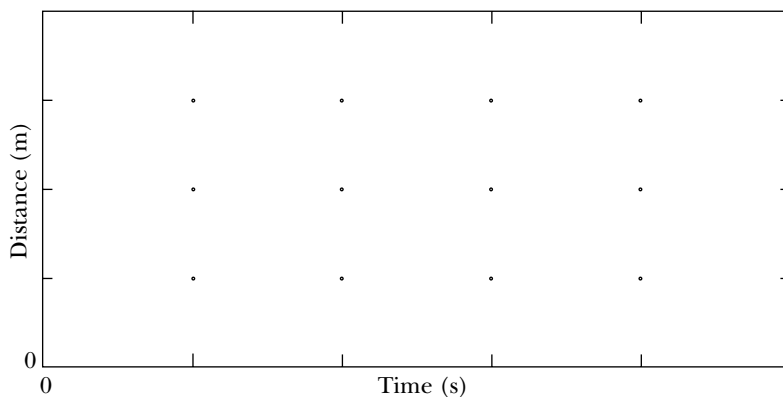


2. Analyze the shape of your function.

Quit the program. View the graph in the graphing window of your calculator. Use the TRACE feature to help answer the following questions.

- a. Consider the time intervals where your function is increasing.
  - (1) Below the horizontal axis on your graph, label each interval where your function is increasing.
  - (2) Express each interval using open interval notation.

5. Investigate what happens when you stand still in front of the detector—that is, when you neither increase nor decrease your distance.
  - a. Stand 1.5 meters from the detector for 5 seconds. Sketch the associated distance versus time graph.



- b. What is your rate of change in this case?
- c. Model your motion with an equation.

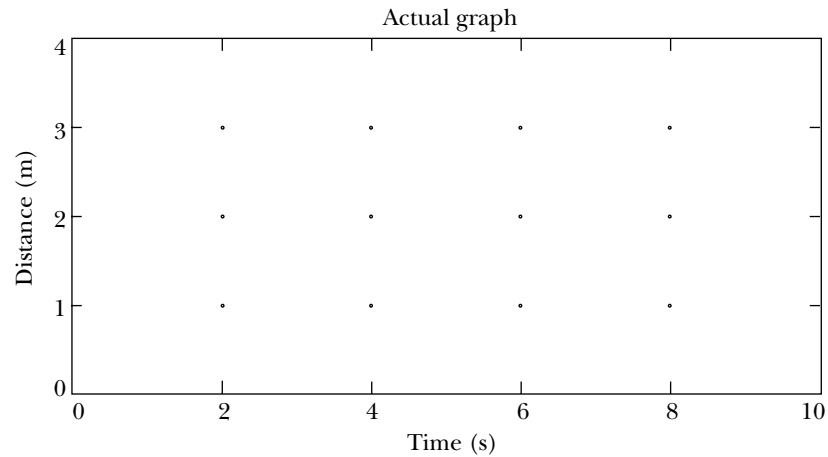
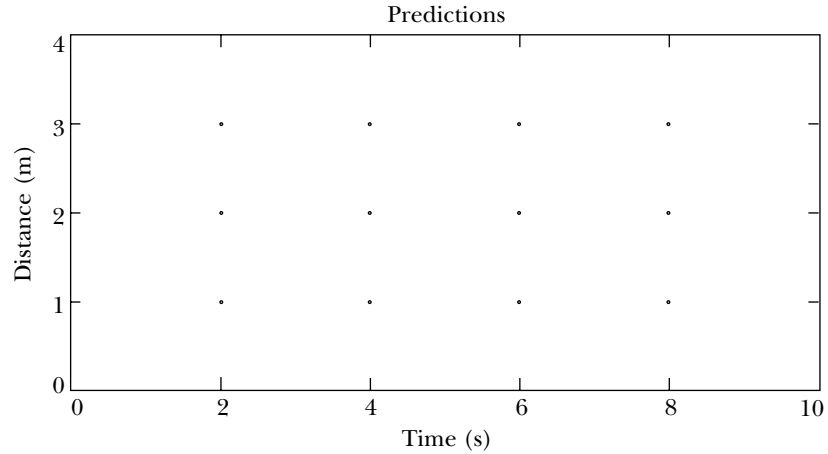
When you created each of the graphs in the last task, your rate of change was constant. Consequently, the graphs representing your motion could be modeled by a line. What happens if you vary your rate of change? How does this affect the shape of the associated graph? In the next section, you will explore the answers to these questions when your rate of change varies continuously, for example, when you walk faster and faster and faster. First examine the situation where you move at one constant rate of change, followed by another constant rate, followed by another. In this case, the graph representing your motion consists of a bunch of line segments. The graph is said to be *piecewise-linear*, since each piece of the graph is itself a line. In the next task, you will convert among verbal, graphic and symbolic representations of piecewise-linear functions, and use them to model situations.

---

### Task 1-5: Examining Piecewise-Linear Functions

1. Convert from a verbal representation to a graphical representation and model the function.
  - a. Predict the graph that would be produced if you were to start at the 1-meter mark, increase your distance from the detector walking at a slow constant pace for 4 seconds, stop for 4 seconds, and then decrease your distance walking at a rapid constant pace for 2 seconds. Use a dotted line to draw your prediction on the first pair of axes given on the next page.

- b. Compare predictions with the rest of your group. See if you can all agree. Use a solid line to draw your group’s prediction on the first pair of axes. (Do not erase your original prediction.)
- c. Try it. Move in the described way and graph your motion. When you are satisfied with your group’s final result, sketch it on the second pair of axes given below.



- d. Does your prediction have the same graph as the final result? If not, describe how you would move to make a graph that looks like your prediction.

- e. For each of the ways that you moved, find an expression that represents the function during that particular time period by fitting a line to the graph. That is:

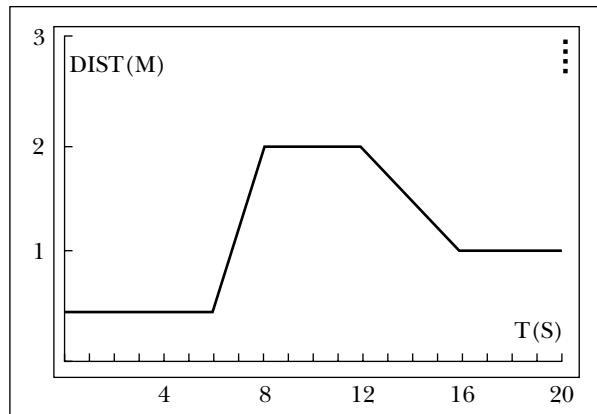
(1) Find an expression that models your movement when you started at the 1-meter mark and increased your distance from the de-

tector by walking slowly and steadily for 4 seconds. State the domain for this piece of your function.

- (2) Find an expression that models your movement when you stopped for 4 seconds. State the domain for this piece of your function.
  
  
  
  
  
  
  
  
  
  
- (3) Find an expression that models your movement when you decreased your distance by walking quickly for 2 seconds. State the domain for this piece of your function.
  
  
  
  
  
  
  
  
  
  
- (4) Represent your function by a *piecewise-defined* function, by combining your expressions in parts (1)–(3) using “curly bracket” notation.

2. Match a given distance versus time graph.

- a. Open the program DISMATCH. The distance versus time graph shown below will appear on your calculator screen.



- b. Try to duplicate the **Distance Match** graph. Sketch your best match on the axes given above.
- c. Complete the following verbal description of the motion depicted by the **Distance Match** function.

First you stand still at a distance of \_\_\_\_\_ meters for \_\_\_\_\_ seconds. Then you ...

- d. During which time interval is the rate of change the greatest?
- e. Analyze the shape of the **Distance Match** function. Express your conclusions using open interval notation.
- (1) Find all the time intervals (if any) where the **Distance Match** function is *increasing*—that is, find the intervals where you increased your distance from the detector as you tried to duplicate the graph.
  - (2) Find all the time intervals (if any) where the **Distance Match** function is *decreasing*—that is, where you decreased your distance from the detector.

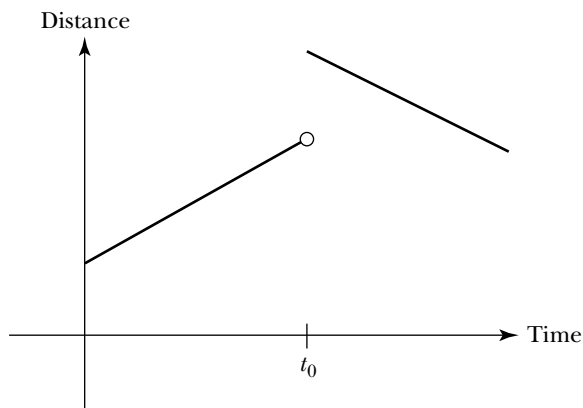


(3) Find all the time intervals (if any) where the **Distance Match** function is *constant* — that is, where your distance from the detector was constant.

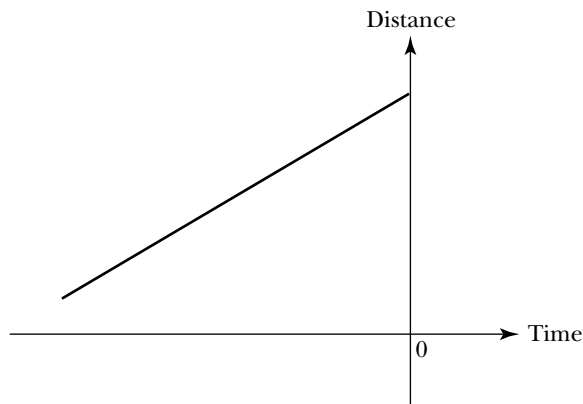
f. Use “curly bracket” notation to represent the **Distance Match** function by a piecewise-defined function.

3. Explain why a person cannot replicate each of the following graphs by walking in front of the motion detector.

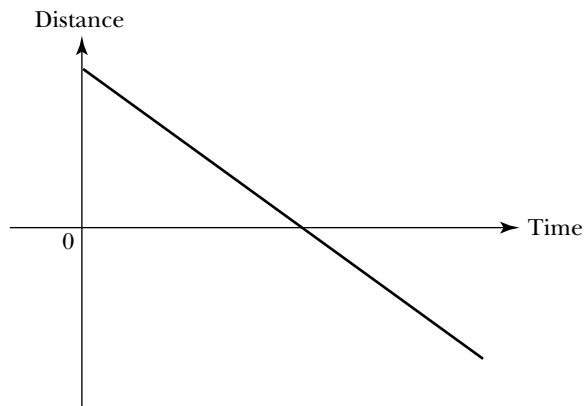
a.



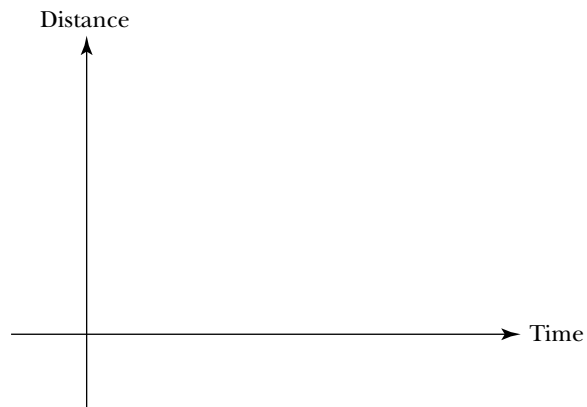
b.



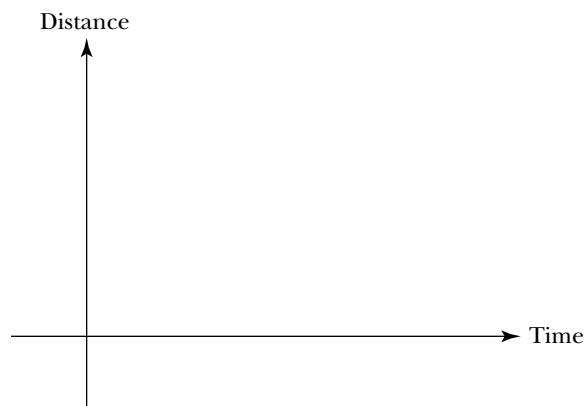
c.



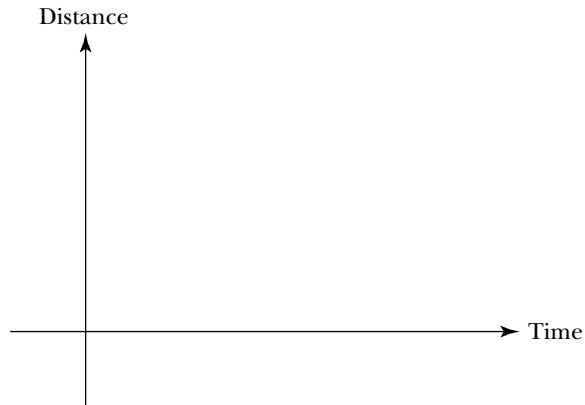
4. For each of the following scenarios, sketch a distance versus time graph that represents the situation. Label the time of specific events, such as “stopped to tie shoe,” “stepped in puddle,” “ran to class,” on the horizontal axis. Label specific locations, such as “leave dorm,” “arrive at class,” along the vertical axis.
- a. You left your dorm and started walking slowly to class when you realized that you had forgotten your books, so you ran back to your dorm, grabbed your books and sprinted to class.



- b. As you were walking slowly to lunch, you realized that you were late for your meeting with Christa so you started to run.

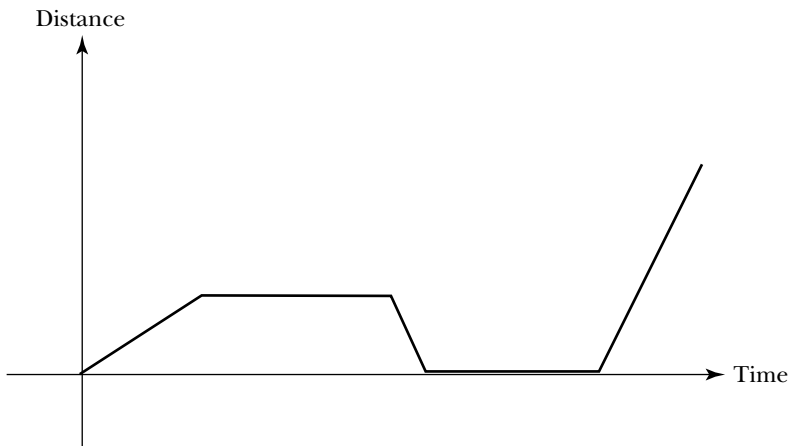


- c. On your way to soccer practice you ran into Greta. You stopped to chat before continuing on to the gym.

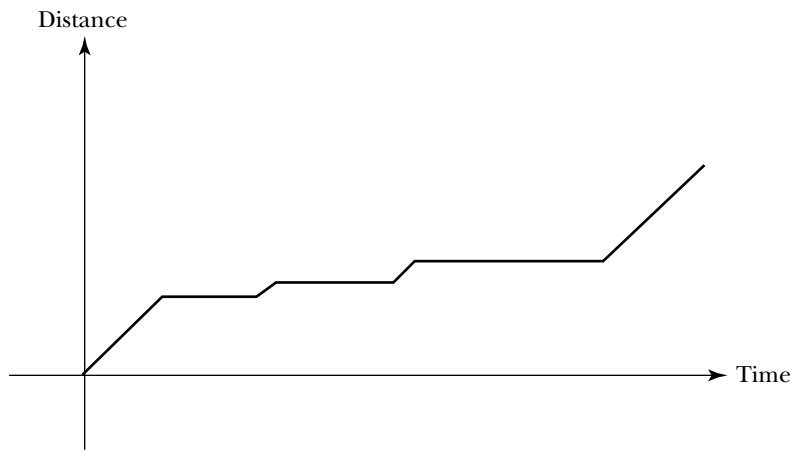


5. For each of the following distance versus time graphs, describe a situation that the graph might represent.

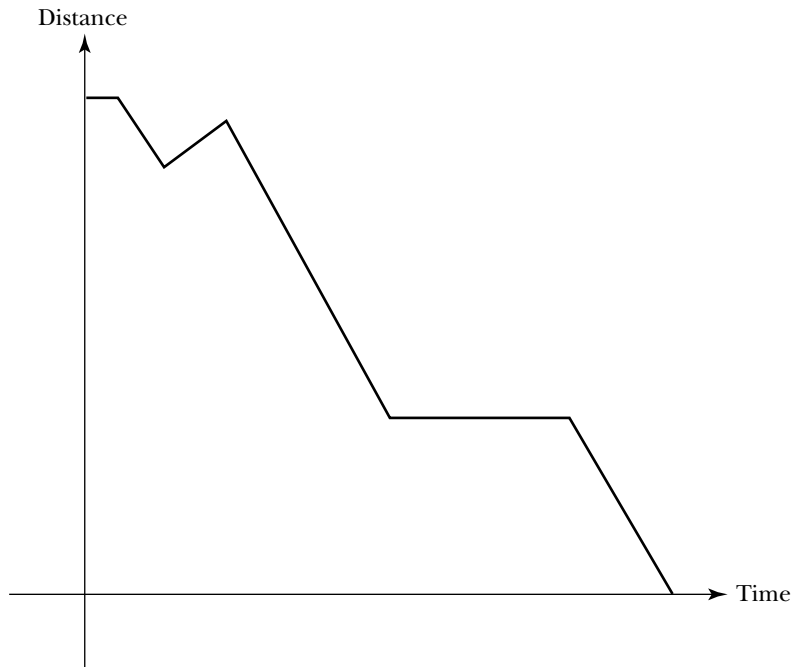
a.



b.



c.



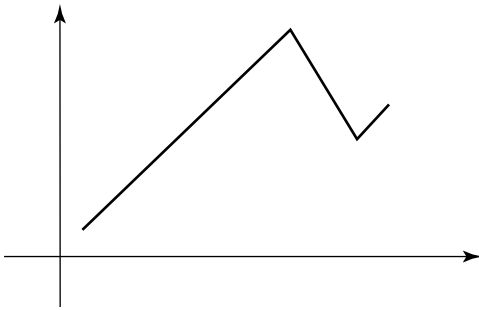
**Unit 1 Homework After Section 2**

- Complete the tasks in Section 2. Be prepared to discuss them in class.

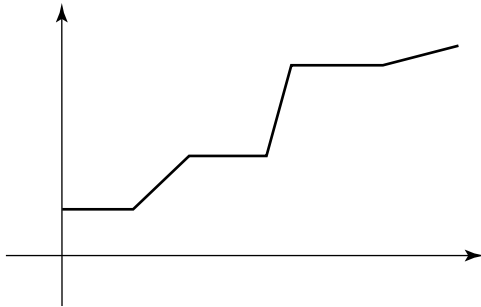
**HW1.2** Analyze the shape of some functions.

For each of the following graphs, identify where the graph is increasing, decreasing, and constant. Note: Label the appropriate points on the horizontal axis  $a$ ,  $b$ ,  $c$ , and so on. Use these labels to express your answers using open interval notation.

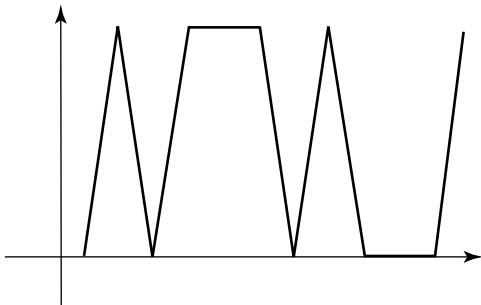
1.



2.



3.



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**HW1.3** Model some situations where you know the exact rate of change.

1. You start at the 1-meter mark. For 3 seconds you increase your distance from the detector at the rate of 2 meters per second. You stand still for 4 seconds.
  - a. Draw a precise distance versus time graph corresponding to this situation. Carefully label the axes.
  - b. Model your motion by a piecewise-defined function.
  - c. Fill in the following table.

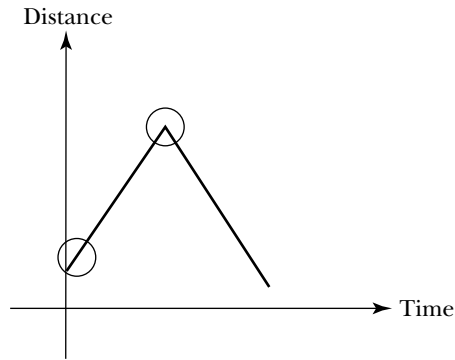
Time	0	1	2	2.5	3	5.5	7
Distance							

2. Starting at the 10-meter mark you walk at a constant rate of  $-2$  meters per second. In other words, you decrease your distance by 2 meters per second.
  - a. What is your distance when  $t = 3$  seconds?
  - b. At what time are you 1 meter from the detector?
  - c. How many seconds must you walk before you reach the detector?
  - d. Model your motion with a linear expression.
  - e. In this example the rate of change is negative. What does this tell you about the behavior of the graph of the associated distance versus time graph?

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**HW1.4** Explain what really happens

Actually, we have been a little sloppy with our graphs. It is not possible to duplicate exactly a graph such as the one given on page 27, by walking in front of the motion detector. There are two places where we have difficulty: the initial portion and the corner (peak), both of which are circled. Explain what actually happens at these places when you try to duplicate them by walking in front of the motion detector. Make a more realistic sketch.




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**HW1.5** Examine the concept of independent and dependent variables.

When creating a distance versus time graph, your distance is determined by the value of the time—that is, your distance “depends on” the time or your distance is a “function of” time. Consequently, distance is called the *dependent* variable, whereas time is called the *independent* variable. In general, the quantity which varies freely is the independent variable, whereas the quantity that depends on the independent variable is the dependent variable.

For each of the following functional descriptions:

- i. Identify the two quantities that vary. Give each quantity a variable name and indicate the unit of measurement for the variable. For example, time (seconds) or speed (miles/hour).
  - ii. Determine which quantity is the independent variable and which is the dependent variable.
  - iii. Illustrate the basic relationship between the variables by sketching the general shape of the graph. Label each axis with its variable name. Note: It is customary to place the independent variable on the horizontal axis and the dependent variable on the vertical axis.
1. The temperature rises from 6 A.M. until 6 P.M. and then falls from 6 P.M. to 6 A.M.
  2. The number of bacteria in the petri dish doubles every 2 hours.
  3. Dan works at his office 12 hours on Monday and Friday, 4 hours on Tuesday and Thursday, and 8 hours on Wednesday. He does not work at his office over the weekend.
  4. The water continually rises until high tide occurs and then falls until low tide occurs.
  5. The higher the cost of an item, the fewer people are willing to buy.
  6. As you increase the rate at which you do push-ups, your heart rate increases.

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**HW1.6** In your past study of mathematics, you may have used the “Vertical Line Test” to determine if a given graph represents a function. This test claims:

A graph represents a function if and only if every vertical line in the  $xy$ -plane intersects the graph in at most one point.

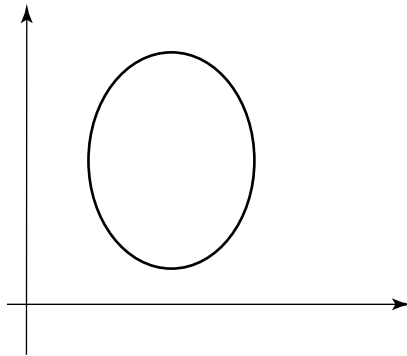
Explain why this test makes sense. State your explanation in terms of “inputs” and “outputs,” keeping in mind that for each output a function returns exactly one input. No more. No less. Support your explanation with diagrams of two curves, one which represents a function and one which does not.

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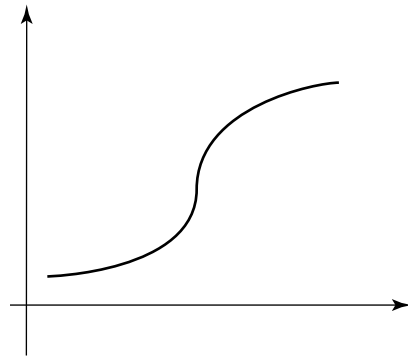
**HW1.7** For each of the following graphs, use the Vertical Line Test to determine if the graph represents a function. If the graph fails the test, draw a vertical line which shows that the graph does not have one output for each input.

*Note: An open circle indicates a “hole” in the graph, whereas a closed circle indicates a point on the graph.*

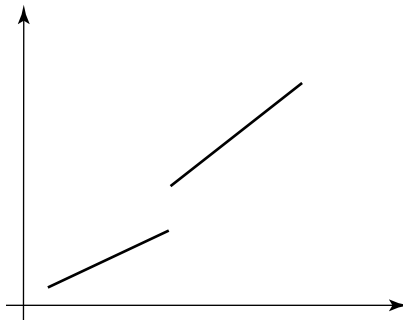
1.



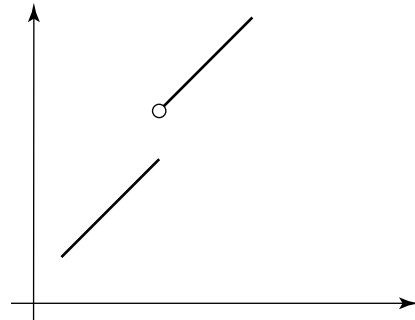
2.



3.

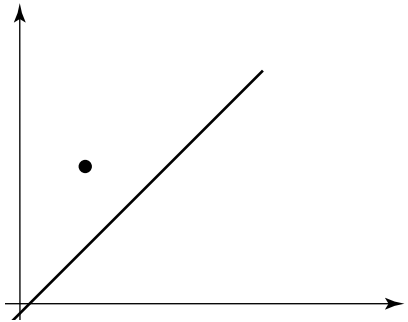


4.

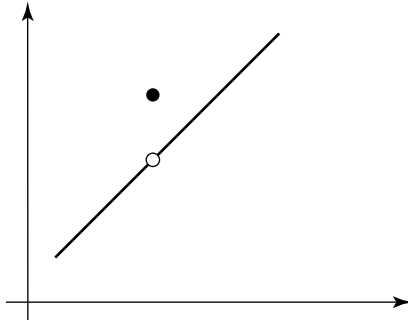




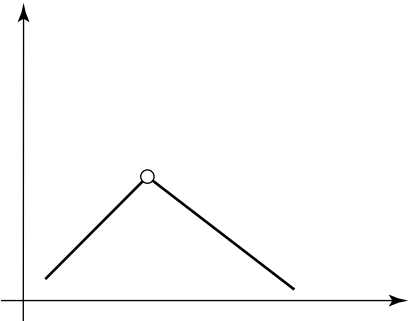
5.



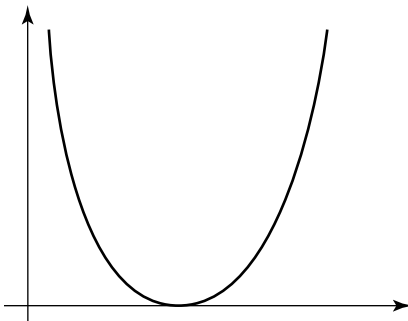
6.



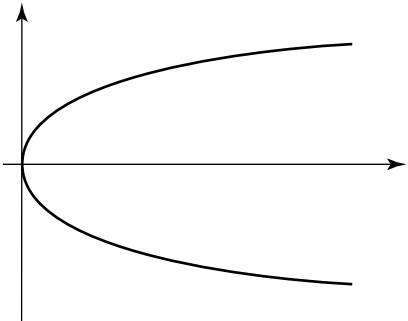
7.



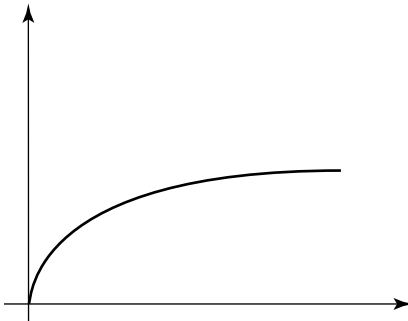
8.



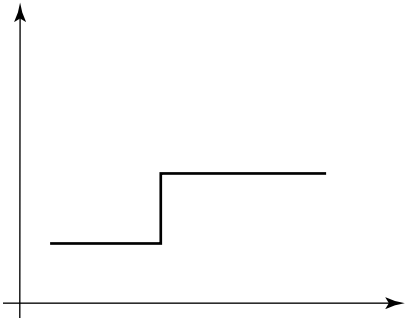
9.



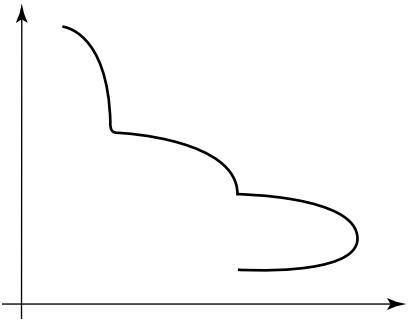
10.



11.



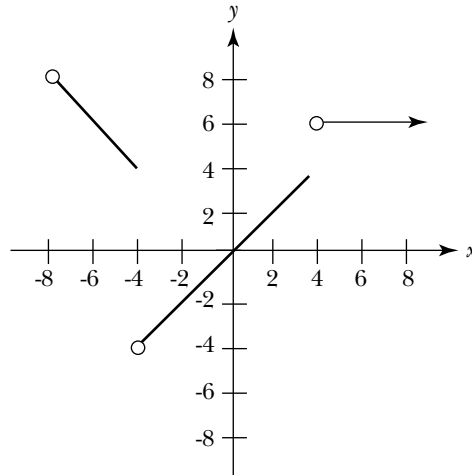
12.



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**HW1.8** Examine some piecewise-linear functions that have “jumps.”

1. Represent the following function using curly bracket notation. Be careful at the endpoints of the subdomains.



2. Sketch the following function. Use an open circle to denote an endpoint which is not on the graph.

$$s(t) = \begin{cases} -t + 3, & \text{if } -1 \leq t \leq 1 \\ t + 3, & \text{if } 1 < t < 4 \\ t, & \text{if } 4 \leq t < 10 \end{cases}$$

---

**HW1.9** Try to match some other piecewise-linear distance versus time graphs.

Connect a motion detector to your calculator and run the program DTMATCH<sup>2</sup> several times. For each function generated by your calculator:

- i. Try to match the graph.
- ii. Make a sketch of the graph.
- iii. List the intervals where the function is increasing, where it is decreasing, and where it is constant.
- iv. Represent the function using curly bracket notation.

---

<sup>2</sup>See CBL System Experiment Workbook, published by Texas Instruments Incorporated, 1994, p. 96.

Consequently, the point  $P(4, -5)$  lies on the graph of  $f$ . Moreover, since the value of  $f(4)$  is negative,  $P$  lies below the horizontal axis.

The *sign of  $f(x)$*  is the sign of the output value corresponding to  $x$ . For instance, the sign of  $f$  is negative ( $-$ ) when  $x = 4$ , since  $f(4) = -5 < 0$ . Similarly, the sign of  $f$  is positive ( $+$ ) when  $x = 6$ , since  $f(6) = 5 > 0$ . You can record this information on the sign chart for  $f$  as follows:



But  $x = 4$  and  $x = 6$  are only two of the values in the domain of  $f$ . What happens at the other values in the domain? What does the sign chart tell you about the location of the graph of  $f$ ? You will explore answers to these questions in the next task.

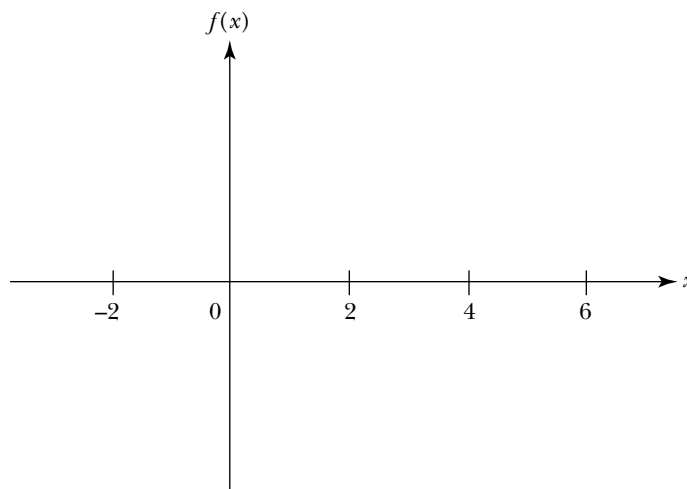
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### Task 1-9: Interpreting Sign Charts

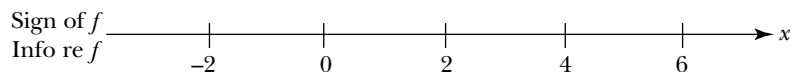
1. Create some sign charts.

a. Consider the function  $f(x) = 2x - 2$ .

(1) Sketch the graph of  $f$  on the pair of axes given below. Label the vertical axis.



- (2) Find all the values of  $x$  where  $f(x) < 0$ . In other words, find all the values of  $x$  where the graph of  $f$  dips below the horizontal axis.
  
- (3) Find all the values of  $x$  where  $f(x) = 0$ . In other words, find all the values of  $x$  where the graph of  $f$  touches the horizontal axis.
  
- (4) Find all the values of  $x$  where  $f(x) > 0$ . In other words, find all the values of  $x$  where the graph of  $f$  lies above the horizontal axis.
  
- (5) Fill in the sign chart for  $f$  and indicate what the information on the sign chart says about the behavior of the graph of  $f$  by completing steps i and ii given below.



*Note: We have used the same labeling for the horizontal axis in the graph of  $f$  and for the sign chart of  $f$ . Place the sign chart under the graph, and line up the labels. Doing this should help you see the connection between the chart and the graph.*

*Step i.* Fill in your observations about the sign of  $f$ . **Over** the axis for the sign chart:

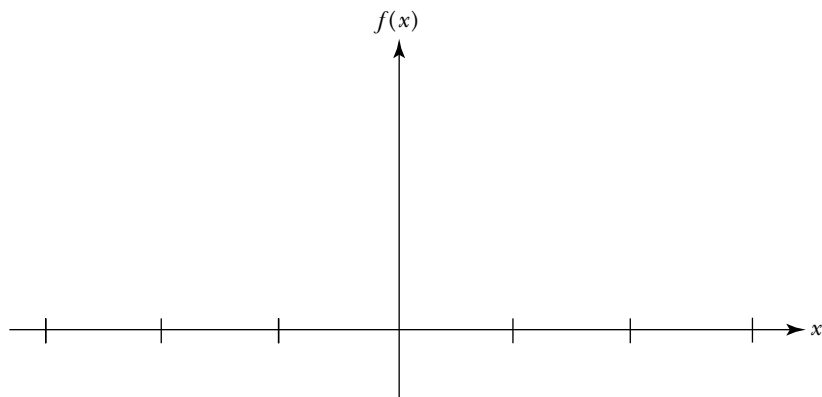
- Indicate the intervals in the domain where  $f(x) < 0$  by placing a string of “-”s over each interval.
- Indicate the values of  $x$  where  $f(x) = 0$  by placing a “0” over each value.
- Indicate the intervals in the domain where  $f(x) > 0$  by placing a string of “+”s over each interval.

*Step ii.* Fill in the information about (info re) the graph of  $f$ . **Beneath** the axis for the sign chart:

- Indicate the intervals in the domain where the graph of  $f$  dips below the horizontal axis by writing “below” under each interval.
- Indicate the values of  $x$  where the graph of  $f$  intersects the horizontal axis by writing “intersects” under each value.
- Indicate the intervals in the domain where the graph of  $f$  lies above the horizontal axis by writing “above” under each interval.

b. Consider the function  $f(x) = x^2$ .

- (1) Sketch a graph of  $f$  on the pair of axes given below for  $-3 \leq x \leq 3$ . Label the axes.



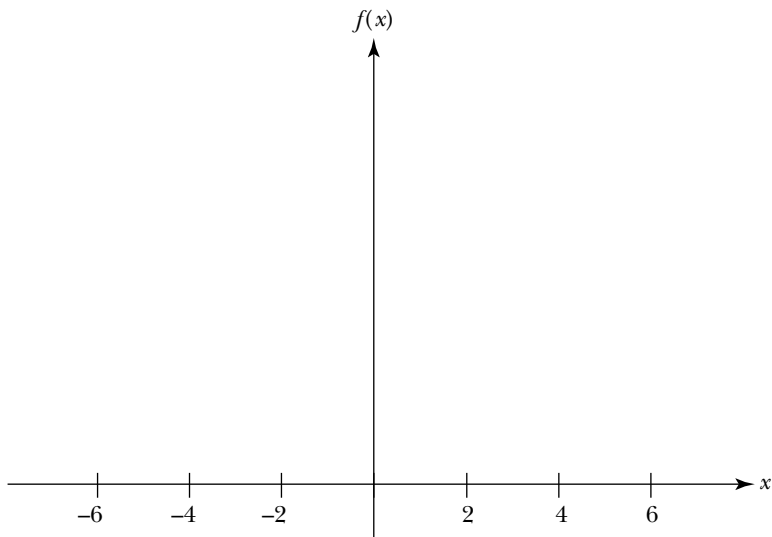
- (2) Label the  $x$  values on the sign chart given below. Fill in the sign chart for  $f$  and indicate what the information on the sign chart says about the behavior of the graph of  $f$ .



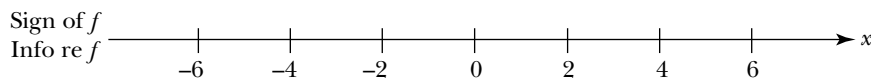
c. Consider the function

$$f(x) = \begin{cases} -1, & \text{if } x < -2 \\ x, & \text{if } -2 \leq x \leq 2 \\ 2, & \text{if } x > 2 \end{cases}$$

- (1) Sketch a graph of  $f$  on the pair of axes given below for  $-6 \leq x \leq 6$ . Label the vertical axis.



- (2) Label the  $x$  values on the sign chart given below. Fill in the sign chart for  $f$  and indicate what the information on the sign chart says about the behavior of the graph of  $f$ .

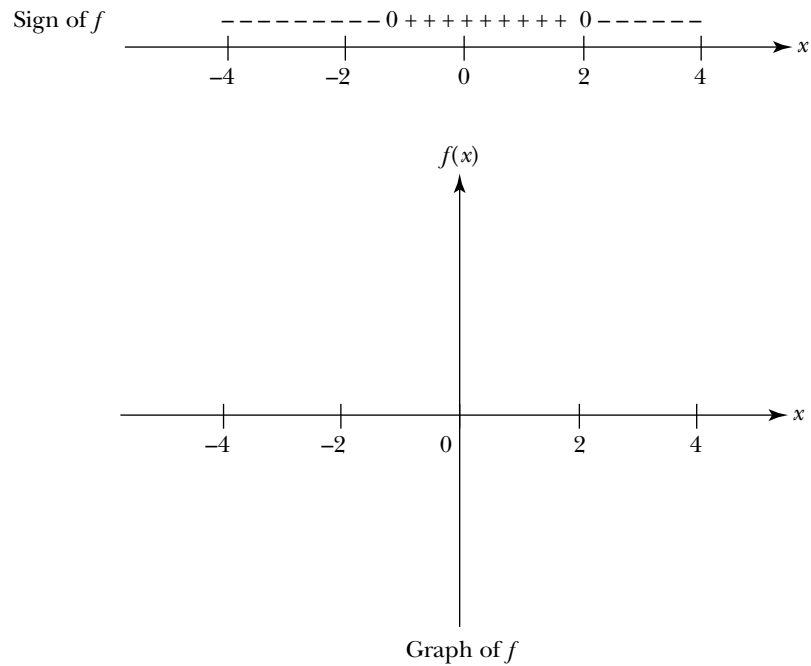


2. In the last part, you sketched the graph and then created the sign chart. In this part, reverse the process and use the chart (along with some other information) to sketch the graph.

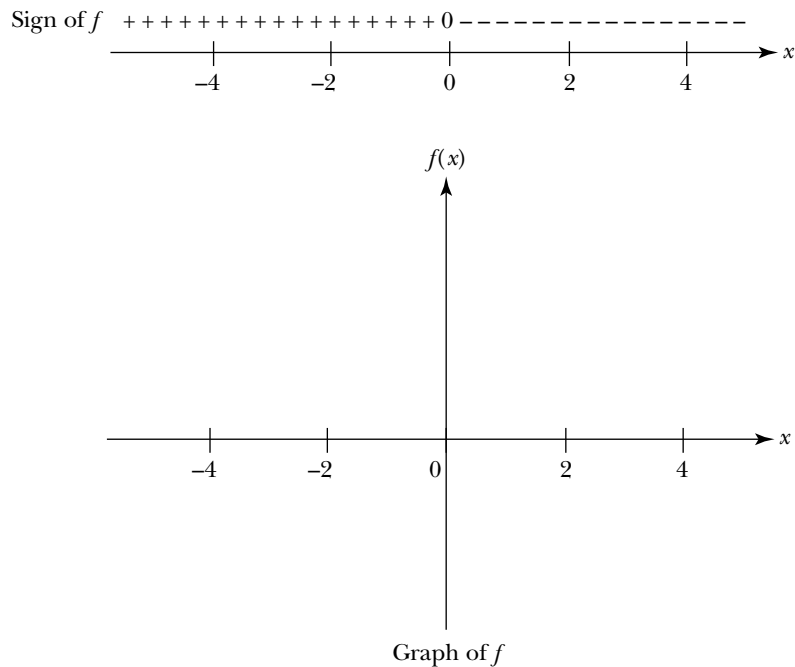
*Note: Before you sketch a graph, first indicate the information that the sign chart gives about the graph on the sign chart.*

For each list of conditions given below, sketch a graph of a function satisfying all the conditions in the list.

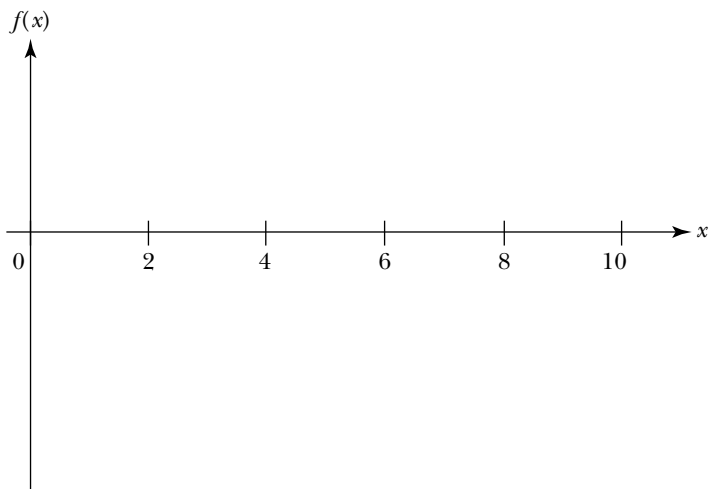
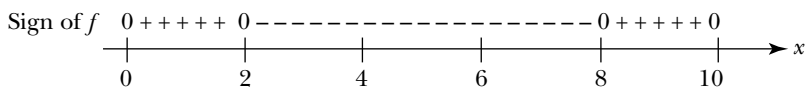
- a. The domain of  $f$  is all real numbers between  $-4$  and  $4$ .  
 $f$  has a local minimum at  $x = -2$ .  
 $f$  has local maxima at  $x = 1/2$  and  $x = -3$ .



- b. The domain of  $f$  is the set of all real numbers.  
 $f$  is increasing for  $-\infty < x < -2$  and  $2 < x < \infty$ .  
 $f$  is decreasing for  $-2 < x < 2$ .



- c. The domain of  $f$  is the set of all  $x$  where  $0 \leq x \leq 10$ .  
 $f$  has a turning point at each odd integer between 0 and 10.



Graph of  $f$

3. Summarize what is happening here by filling in the missing information in the table given below.

Sign of $f$	Graph of $f$	Sign chart
$f(x) < 0$	Below $x$ -axis	—
$f(x) = 0$		
	Above $x$ -axis	

Unit 1 Homework After Section 3

- Complete the tasks in Section 3. Be prepared to discuss them in class.

**HW1.10** Analyze some functions.

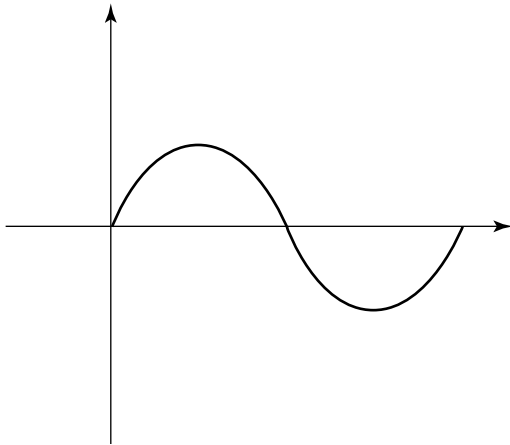
For each of the following functions:

- Describe the behavior of the function. Label important points on the horizontal axis— $a$ ,  $b$ ,  $c$ , and so on—and use these points to identify:

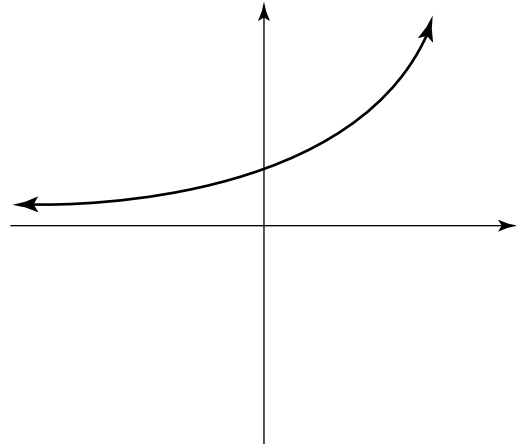


- a. The intervals where the function is increasing.
  - b. The intervals where the function is decreasing.
  - c. The local maxima.
  - d. The local minima.
  - e. The intervals where the function is concave up.
  - f. The intervals where the function is concave down.
  - g. The points of inflection.
- ii. Sketch the sign chart for the function.

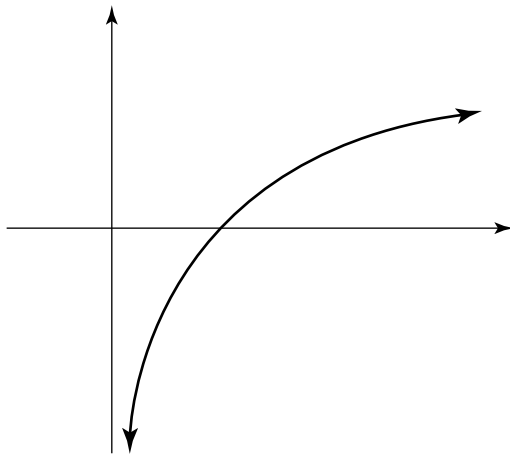
1.



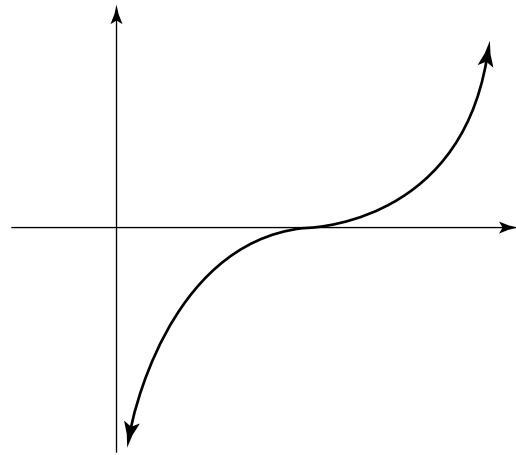
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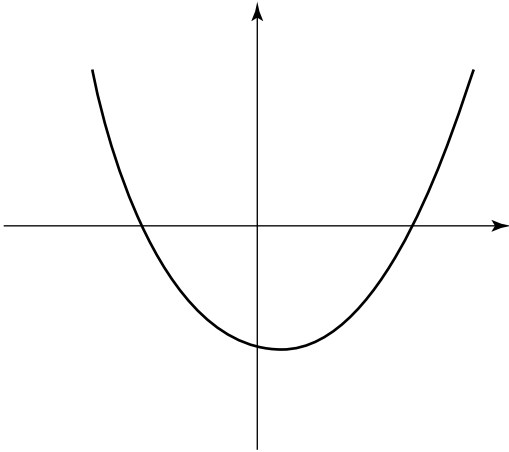
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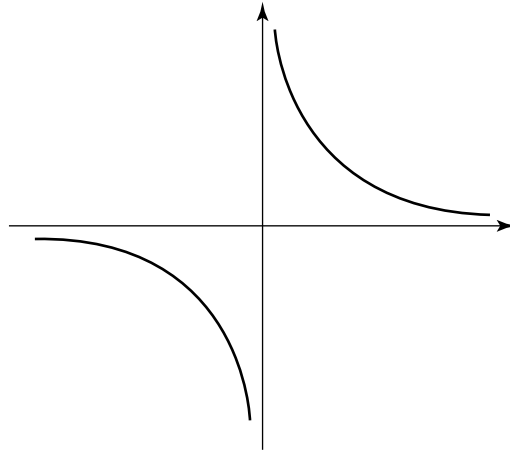
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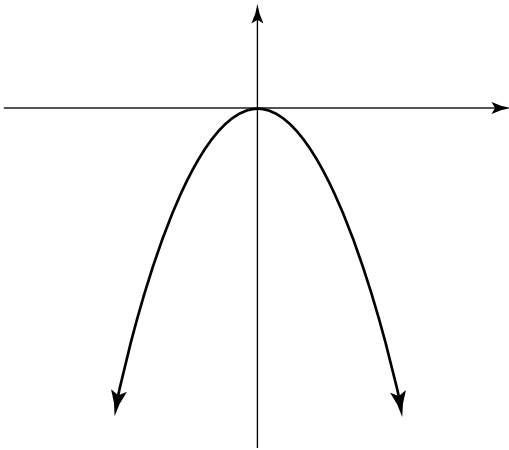
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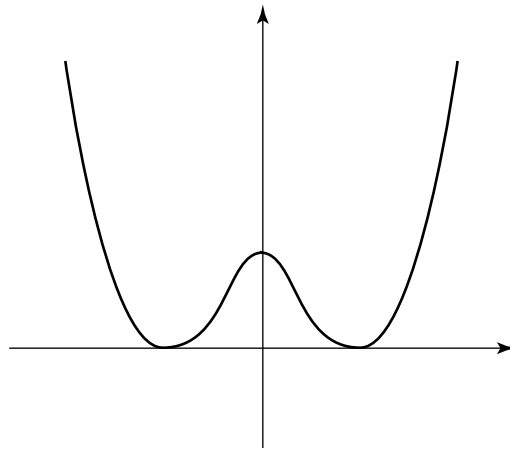
6.



7.



8.




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**HW1.11** Use inequalities to express the definitions of increasing and decreasing.

1. Sketch a curve with one turning point. Assume  $s$  is the name of the function and  $t$  is the name of the independent variable.
2. Consider the interval where  $s$  is increasing. Let  $t_1$  and  $t_2$  be any two points in this interval where  $t_1 < t_2$ . Mark  $t_1$  and  $t_2$  on the horizontal axis and the associated values of  $s(t_1)$  and  $s(t_2)$  on the vertical axis of your graph. Use an inequality to indicate the relationship between  $s(t_1)$  and  $s(t_2)$ , when  $s$  is increasing.

3. Consider the interval where  $s$  is decreasing. Let  $t_3$  and  $t_4$  be any two points in this interval where  $t_3 < t_4$ . Use an inequality to indicate the relationship between  $s(t_3)$  and  $s(t_4)$ , when  $s$  is decreasing.

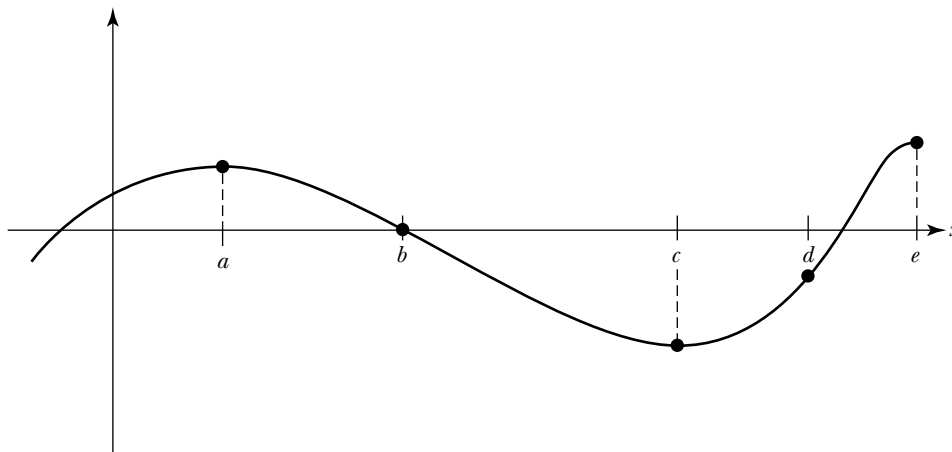
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**HW1.12** Show that you understand the concepts of domain, range, increasing, decreasing, concave up, and concave down by developing your own examples.

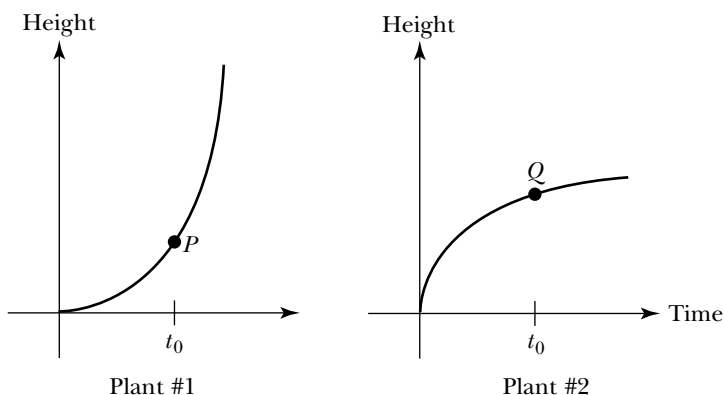
1. Draw the graph of a function whose domain is the set of all real numbers, which is increasing on the open intervals  $(-4.5, 0)$  and  $(3, 8.5)$  and decreasing everywhere else.
2. Give the table for a function whose domain is the set of integers from 1 to 10 and whose range is contained in the set of real numbers between 0 and 1.
3. Draw the graph of a function whose domain is the closed interval  $[-5, 5]$ , whose range is the closed interval  $[0, 4]$ , and which is always increasing.
4. Draw the graph of a function, whose domain is the open interval  $(-5, 5)$  whose range is the open interval  $(0, 4)$  and where the slope of the tangent line is always negative.
5. Draw the graph of a function whose domain and range are the set of real numbers and which has exactly two turning points.
6. Draw the graph of a function whose domain is the closed interval  $[-10, 8]$ , which has a turning point at each odd integer in its domain, and has a local maximum at 0.
7. Draw the graph of a function whose domain is the open interval  $(-30, 30)$ , where the slope of the tangent line to the graph at  $x = -25, -15, -5, 5, 15,$  and  $25$  equals 0.
8. Draw the graph of a function whose domain and range are the set of real numbers and where the graph is always concave up.
9. Draw the graph of a function whose domain and range are the positive real numbers and where the graph is concave up for all  $x < 50$  and concave down for all  $x > 50$ .
10. Draw the graph of a function whose domain is the open interval  $(-30, 30)$ , which is always increasing, and where the slope of the tangent line to the graph at  $x = -25, -15, -5, 5, 15,$  and  $25$  equals 0.
11. Draw the graph of a function whose domain is the open interval  $(-10, 8)$ , whose range is the set of all real numbers, and which has local maxima at  $x = -7, -4, 0, 5,$  and  $7$ .
12. Draw the graph of a function whose domain is the set of real numbers and where the slope of the tangent line is always 0.

**HW1.13** Think in terms of “rate of change.”

1. Consider the graph given below.



- a. At each of the labeled points, determine if the rate of change of the function—that is, the slope of the tangent line to the graph—is positive, negative, or zero.
  - b. At which of the labeled points is the rate of change the greatest? Justify your response.
2. Consider the graphs modeling the growth of Plant #1 and Plant #2 which are described in the discussion preceding Task 1.7. Note: The scales for the graphs and the value of  $t_0$  are the same.



- a. Describe the rate of growth—that is, the rate of change—of each plant by describing how the slope of the tangent line varies over the given time period.
- b. Explain how you can tell by examining the graphs that at time  $t_0$  Plant #1 is growing more rapidly than Plant #2.

- c. Even though at time  $t_0$  Plant #1 is growing at a faster rate than Plant #2, it is shorter than the other plant. Use a rate-of-growth argument to explain why this is the case.
- d. At some point in time, Plant #1 catches up with Plant #2 and thereafter is taller than Plant #2. Indicate on the graphs the time when this change occurs.

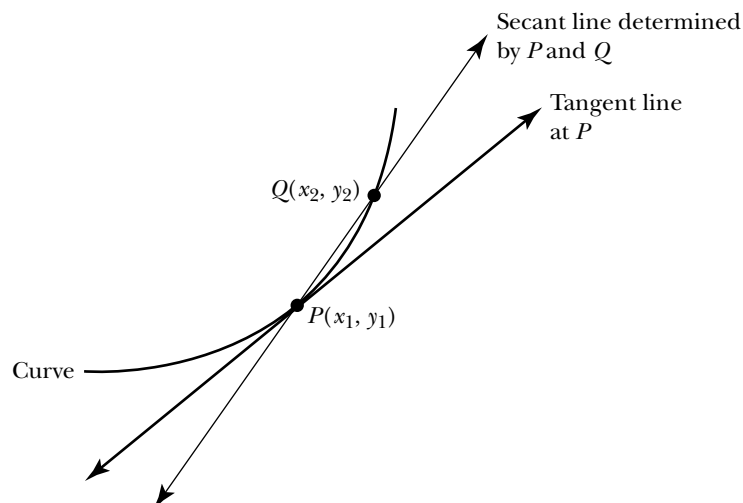
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**HW1.14** You cannot use the traditional two-point formula from algebra to find the slope of the tangent line to a smooth curve, since you only have one point, namely the point where the tangent line touches the curve. You can, however, approximate the value of the slope by considering a nearby point on the curve. Investigate how this might be done.

The basic idea is this: Assume you have a function represented by an expression, and you have a point on the graph of the function,  $P$ . You want to calculate the slope of the tangent line to the graph at  $P$ . In order to be able to use the standard equation for finding the slope, namely

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

you need to know the  $(x,y)$ -coordinates for two different points *on the tangent line*. Since  $P$  is on both the tangent line and the graph, you can find the  $xy$ -coordinates of  $P$  by substituting in the expression for the function. This gives you  $P(x_1, y_1)$ . The problem is that you don't know the coordinates of a second point on the tangent line. You can, however, use the expression for the function to find the  $(x,y)$ -coordinates for a point *on the curve* which is *near*  $P$ , say  $Q(x_2, y_2)$ . The line through  $P$  and  $Q$  is called a *secant line*. Since you have two points, you can calculate the slope of secant line determined by  $P$  and  $Q$ . This provides you with an approximation of the slope of the tangent line at  $P$ .



1. Before trying this approach on a specific example, study the diagram given above and make sure you understand the underlying idea.
2. Consider the parabola  $y = x^2 + 1$ . Use the secant line approach to approximate the slope of the tangent line—and hence the rate of change—of the function at the point  $P(2,5)$ .
  - a. Sketch a large-scale graph of the parabola for  $0 \leq x \leq 3$ . Label the point  $P(2,5)$  on the graph. Sketch the tangent line to the parabola at the point  $P$ .
  - b. Choose a point  $Q$  on the parabola, which is near  $P$ . For instance, let  $x = 2.25$ . Use the expression for the parabola to find the second coordinate of  $Q$ . Label  $Q$  on the graph. Sketch the secant line determined by  $P$  and  $Q$ .
  - c. Use the slope formula to find the slope of the secant line determined by  $P$  and  $Q$ .
  - d. Your answer to part c gives you an approximation for the slope of the tangent line to the parabola at  $P$ . You can improve the approximation by considering a point  $Q$  that is closer to  $P$ , say  $x = 2.1$ . Try this.
3. Use the secant line approach to approximate the rate of change of the function  $y = x^2 + 1$  at the point  $P(-2,5)$ .

---

**HW1.15** Do you feel comfortable using your graphing calculator? If so, use it to analyze the behavior of the tangent line as it travels along a curve.

Before beginning this problem, you might want to review how to graph a function and how to draw a tangent line on a graph, in the appendix for your graphing calculator.

1. Graph  $y = \sin(x)$  on your calculator, where  $0 \leq x \leq 2\pi$  and  $-1.5 \leq y \leq 1.5$ . Make a large scale sketch of the graph.
2. For each value of  $x$  listed in the table below:
  - a. Use the DRAW feature to draw the associated tangent line on the graph of the sine function. Sketch the tangent line on your graph.
  - b. Give the  $y$ -value corresponding to  $x$ .
  - c. Give the slope of the tangent line through  $P(x, y)$ .
  - d. Describe the shape of the graph at  $P(x, y)$ —for example, “decreasing, concave up,” “local maximum, concave down,” “increasing, inflection point,” and so on.

$x$	$y$	Slope	Shape of the graph of the function
$0.2\pi$			
$0.4\pi$			
$0.5\pi$			
$0.6\pi$			
$0.8\pi$			
$\pi$			
$1.2\pi$			
$1.4\pi$			
$1.5\pi$			
$1.6\pi$			
$1.8\pi$			

3. Describe the behavior of the tangent line as it travels along the curve.
  - a. List the intervals where the slope of the tangent line is positive.
  - b. List the intervals where the slope of the tangent line is negative.
  - c. Give the  $(x, y)$ -coordinates of all the points on the graph where the slope of the tangent line is 0.
  - d. List the intervals where the graph lies above the tangent line.
  - e. List the intervals where the graph lies below the tangent line.
  - f. Give the  $(x, y)$ -coordinates of the points on the graph where the tangent passes through the graph.

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**HW1.16** At the end of each unit you will be asked to write a journal entry reflecting on what you learned in the unit and on the learning environment for the course. Before you begin to write, review the material in the unit. Think about how it all fits together. Try to identify what, if anything, is still causing you trouble.

Write your journal entry for Unit 1.

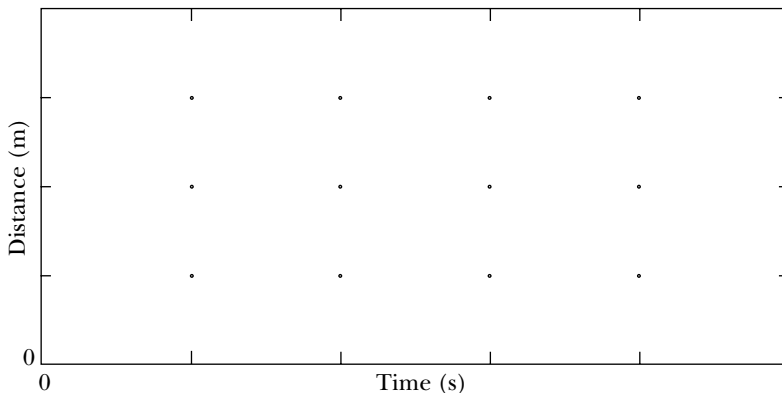
1. Reflect on what you have learned in this unit. Describe in your own words the concepts that you studied and what you learned about them. How do they fit together? What concepts were easy? Hard? What were the main (important) ideas? Give some examples of the main ideas.
2. Reflect on the learning environment for the course. Describe the aspects of this unit and the learning environment that helped you understand the concepts you studied. What activities did you like? Dislike?

order to use the slope formula which you learned in algebra, you need two points, and in the case of a tangent line to a curve, you have only one, namely  $P$ . The good news is that calculus will come to your rescue and provide you with a way of doing this. In the meantime, you can use your intuitive understanding of tangent line which you developed in the last task. You know the circumstances under which it exists. You know what it looks like, and given a point, you can sketch a reasonable approximation. The motion detector can help you develop a feel for what rate of change means, since its value depends on how fast you are moving and its sign depends on whether you are increasing or decreasing your distance from the detector. In the following task, you will use the motion detector to discover some of the relationships between the shape of a graph and the behavior of the tangent line as it moves along the curve.

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### Task 1-7: Investigating the Behavior of the Tangent Line Near a Turning Point

1. Create a smooth curve by walking back and forth in front of the motion detector for 10 seconds. Change your direction at least two times. Sketch your graph below. Label the axes.



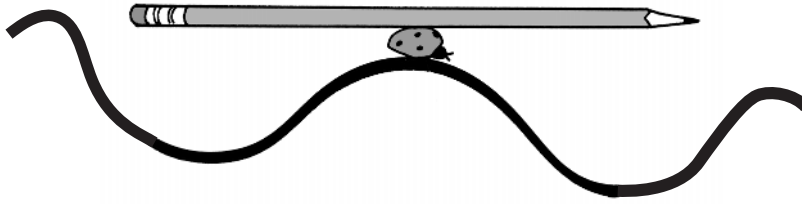
2. Analyze the shape of your function.

Quit the program. View the graph in the graphing window of your calculator. Use the TRACE feature to help answer the following questions.

- a. Consider the time intervals where your function is increasing.
  - (1) Below the horizontal axis on your graph, label each interval where your function is increasing.
  - (2) Express each interval using open interval notation.



- b. Consider the time intervals where your function is decreasing.
- (1) Below the horizontal axis on your graph, label each interval where your function is decreasing.
  - (2) Express each interval using open interval notation.
- c. Find the first and second coordinate of each point where you changed direction—that is, find the coordinates of the *turning points* of your graph.
- d. Consider the turning points where your graph opens down—that is, where your function has a *local maximum*.
- (1) Mark each point on your graph where your function has a local maximum.
  - (2) Describe the behavior of your function to the left and right of a local maximum in terms of increasing and decreasing.
- e. Consider the turning points where your graph opens up—that is, where your function has a *local minimum*.
- (1) Mark each point on your graph where your function has a local minimum.
  - (2) Describe the behavior of your function to the left and right of a local minimum in terms of increasing and decreasing.
3. One way to think about a tangent line to a graph is to think about a teeny-tiny bug crawling along the curve with a pencil strapped to its back. The pencil corresponds to the tangent line at each point where the bug touches the curve.

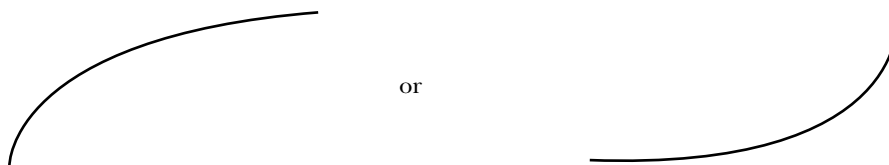


Analyze the behavior of the tangent line as it travels along your graph, moving from left to right.

- a. What can you conclude about the sign of the slope of the tangent line in the intervals where your function is increasing?
  
- b. What can you conclude about the sign of the slope of the tangent line in the intervals where your function is decreasing?
  
- c. What can you conclude about the value of the slope of the tangent line at each of the turning points on your graph?
  
- d. Consider a turning point where your function has a local maximum.
  - (1) Sketch the general shape of a smooth function near a local maximum. Sketch a tangent line to the left, to the right, and at the turning point.
  
  - (2) Describe the behavior of the slope of the tangent line as it moves along the curve from left to right, through the local maximum. In other words, what happens to the rate of change near a local maximum?

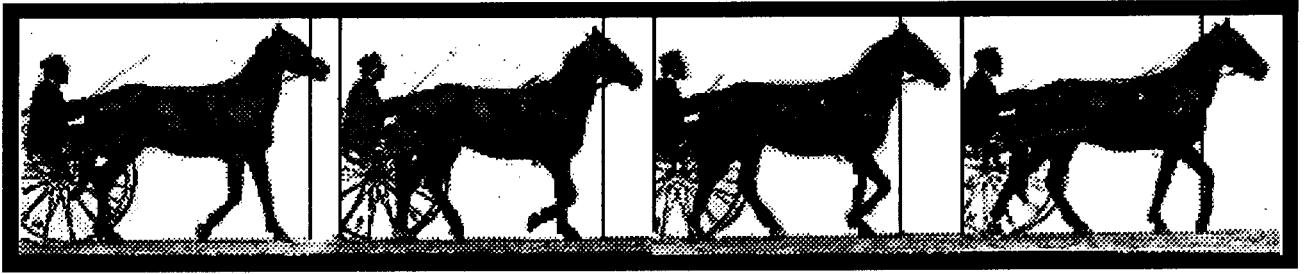
- e. Consider a turning point where your function has a local minimum.
- (1) Sketch the general shape of a smooth function near a local minimum. Sketch a tangent line to the left, to the right, and at the turning point.
  - (2) Describe the behavior of the slope of the tangent line as it moves along the curve from left to right, through a local minimum. In other words, what happens to the rate of change near a local minimum?

When a smooth function is increasing, its rate of change is positive, the slope of the tangent line to the curve is positive, the graph rises from left to right, and the function is growing. On the other hand, when a smooth function is decreasing, its rate of change is negative, the slope of the tangent line to the curve is negative, the graph falls from left to right, and the function is decaying. At a local extremum—that is, at a turning point—the rate of change is zero, thus the slope of the tangent line is zero. This provides a lot of information about the rise and fall of the curve, but does not tell you anything about the *concavity* of a function—that is, when it is shaped like



Notice that both these functions are increasing. The curve on the left, however, opens down and is said to be *concave down*, whereas the one on the right opens up and is said to be *concave up*.

In the next task you will examine the relationship between the concavity of a function and your velocity when you move in front of a motion detector. You will examine also how the concavity of a function is related to the location and behavior of the tangent line as it travels along its graph.



**Figure 1:** Selections from a photo sequence by Eadweard Muybridge. The lines near the horse’s head (which have been enhanced for clarity) indicate the marking pole used by the photographer. The other lines indicate the breaks between successive frames. *Sulky Horse Walking* (Plate 588). The Trout Gallery, Dickinson College. Gift of Samuel Moyerman, 87.4.65.

four times. You will then describe—or represent—this information which defines a function in several ways.

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### Task 1-1: Relating Position and Time

1. One of the objectives of this task is to calculate the “real” distance from the horse’s nose to the marker pole in each frame. You can use a centimeter ruler to measure the distances in the photograph, but then you will need to convert these results to actual centimeters. To do this, you need to know the *scale* of Muybridge’s photographs—for example, the scale for a given map might be that  $1/4$ " (on the map) = 1 “real” mile. Use the assumption that the actual length of the horse’s head is 60 centimeters to calculate the scale for the photographs.
  - a. 1 cm (on the photograph) = \_\_\_\_\_ “real” centimeters
  - b.  $1/10$  cm (on the photograph) = \_\_\_\_\_ “real” centimeters
2. Determine the real distance from the tip of the horse’s nose to the marker pole in each of the four frames.
  - a. Frame 1:  $t = 0.00$  seconds
  
  - b. Frame 2:  $t = 0.20$  seconds
  
  - c. Frame 3:  $t = 0.40$  seconds
  
  - d. Frame 4:  $t = 0.60$  seconds



At any given time between 0 and 0.6 seconds, the horse's nose is at one and only one position (with respect to the marker pole). Moreover, you can describe a process for determining the position corresponding to a particular time. Consequently, you say that there is a *functional relationship* between position and time or that position is a *function of time*. The important observation is that corresponding to each input (time), there is exactly one output (position).

Apply the idea of a function to another situation. Suppose you have collected \$10 to buy pizza for your hungry friends. An 8" pizza costs \$5 and a 12" pizza costs \$10. Would it be better to buy two 8" pizzas for the gang or one 12" pizza? Is area a function of diameter in the mathematical sense? In the following task, you will find a function that *models* this situation and then use the function to determine which combination gives you more for your money.

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### Task 1-3: Using the Concept of Function to Buy Pizza

1. Assuming you have collected \$10 to buy pizza and that an 8" pizza costs \$5 and a 12" pizza costs \$10, offhand what do you think will give you the most pizza to eat? One 12" pizza or two 8" pizzas?
2. Use a formula to represent the functional relationship between the area  $A$  of a circle and its diameter  $d$ .
3. Give a verbal description of a process you can use to find the area of a pizza if you know its diameter.
4. In a functional relationship there is exactly one output corresponding to each input.
  - a. Give the input quantity in the model for the pizza dilemma.

- b. Give the corresponding output quantity.
- c. Explain why there can be only one output for each given input. In other words, explain why the area formula for a pizza represents a function.
5. Now that you have used a function to model this situation, what pizza combination would you buy for your friends? One 12" pizza or two 8" pizzas? Use your function to justify your decision.

## Unit 1 Homework After Section 1

- Complete the tasks in Section 1. Be prepared to discuss them in class.
- As a general rule, do your homework on separate sheets of paper and place the sheets in your activity guide at the end of the appropriate section. Please don't try to scrunch your answers in with the description of the homework problem!

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**HW1.1** There are a few basic mathematical ideas that you need to recall before you go on to the next section. Refresh your memories as you do the following problems. Work together with the members of your group. Answer each other's questions. Share your results. Go through the problems quickly. If you need special help, see your instructor.

1. Lines and slopes. First recall a few definitions.

If  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are any two points on a given line, then the *slope*  $m$  of the line through  $P$  and  $Q$  is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

If  $m$  is the *slope* of a given line and  $b$  is the *y*-intercept, then the *slope-intercept equation* for the line is given by

$$y = mx + b.$$

If  $p(x_1, y_1)$  is any point on the line, then the *point-slope equation* of the line is given by

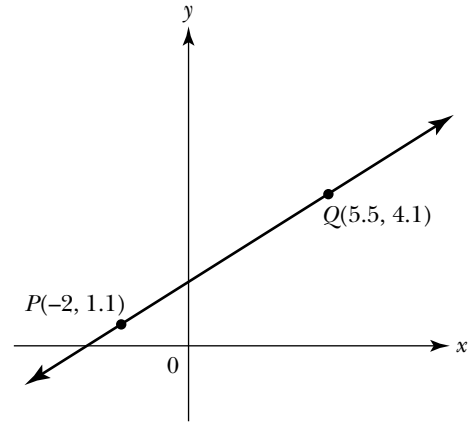
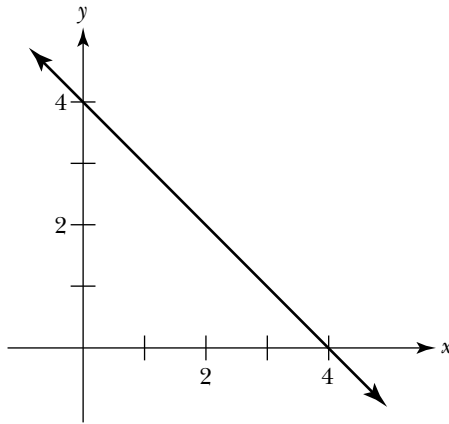
$$y - y_1 = m(x - x_1) \text{ or } y = m(x - x_1) + y_1.$$

a. Find the slope of the line through the following pairs of points.

- (1)  $P(1,2)$  and  $Q(3,6)$
- (2)  $R(0.5,1.6)$  and  $T(-2,1.6)$
- (3)  $M(-1,-3)$  and  $P(-4,5)$

b. Find the equation of each of the following lines.

- (1) The line through the points  $P(1,2)$  and  $Q(3,6)$
- (2) The line through the point  $S(\frac{1}{2}, \frac{1}{2})$  with slope  $\frac{1}{2}$
- (3) The line with slope  $-3$  and  $y$ -intercept  $-10.67$
- (4) The line parallel to the horizontal axis that passes through the point  $P(-4,5)$
- (5) The line through the point  $K(0, -3.4)$  that is parallel to the line  $y = 9.8x + 6$
- (6)
- (7)



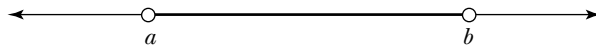
c. Sketch a graph of the following functions.

- (1)  $y = x$
- (2)  $y = -x$
- (3)  $x + y = 2$
- (4)  $y = 0$
- (5)  $y = \begin{cases} 2, & \text{if } x \leq 3 \\ x - 1, & \text{if } x > 3 \end{cases}$



2. Ways of denoting an interval.

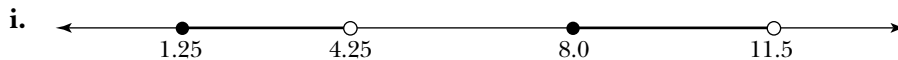
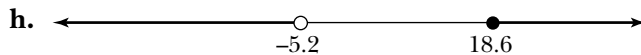
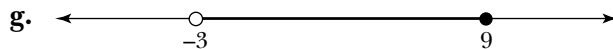
Recall that there are several ways to denote an interval of real numbers. One way is to use a *number line*, where



denotes the set of all real numbers between  $a$  and  $b$ , but not including either  $a$  or  $b$ . This set can also be denoted using *open interval notation*  $(a,b)$  or using the *inequality*  $a < x < b$ , where  $x$  is any value between  $a$  and  $b$ . If you want to include  $a$  and/or  $b$  in the interval, draw a closed circle instead of an open circle on the number line; use a square bracket  $[$  and/or  $]$  instead of an open bracket with the interval notation; or use a less than or equal to sign  $\leq$  in the inequality. Intervals that include either  $a$  or  $b$ , but not both, are called *half-open intervals*. Intervals that include both  $a$  and  $b$  are said to be *closed intervals*.

Practice converting between the ways. Each of the following intervals is represented using one of the three ways—that is, using a number line, interval notation, or an inequality. Express each interval two other ways.

- a.  $-6 \leq x \leq 3$
- b.  $0 < x \leq 2.5$
- c.  $x < -5$  or  $x \geq 16$
- d.  $[-7,8)$
- e.  $(-2.1,6)$  or  $(10,12]$
- f.  $(-\infty,0]$  or  $(1.5,\infty)$



3. A few important facts.

Briefly explain or show why each of the following statements makes sense to you.

- a. You cannot divide a real number by 0 (and get a real number).
- b. You cannot find the square root of a negative number (and get a real number).
- c. The slope of a horizontal line is always 0.
- d. The slope of a line which rises from left to right is always positive.
- e. The slope of a line which falls from left to right is always negative.
- f. The slope of a vertical line is undefined.

## SECTION 2

### Analyzing Linear Functions

You will begin your study of functions by using a motion detector to create your own functions. As you walk, jump, or run in front of the motion detector, it sends out a series of ultrasonic pulses which bounce off an object in its path (in this case, you) and back to the detector. The computer or calculator records the time it takes each sound pulse to make the round-trip between the detector and the object, uses this information to calculate the object's distance from the detector, and instantaneously displays the distance versus time graph generated by the motion on the screen.

In the first task<sup>1</sup> you will use the motion detector to create some *linear* functions—that is, functions whose graphs are straight lines—and then find an expression that models your curve. You will create increasing and decreasing functions and examine how varying the rate at which you move affects the shape of the graph representing your motion. You will examine some ways of representing a function and think about the meaning of domain and range. Before beginning the task, review the calculator basics, how to use the CBL tools, and how to analyze data collected by the CBL System in the appendix for your graphing calculator.

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#### Task 1-4: Creating Linear Functions

*As you go through this task, read each of the questions very carefully before you begin. Try to do exactly what it says. Work as a team. Share the calculations. Compare your results. Talk about what is going on. Help each other out. Make sure each person takes a turn using the motion detector. You may try a number of times. Get the times right. Get the distances right.*

Connect a motion detector to your graphing calculator and open the program MOTION.

1. Create two linear graphs while increasing your distance from the detector.

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<sup>1</sup>The motion detector investigations are based on activities developed by Ron Thornton of Tufts University, David Sokoloff of the University of Oregon, and Priscilla Laws of Dickinson College.

Try not to skip this step. Doing this should help you to:

- envision a function taking an input object and transforming it into a corresponding output object, and
- construct new functions.

Your calculator is a powerful tool with many different features. The appendix for your calculator will help you learn to use the features that are needed for Workshop Calculus. Read the appropriate sections carefully and interactively. Experiment. Play. The appendix is a valuable resource. Turn to it when you have questions. Become familiar with it. If the appendix does not answer your questions, look in the guidebook for your calculator.

In the next task, you will examine how to evaluate an expression, assign a value to a variable, and use some predefined constants and functions. Before starting the task, review the “Calculator Basics” and read about the following concepts in the appendix for your graphing calculator:

- Predefined constants and functions
- Editing and re-executing lines of code
- Arithmetic operations and expressions
- Specifying the number of digits displayed to the right of the decimal point of a real number
- Using variables
- Relational and Boolean operators

---

### Task 2-1: Becoming Familiar with Your Calculator

1. Enter the following arithmetic expressions and see what happens. Predict what your calculator is going to return *before* pressing the ENTER key.

Calculator Code	Predicted Output	Actual Output
$2.01 + 3.24$		
$2*(6 - 7)/4$		
$3^2$		
$23.8 - 1.2$		
$2^{(3^2)}$		

2. Next, assign values to variables and evaluate some expressions. As usual, predict the output before you press the ENTER key. If you make an error, edit your code.

Calculator Code	Predicted Output	Actual Output
$2 \rightarrow X$		
$X^4$		
$6.0 \rightarrow Y$		
$Y$		
$-5X + Y/2$		
$Y^X$		
$Z$		
$-35.4 \rightarrow Z$		
$Z$		
$Z \geq 0.0$		
$Z \neq -100$		
$Z < -35$		
$(-50 \leq Z) \text{ and } (Z \leq -30)$		
$(Z < -40.0) \text{ or } (Z > 0)$		

3. Use some predefined constants and functions. See if you can figure out what each function does.

Calculator Code	Predicted Output	Actual Output
$16 \rightarrow A$		
$\sqrt{A}$		
$\sqrt{-A + 30}$		
$\text{abs}(-45.8)$		
$\text{abs}(3A)$		
$\sin(0)$		
$\pi \rightarrow T$		
$\sin(T/2)$		

The next task will introduce you to the calculator syntax for representing functions defined by one or more expressions. Before starting the task, read through “Representing a Function by an Expression” and “Representing a Function by a Piecewise-Defined Function” in the appendix for your graphing calculator. Determine how to:

- Define a function
- Specify the domain of a function
- Evaluate a function at an input value
- Select and de-select functions

---

### Task 2-2: Implementing Functions Using Expressions

1. Use your calculator to define some functions that are represented by expressions.
  - a. Define and evaluate some functions using your calculator. Each time you ask your calculator to evaluate a function at a particular input, think about how the calculator will process the input. Take your time. Think about what’s happening before you press the ENTER key. Form an image in your mind of the function processing the information.

Calculator Code	Predicted Output	Actual Output
$Y_1(X) = (X^2)/(X > 0)$		
$Y_1(4)$		
$Y_1(-5.57)$		
$6 \rightarrow X$		
$Y_1(X)$		
$X/2$		
$Y_1(X/2)$		
$8.0 \rightarrow R$		
$Y_1(R)$		
$Y_1(2R + 4)$		
$Y_1(-R)$		
$Y_2(X) = 3X + 8$		
$Y_2(10)$		

If the domains of the underlying functions are the same, then the domain of the new function is usually the same set, except in the case of a quotient where you have to be careful not to divide by zero. On the other hand, if the domains of the functions are not the same, then the domain of the new function is the set of values where all the underlying functions are defined. You will have to figure out what this is by comparing their domains.

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### Task 2-6: Combining Functions

In the following exercises, represent the specified combination using the same representation as the underlying functions. For instance, if you are asked to find a representation for  $f + g$ , where  $f$  and  $g$  are represented by tables, then use the definition of sum to find a table representing  $f + g$ .

Determine the domain of the combination by comparing the domains of the underlying functions. What values do they have in common? Do you have to worry about dividing by zero?

*Note: Although this is a pencil and paper exercise you may want to use your calculator to check your results. See "Combining Functions" in the appendix for your graphing calculator.*

1. Consider the functions:

$$f(x) = x^2 - 1, \quad \text{where } x \text{ is any real number}$$

$$g(x) = \sqrt{x}, \quad \text{where } x \geq 0$$

Represent each of the following combinations by an expression. Give the domain of the combination.

- a.  $f + g$

(1) Representation:

$$(f + g)(x) =$$

(2) Domain:

- b.  $2f$

(1) Representation:

$$(2f)(x) =$$

(2) Domain:

c.  $\frac{f}{g}$

(1) Representation:

$$\left(\frac{f}{g}\right)(x) =$$

(2) Domain:

2. Consider the functions:

x	-2	-1	0	1	2	3	4
f(x)	0	5	3	1	-1	4	-1

x	-2	0	2	4	6	8	10
g(x)	3	-4	2	0	6	3	1

Represent each of the following combinations by a table. Give the domain of the combination.

a.  $f + g$

(1) Representation:

x	
$(f + g)(x)$	

(2) Domain:

b.  $-4fg$

(1) Representation:

x	
$(-4fg)(x)$	

(2) Domain:

c.  $\frac{f}{g}$

(1) Representation:

$x$	
$\left(\frac{f}{g}\right)(x)$	

(2) Domain:

3. Consider the functions:

$$f(x) = \begin{cases} 2x + 6, & \text{if } x \geq 0 \\ x - 8, & \text{if } x < 0 \end{cases}$$

$$g(x) = x^3, \text{ where } x \text{ is any real number}$$

Represent each of the following combinations by a piecewise-defined function. Give the domain of the combination.

a.  $f - g$

(1) Representation:

$$(f - g)(x) = \left\{ \right.$$

(2) Domain:

b.  $\frac{f}{g}$

(1) Representation:

$$\left(\frac{f}{g}\right)(x) = \left\{ \right.$$

(2) Domain:



c.  $g + 0.5f$

(1) Representation:

$$(g + 0.5f)(x) = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

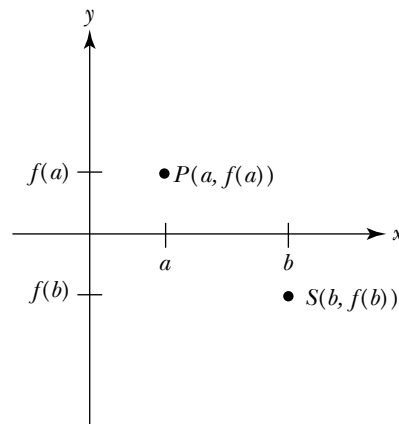
(2) Domain:

4. Represent a combination of functions by a graph.

a. Represent the scalar multiple of a function — which is represented by a graph — by a graph.

(1) Recall that by definition  $(3f)(x) = 3f(x)$ .

Suppose  $P$  is the point on the graph of  $f$  corresponding to  $x = a$ . Locate the point on the graph of  $3f$  corresponding to  $x = a$ . Label the point  $R$ .



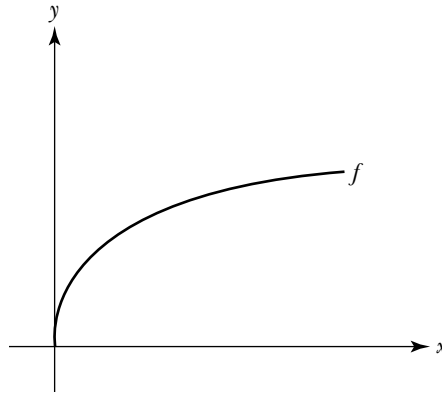
Similarly, suppose  $S$  is a point on the graph of  $f$  corresponding to  $x = b$ . Locate the point on the graph of  $3f$  corresponding to  $x = b$ . Label the point  $U$ .

(2) Explain how in general you can use the graph of  $f$  to find the graph of  $c f$  when  $c$  is a constant greater than 1.

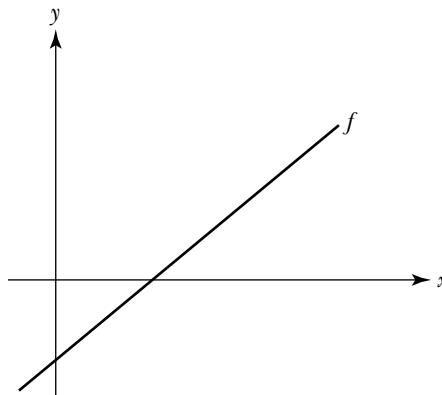
(3) Describe the relationship between the graph of  $f$  and the graph of  $c f$  when  $c$  is a constant between 0 and 1.

- (4) Try some. Consider the following graphs. Sketch the graphs representing  $2f$  and  $\frac{1}{2}f$ . In each case,
- Mark several  $x$  values on the horizontal axis. For each value, indicate the values of  $2f(x)$  and  $\frac{1}{2}f(x)$ .
  - Using the points from part i, sketch the graphs representing  $2f$  and  $\frac{1}{2}f$  on the same axes as  $f$ . Label the two graphs.

(a)



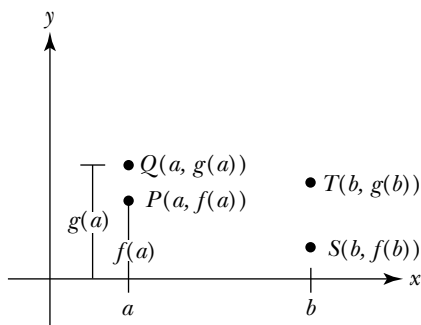
(b)



- b. Represent the sum of two functions—which are represented by graphs—by a graph.

(1) Recall that by definition  $(f + g)(x) = f(x) + g(x)$ .

Suppose  $P(a, f(a))$  is the point on the graph of  $f$ , and  $Q(a, g(a))$  is the point on the graph of  $g$  corresponding to  $x = a$ . Approximate the value of  $f(a) + g(a)$ . Use this approximation to locate the point on the graph of  $f + g$  corresponding to  $x = a$ . Label the point  $R$ .



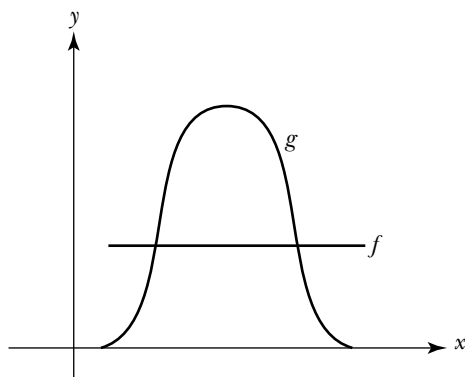
Similarly, suppose  $S$  and  $T$  are points on the graphs of  $f$  and  $g$  respectively corresponding to  $x = b$ . Locate the point on the graph of  $f + g$  corresponding to  $x = b$ . Label the point  $U$ .

- (2) Explain how in general you can use the graphs of  $f$  and  $g$  to determine the graph of  $f + g$ .

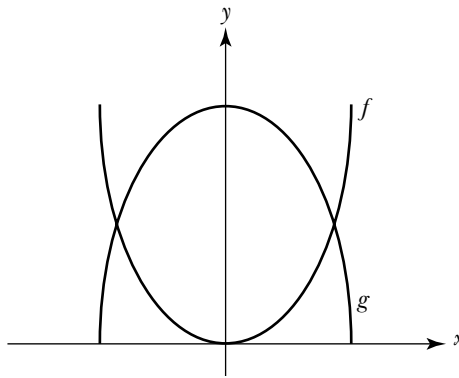
- (3) Try it. Suppose the functions  $f$  and  $g$  are represented by graphs. In each case,

- i. Mark several  $x$  values on the horizontal axis. For each value, indicate the value of  $f(x)$ ,  $g(x)$ , and  $(f + g)(x)$ .
- ii. Using the points from part i, sketch the graph representing  $f + g$  on the same axes as  $f$  and  $g$ .

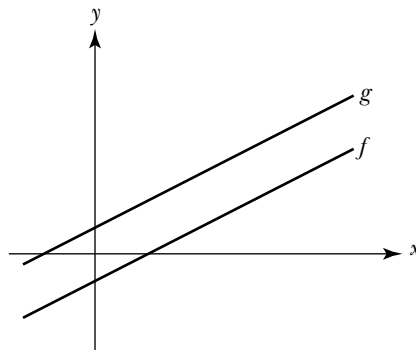
(a)



(b)



(c)



c. Use your calculator to graph some combinations of functions.

Consider  $f(x) = -x^2 + 9$  and  $g(x) = -5$ , where  $-4 \leq x \leq 4$ .

(1) Graph  $f$ ,  $g$  and  $f + g$  on the same pair of axes.

(a) Use the TRACE feature to evaluate  $f(x)$ ,  $g(x)$  and  $(f + g)(x)$  when  $x = -1$ . Observe that  $f(x) + g(x) = (f + g)(x)$ . Repeat the process for  $x = -2, 0, 1$  and  $2$ .

(b) Sketch the graphs of  $f$ ,  $g$  and  $f + g$  in the following space. Label the graphs and label the axes.

- (2) Graph  $f$ ,  $g$  and  $fg$  on the same pair of axes.
- (a) Use the TRACE feature to evaluate  $f(x)$ ,  $g(x)$  and  $(fg)(x)$  when  $x = -1$ . Observe that  $f(x)g(x) = (fg)(x)$ . Repeat the process for  $x = -2, 0, 1$  and  $2$ .
  - (b) Sketch the graphs of  $f$ ,  $g$  and  $fg$  in the following space. Label the graphs and label the axes.

In the previous tasks, you formed new functions by adding, subtracting, multiplying, and dividing old functions. To evaluate these combinations, you evaluated the underlying functions and then combined the results using the appropriate operation. Note two things: first, all the underlying functions are evaluated at the same input, and second, they can be evaluated in any order; in fact, they can even be evaluated simultaneously. For instance, think again about modeling your corporate earnings based on the earnings of your individual companies. To calculate your corporate earnings at the end of 1998, you could find the earnings of each company at the end of 1998 (in any order), and then add the results.

There are many situations, however, which can be modeled by performing one action and then performing a second action on the *output* of the first action. This type of situation can be represented by *composing* two functions. For instance, suppose you are teaching a physics class and need to calculate the average of each student's 12 lab grades at the end of the semester. As you know, this involves performing two actions: first you add a student's 12 lab grades, and then divide the sum by 12. One way to express this mathematically is to define functions called *sum* and *div*. The domain of *sum* contains a list of lab grades for each student. For each input, *sum* adds the 12 grades and returns their sum. The domain of *div* is the set of non-negative integers. For each input, *div* divides the given integer by 12 and returns the quotient rounded up to the nearest integer. Suppose  $x$  is a sequence of 12 lab grades. Then, to calculate the average of  $x$ , you evaluate *sum* at  $x$ , then evaluate *div* at  $sum(x)$ , or

$$ave(x) = div(sum(x))$$

The important thing to note is that you perform the first process *sum* and then apply the second process *div* to the output of *sum*. In other words, you *compose* *div* with *sum*.

In general, the composition of two functions  $g$  and  $f$  is denoted by  $g \circ f$  and is read as " $g$  composed with  $f$ ." To evaluate a composition, evaluate the

inner or rightmost function  $f$  at a given input and then evaluate  $g$  at the output from  $f$ , or

$$g \circ f(x) = g(f(x))$$

What is the domain of  $g \circ f$ ? Well, since composing  $g$  and  $f$  involves evaluating  $g$  at an output from  $f$ , any value of  $x$  in the domain of  $f$  such that  $f(x)$  is in the domain of  $g$  would be in the domain of  $g \circ f$ .

Think about all this in the next task.

---

### Task 2-7: Composing Functions

1. Evaluate the composition of two functions at a given input.
  - a. Consider the functions

$$f(x) = x^2 - 1, \text{ where } x \text{ is any real number}$$

$$g(x) = \sqrt{x}, \text{ where } x \geq 0$$

If possible, evaluate  $g \circ f$  at the given input. Show how you arrived at your result. If  $g \circ f(x)$  does not exist, explain why.

(1)  $g \circ f(1)$

(2)  $g \circ f(-4)$

(3)  $g \circ f(-0.5)$

(4)  $g \circ f(4)$

(5)  $g \circ f(0)$

inner or rightmost function  $f$  at a given input and then evaluate  $g$  at the output from  $f$ , or

$$g \circ f(x) = g(f(x))$$

What is the domain of  $g \circ f$ ? Well, since composing  $g$  and  $f$  involves evaluating  $g$  at an output from  $f$ , any value of  $x$  in the domain of  $f$  such that  $f(x)$  is in the domain of  $g$  would be in the domain of  $g \circ f$ .

Think about all this in the next task.

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### Task 2-7: Composing Functions

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(2)  $g \circ f(-4)$

(3)  $g \circ f(-0.5)$

(4)  $g \circ f(4)$

(5)  $g \circ f(0)$

b. Consider the functions

$x$	-2	-1	0	1	2	3	4
$f(x)$	0	5	3	1	-1	4	-1

$x$	-2	0	2	4	6	8	10
$g(x)$	3	-4	2	0	6	3	1

If possible, evaluate  $g \circ f$  at the given input, showing how you arrived at your result. If  $g \circ f(x)$  does not exist for the given value of  $x$ , explain why.

(1)  $g \circ f(-2)$

(2)  $g \circ f(4)$

(3)  $g \circ f(0)$

(4)  $g \circ f(8)$

(5)  $g \circ f(3)$

c. Consider the functions

$$f(x) = \begin{cases} 2x + 6, & \text{if } x \geq 0 \\ x - 8, & \text{if } x < 0 \end{cases}$$

$$g(x) = x^3, \quad \text{where } x \text{ is any real number}$$

If possible, evaluate  $g \circ f$  at the given input. Show how you arrived at your result. If  $g \circ f(x)$  does not exist, explain why.



(1)  $g \circ f(-2)$

(2)  $g \circ f(4)$

(3)  $g \circ f(0)$

(4)  $g \circ f(8)$

## 2. Represent a composition and find its domain.

In the following exercises, represent the composition using the same representation as the underlying functions. For instance, if  $f$  and  $g$  are represented by tables, represent  $g \circ f$  by a table.

Determine the domain of the composition. Note that the domain of  $g \circ f$  is contained in the domain of  $f$ , since  $f$  is evaluated first. The question then is: What values in the domain of  $f$  are not in the domain of the composition? In other words, what values must be discarded? By definition of composition,  $g$  is evaluated at  $f(x)$ . Therefore,  $f(x)$  must be in the domain of  $g$ . Consequently, to find the domain of the composition, start with the domain of  $f$  and then toss out all the values of  $x$  where  $f(x)$  is not in the domain of  $g$ .

## a. Consider the functions

$$f(x) = x^2 - 1, \quad \text{where } x \text{ is any real number}$$

$$g(x) = \sqrt{x}, \quad \text{where } x \geq 0$$

(1) Represent  $g \circ f$  by an expression. Show your work.(2) Find the domain of  $g \circ f$ .

## b. Consider the functions

$x$	-2	-1	0	1	2	3	4
$f(x)$	0	5	3	1	-1	4	-1

$x$	-2	0	2	4	6	8	10
$g(x)$	0	5	3	1	-1	4	-1

(1) Represent  $g \circ f$  by a table.

$x$	
$(g \circ f)(x)$	

(2) Find the domain of  $g \circ f$ .

c. Consider the functions

$$f(x) = \begin{cases} 2x + 6, & \text{if } x \geq 0 \\ x - 8, & \text{if } x < 0 \end{cases}$$

$$g(x) = x^3, \text{ where } x \text{ is any real number}$$

(1) Represent  $g \circ f$  by a piecewise-defined function. Show your work.

$$g \circ f(x) = \left\{ \begin{array}{l} \end{array} \right.$$

(2) Find the domain of  $g \circ f$ .

3. Represent a given function as the composition of two functions.

You know how to combine functions to get new functions. Sometimes, however, it is helpful to reverse the process and pull a function apart and represent it in terms of other functions.

Represent the following functions as a composition of two functions; that is, for each function  $h$  define functions  $f$  and  $g$  such that

$$h(x) = g \circ f(x) = g(f(x))$$

a.  $h(x) = (x^2 - 2.5x + 4.9)^5$ , where  $x$  is any real number.

b.  $h(x) = \sin(x/2 + 1)$ , where  $x$  is any real number.

c.  $h(x) = e^{-x^2}$ , where  $x$  is any real number.

d.  $h(x) = |x - 1.5|$ , where  $x$  is any real number.

e.  $h(x) = \sqrt{2x - 1}$ , where  $x \geq 0.5$ .

## Unit 2 Homework After Section 2

- Complete the tasks in Section 2. Be prepared to discuss them in class.

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**HW2.6** Evaluate some combinations and find some representations.

1. Consider the functions

$$f(x) = \frac{1}{x}, \quad \text{where } x \neq 0 \quad \text{and} \quad g(x) = \begin{cases} x + 1, & \text{if } x < 3 \\ -x, & \text{if } x > 3 \end{cases}$$

- a. Evaluate each of the following combinations. If it is not possible, explain why.

(1)  $(f + 3g)(1)$

(2)  $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$

(3)  $(g \circ f)\left(\frac{1}{3}\right)$

- b. Represent each of the following combinations by a piecewise-defined function.

(1)  $\frac{2g}{f}$

(2)  $f \circ g$

2. Consider the functions

$r$	-1	0	1	2	3	4	5
$h(r)$	3	4	5	6	0	1	3

and

$r$	0	1	2	3	4	5	6
$m(r)$	0	-4	1	4	2	-1	3

- a. Evaluate each of the following combinations. If it is not possible, explain why.

(1)  $(m - 5h)(4)$

(3)  $\left(\frac{2h}{m}\right)(0)$

(2)  $(hm)(3)$

(4)  $(m \circ h)(2)$

- b. Represent each of the following combinations by a table.

(1)  $m + h$

(2)  $h \circ m$

3. Represent the following functions as the composition of two functions.

a.  $a(x) = \frac{1}{x^2 + 3x + 2}$

c.  $c(x) = \cos(x^2 + 7)$

b.  $b(x) = (x^4 + 4x^2 + 2x)^3$

d.  $d(x) = 9 - (x + 9)^9$

4. Let  $f(x) = x^3$ ,  $g(x) = \sqrt{x}$ ,  $h(x) = x - 4$ , and  $j(x) = 2x$ . Express each of the following functions as a composition of three of these four functions:

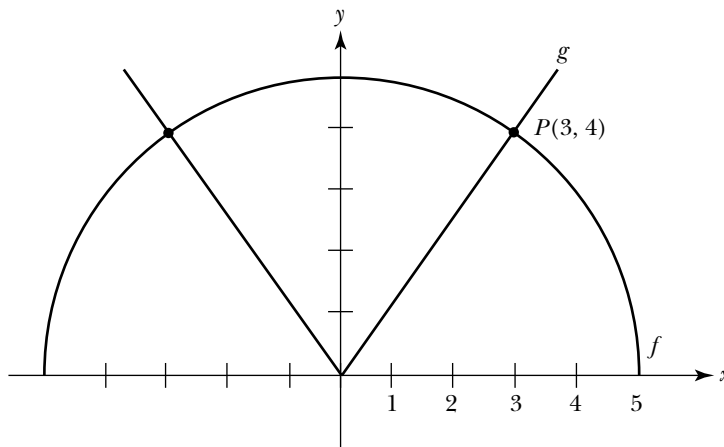
a.  $k(x) = 2(x - 4)^3$

c.  $k(x) = (2x - 8)^3$

b.  $k(x) = \sqrt{(x - 4)^3}$

d.  $k(x) = 2\sqrt{x - 4}$

5. Consider the following diagram:



a. Find:

- (1)  $(f + g)(0)$       (3)  $(f + g)(5)$   
 (2)  $(f + g)(3)$       (4)  $(f + g)(-3)$

b. Find the values of  $x$  such that:

- (1)  $(f - g)(x) = 0$       (3)  $(f - g)(x) = 5$   
 (2)  $(f - g)(x) > 0$       (4)  $(f + g)(x) = 8$

c. Sketch the following combinations.

- (1)  $f(x) + 1$       (3)  $-f$   
 (2)  $2g$       (4)  $f + g$

**HW2.7** Illustrate how the arrow diagram for  $g \circ f$  is related to the arrow diagrams for  $f$  and  $g$ . (Note: Arrow diagrams were introduced in HW2.2.)

Consider functions  $f$  and  $g$  which are represented by sets of ordered pairs:

$$f = \{(1, "a"), (2, "b"), (3, "c"), (4, "d"), (5, "e")\}$$

$$g = \{("b", 1.2), ("c", 2.5), ("d", 1.2), ("e", 0.0), ("f", 2.5)\}$$

1. Represent  $f$  and  $g$  by arrow diagrams. Place the arrow diagrams for the two functions side by side, with the one for  $f$  on the left and the one for  $g$  on the right.
2. Represent  $g \circ f$  by a set of ordered pairs.
3. Represent  $g \circ f$  by an arrow diagram.
4. Explain how you can use the arrow diagrams for  $f$  and  $g$  to find the arrow diagram for  $g \circ f$ .

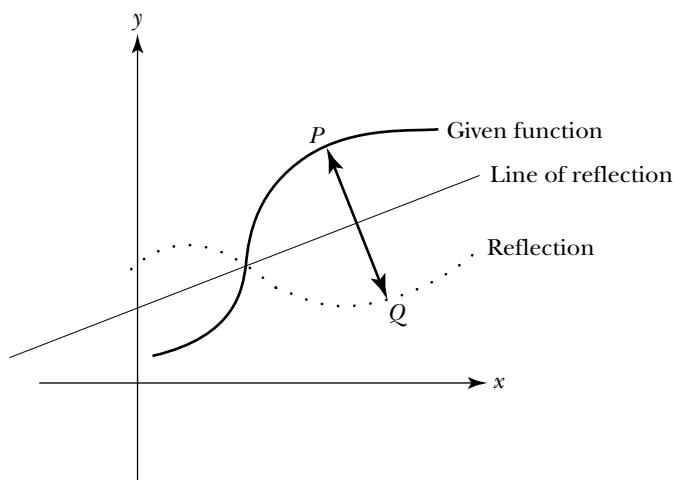
**HW2.8** Represent some given situations by combinations of functions.

1. Suppose you are the chief financial officer (CFO) of a corporation consisting of 10 companies. Let  $E_i$  model the earnings of  $i$ th company over time — in other words,  $E_1$  models the earnings of Company #1, and so on. Model your corporate earnings using a combination of the functions  $E_i$ ,  $1 \leq i \leq 10$ .
2. Suppose  $W(x)$  gives the weight in ounces of item  $x$ . Suppose  $T(w)$  gives the time in minutes:seconds required to defrost an item with weight  $w$ . Model the amount of time required to defrost an item  $x$  in the microwave using a combination of the functions  $W$  and  $T$ .
3. Assume  $F(x)$  is the father of  $x$  and  $M(x)$  is the mother of  $x$ .
  - a. Represent the maternal grandmother of  $x$  by an expression.
  - b. Represent the paternal grandmother of  $x$  by an expression.

## SECTION 3

### Reflecting Functions

Another way to create a new function from an old one is to *reflect* the old function — or find its mirror image — through a fixed line. To reflect a given function through a given line, you flip the graph of the function over the line, so that if the diagram were folded along the line of reflection, each point  $P$  on the graph of the given function would match with an associated point  $Q$  on the graph of its reflection, as illustrated in the figure below.



of points in the domain and evaluating the function at these points. It cannot possibly consider every point in the domain since it is a finite—discrete—machine. The question then is: Do the points which the machine has chosen provide an accurate reflection of the behavior of the function? Unfortunately, a finite sample cannot be used to predict with absolute certainty what happens throughout the domain. Connecting points in a finite sample may connect points that should not be connected, thereby giving a misleading picture of what is really going on. It is possible, for instance, that between two sample values the function “blows up,” has a “hole,” or oscillates rapidly. These types of behaviors may be missed. Consequently, the dot mode shows you what happens at the points in the finite sample and does not jump to any conclusions about what happens in between these values.

The domains of the functions which you considered in the last task contained an infinite number of points, such as the set of all real numbers or the set of all numbers between  $-3$  and  $10$ . The graphs of these functions consisted of lines and curves. The graph of a *discrete* function, on the other hand, consists of a collection of distinct, unconnected points, which is called a *scatter plot*. A discrete function whose domain contains a finite number of values can be represented by a table, a set of ordered pairs, or a scatter plot.

In the next task, you will investigate how to construct discrete functions and how to convert back and forth between their calculator representations and their mathematical representations. Before beginning the task, read the section “Representing a Discrete Function” in the appendix for your graphing calculator.

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### Task 2-4: Constructing Discrete Functions

1. Consider the discrete function

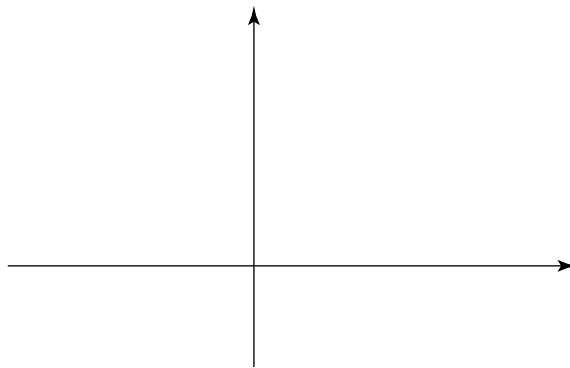
$$p = \{(5, 22), (4, 11), (-3, 13), (0, 5), (1, 7), (-1, 4), (-5, 26)\}$$

which is represented by a set of ordered pairs.

- a. The domain of a discrete function is the set of all first components of the ordered pairs. Give the domain of  $p$ .
- b. The range of a discrete function is the set of all second components of the ordered pairs. Give the range of  $p$ .
- c. The process of the function is implemented by assigning the second value in an ordered pair to the first. Sometimes you can see a pattern indicating how the function is defined, and sometimes you can't. In the case of  $p$ , there does not appear to be a pattern.

Use pencil and paper to evaluate  $p$  at the given input values. If  $p$  is not defined, write “und.”

- (1)  $p(-3)$
  - (2)  $p(4)$
  - (3)  $p(11)$
  - (4) Suppose  $t = 5$ . Find  $p(t - 6)$ .
  - (5) Suppose  $m = \{(1,2), (4,-5), (11,4), (7,1), (-3,5)\}$ . Find:
    - (a)  $2p(-3) - 4m(-3)$
    - (b)  $p(4)m(4)$
    - (c)  $p(m(4))$
    - (d)  $m(p(-5))$
- d. Use your calculator to represent  $p$  as a table. Copy the table in the following space.
- e. Use your calculator to represent  $p$  by a scatter plot. Use the TRACE feature to move the cursor through the points on the scatter plot.  
Sketch the scatter plot and label the axes.



2. Represent some discrete functions.
- a. The domain of the function is the odd integers between  $-11$  and  $-1$  (inclusive). The function returns the square of each input value.
    - (1) Give the domain of the function.
    - (2) Give the range of the function.



(3) Represent the function as a set of ordered pairs.

b. Consider

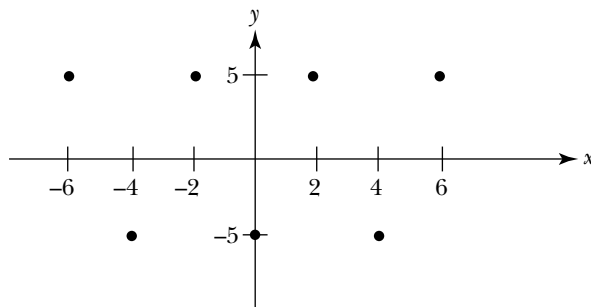
x	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5
y	3.4	-4.8	0	3.4	5.6	0	7.1

(1) Give the domain of the function.

(2) Give the range of the function.

(3) Represent the function as a scatter plot. Label the axes.

c. Consider the function represented by the following scatter plot.



(1) Give the domain of the function.

(2) Give the range of the function.

(3) Represent the discrete function by a table.

d. Suppose  $g(r) = 3r$ , where  $r = -2, -1.5, -0.5, 0, 0.5, 1, 1.5, 2$ .

(1) Represent  $g$  by a table.

- (2) Give the domain of  $g$ .
- (3) Give the range of  $g$ .
- (4) Represent  $g$  by a set of ordered pairs.
- (5) Represent  $g$  by a scatter plot. Sketch the graph in the following space. Label the axes.

## Unit 2 Homework After Section 1

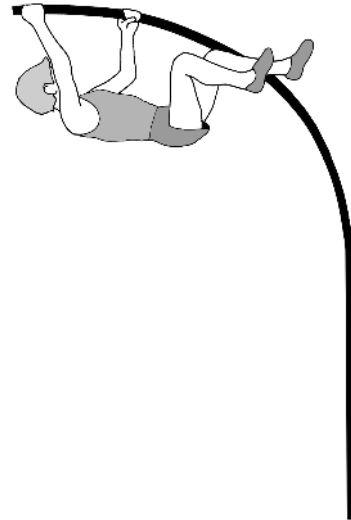
- Complete the tasks in Section 1. Be prepared to discuss them in class.

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**HW2.1** Implement some functions on your calculator and analyze their behavior.

1. According to Podunk's income tax schedule, a person who earns less than \$10,000 does not have to pay any taxes. A person who earns at least \$10,000 but less than \$50,000 must pay 15% of the amount he or she earns in taxes. Anyone who earns \$50,000 or more must pay \$6,000 plus 27% of the amount earned over \$50,000. Model Podunk's tax schedule, and use the model to determine if the schedule is fair.
  - a. Calculate the tax bill for a person who earned:
    - (1) \$6,798
    - (2) \$23,456
    - (3) \$74,892
  - b. Suppose  $T$  is a function that accepts the amount of money an individual earns and returns the person's tax bill. Give a mathematical description of  $T$ . State the domain of  $T$ .
  - c. Define  $T$  in your calculator. Give the code for your function. Use your function to check your answers to part a.
  - d. Represent  $T$  by a graph. Sketch the graph and label the axes.
  - e. Find the range of  $T$ .
  - f. Is the tax scheme fair? Justify your response.

2. “The Pole-Vault Principle: If every pole-vaulter’s technique and strength were equal, judges could skip the actual vault and determine the height an athlete would reach by clocking his speed at the point he plants the pole. The equation is straightforward: height equals the square of the velocity divided by twice the gravitational constant ( $9.8 \text{ m/sec}^2$ ). But given a particular running speed, explains NYU’s Richard Brandt, a taller vaulter has a bit of an edge because his center of mass is higher and so requires a bit less energy to reach a particular height. Lesson: Be fast; be tall; hang on.” (*Newsweek*, 27 July 1992, p. 59)



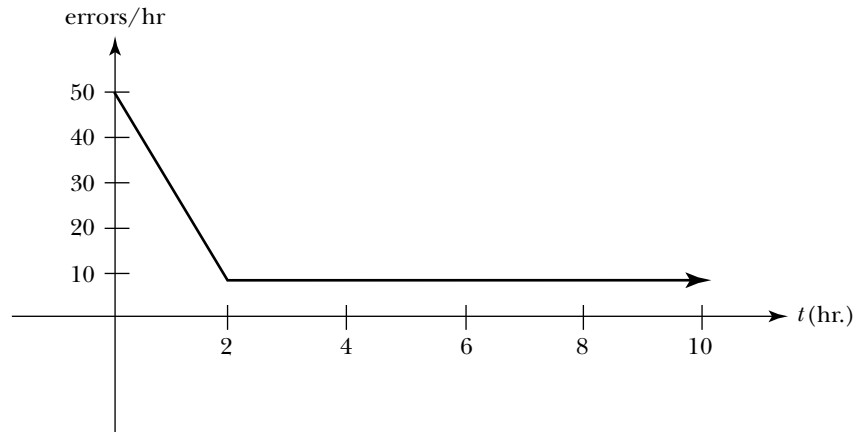
Assume every pole-vaulter’s technique and strength are equal and their velocity varies from 0 to 15 meters per second (m/sec).

- Find a mathematical expression which determines the height an athlete at a particular velocity. State the domain of the function.
  - Find the height an athlete would reach if his or her velocity is:
    - 8 m/sec
    - 9 m/sec
    - 10 m/sec
  - Define the function that determines the height an athlete would reach at a particular speed using your calculator. Give the code for your function. Use your function to check your responses to part b.
  - Find the range of the function.
  - Represent the function by a graph. Sketch the graph.
3. Your uncle sells cold beer at the Baltimore Orioles’ ballpark, Camden Yards. Based on his past experience (and very careful record-keeping), he knows that at a Saturday game, for each degree the temperature increases he sells eight more liters of beer than when it’s  $62^\circ\text{F}$ . He sells 354 liters when it’s  $62^\circ\text{F}$ , and the temperature varies between  $62^\circ\text{F}$  and  $103^\circ\text{F}$ .



- Calculate the number of liters he sells when the temperature is:
  - $65^\circ\text{F}$
  - $82^\circ\text{F}$
  - $97^\circ\text{F}$
- Find a mathematical expression that determines the number of liters of beer he sells for a given temperature. State the domain of the function.
- Represent your uncle’s situation by a function using your calculator. Give the code for your function. Use your function to check your answers to part a.

- d. Represent your function by graph. Sketch the graph.
- e. Find the range of the function.
4. Carl has all sorts of problems learning to use the computer. In the beginning, everything that can go wrong goes wrong. But Carl never gives up on anything, so he hangs in there. The following is a graph of his learning curve, giving the rate of his errors per hour versus time in hours.



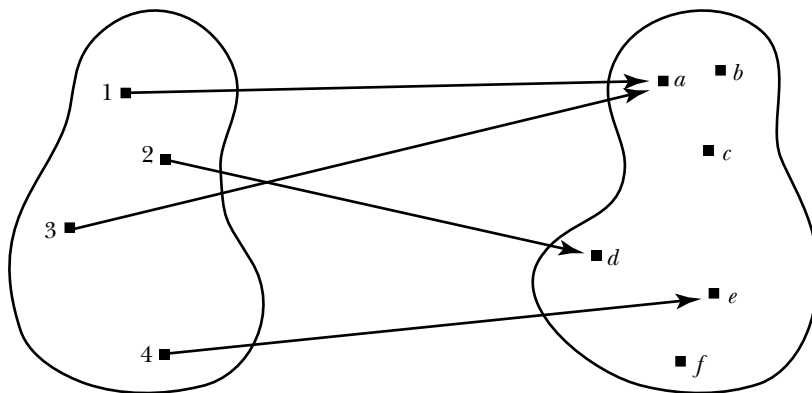
- a. Find Carl's errors per hour at
- (1) 30 minutes
  - (2)  $2\frac{1}{2}$  hours
  - (3) 75 minutes
- b. Find the difference between Carl's error rate at 1 hour and his error rate at 5 hours.
- c. Describe Carl's improvement as time goes by.
- d. Model Carl's error rate at a given time using a piecewise-defined function. State the domain of the function.
- e. Find the range of the function.
- f. Define the function using your calculator. Give the code for your function. Use your function to check your answers to part a.

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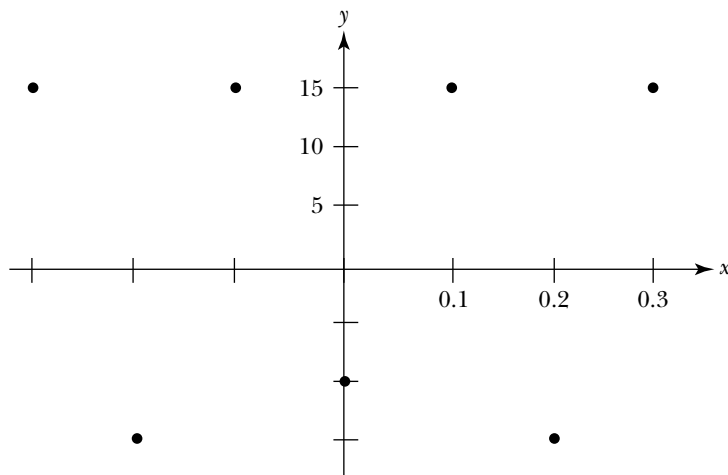
**HW2.2** As you know, a discrete function whose domain and range are both finite sets can be represented by a table, set of ordered pairs and a scatter plot. It can also be represented by an "arrow diagram."

The general idea underlying an arrow diagram is to sketch two closed, wiggly regions, one corresponding to the domain of the function and the other containing the range. The process of a discrete function can be implemented by drawing an arrow from each point in the domain to its cor-

responding value in the range. For example, a function whose domain is  $\{1,2,3,4\}$  and whose range is contained in the set  $\{a,b,c,d,e,f\}$  can be represented as follows:



1. Assume  $f$  is the name of the function represented by the arrow diagram given above.
  - a. Find  $f(2)$  and  $f(4)$ .
  - b. Find the range of  $f$ .
  - c. Represent  $f$  by a table.
2. Represent each of the following discrete functions by an arrow diagram.
  - a.



- b. NumVowels =  $\{(\text{"Anne"},2), (\text{"Elizabeth"},4), (\text{"Seth"},1), (\text{"Jamal"},2), (\text{"Tad"},1)\}$
- c. The function assigns to each of the twelve months in a non-leap-year the number of days in the month.

**HW2.3** In Unit 1 you used the Vertical Line Test to determine if a graph represents a function. This approach does not apply, however, if the function is represented by something other than a graph. In general, to determine if a relationship defines a function you have to check whether each item in the domain corresponds to exactly one item in the range. None of the relationships given in this exercise represents a function. Your job is to explain why.

Explain why each of the following relationships does not represent a function.

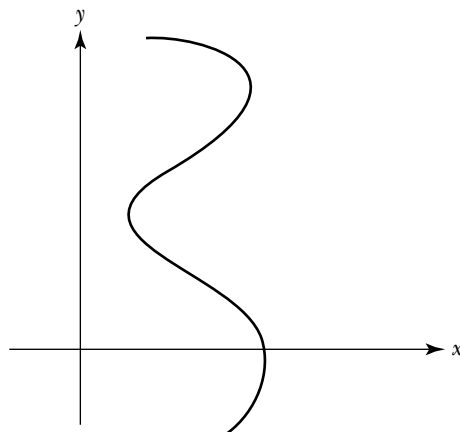
1. The relationship defined by the table:

$x$	"a"	"z"	"l"	"a"	"t"	"o"	"?"	"c"	"m"
$y$	6	-2	1	0	-4	1	8	6	6

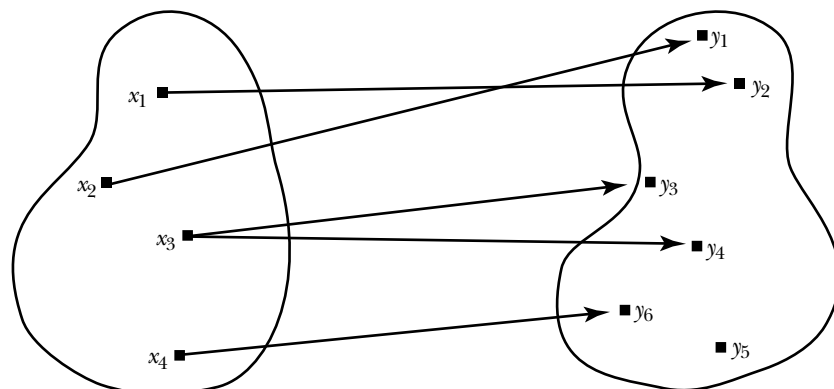
2. The relationship defined by the set of ordered pairs:

$$F = \{(2.3, 5.8), (6.7, 9.1), (5.8, 2.3), (7.1, 5.8), (6.7, 2.3)\}$$

3. The relationship defined by the graph:



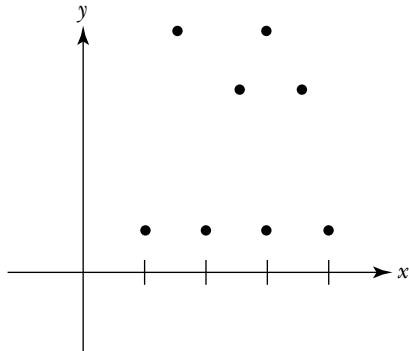
4. The relationship defined by the arrow diagram:



5. The relationship given by the piecewise definition:

$$h(r) = \begin{cases} 6, & \text{if } -3 \leq x \leq 0 \\ x^2, & \text{if } 0 \leq x \leq 4 \\ 2x, & \text{if } 4 \leq x \leq 6 \end{cases}$$

6. The relationship defined by the scatter plot:



7. The relationship defined by the verbal description:

Ladd loves chocolate covered almonds, but he knows that he has to exercise self-control or else he will eat too many. So on Sunday he allows himself to have 20, and then 2 less on each successive day of the week; unless he does not have a date for Saturday evening, in which case he lets himself have 25 on Saturday. Assume the domain is the 7 days of the week, Sunday through Saturday.

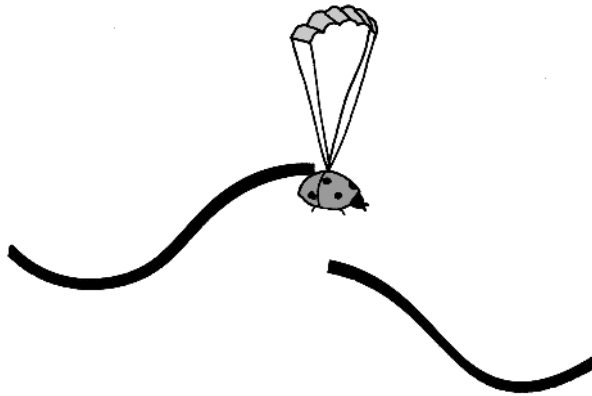


8. The graph given by the equation:

$$x^2 + y^2 = 1$$

**HW2.4** In this exercise, you will develop an intuitive understanding of continuity and classify points of discontinuity.

The word “continuous” means going on without interruption. An intuitive way to test if a function is *continuous* over its domain is to trace the graph representing the function with a pencil. If you can do so without lifting the pencil, it is a continuous function. On the other hand, if the function is *discontinuous* at a point in the domain, then you must lift the pencil at that point since you cannot continue “without interruption.”



A function is said to have

- a *removable discontinuity* or a *hole* if the function can be redefined so that it is continuous at that point;
- a *jump discontinuity* if the graph has a vertical gap at that point (see the diagram given above);
- a *blow-up discontinuity* if the function has a vertical asymptote at that point—that is, if the values of the function explode as you approach the point from either the left and/or the right.

In order to use the do-I-or-don't-I-have-to-lift-the-pencil approach you need to know what the graph of the function looks like, which is not always the case. Moreover, to determine whether a function is continuous throughout its domain, you need to know what the entire graph looks like, which may not be possible if, for example, its domain is the set of all real numbers. To actually prove that a function is continuous you need another approach. You will use the tools of calculus to do this. In the meantime, use the intuitive approach.

1. For each of the following functions:

- Graph the function. Identify the points in the domain (if any) where the function fails to be continuous by looking at the graph.
- Classify each point where the function is discontinuous as a removable, jump, or blow-up discontinuity.

$$\text{a. } f(x) = \begin{cases} -x + 2, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

$$\text{b. } s(t) = \begin{cases} t^2, & \text{if } t < 0 \\ 4, & \text{if } t = 0 \\ t, & \text{if } t > 0 \end{cases}$$

$$\text{c. } h(t) = \begin{cases} t^2, & \text{if } t \leq 0 \\ -t, & \text{if } 0 < t \leq 4 \\ -5, & \text{if } t > 4 \end{cases}$$

$$\text{d. } f(x) = \begin{cases} 1/x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$



2. The *greatest integer function* assigns to each real number the largest integer less than or equal to that number. Denote the greatest integer in  $x$  by  $\lfloor x \rfloor$ .
  - a. Find  $\lfloor 3.14 \rfloor$ ,  $\lfloor 7 \rfloor$ , and  $\lfloor -2.6 \rfloor$ .
  - b. Sketch  $f(x) = \lfloor x \rfloor$  for  $-10 \leq x \leq 10$ .
  - c. Give the range of  $f$ .
  - d. List all the points where  $f$  is discontinuous. What types of discontinuity occur?
  - e. The greatest integer function is also called the *floor* function. Explain why naming the function “floor” makes sense.
3. Similarly,  $\lceil x \rceil$  returns the *smallest integer* greater than or equal to  $x$ .
  - a. Find  $\lceil 3.14 \rceil$ ,  $\lceil 7 \rceil$ , and  $\lceil -2.6 \rceil$ .
  - b. Sketch the graph of  $c$  for  $-10 \leq x \leq 10$ .
  - c. Give the range of  $c$ .
  - d. List all the points where  $c$  is discontinuous. What types of discontinuity occur?
  - e.  $c$  is also called the *ceiling* function. Explain why naming the function “ceiling” makes sense.
4. The cost of parking your car at Workshop International Airport (WIA) is free for 0 to 60 minutes, \$4 for 61 minutes to 2 hours, \$2 for each additional half-hour or fraction thereof, to a maximum of \$10 for the first 24 hours.
  - a. How much does it cost to park for 3 hr. 45 min.? 6 hr. 20 min.?
  - b. Sketch a graph of the cost function for the first day—that is, for  $0 \leq t \leq 24$  hours.
  - c. Give the range of the function.
  - d. List all the points where the function is discontinuous. Classify each point of discontinuity.
  - e. Represent the cost function for parking at WIA with a piecewise-defined function.

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**HW2.5** What is a function?

Write an essay explaining what a function is. Your essay should include descriptions of (1) independent versus dependent variables, (2) domain and range, (3) discrete versus continuous functions, and (4) at least three different ways to represent a function and how to determine if each representation is a function.

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## Task 2-5: Evaluating Combinations of Functions

1. Consider the functions

$$f(x) = x^2 - 1, \text{ where } x \text{ is any real number}$$

$$g(x) = x, \text{ where } x \geq 0$$

- a. Evaluate the following combinations of  $f$  and  $g$  at the given points. Show how you arrive at your results. For example, to evaluate  $(f + g)(4)$ , you might write

$$\begin{aligned} (f + g)(4) &= f(4) + g(4) && \text{use definition of sum} \\ &= 15 + 4 && \text{evaluate } f \text{ and } g \text{ at } x = 4 \end{aligned}$$

$$(f + g)(4) = 19 \quad \text{simplify}$$

(1)  $(f - 4g)(25)$

(2)  $\left(\frac{f}{g}\right)(9)$

(3)  $(fg)(6)$

(4)  $(6.5g)(100)$

- b. Explain why it is not possible to evaluate each of the following combinations.

(1)  $(gf)(-1)$

(2)  $\left(\frac{g}{f}\right)(1)$

2. Consider the functions

$x$	-2	-1	0	1	2	3	4
$f(x)$	0	5	3	1	-1	4	-1

$x$	-2	0	2	4	6	8	10
$g(x)$	3	-4	2	0	6	3	1

a. Evaluate the following combinations. Use the definitions of the various combinations to show how you arrive at your results.

(1)  $(f - g)(-2)$

(2)  $(2f + 10g)(0)$

(3)  $\left(\frac{g}{6f}\right)(4)$

(4)  $(gf)(-2)$

b. Explain why it is not possible to evaluate the following combinations.

(1)  $(f + g)(8)$

(2)  $\left(\frac{f}{g}\right)(1)$

(3)  $\left(\frac{f}{g}\right)(4)$

3. Consider the functions

$$f(x) = \begin{cases} 2x + 6, & \text{if } x \geq 0 \\ x - 8, & \text{if } x < 0 \end{cases}$$

$$g(x) = x^3, \quad \text{where } x \text{ is any real number}$$

a. Evaluate the following combinations. Use the definitions to show how you arrive at your results.

(1)  $(f - g)(-2)$

(2)  $(2f + 10g)(0)$

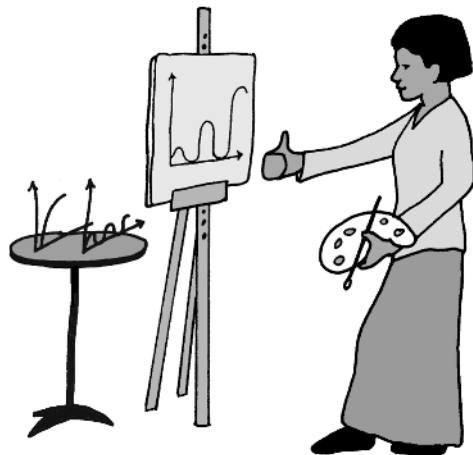
(3)  $\left(\frac{g}{6f}\right)(4)$

(4)  $(gf)(-2)$

b. Explain why it is impossible to evaluate  $\left(\frac{f}{9g}\right)(0)$ .

In the last task you found the value of a combination of functions at a particular input. In the next task, you will find a representation for the combination—which, of course, you can use to evaluate the new function at a given input. You will also determine the domain of the combination.

If  $f$  and  $g$  are represented by expressions, finding an expression to represent the new function is fairly straightforward: You combine the expressions (using the indicated operation) and, if possible, simplify. If  $f$  and  $g$  are represented by tables, sets of ordered pairs, or graphs, the situation is a little more interesting since you have to decide what it means to “add two tables” and so on.



If the domains of the underlying functions are the same, then the domain of the new function is usually the same set, except in the case of a quotient where you have to be careful not to divide by zero. On the other hand, if the domains of the functions are not the same, then the domain of the new function is the set of values where all the underlying functions are defined. You will have to figure out what this is by comparing their domains.

---

### Task 2-6: Combining Functions

In the following exercises, represent the specified combination using the same representation as the underlying functions. For instance, if you are asked to find a representation for  $f + g$ , where  $f$  and  $g$  are represented by tables, then use the definition of sum to find a table representing  $f + g$ .

Determine the domain of the combination by comparing the domains of the underlying functions. What values do they have in common? Do you have to worry about dividing by zero?

*Note: Although this is a pencil and paper exercise you may want to use your calculator to check your results. See "Combining Functions" in the appendix for your graphing calculator.*

1. Consider the functions:

$$f(x) = x^2 - 1, \quad \text{where } x \text{ is any real number}$$

$$g(x) = \sqrt{x}, \quad \text{where } x \geq 0$$

Represent each of the following combinations by an expression. Give the domain of the combination.

- a.  $f + g$

(1) Representation:

$$(f + g)(x) =$$

(2) Domain:

- b.  $2f$

(1) Representation:

$$(2f)(x) =$$

(2) Domain:

The next task will introduce you to the calculator syntax for representing functions defined by one or more expressions. Before starting the task, read through “Representing a Function by an Expression” and “Representing a Function by a Piecewise-Defined Function” in the appendix for your graphing calculator. Determine how to:

- Define a function
- Specify the domain of a function
- Evaluate a function at an input value
- Select and de-select functions

---

### Task 2-2: Implementing Functions Using Expressions

1. Use your calculator to define some functions that are represented by expressions.
  - a. Define and evaluate some functions using your calculator. Each time you ask your calculator to evaluate a function at a particular input, think about how the calculator will process the input. Take your time. Think about what is happening before you press the ENTER key. Form an image in your mind of the function processing the information.

Calculator Code	Predicted Output	Actual Output
$Y_1(X) = (X^2)/(X > 0)$		
$Y_1(4)$		
$Y_1(-5.57)$		
$6 \rightarrow X$		
$Y_1(X)$		
$X/2$		
$Y_1(X/2)$		
$8.0 \rightarrow R$		
$Y_1(R)$		
$Y_1(2R + 4)$		
$Y_1(-R)$		
$Y_2(X) = 3X + 8$		
$Y_2(10)$		

$Y_1(10)$		
$Y_1(10) + Y_2(10)$		
$2 Y_1(10) + Y_2(10)$		
$Y_2(4)/Y_1(4)$		

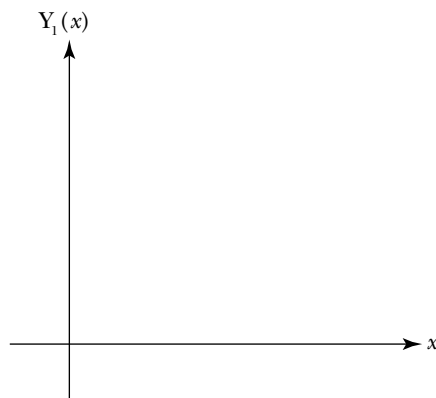
**b.** Answer some questions about the function  $Y_1$  defined in the previous session.

(1) Give the domain of  $Y_1$ .

(2) Give a verbal description of the process  $Y_1$  uses to transform an input item into a new item.

(3) Give a mathematical description of  $Y_1$ —that is, represent  $Y_1$  by an expression.

(4) Sketch a graph of  $Y_1$ . Label your axes carefully.



(5) Give the range of  $Y_1$ .

c. Construct some functions.

(1) Let  $h(t) = \frac{1}{t}$  for  $t \neq 0$ .

(a) Fill in the following table by evaluating  $h$  by hand. Express your results in decimal form. If  $h(t)$  does not exist, write “und”—for undefined—in the table.

$t$	-10	-1/10	0	0.1	1/2	5	1,000
$h(t)$							

(b) Define  $h$  in your calculator. Use your function to check your answers to part (a). Record the code for  $h$  below.

(2) Consider  $g(r) = \sqrt{r-1}$  for  $r \geq 1$ .

(a) Evaluate  $g$  (by hand) at the specified input values. If  $g(r)$  is undefined, write “und” in the table.

$r$	0	1	65	100	1.25	101/100
$g(r)$						

(b) Define  $g$  in your calculator. Use your function to check your answers to part (a). Record the code for  $g$  below.

(3) Suppose  $f$  accepts any negative real number as input.  $f$  cubes the input value, changes the sign of the result, and then returns its reciprocal.



(a) Give the domain of  $f$ .

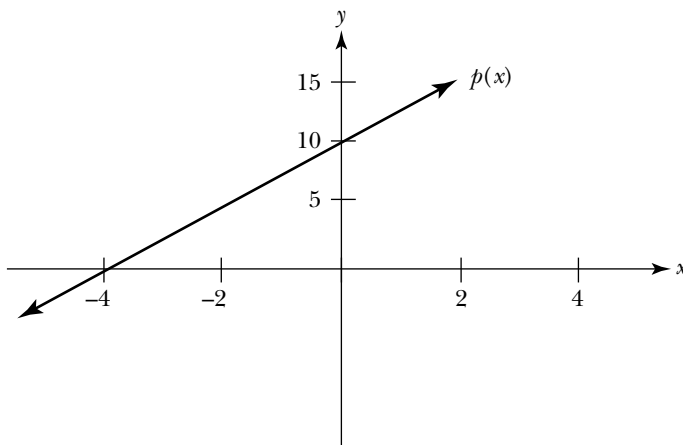
(b) Give a mathematical description of  $f$ .

(c) Evaluate  $f$  at the input values in the following table.

$x$	-10	-1	$-1/2$	-0.01	0	10.7
$f(x)$						

(d) Define  $f$  in your calculator. Use your function to check your answers to part (c). Record the code for your function below.

(4) Consider the function  $p$  represented by the following graph.



Assume the domain of  $p$  is the set of all real numbers.

(a) Give a mathematical description of the function  $p$ .

(b) Evaluate  $p$  at the given input values.

$x$	-6	-4	0	$1/4$	10	40
$p(x)$						

(c) Define  $p$  in your calculator, noting that there are no restrictions on the domain of  $p$ . Use your function to check your answers to part (b). Record the code for your function below.

2. Represent some piecewise-defined functions using your calculator.
- a. Define and evaluate some functions using your calculator. As usual, pay careful attention to the syntax. Whenever you instruct your calculator to evaluate a function at an input value, predict the calculator's output before pressing the ENTER key.

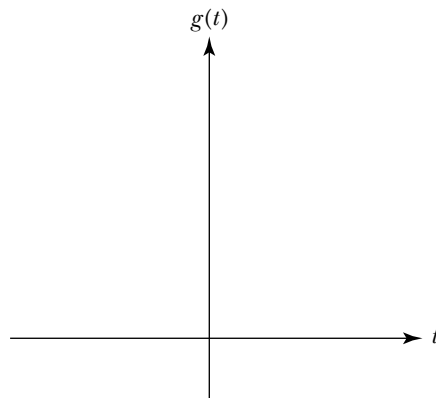
Calculator Code	Predicted Output	Actual Output
<p>Let <math>g</math> be the piecewise-defined function which is defined as follows:</p> $Y_1(X) = (-X + 1)/(-10 < X \text{ and } X < 0)$ $Y_2(X) = (5)/(X = 0)$ $Y_3(X) = (X - 1)/(X > 0)$ <p>Use the appropriate subfunction to evaluate <math>g</math> at the input values listed as follows. If <math>g</math> is not defined at the indicated value, write "und."</p>		
$g(-15.6)$		
$g(-2.7)$		
$g(0)$		
$2 \rightarrow S$		
$g(S)$		
$g(S^2 + 5)$		
<p>Let <math>h</math> be the function which is defined as follows:</p> $Y_4(X) = 2(X^3)$ <p>Use <math>h</math> to evaluate the following expressions:</p>		
$h(4)$		
$g(h(4))$		
$g(6.0)$		
$h(g(6.0))$		
$6 (g(0) + h(0))$		
$(3 h(-1) - 10 g(-1))/13$		

Note: We have chosen to represent each part of the piecewise-defined function by a separate subfunction. To evaluate the function, you will need to evaluate the appropriate subfunction.

b. Answer some questions about the piecewise-defined function  $g$  which is defined in the previous calculator session.

- (1) Give a verbal description of the process  $g$  uses to transform an input object into a new item.
- (2) Give a mathematical description of  $g$  by representing  $g$  by a piecewise-defined function using curly bracket notation.

(3) Sketch a graph of  $g$ . Label your axes carefully.



(4) Give the domain of  $g$ .

(5) Give the range of  $g$ .

c. Define some piecewise-defined functions using your calculator.

(1) Consider

$$f(t) = \begin{cases} t^2, & \text{if } t < 0 \\ -t, & \text{if } 0 \leq t < 5 \\ -2, & \text{if } t \geq 5 \end{cases}$$

(a) Evaluate  $f$  (by hand) at the values given in the following table.

$t$	-20	-0.25	0	3.75	5	40
$f(t)$						

- (b) Define  $f$  in your calculator. Use your function to check your answers to part (a). Record the code for your function below.

- (2) Suppose the function finds the reciprocal of the input value, if the input value is negative; otherwise it returns the square root of the input value.

(a) Give the domain of the function.

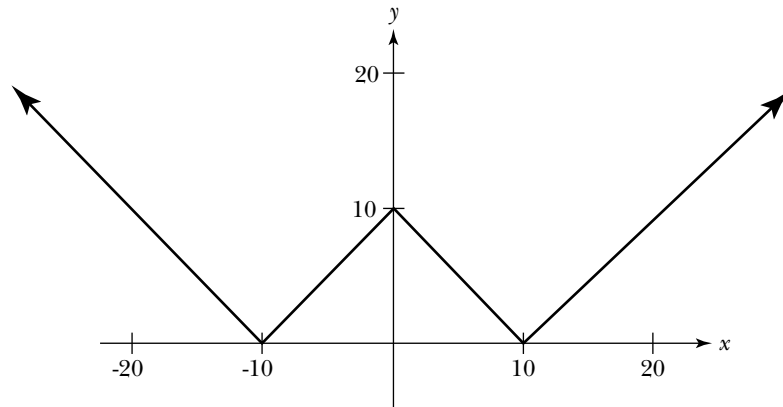
(b) Give a mathematical description of the function.

(c) Evaluate the function at the input values in the table.

$x$	-4	-0.25	0	0.25	9	11.8
$f(x)$						

- (d) Define the function in your calculator. Use your function to check your answers to part (c). Record the code for your function below.

- (3) Consider the function represented by the following graph.



(a) Give a mathematical description of the function.

(b) Evaluate the function at the given input values.

$x$	-20	-10	-5	0	5	10	20
$f(x)$							

(c) Define the function in your calculator. Use your function to check your answers to part (b). Record the code for your function below.

Thus far, you have examined two ways to construct a function using your calculator. First, you investigated how a function is represented by an expression. In this case, the function accepts an input value and tests if it is in the domain. If it is, the function implements the process of the function by substituting the value into the expression, performing the indicated computations, and returning the result. If the value is not in the domain, then the function returns an error message.

You also investigated how a *piecewise-defined* function can be represented using a subfunction for each part of the piecewise-defined function. To evaluate the function, you evaluate the appropriate subfunction.

In the next task you will examine how to represent a function by a graph. Before starting the task, read the sections concerning representing a function by a graph in the appendix for your graphing calculator. Learn how to:

- Select the functions to be graphed
- Choose the display styles
- Choose connected or dot mode
- Adjust the viewing window
- Turn the grid on and off
- Turn the labels on and off
- Use the TRACE feature
- Use the ZOOM feature

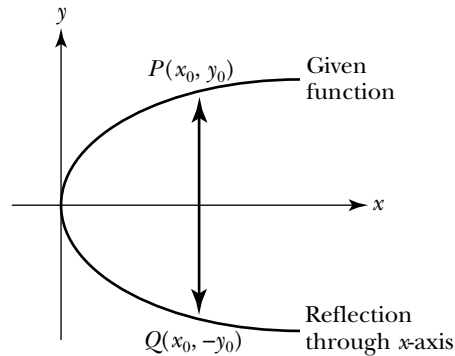
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### Task 2-3: Representing Functions by Graphs

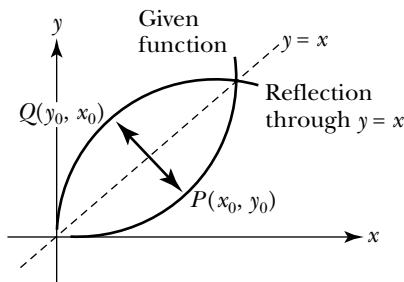
1. Consider the following function:

$$f(t) = \begin{cases} \sqrt{-t}, & \text{if } t \leq 0 \\ \sqrt{t}, & \text{if } t > 0 \end{cases}$$

When you *reflect a function through the  $x$ -axis*, each point  $P(x_0, y_0)$  on the graph of a given function corresponds to the point  $Q(x_0, -y_0)$  on the graph of its reflection. To represent the reflection by an expression, you replace each  $y$  in the expression for the given function with  $-y$  and solve for  $y$ , or if the function has a name, you find  $-f(x)$ .



Finally, when you *reflect a function through the line  $y = x$* , each point  $P(x_0, y_0)$  on the graph of a given function corresponds to the point  $Q(y_0, x_0)$  on the graph of its reflection. To represent the reflection by an expression, you interchange  $x$  and  $y$  in the expression for the given function and solve for  $y$ .



The reflection through the line  $y = x$  is probably the most useful type of reflection since it enables you to reverse, or undo, the process of a function. There is one problem, however: The reflection of a function through the line  $y = x$  is not necessarily itself a function. For example, the reflection of  $y = x^3$  is a function, whereas the reflection of  $y = x^2$  is not. What is it about these two functions that causes this to happen?

In the next task, you will discover conditions guaranteeing that the reflection of a function through  $y = x$  is itself a function. This new function is said to be the *inverse* of the given function. Since the inverse of a function reverses the process of the original function, composing a function with its inverse gets you back to where you started.

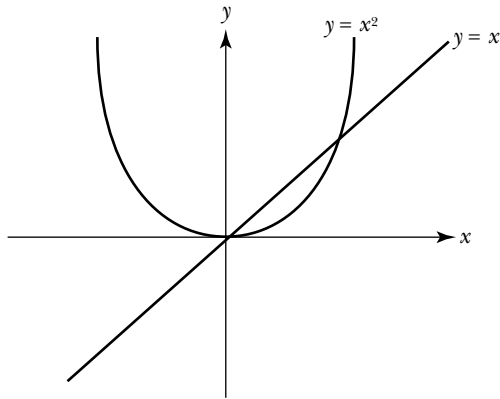
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### Task 2-10: Investigating Inverse Functions

1. In the following parts, examine some situations where the reflection through  $y = x$  is not a function. Think about what causes this to occur.

Recall that a relationship defines a function if corresponding to each input ( $x$  value) there exists exactly one output ( $y$  value). A graph represents a function if it passes the Vertical Line Test—that is, if every vertical line passes through at most one point on the graph.

Consider the graph of the function  $y = x^2$ .



- On the graph given above, sketch the reflection of  $y = x^2$  through the line  $y = x$ .
- Explain why the graphs show that the reflection of  $y = x^2$  through the line  $y = x$  is not a function.
- Find the coordinates of the reflection of some given points through the line  $y = x$  by filling in the chart given below.

$y = x^2$	Reflection
$P(-2,4)$	
$P(2,4)$	
$P(-3,9)$	
$P(3,9)$	
$P(-5,25)$	
$P(5,25)$	

- Explain why the chart shows that the reflection of  $y = x^2$  through the line  $y = x$  is not a function.

- e. In the case of  $y = x^2$ , except for the origin, each  $y$  value corresponds to two different  $x$  values. Consequently, when you interchange the  $x$  and  $y$  values, the reflection has two different output values corresponding to the same input value, and the reflection is not a function.

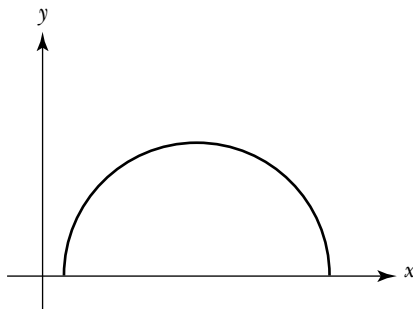
*Note: There only needs to be one instance of a  $y$  value corresponding to two different  $x$  values for the reflection to fail to be a function.*

The discussion above describes when the reflection of a function fails to be a function. Say the same thing, but say it in a positive sense. In other words, what condition must the given function satisfy to guarantee that its reflection through the line  $y = x$  is also a function?

2. If the reflection of a given function through the line  $y = x$  is also a function, the reflection is called the *inverse* of the given function. In this case, the given function is said to be a *one-to-one* function, since not only does each  $x$  value correspond to exactly one  $y$  value, but each  $y$  value corresponds to exactly one  $x$  value.

Determine if each of the following functions has an inverse—that is, if the given function is one-to-one. If it is not, show why not. If it is, represent the inverse using the same form as the given function.

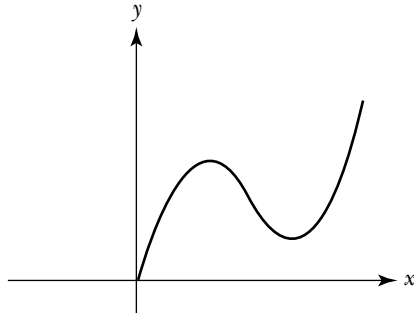
a.



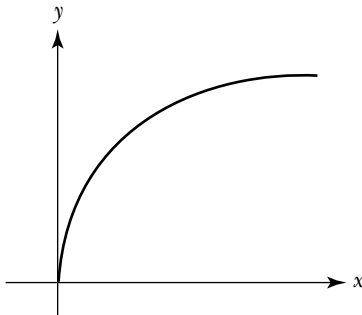
- b.  $y = -2x$ , where  $0 \leq x \leq 3$ .



c.



d.



3. The inverse of a function reverses, or undoes, the process of the given function. As a result, the composition of a function and its inverse returns the original input value. Moreover, if  $g$  is the inverse of  $f$ , then  $f$  is the inverse of  $g$ . Consequently,

$$f \circ g(x) = f(g(x)) = x, \text{ for all } x \text{ in the domain of } g$$

and

$$g \circ f(x) = g(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

- a. Show that  $f(x) = 3x - 1$  and  $g(x) = \frac{1}{3}x + \frac{1}{3}$  are inverses of each other by showing that  $f \circ g(x) = x$  and  $g \circ f(x) = x$ , for any real number.

**b.** The formula to convert a temperature  $t$  from Celsius to Fahrenheit is given by  $F(t) = \frac{9}{5}t + 32$ .

(1) Find the formula for converting a temperature  $t$  from Fahrenheit to Celsius.

$$C(t) =$$

(2) Show that  $F$  and  $C$  are inverses of each other.

**4.** If a function  $f$  has an inverse, its inverse is denoted by  $f^{-1}$ . Consequently,  $f^{-1} \circ f(x) = x$  and  $f \circ f^{-1}(x) = x$ .

*Caution: Do not confuse inverse functions with reciprocal functions. Just as  $4^{-1} = \frac{1}{4}$ ,  $(f(x))^{-1}$ , denotes the reciprocal of  $f(x)$ ; that is, in contrast,  $f^{-1}(x)$  denotes the inverse of  $f$  evaluated at  $x$ . In general, these are different functions. Be careful not to mix up their notations. To determine which is which, observe where the superscript “-1” appears.*

$$(f(x))^{-1} = \frac{1}{f(x)}$$

Suppose  $f$  has an inverse, where  $f(0) = -6$  and  $f(-6) = 2$ . Find the value of:

**a.**  $f^{-1}(-6)$

**b.**  $(f(-6))^{-1}$

**c.**  $f^{-1}(f(0))$

**d.**  $f \circ f^{-1}(2)$

e.  $f^{-1} \circ f(10)$

f.  $-(f(0))^{-1}$

### Unit 2 Homework After Section 3

- Complete the tasks in Section 3. Be prepared to discuss them in class.

**HW2.9** Consider the four functions:

- i.  $f(x) = x^2 + 1$ , where  $x \geq 0$ .
  - ii.  $f(x) = x$ , where  $x$  is any real number.
  - iii.  $f(x) = \sqrt{x} - 1$ , where  $x \geq 0$ .
  - iv.  $f(x) = -1$  where  $x \geq 0$ .
1. Recall that  $y = f(x)$ . Reflect each of the functions given above in i–iv through the  $y$ -axis.
    - a. Find a formula for the reflection.
    - b. Sketch the function and its reflection on a single pair of axes. Label the two graphs.
    - c. Give the domain of the reflection.
  2. Reflect each of the functions given above in i–iv through the  $x$ -axis.
    - a. Find a formula for the reflection.
    - b. Sketch the function and its reflection on a single pair of axes. Label the two graphs.

**HW2.10** Consider the functions:

- i.  $f(x) = -2x + 4$ , where  $x$  is any real number.
- ii.  $f(x) = 1$ , where  $x \leq 0$ .

For each of the functions given above in i–ii:

1. Sketch the function and its reflection through the line  $y = x$  on a single pair of axes. Label the two graphs.

2. Determine whether or not  $f$  has an inverse. If it does not, explain why. If  $f^{-1}$  does exist:
  - a. Represent  $f^{-1}$  by an expression and find the domain of  $f^{-1}$ .
  - b. Show that  $f \circ f^{-1}(x) = x$ , for all  $x$  in the domain of  $f^{-1}$ .
  - c. Show that  $f^{-1} \circ f(x) = x$ , for all  $x$  in the domain of  $f$ .

**HW2.11** Think in general about reflecting functions.

1. Describe the shape of a function whose reflection through the horizontal axis is the function itself—that is, when you reflect the function through the horizontal axis, you get the same function. Support your description by sketching the graph of a few functions where this is the case.
2. Describe the shape of a function whose reflection through the vertical axis is the function itself. Support your description by sketching the graph of a few functions where this is the case.
3. Describe the shape of a function whose reflection through the horizontal axis is the same as the reflection of the function through the vertical axis. Support your description by sketching the graph of a few functions where this is the case.
4. If the domain of a given function is the set of all numbers greater than or equal to zero, find:
  - a. the domain of the reflection of the function through the  $y$ -axis.
  - b. the domain of the reflection of the function through the  $x$ -axis.
5. Suppose  $f$  is a constantly increasing function.
  - a. Explain why  $f$  must have an inverse.
  - b. Is  $f^{-1}$  an increasing or a decreasing function? Justify your answer and support it with two examples.
  - c. If  $f$  is concave up, is  $f^{-1}$  concave up or concave down?
6. Describe the arrow diagram for a function that has an inverse.

**HW2.12** Invert some functions that are not represented by expressions.

To reflect a function represented by an expression through the line  $y = x$ , you interchange the values of  $x$  and  $y$ . This approach can be generalized

to functions represented in other forms by interchanging the associated input and output values.

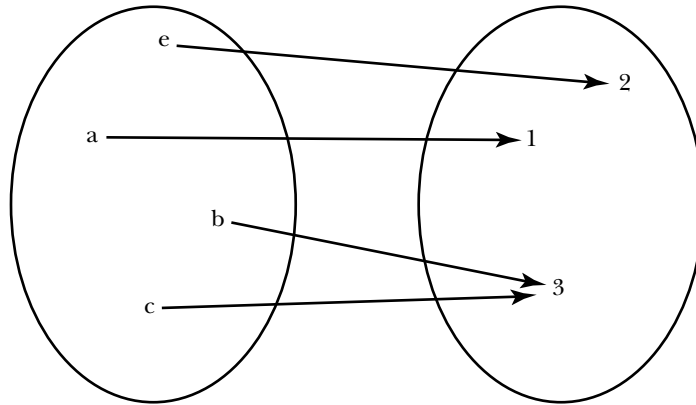
- Invert the functions given in parts a–e. Represent the reflection using the same representation as the given function. For instance, in part a,  $f$  is represented by a table; represent its reflection by a table.

a.

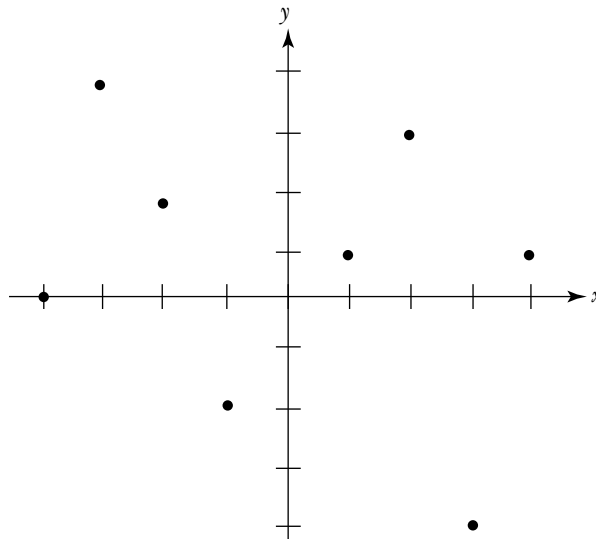
$x$	"a"	"c"	"f"	"i"	"m"	"u"	"z"
$f(x)$	5.4	-0.2	2.6	-1.5	2.4	0.2	8.5

b.  $f = \{(1,2), (3,4), (5,-9), (7,-7), (9,-5)\}$

c. The function represented by the arrow diagram:



d. The function represented by the scatter plot:



- Which reflections in part 1 are functions?

**HW2.13** The Vertical Line Test gives a way of testing whether or not a given graph represents a function. The Horizontal Line Test, on the other hand, gives a way of testing whether or not a given graph represents a function that is one-to-one—that is, if each  $y$  value corresponds to exactly one  $x$  value.

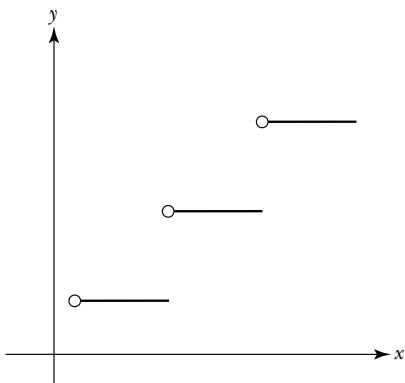
**1.** According to the Horizontal Line Test:

A graph represents a function that is one-to-one if and only if every horizontal line in the  $xy$ -plane intersects the graph in at most one point.

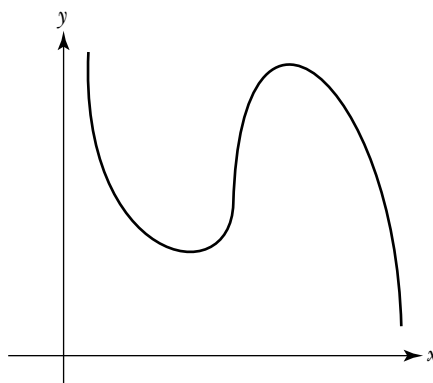
Explain why this makes sense. Support your explanation by sketching two appropriate diagrams (one which represents a one-to-one function and one which does not).

**2.** For each of the following graphs, use the Horizontal Line Test to determine whether the function is one-to-one and, consequently, whether it has an inverse.

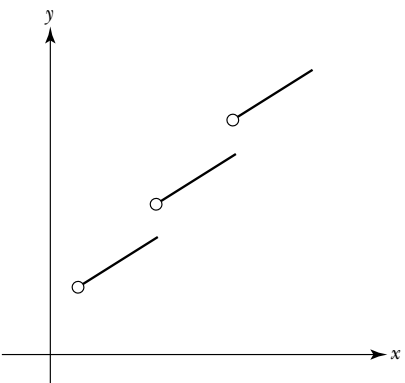
**a.**



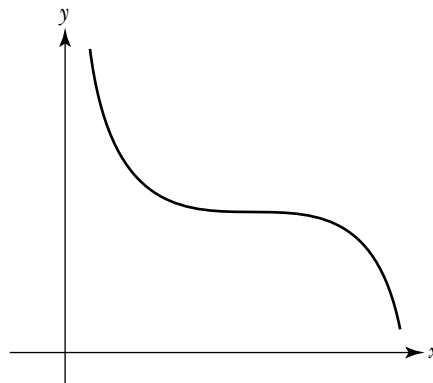
**c.**



**b.**



**d.**



**3.** If  $g(-4) = 1$  and  $g(0) = 1$ , use the Horizontal Line Test to explain why  $g$  does not have an inverse.

4. Remember your beer-selling uncle (see HW2.1, part 3)? The number of liters of beer he sells at a Saturday game is a function of the temperature, where

$$\text{NumLiters}(t) = 354 + 8(t - 62), \text{ for } 62^\circ\text{F} \leq t \leq 103^\circ\text{F}.$$

- a. Explain why  $\text{NumLiters}$  has an inverse.
- b. Find an expression for the inverse of  $\text{NumLiters}$ . Call it  $\text{Temp}$ .
- c. Find  $\text{Temp}(498)$ . Describe what this value represents.
- d. Find the domain and range of  $\text{Temp}$ .
- e. Prove that your expression for  $\text{Temp}$  is correct by showing that
  - (1)  $\text{Temp} \circ \text{NumLiters}(t) = t$ , for  $62^\circ\text{F} \leq t \leq 103^\circ\text{F}$ .
  - (2)  $\text{NumLiters} \circ \text{Temp}(t)$ , for all  $t$  in the domain of  $\text{Temp}$ .

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**HW2.14** A function is one-to-one if each output value corresponds to exactly one input value. Determine whether or not each of the following verbal descriptions represents a function that is one-to-one. Justify your response.

1. You manage a shop that sells 200 different small items. You never keep more than 100 of a particular item in stock. Your inventory control function maps each item in your shop to the number of that item that you currently have in stock.
2. The domain of the function is the set of all people in Pennsylvania, and the range is a subset of the positive integers. The function assigns each person to his or her age.
3. The domain of the function is the set of all people in the world. The range is a subset of all pairs of men and women. The function maps a person to his or her natural-parents.
4. The domain of the function is the set of all people. The range is the set of all right thumb prints. The function maps a person to his or her right thumb print.
5. The domain of the function is the set of all current students at Kirch College. The range is the set of all student mailbox numbers. The function assigns a student to his or her mailbox.
6. The function represents your movement in front of a motion detector over a 10-second time interval. Starting  $\frac{1}{2}$  meter from the detector, you slowly increase your distance from the detector for 5 seconds; you stand still for 3 seconds; and then move rapidly towards the detector during the remaining 2 seconds.

7. The function represents your movement in front of a motion detector over a 7-second time interval. Starting 4 meters from the detector, you decrease your distance from the detector. You start walking very slowly, and then walk faster and faster.

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**HW2.15** Write your journal entry for Unit 2. Before you begin to write, review the material in the unit. Think about how it all fits together. Try to identify what, if anything, is still causing you trouble.

1. Reflect on what you have learned in this unit. Describe in your own words the concepts that you studied and what you learned about them. How do they fit together? What concepts were easy? Hard? What were the main (important) ideas? Give some examples of the main ideas.
2. Reflect on the learning environment for the course. Describe the aspects of this unit and the learning environment that helped you understand the concepts you studied. What activities did you like? Dislike?



(a) Give a mathematical description of the function.

(b) Evaluate the function at the given input values.

$x$	-20	-10	-5	0	5	10	20
$f(x)$							

(c) Define the function in your calculator. Use your function to check your answers to part (b). Record the code for your function below.

Thus far, you have examined two ways to construct a function using your calculator. First, you investigated how a function is represented by an expression. In this case, the function accepts an input value and tests if it is in the domain. If it is, the function implements the process of the function by substituting the value into the expression, performing the indicated computations, and returning the result. If the value is not in the domain, then the function returns an error message.

You also investigated how a *piecewise-defined* function can be represented using a subfunction for each part of the piecewise-defined function. To evaluate the function, you evaluate the appropriate subfunction.

In the next task you will examine how to represent a function by a graph. Before starting the task, read the sections concerning representing a function by a graph in the appendix for your graphing calculator. Learn how to:

- Select the functions to be graphed
- Choose the display styles
- Choose connected or dot mode
- Adjust the viewing window
- Turn the grid on and off
- Turn the labels on and off
- Use the TRACE feature
- Use the ZOOM feature

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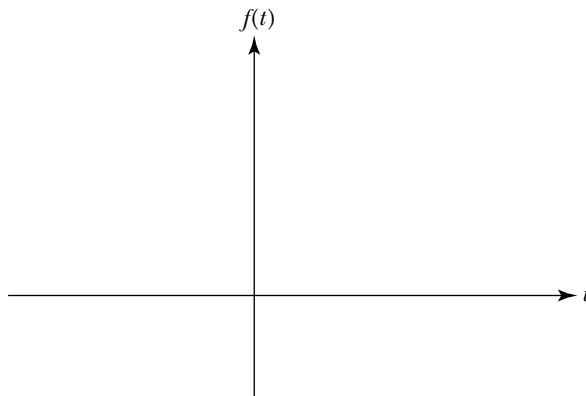
### Task 2-3: Representing Functions by Graphs

1. Consider the following function:

$$f(t) = \begin{cases} \sqrt{-t}, & \text{if } t \leq 0 \\ \sqrt{t}, & \text{if } t > 0 \end{cases}$$

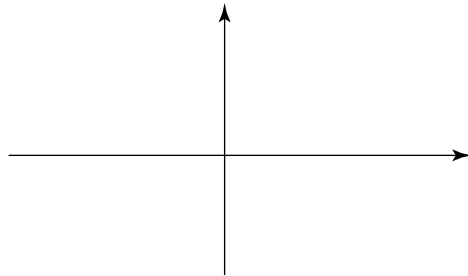
- a. Define the function in your calculator.
- b. Graph the function. Experiment with the various graphing options.
  - (1) Consider different sizes for the viewing window, such as:
    - $-16 \leq t \leq 16$  and  $0 \leq f(t) \leq 4$
    - $-25 \leq t \leq 0$  and  $0 \leq f(t) \leq 5$
    - $-10 \leq t \leq 100$  and  $0 \leq f(t) \leq 10$ .
  - (2) Use the TRACE feature to move the cursor along both pieces of the graph.
    - Observe what happens to the value of  $y$  when you move outside the domain of a subfunction.
    - Change the number of digits displayed to the right of the decimal point.
  - (3) ZOOM in on the graph at the origin. ZOOM back out.
  - (4) Turn the axes labels on and off.
  - (5) Turn the grid on and off.
  - (6) Use different display styles for the pieces of the graph.
  - (7) Change from connected mode to dot mode and vice versa.
- c. Sketch a graph of  $f$ . Label the horizontal and vertical axes.

*Note: To determine appropriate labels for the axes, use the TRACE feature with the grid on.*



2. Answer some questions about representing a function by a graph.
  - a. Describe how to determine the domain of a function by looking at its graphical representation.

- b. Describe how to determine the range of a function by looking at its graphical representation.
- c. Explain how you use a graph to find the output that corresponds to a given input.
3. Represent some functions using graphs. Analyze the behavior of the functions by observing their graphs.
- a. Let  $f(x) = \sin(x)$ , where  $-2\pi \leq x \leq 2\pi$ .
- (1) Give the domain of  $f$ .
- (2) Represent  $f$  by a graph. Sketch the graph on the following axes. Label the axes.



- (3) Use the TRACE feature to help analyze the behavior of  $f$ . Approximate:
- (a) The range of  $f$ .
- (b) The intervals where  $f$  is increasing.
- (c) The intervals where  $f$  is decreasing.

(d) The coordinates of the local maxima of  $f$ .

(e) The coordinates of the local minima of  $f$ .

(f) The intervals where the graph of  $f$  is concave up.

(g) The intervals where the graph of  $f$  is concave down.

b. Let 
$$h(r) = \begin{cases} -r, & \text{if } r < -5 \\ r - 3, & \text{if } -5 < r < 5 \\ -3r + 20, & \text{if } r > 5 \end{cases}$$

(1) Represent  $h$  by a graph. Sketch the graph in the following space. Label the axes. Place small open circles at endpoints of the graph where  $h$  is not defined.

(2) Describe the behavior of the graph of  $h$  when  $r = -5$  and when  $r = 5$ .

The domain of a function represented by a graph is determined by drawing a vertical line from each point on the graph to the horizontal axis. The process of the function is implemented by starting at a value in the domain, traveling in a vertical direction until you reach the graph, and then traveling in a horizontal direction to the corresponding value on the vertical axis.

Although most of the points on the graph of a function represented by an expression or a piecewise-defined function are connected, one question you need to answer is whether or not to ask your calculator to connect them. A calculator determines the shape of a graph by sampling a finite number

of points in the domain and evaluating the function at these points. It cannot possibly consider every point in the domain since it is a finite—discrete—machine. The question then is: Do the points which the machine has chosen provide an accurate reflection of the behavior of the function? Unfortunately, a finite sample cannot be used to predict with absolute certainty what happens throughout the domain. Connecting points in a finite sample may connect points that should not be connected, thereby giving a misleading picture of what is really going on. It is possible, for instance, that between two sample values the function “blows up,” has a “hole,” or oscillates rapidly. These types of behaviors may be missed. Consequently, the dot mode shows you what happens at the points in the finite sample and does not jump to any conclusions about what happens in between these values.

The domains of the functions which you considered in the last task contained an infinite number of points, such as the set of all real numbers or the set of all numbers between -3 and 10. The graphs of these functions consisted of lines and curves. The graph of a *discrete* function, on the other hand, consists of a collection of distinct, unconnected points, which is called a *scatter plot*. A discrete function whose domain contains a finite number of values can be represented by a table, a set of ordered pairs, or a scatter plot.

In the next task, you will investigate how to construct discrete functions and how to convert back and forth between their calculator representations and their mathematical representations. Before beginning the task, read the section “Representing a Discrete Function” in the appendix for your graphing calculator.

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### Task 2-4: Constructing Discrete Functions

1. Consider the discrete function

$$p = \{(5, 22), (4, 11), (-3, 13), (0, 5), (1, 7), (-1, 4), (-5, 26)\}$$

which is represented by a set of ordered pairs.

- a. The domain of a discrete function is the set of all first components of the ordered pairs. Give the domain of  $p$ .
- b. The range of a discrete function is the set of all second components of the ordered pairs. Give the range of  $p$ .
- c. The process of the function is implemented by assigning the second value in an ordered pair to the first. Sometimes you can see a pattern indicating how the function is defined, and sometimes you can't. In the case of  $p$ , there does not appear to be a pattern.

Use pencil and paper to evaluate  $p$  at the given input values. If  $p$  is not defined, write “und.”

### Task 2-9: Representing Reflections by Expressions

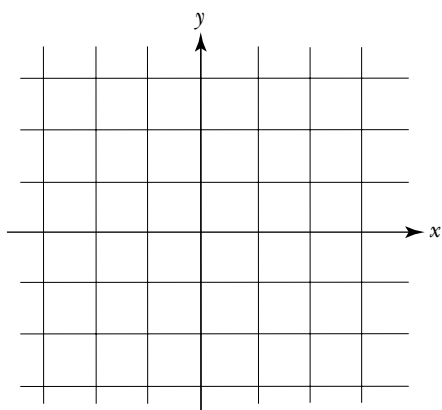
1. Reflect a function through the vertical or  $y$ -axis.

a. Reflect some points.

(1) Start by reflecting some specific points through the  $y$ -axis and finding their coordinates. Consider the points  $P_1(-2,2)$ ,  $P_2(-1,0)$ ,  $P_3(0,3)$ ,  $P_4(1,-3)$ ,  $P_5(2,1)$ , and  $P_6(3,-2)$ .

(a) Carefully plot the points  $P_1$  to  $P_6$  on the axes given below. Label the axes and label each point.

(b) Suppose the points  $Q_1$  to  $Q_6$  are the reflections of  $P_1$  to  $P_6$  through the  $y$ -axis. Find the coordinates of the reflections. Record your information in the table.



Point on Given Function	Reflection Through $y$ -axis
$P_1(-2,2)$	$Q_1( \quad , \quad )$
$P_2(-1,0)$	$Q_2( \quad , \quad )$
$P_3(0,3)$	$Q_3( \quad , \quad )$
$P_4(1,-3)$	$Q_4( \quad , \quad )$
$P_5(2,1)$	$Q_5( \quad , \quad )$
$P_6(3,-2)$	$Q_6( \quad , \quad )$

(2) Generalize your observations.

(a) Suppose the point  $Q$  is the reflection of the point  $P(x_0,y_0)$  through the  $y$ -axis. Find the coordinates of  $Q$  in terms of  $x_0$  and  $y_0$ ; that is, fill in

$$P(x_0,y_0) \leftrightarrow Q( \quad , \quad )$$

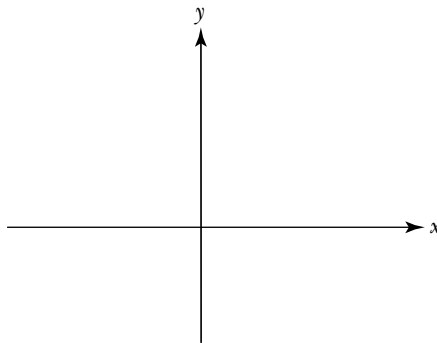
(b) Describe the relationship between the coordinates of a point and the coordinates of its reflection through the  $y$ -axis.

b. Find an expression for the reflection of a function through the vertical axis and examine how this expression is related to the expression representing the given function.

Based on your work in part a, you know that you can find the coordinates of the reflection of any point through the vertical axis by negating its first coordinate—that is, by replacing the value of  $x$  with

$-x$ . You can use the same approach to find the expression for the reflection; replace every  $x$  in the given expression with  $-x$ , and simplify (if possible). Try it.

- (1) Consider  $y = 2x + 1$ . Show that  $y = 2(-x) + 1$ , or  $y = -2x + 1$ , is the reflection of  $y = 2x + 1$  through the  $y$ -axis by graphing the two functions on the same axes.

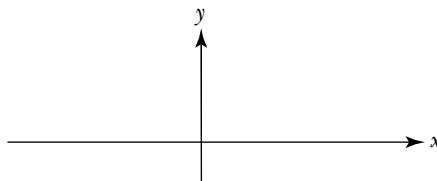


- (2) Consider  $y = \sqrt{x}$ , where  $x \geq 0$ .

- (a) Find an expression for the reflection of the given function through the  $y$ -axis.

*Caution: Be careful with the domain. You also have to replace  $x$  with  $-x$  here.*

- (b) Draw rough sketches of the given function and its reflection through the  $y$ -axis on the axes given below. Label the graphs.



- c. Remember that the graph of a function satisfies the Vertical Line Test. Is it possible for the reflection of a function through the  $y$ -axis to fail this test? In other words, is the reflection of a function through the  $y$ -axis always a function? Justify your response.

- d. In calculus, functions are usually given a name; for example,  $f(x) = 2x + 1$  or  $f(x) = \sqrt{x}$ . In this case, replacing  $x$  with  $-x$  is equiv-

alent to finding  $f(-x)$ . Represent the reflection of the following functions through the  $y$ -axis by an expression. Give the domain of the reflection.

(1)  $f(x) = 3x^2 - 6x + 9$ , where  $x$  is any real number.

(2)  $f(x) = x^2$ , where  $x \geq 0$ .

(3)  $f(x) = \frac{1}{x+1}$ , where  $x \neq -1$ .

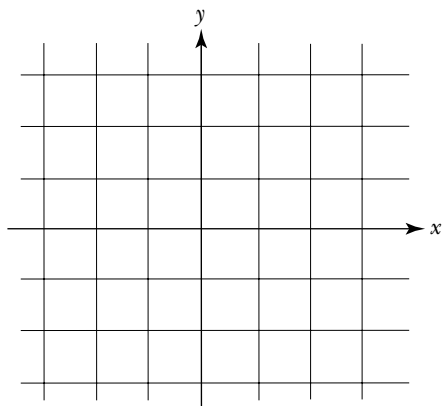
2. Next, instead of reflecting a function over the vertical axis, reflect it over the horizontal or  $x$ -axis.

a. Reflect some points.

(1) Start by reflecting some specific points through the  $x$ -axis and finding their coordinates. Consider the points  $P_1(-2,2)$ ,  $P_2(-1,0)$ ,  $P_3(0,3)$ ,  $P_4(1,-3)$ ,  $P_5(2,1)$ , and  $P_6(3,-2)$ . Note: These are the same points you considered in part 1.

(a) Replot  $P_1$  to  $P_6$  on the axes given below. Label the axes and label each point.

(b) This time, let  $Q_1$  to  $Q_6$  be the reflections of  $P_1$  to  $P_6$  through the  $x$ -axis. Find the coordinates of the reflections. Record your information in the table.



Point on Given Function	Reflection Through $x$ -axis
$P_1(-2,2)$	$Q_1( \quad , \quad )$
$P_2(-1,0)$	$Q_2( \quad , \quad )$
$P_3(0,3)$	$Q_3( \quad , \quad )$
$P_4(1,-3)$	$Q_4( \quad , \quad )$
$P_5(2,1)$	$Q_5( \quad , \quad )$
$P_6(3,-2)$	$Q_6( \quad , \quad )$



- (2) Generalize your observations.
- (a) Suppose  $Q$  is the reflection of the point  $P(x_0, y_0)$  through the  $x$ -axis. Find the coordinates of  $Q$  in terms of  $x_0$  and  $y_0$ .

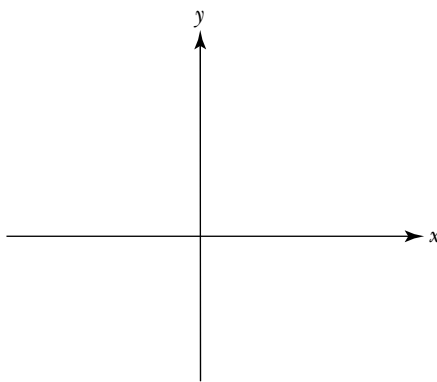
$$P(x_0, y_0) \leftrightarrow Q( \quad , \quad )$$

- (b) Describe the relationship between the coordinates of a point and the coordinates of its reflection through the  $x$ -axis.
- b. Find an expression for the reflection of a function through the horizontal axis and examine how this expression is related to the expression representing the given function.

You can find the coordinates of the reflection of any point through the horizontal axis by negating its second coordinate—that is, by replacing the value of  $y$  with  $-y$ . You can use the same approach to find an expression for the reflection; namely, replace  $y$  with  $-y$  in the given expression and solve for  $y$ .

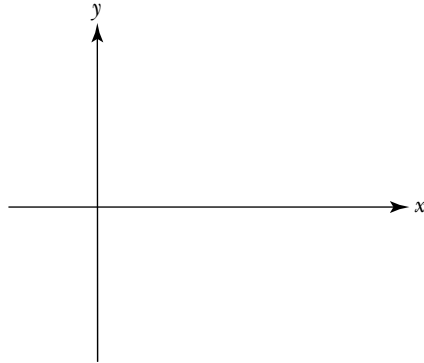
*Note: The following functions are the same functions you considered in part 1, only this time you are investigating their reflection through the horizontal axis.*

- (1) Consider  $y = 2x + 1$  once more. Show that  $-y = 2x + 1$ , or  $y = -2x - 1$ , is the reflection of  $y = 2x + 1$  through the  $x$ -axis by sketching graphs of the two functions on the same axes.



- (2) Consider  $y = \sqrt{x}$ , where  $x \geq 0$ .
- (a) Find an expression for the reflection of the given function through the  $x$ -axis. What is the domain of the reflection in this case?

- (b) Draw rough sketches of the given function and its reflection through the  $x$ -axis on the axes given below. Label the graphs.

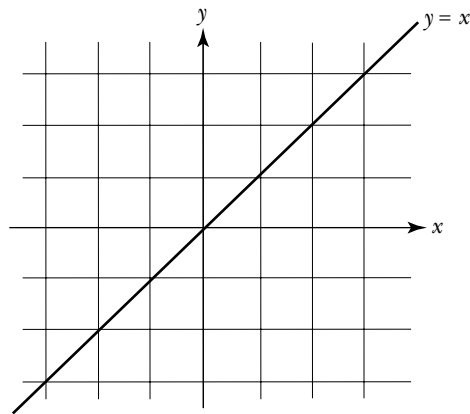


- c. The domain of the reflection of a function through the horizontal axis is always the same as the domain of the given function. Explain why this is the case.
- d. Is the reflection of a function through the  $x$ -axis always a function? Justify your response.
- e. If the function has a name, such as  $f$ , then  $y = f(x)$ . In this case, replacing  $y$  with  $-y$  is equivalent to finding  $-f(x)$ . Represent the reflection of the following functions through the  $x$ -axis by an expression.
- (1)  $f(x) = 3x^2 - 6x + 9$ , where  $x$  is any real number.
  - (2)  $f(x) = x^2$ , where  $x \geq 0$ .
  - (3)  $f(x) = \frac{1}{x+1}$ , where  $x \neq -1$ .

3. Finally, consider the reflection of a function through the line  $y = x$ .

a. Reflect some points.

- (1) Determine the relationship between the coordinates of a given point and the coordinates of the reflection of the point through the line  $y = x$ . Choose your own points. Find their reflections. Record your data on the graph and in the table given below. Figure out what happens in this case.



Point on Given Function	Reflection Through $y = x$

(2) Generalize your observations.

- (a) Suppose  $Q$  is the reflection of the point  $P(x_0, y_0)$  through the line  $y = x$ . Find the coordinates of  $Q$  in terms of  $x_0$  and  $y_0$ .

$$P(x_0, y_0) \leftrightarrow Q( \quad , \quad )$$

- (b) Describe the relationship between the coordinates of a point and the coordinates of its reflection through the line  $y = x$ .

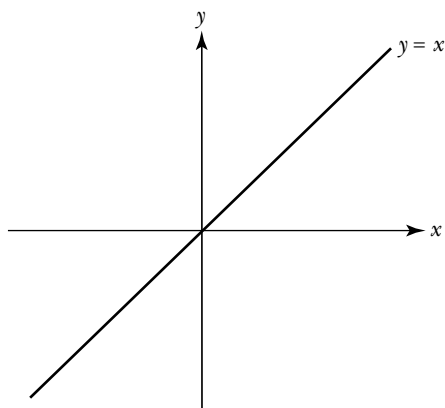
b. Find an expression for the reflection of a function through the line  $y = x$ .

- (1) Suppose you have a function which is represented by an expression. Based on your observations in part a, describe how you can find an expression for the reflection of this function through the line  $y = x$ .

(2) Consider  $y = 2x + 1$  once more.

(a) Use the approach you described above to find an expression for the reflection of  $y = 2x + 1$  through the line  $y = x$ .

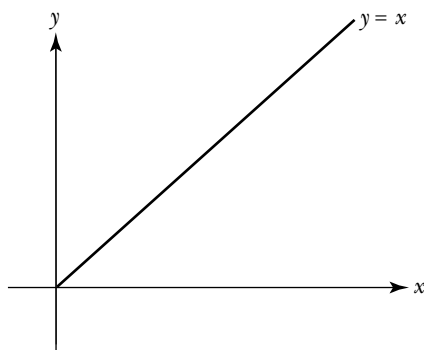
(b) Show that your approach works by sketching the graphs of  $y = 2x + 1$  and its reflection.



(3) Consider  $y = \sqrt{x}$ , where  $x \geq 0$ .

(a) Find an expression for the reflection of the given function through the line  $y = x$ . What is the domain of the reflection?

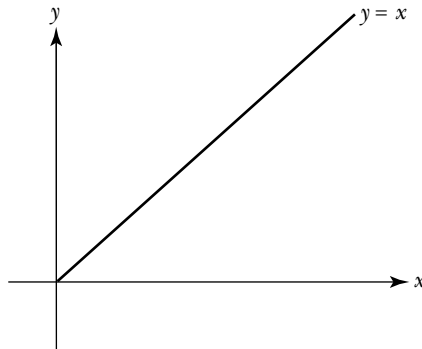
(b) Draw rough sketches of the given function and its reflection through the line  $y = x$ . Label the graphs.



(4) Consider  $y = 3$ .

(a) Find an expression for the reflection of the given function through the line  $y = x$ .

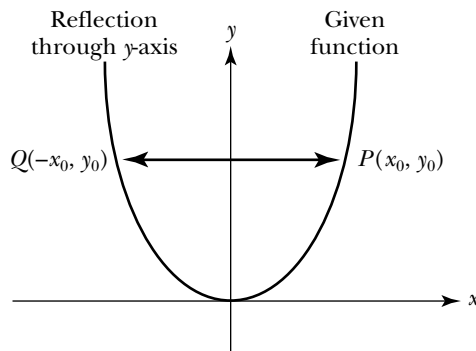
(b) Sketch the given function and its reflection through the line  $y = x$ . Label the graphs.



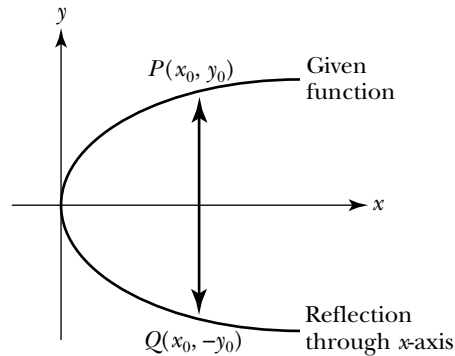
(c) Is the reflection of  $y = 3$  through the line  $y = x$  a function? Explain your response.

Before continuing on, let's summarize what you discovered in the last task.

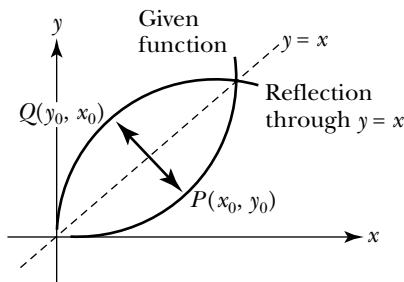
When you *reflect a function through the y-axis*, each point  $P(x_0, y_0)$  on the graph of a given function corresponds to the point  $Q(-x_0, y_0)$  on the graph of its reflection, as shown in the diagram below. To represent the reflection by an expression, you replace each  $x$  in the expression for the given function with  $-x$ , or if the function has a name, you find  $f(-x)$ .



When you *reflect a function through the  $x$ -axis*, each point  $P(x_0, y_0)$  on the graph of a given function corresponds to the point  $Q(x_0, -y_0)$  on the graph of its reflection. To represent the reflection by an expression, you replace each  $y$  in the expression for the given function with  $-y$  and solve for  $y$ , or if the function has a name, you find  $-f(x)$ .



Finally, when you *reflect a function through the line  $y = x$* , each point  $P(x_0, y_0)$  on the graph of a given function corresponds to the point  $Q(y_0, x_0)$  on the graph of its reflection. To represent the reflection by an expression, you interchange  $x$  and  $y$  in the expression for the given function and solve for  $y$ .



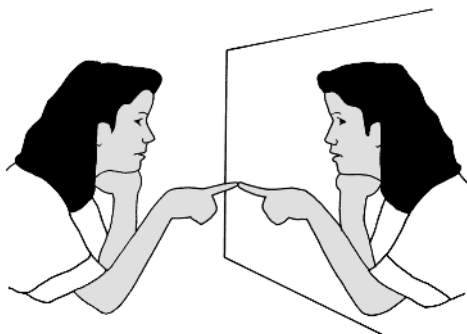
The reflection through the line  $y = x$  is probably the most useful type of reflection since it enables you to reverse, or undo, the process of a function. There is one problem, however: The reflection of a function through the line  $y = x$  is not necessarily itself a function. For example, the reflection of  $y = x^3$  is a function, whereas the reflection of  $y = x^2$  is not. What is it about these two functions that causes this to happen?

In the next task, you will discover conditions guaranteeing that the reflection of a function through  $y = x$  is itself a function. This new function is said to be the *inverse* of the given function. Since the inverse of a function reverses the process of the original function, composing a function with its inverse gets you back to where you started.

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### Task 2-10: Investigating Inverse Functions

1. In the following parts, examine some situations where the reflection through  $y = x$  is not a function. Think about what causes this to occur.



Reflection is a powerful way to create new functions. For instance, reflecting functions through the diagonal line  $y = x$  enables you to reverse a process; for example, if the original function converts temperatures from Celsius to Fahrenheit, its reflection converts from Fahrenheit to Celsius. In the next unit, the ideas underlying the reflection of functions will help you see the connection between exponential growth and decay functions and between logarithmic and exponential functions.

In this section you will consider what happens when you reflect a given function through each of the axes and through the line  $y = x$ . You will investigate how the graphs of a given function and its reflection are related and how their expressions are related. It turns out that the reflection of a function through the line  $y = x$  may not be a function; you will find conditions guaranteeing that a reflection of this type is also a function.

Start by sketching the graphs of some reflections in the next task.

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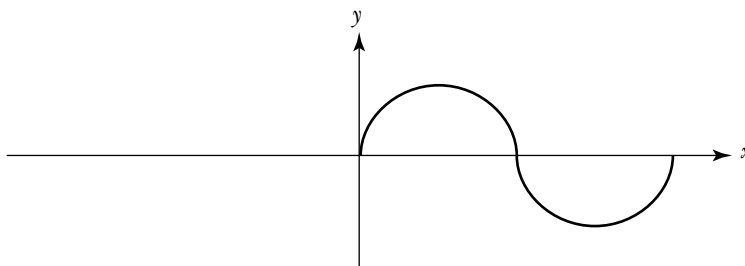
### Task 2-8: Sketching Reflections

1. Before beginning this task, look carefully at the figure given in the discussion at the beginning of this section. Observe that each point  $P$  on the graph of the given function corresponds to a point  $Q$  on the graph of the reflection, where  $Q$  can be obtained by moving in a direction perpendicular to the line of reflection and where the distance from  $P$  to the line of reflection equals the distance from  $Q$  to the line of reflection.

In parts a through j below:

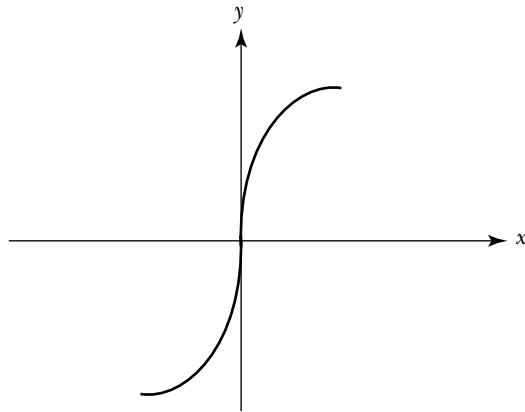
- i. Find the reflection of a few points on the given graph.
- ii. Sketch the graph of the reflection. Label the reflection.

a.



- (1) Reflect the given function through the  $x$ -axis. Label.
- (2) Reflect the given function through the  $y$ -axis. Label.

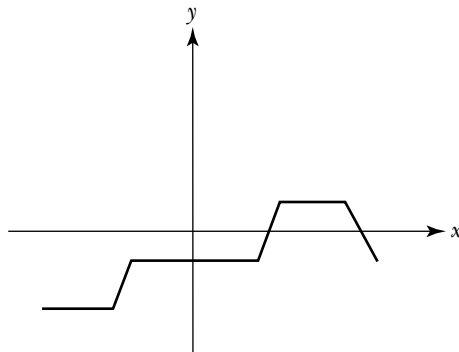
b.



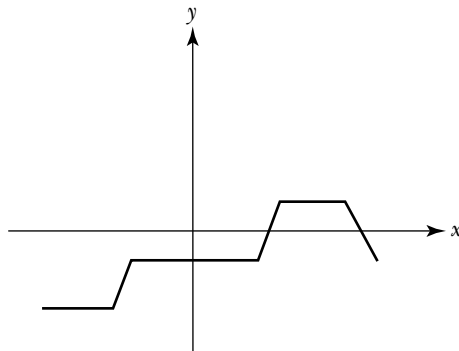
(1) Reflect the given function through the  $x$ -axis. Label.

(2) Reflect the given function through the  $y$ -axis. Label.

c. Reflect the following function through the  $x$ -axis.

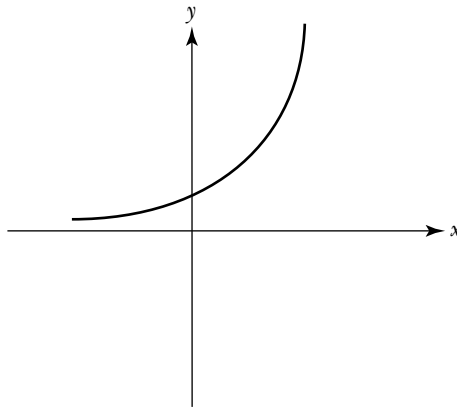


d. Consider the function given in part c again. Reflect the function through the  $y$ -axis.





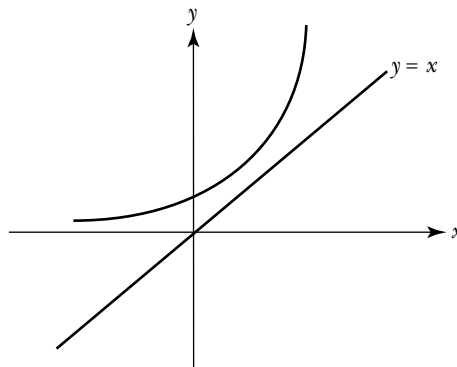
e.



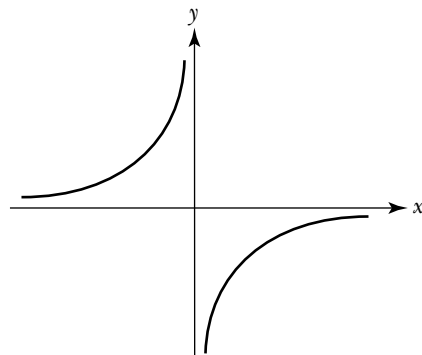
(1) Reflect the given function through the  $x$ -axis. Label.

(2) Reflect the given function through the  $y$ -axis. Label.

f. Consider the function given in part e again. Reflect the function through the line  $y = x$ . Be sure to show the reflection of several points on the given graph.



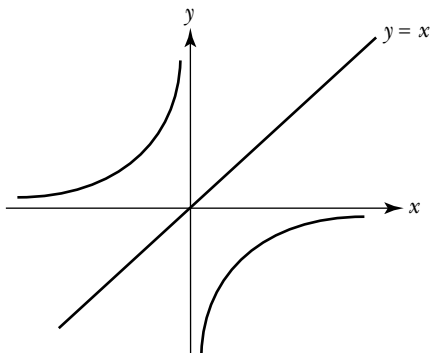
g.



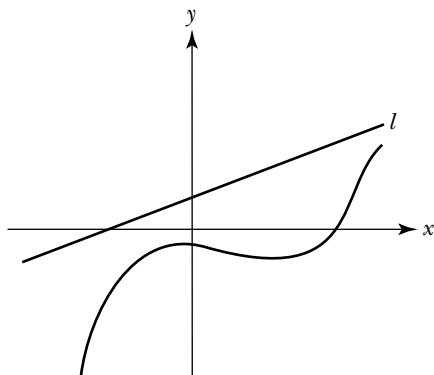
(1) Reflect the given function through the  $x$ -axis. Label.

(2) Reflect the given function through the  $y$ -axis. Label.

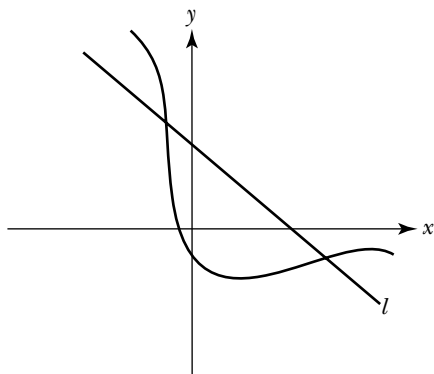
- h.** Consider the function given in part g again. Reflect the function through the line  $y = x$ .



- i.** Reflect the following function through the line  $l$ .



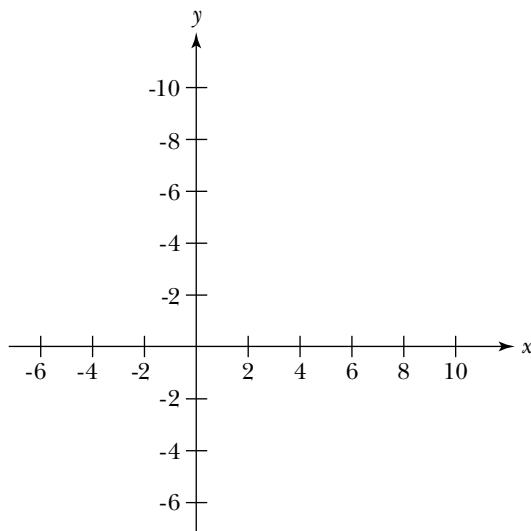
- j.** Reflect the following function through the line  $l$ .



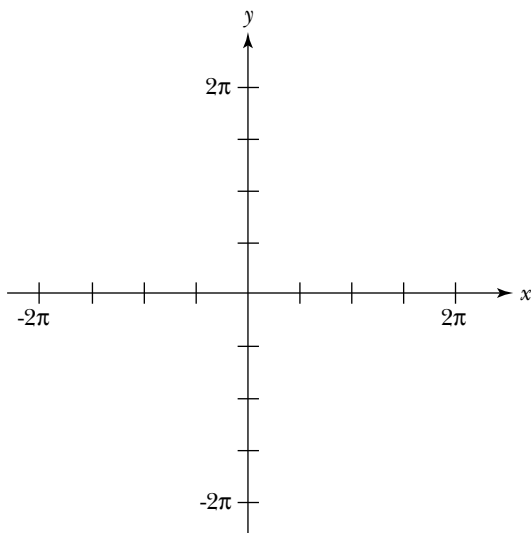
- 2.** Use your calculator to draw the reflections of some functions through the line  $y = x$ . Before you begin, read “Drawing the Reflection through the Line  $y = x$ ,” in the appendix for your calculator.

*Note: Since the range of the reflection equals the domain of the original function, set Ymin equal to Xmin and Ymax equal to Xmax in the graphing window.*

- a. Consider  $f(x) = x^3 + 3x^2 + 3$ , for  $-6 \leq x \leq 10$ . Draw the reflection of  $f$  through the line  $y = x$  on your calculator. Sketch the graphs below.



- b. Consider  $f(x) = \sin(x)$ , where  $-2\pi \leq x \leq 2\pi$ . Draw the reflection of  $f$  through the line  $y = x$  on your calculator. Sketch the graphs below.



In the last task, you examined how the graphs of a given function and its reflection are related. In the next task, you will examine how their expressions are related. You will explore questions such as: How can you go back and forth between the expression for the given function and its reflection? Is a reflection always a function? If not, what conditions guarantee that the reflection of a function is a function?

The impact, however, of this zero in the denominator on the graphs of  $r$  and  $s$  is quite different; one graph has a *hole*, whereas the other has a *vertical asymptote*. In the next task, you will investigate why this is the case. You will also learn how to determine whether a zero in the denominator results in a hole or an asymptote by examining the expression for the rational function.

Before starting the task, read “Evaluating a Function Using a Table,” in the appendix for your calculator.

### Task 3-2: Analyzing Rational Functions

- Examine the behavior of a rational function near a hole.

Consider

$$r(x) = \frac{x^2 - 2x - 3}{x - 3}$$

for  $x$  near 3.

- Give the domain of  $r$ .
- Examine the limiting behavior of  $r$  as  $x$  approaches 3 from either side.
  - Find some values of  $r(x)$  as  $x$  approaches 3 from the left by using your calculator to fill in the following table. Consider at least four digits to the right of the decimal place.

$x$	$r(x)$
2.0	
2.5	
2.75	
2.9	
2.99	
2.999	
2.9999	
3.00	und

Note: “und” indicates that  $r$  is not defined at  $x = 3$ .

- Find some values of  $r(x)$  as  $x$  approaches 3 from the right by filling in the following chart.

$x$	$r(x)$
4.0	
3.5	
3.25	
3.1	
3.01	
3.001	
3.0001	
3.00	und

(3) Based on the information in parts (1) and (2), what appears to be happening to the value of  $r(x)$  as  $x$  gets close to 3 from either side?

- c. Although  $r(x)$  is not defined at  $x = 3$ , you should have discovered that the value of  $r(x)$  appears to get closer and closer to 4 as  $x$  gets closer and closer to 3. This tells you that the graph of  $r$  has a *hole* at  $x = 3$ . In fact, it even tells you where the hole is, namely at the point  $P(3,4)$ . Another way to show that  $r$  has a hole at  $x = 3$  is to observe that *both* the numerator and the denominator of  $r$  have a zero at 3. Consequently, you can simplify the expression representing  $r$  by canceling the  $x - 3$  term.

Simplify  $r$  and observe what happens.

*Caution: The domain of the function is fixed at the outset. Simplifying the expression does not change the function's domain.*

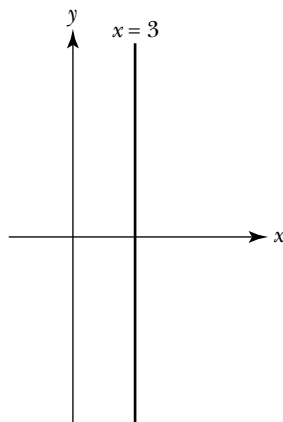
(1) Give the domain of  $r$  (again).

(2) Simplify the expression representing  $r$  by factoring the numerator and the denominator and canceling like terms.



$x$	$s(x)$	$x$	$s(x)$
2.5		3.5	
2.75		3.25	
2.9		3.1	
2.99		3.01	
2.999		3.001	
2.9999		3.0001	
3.0		3.0	

- (2) Describe what happens to the value of  $s(x)$  as  $x$  approaches 3 from the left.
- (3) Describe what happens to the value of  $s(x)$  as  $x$  approaches 3 from the right.
- d.  $s$  is not defined at  $x = 3$ . Moreover, the value of  $s(x)$  “explodes” as  $x$  gets closer and closer to 3. This tells you that the graph of  $s$  has a *vertical asymptote* at  $x = 3$ . The values in the table in part c also indicate how the function  $s$  behaves as  $x$  gets closer and closer to the asymptote. In particular, when the asymptote is approached from the left, the function is decreasing since the value of  $s(x)$  is approaching  $-\infty$ . On the other hand, when the asymptote is approached from the right, the function is increasing, since the value of  $s(x)$  is approaching  $+\infty$ .
- Sketch the graph of  $s$  for  $x$  near 3 on the axes given below.



3. Not only can a rational function have vertical asymptotes, it can also have a *horizontal asymptote*. In this case, the values of the function get closer and closer to a fixed number as  $x$  gets farther and farther away from 0 in the positive direction and/or the negative direction.

It turns out that the function

$$s(x) = \frac{x + 2}{x - 3}$$

which you analyzed in part 2 has a horizontal asymptote. Investigate the behavior of  $s$  as  $x$  gets larger and larger in both a positive and negative sense.

- a. Find some values of  $s(x)$  as  $x$  approaches  $+\infty$ . Calculate the values of  $s(x)$  to at least four decimal places.

$x$	10	100	1,000	10,000	100,000
$s(x)$					

- b. Find some values of  $s(x)$  as  $x$  approaches  $-\infty$ . Calculate the values of  $s(x)$  to four decimal places.

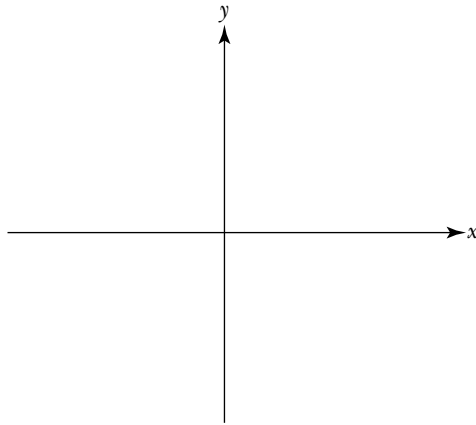
$x$	-10	-100	-1,000	-10,000	-100,000
$s(x)$					

- c. Based on the two tables, what appears to be happening to the values of  $s(x)$  as  $x$  gets larger and larger, in both the positive and negative directions?

4. Sketch the graph of the rational function  $s$ .
- a. Reiterate your observations from the work you did in the previous two parts.
- (1)  $s$  has a vertical asymptote at  $x = \underline{\hspace{2cm}}$ .
- (2)  $s$  has a horizontal asymptote at  $y = \underline{\hspace{2cm}}$ .
- b. Find the value of  $y$  where the graph of  $s$  intersects the  $y$ -axis—that is, find the  $y$ -intercept.



- c. Find the values of  $x$ , if any, where the graph of  $s$  intersects the  $x$ -axis—that is, find the roots of  $s$  or the  $x$ -intercepts.
- d. Sketch the graph of  $s$ , using the information from parts a, b, and c. Use dashed lines to indicate the locations of the vertical and horizontal asymptotes.



5. Summarize your observations.

Assume  $f$  is a rational function, where the numerator and denominator of  $f$  have been factored; that is, where the zeros of the numerator and the denominator are known. Explain how you can use the zeros to analyze the behavior of the function.

- a. Describe a general method for finding the domain of  $f$  by considering the zeros of the denominator of  $f$ .
- b. Describe a general method for finding the holes of  $f$  by comparing the zeros of the numerator and the denominator of  $f$ .
- c. Describe a general method for finding the vertical asymptotes of  $f$  by comparing the zeros of the numerator and the denominator of  $f$ .

- d. Describe a general method for finding the  $x$ -intercepts of  $f$  by using the zeros of the numerator of  $f$ .
- e. Describe a general method for finding the  $y$ -intercept of  $f$ .

Analyzing the behavior of polynomial and rational functions involves doing a lot of symbolic manipulations and calculations. Fortunately many of the calculations can be done using your graphing calculator. Using your calculator will save you from getting bogged down in all the nitty-gritty details. It will force you, however, to think at a higher level, since in order to use a calculator you must be able to articulate a procedure for solving a problem. For instance, if you want to find the location of all the holes and vertical asymptotes of a given rational function what might you do? First, you could find the candidates for the holes and vertical asymptotes by finding all the values where the denominator of the given function equals 0. Next you need to decide if a particular value results in a hole or an asymptote. Continuing to think generally, one approach is to find all the zeros of the numerator and compare them to the zeros of the denominator. If the numerator and the denominator have a common zero, then the associated terms cancel and the function has a hole at the zero. However, if a value appears in the list of zeros for the denominator, but is not in the list for the numerator, then the associated terms do not cancel and the function has a vertical asymptote at the zero.

To use your calculator to do this, you do need to know that instructing your calculator to find the zeros is an appropriate thing to do. You then need to be able to interpret the results. In addition, you need to be able to judge if the results that the calculator gives you are reasonable. Be careful not to blindly accept what the calculator tells you. Always question the output from the machine to make sure that it makes sense.



sidered do not contain any negative numbers, since in your friend's experiment it doesn't make sense to talk about negative time. Exponential functions, however, can be defined for all real numbers.

In general, if  $a$  is a fixed positive number, not equal to 1, then

$$f(x) = c \cdot a^x, \quad \text{where } -\infty < x < +\infty$$

is called an *exponential function* with base  $a$  and initial condition  $c$ . When  $0 < a < 1$ ,  $f$  is an *exponential decay* function. When  $a > 1$ ,  $f$  is an *exponential growth* function. Since  $f(0) = c$ , the value of  $c$  determines where the graph crosses the vertical axis.

In the next task, you will compare the behavior of exponential growth functions and the behavior of exponential decay functions for different values of the base. You will also compare the behavior of an exponential growth function, such as  $g(x) = 3^x$ , and its associated exponential decay function,  $d(x) = (\frac{1}{3})^x$ .

Before beginning Task 3-9, recall two rules for manipulating exponents:

$$\left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x}$$

and

$$a^{m/n} = \sqrt[n]{a^m}$$

According to these rules,  $(\frac{1}{8})^x$  and  $\frac{1}{8^x}$  and  $8^{-x}$  are three different ways of writing the same thing, and

$$8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

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### Task 3-9: Comparing Exponential Functions

1. Analyze how the value of the base affects the shape and location of an exponential growth function.
  - a. Consider the behavior of three exponential growth functions whose bases are  $a = 2$ ,  $a = 3$ , and  $a = 5$ . Let

$$f(x) = 2^x$$

$$g(x) = 3^x$$

$$h(x) = 5^x$$

- (1) Compare some output values for  $f$ ,  $g$ , and  $h$ .

Evaluate the functions at a variety of inputs, negative and positive, large and small. Record your results in the table given below.

$x$	$f(x) = 2^x$	$g(x) = 3^x$	$h(x) = 5^x$

(2) Compare the graphs of  $f$ ,  $g$ , and  $h$ .

Use your calculator to plot  $f$ ,  $g$ , and  $h$  on same pair of axes. To get a clear idea about what's happening, plot the graphs for  $-1 \leq x \leq 1$ .

Sketch the graphs below. Label each function.

(3) Summarize your observations concerning the behavior of  $f$ ,  $g$ , and  $h$ .

Compare the values of the three functions and describe the

relationship among the graphs, when  $x$  is negative, when  $x$  equals zero, and when  $x$  is positive.

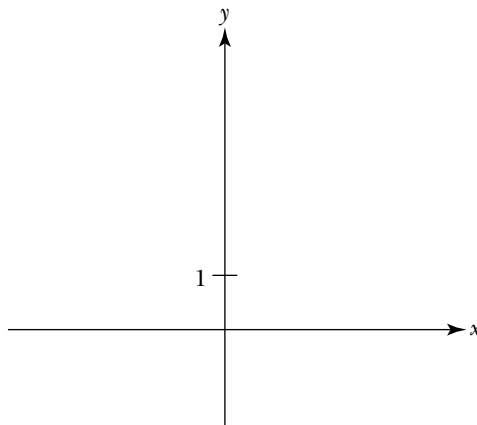
- b. Generalize your observations concerning the impact of the size of the base on the shape and location of the graph of an exponential growth function.

Let  $g$  and  $h$  be two arbitrary functions, having different bases, where

$$g(x) = p^x \quad \text{and} \quad h(x) = r^x, \quad \text{where } 1 < p < r$$

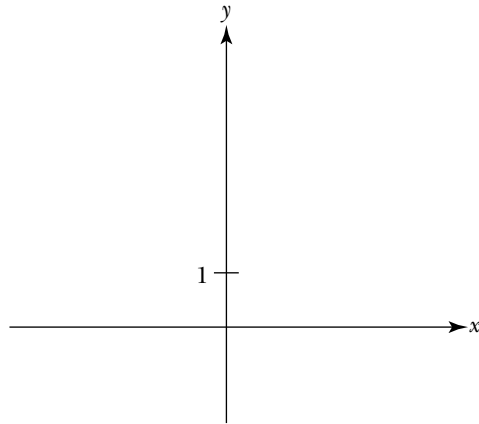
*Note: All you know about the bases  $p$  and  $r$  is that they are two real numbers, they are both greater than 1, and  $p$  is less than  $r$ . You do not know their actual values.*

- (1) Sketch (by hand) representative graphs of  $g$  and  $h$  on the axes below. Label the two graphs.

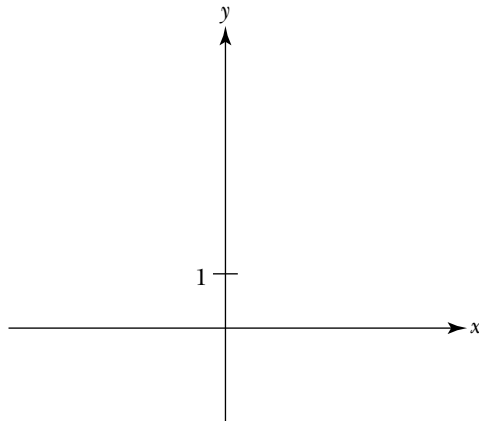


- (2) Describe the set of all  $x$  values such that  $g(x) < h(x)$ —that is, the set of all values where the graph of  $g$  lies below the graph of  $h$ .
- (3) Describe the set of all  $x$  values such that  $g(x) > h(x)$ —that is, the set of all values where the graph of  $g$  lies above the graph of  $h$ .

- (4) Describe the set of all  $x$  values such that  $g(x) = h(x)$ —that is, the set of all values where the two graphs intersect.
2. Investigate how the value of the base affects the shape and location of an exponential decay function.
- a. First, use your calculator to scrutinize some particular decay functions and note their relationships.
- (1) Consider  $g(x) = \left(\frac{1}{4}\right)^x$  and  $h(x) = \left(\frac{1}{2}\right)^x$ . Compare their values when  $x$  is negative, when  $x$  equals zero, and when  $x$  is positive. Compare the shape and location of the two graphs. Sketch the graphs of the functions on the axes given below. Label the graphs, indicating the value of the base for each function.

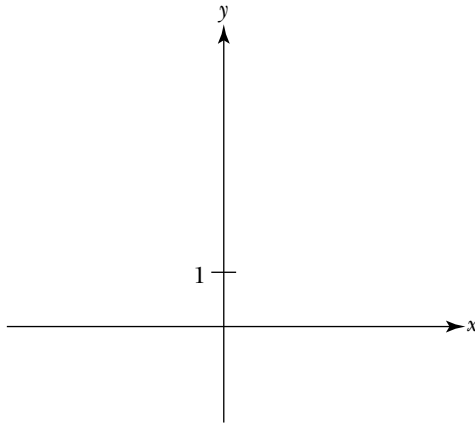


- (2) Repeat the process for another pair of values for the base. For instance, compare the decay functions generated by  $a = \frac{1}{3}$  and  $a = \frac{2}{3}$ . Sketch their graphs on the axes given below. Label the graphs.



b. Generalize your observations from part a.

- (1) On the pair of axes given below, sketch graphs of two arbitrary exponential decay functions  $g$  and  $h$ , where  $g(x) = u^x$ ,  $h(x) = w^x$ , and  $0 < u < w < 1$ . Label the graphs.



- (2) Describe the set of all  $x$  values such that  $g(x) < h(x)$ .

- (3) Describe the set of all  $x$  values such that  $g(x) > h(x)$ .

- (4) Describe the set of all  $x$  values such that  $g(x) = h(x)$ .

3. Compare a growth function to its associated decay function.

Notice that if the value of the base is greater than 1—that is, if  $a > 1$ —then the reciprocal of the base is a positive number less than 1—that is,  $0 < 1/a < 1$ . Consequently, for each value of  $a > 1$ , there exists an exponential growth function,

$$g(x) = a^x, \quad \text{where } -\infty < x < +\infty$$

and a corresponding exponential decay function

$$d(x) = \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x}, \quad \text{where } -\infty < x < +\infty$$

Find the relationship between the graphs of these associated functions.

- a.** For each of the following exponential functions, state whether it is a growth or decay function. If it is a growth function, give the expression for its associated decay function. If it is a decay function, give the expression for its associated growth function.

(1)  $f_1(x) = 3^x$

(2)  $f_2(x) = \frac{1}{8^x}$

(3)  $f_3(x) = 4^{-x}$

(4)  $f_4(x) = 0.25^x$

(5)  $f_5(x) = 2.5^x$

(6)  $f_6(x) = \left(\frac{1}{5}\right)^x$

(7)  $f_7(x) = \left(\frac{11}{2}\right)^x$

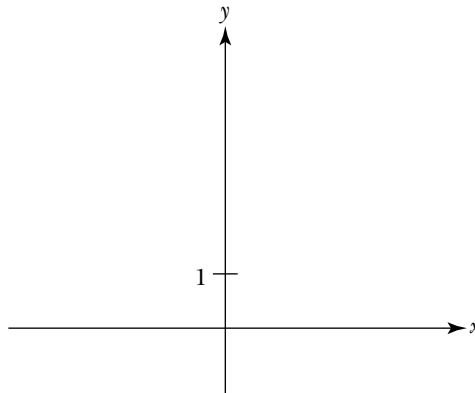
- b.** Compare the graphs of a growth function and its corresponding decay function.

- (1) Consider some particular functions.

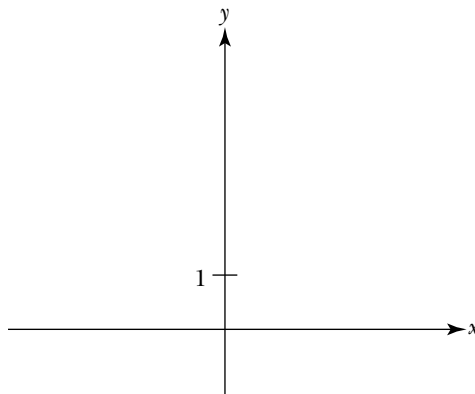
- (a) For instance, let  $a = 2$ . Use your calculator to graph  $g(x) = 2^x$  and  $h(x) = \left(\frac{1}{2}\right)^x$  on the same pair of axes for  $-4 \leq x \leq 4$ . Describe the relationship between the shapes and locations



of the two graphs. Sketch the graphs of the associated functions below. Label the functions.



- (b) Repeat the process for another pair of functions. For instance, consider  $a = 4$ . What do you notice about the shape and location of the graphs for this new value? Sketch the graphs of this pair of associated functions on the axes given below. Label the graphs.



- (2) Suppose  $a > 1$ . Describe the relationship between the graphs of an arbitrary growth function,  $g(x) = a^x$ , and its associated decay function,  $d(x) = (1/a)^x$ .

- c. Compare the expressions for a growth function and its corresponding decay function.

Recall that to represent the reflection of a function through the  $y$ -axis by an expression, you replace each  $x$  in the given expression

with  $-x$ . Use this fact to show that  $g(x) = a^x$  and  $d(x) = (1/a)^x$  are reflections of one another through the vertical axis.

4. Summarize your observations.

$f(x) = a^x$	$0 < a < 1$	$1 < a$
Sketch of typical graph		
Domain		
Range		
x-intercept(s)		
y-intercept		
Intervals where increasing		
Intervals where decreasing		
Intervals where concave up		
Intervals where concave down		
Intervals where one-to-one		

In the last task, you examined the behavior of exponential growth functions and exponential decay functions. You investigated the general shape of the graph of a growth function and the shape of a decay function; you compared the graphs of growth and decay functions that have different bases; you analyzed the relationship between a growth function and its corresponding decay function and observed that the two functions are reflections of one another through the vertical axis.

Now turn your attention to the class of functions called *logarithmic functions*, or *log functions* for short. Just as exponential growth and decay functions are reflections of one another, logarithmic functions and exponential functions are also reflections. In particular, they are inverse functions.

The term “inverse function” is familiar. In Unit 2, Section 3 you learned how to develop new functions from existing ones by performing reflections. In particular, given a one-to-one function  $f$ , you constructed its *inverse function*  $f^{-1}$  by reflecting  $f$  through the line  $y = x$ . Finding the reflection of a graph through  $y = x$  involved flipping the graph over the diagonal line. Representing the reflection of a function through  $y = x$  by an expression involved interchanging the  $x$ 's and  $y$ 's in the expression for the given function and solving for  $y$ .

Since an exponential function is a one-to-one function, it makes sense to find its inverse. This is the approach which you will use in the next task to develop the concept of a logarithmic function.

### Task 3-10: Investigating the Relationship Between Exponential and Logarithmic Functions

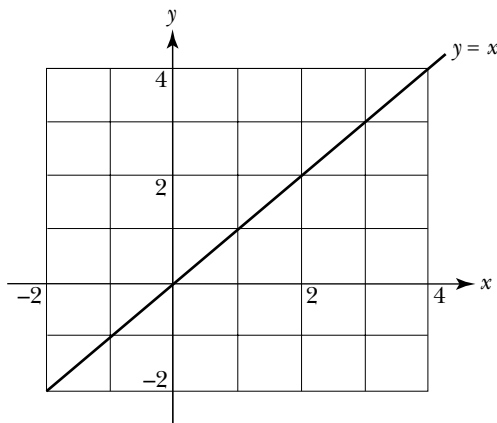
- Examine the reflection of a given exponential function through the line  $y = x$ .

Consider the exponential function with base 2,  $y = 2^x$ , where  $-\infty < x < +\infty$ .

- Find some values of  $y = 2^x$  by filling in the following table.

$x$	-2	-1	0	1	2
$y$					

- Graph  $y = 2^x$  and its reflection through  $y = x$  on the same pair of axes. Use the entries in the table for  $y = 2^x$ , recalling that you can graph the reflection by graphing the ordered pairs in reverse order.



Intervals where concave up		
Intervals where concave down		
Intervals where one-to-one		

In the last task, you examined the relationship between a given exponential function and its associated logarithmic function.

The next task will give you some practice working directly with logarithmic functions. In particular, you will:

- Convert back and forth between the two representations for logs, namely the exponential and logarithmic forms.
- Examine ways to evaluate a logarithmic expression, such as  $\log_2 128$ .
- Examine ways to graph a logarithmic function directly, without first graphing its associated exponential function.

### Task 3-11: Evaluating and Graphing Log Functions

1. Practice using logarithmic notation by converting between exponential and logarithmic forms.

As stated above,  $5^y = x$  is equivalent to  $y = \log_5 x$ , where  $5^y = x$  is in exponential form, and  $y = \log_5 x$  is in logarithmic form. The following table gives some equations in exponential and/or logarithmic forms. Fill in the missing entries.

Exponential form	Logarithmic form
$2^4 = 16$	$\log_2 16 = 4$
$5^3 = 125$	
	$\log_5 1 = 0$
	$\log_7 7 = 1$
$\pi^2 = 9.86 \dots$	$\log_\pi 9.86 \dots = 2$
$2^{-3} = (\frac{1}{8})$	
$e = 2.718 \dots$	$\log_e 2.718 \dots = 1$
$10^4 = 10,000$	
	$\log_6(\frac{1}{6}) = -1$
$\pi^1 = 3.2415 \dots$	
	$\log_2 \sqrt[3]{2} = \frac{1}{3}$

2. Evaluate some logarithmic expressions.

- a. One way to evaluate a logarithmic expression is to write the expression as an equation, convert the equation to exponential form, and then equate the exponents.

For example, to evaluate  $\log_{10} 100$  you would do the following:

$\log_{10} 100$	Given
$\log_{10} 100 = y$	Write as an equation
$10^y = 100$	Convert to exponential form
$10^y = 10^2$	Express right side as a power of the base, namely 10
$y = 2$	Equate exponents

Evaluate the following logs using this method. Use your calculator to check your answers, recalling that  $\log_a x = \ln x / \ln a$ .

(1)  $\log_{10} 1$

(2)  $\log_{10} 0.1$

(3)  $\log_{10} 0.001$

(4)  $\log_{10} 10^6$

(5)  $\log_2 \sqrt[3]{2}$

(6)  $\log_2 0.5$

(7)  $\log_{0.5} 1$

(8)  $\log_{0.5} \left(\frac{1}{2}\right)^{-7}$

$$(9) \log_{0.5} \left( \frac{1}{16} \right)$$

- b. You can also evaluate logs directly without writing an equation and converting its form. The key is knowing how to “read” logs.

For example, in order to evaluate  $\log_{10} 100$ , you could ask yourself:

*Question:* What power do I raise 10 to in order to equal 100?

*Answer:* Since 10 raised to the power 2 equals 100, the answer to the question is 2.

Therefore,  $\log_{10} 100 = 2$ .

Evaluate the following logs by translating the expression to the related question and then finding the answer. Use your calculator to check your answers.

- (1) Evaluate  $\log_2 32$ .

Question:

Answer:

Therefore,  $\log_2 32 =$

- (2) Evaluate  $\log_3 81$ .

Question:

Answer:

Therefore,  $\log_3 81 =$

- (3) Evaluate  $\log_4 64$ .

Question:

Answer:

Therefore,  $\log_4 64 =$

3. Graph some logarithmic functions directly (without sketching the associated exponential function).

A couple of comments before you begin:

- i. It is important to recognize that  $y = \log_a x$  and  $y = a^x$  are inverse functions, whereas  $y = \log_a x$  and  $a^y = x$  are equivalent forms of the same equation.
- ii. When graphing  $y = \log_a x$ , remember that  $x$  is the independent variable and  $y$  the dependent variable.
  - a. Graph a familiar log function using the usual approach.

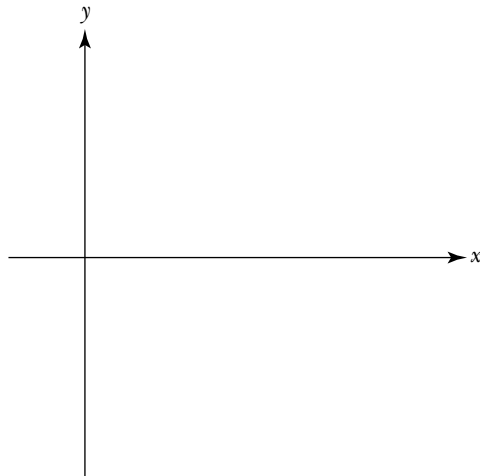
The “usual way” to graph a function is to choose values for the independent variable  $x$ , solve for the corresponding values of the dependent variable  $y$ , and then plot the points. Graph  $y = \log_2 x$  using the usual approach.

- (1) It is to your advantage to choose the  $x$  values “wisely,” such as  $x = 1, 2, 4, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ . Wisely choose four more values for  $x$  and record the corresponding  $y$  values in the table.

$x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4				
$y$										

- (2) Explain what it means to make a “wise” choice.

- (3) Graph the function  $y = \log_2 x$  by plotting the points in your table. Label the axes.



- b. Graph a familiar log function by considering the exponential form.

Another way to graph a log function is to convert the equation  $y = \log_a x$  to exponential form  $a^y = x$ . Then, instead of choosing values for the independent variable  $x$ , you choose values for the dependent variable  $y$  and solve for  $x$ . In other words, you reverse the process used in the usual approach. Graph  $y = \log_2 x$  using this approach.

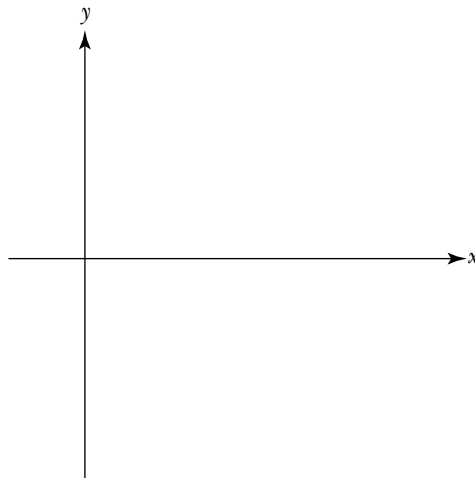
- (1) Convert  $y = \log_2 x$  into exponential form.

- (2) Find ordered pairs that satisfy the equation  $2^y = x$  by choosing values for  $y$  and then solving for  $x$ .

Build a table of 10 ordered pairs that satisfy  $2^y = x$ , where some values for  $y$  are already entered.

$x$										
$y$	-3	-2	-1	0	1	2	3			

- (3) Graph the function  $y = \log_2 x$ , by plotting the points in your table. Label the axes.



- c. Use your calculator to check that your graphs of  $y = \log_2 x$  are correct.
- d. Examine the domain of a logarithmic function.

The domain of  $y = \log_2 x$  appears to be  $x > 0$ . Investigate this a little more closely. Rewrite  $y = \log_2 x$  in its exponential form



$2^y = x$ . Can you find any value for  $y$  that is associated with a negative value of  $x$ ? Can you find a value for  $y$  that is associated with  $x = 0$ ? Justify your responses.

### Unit 3 Homework After Section 3

- Complete the tasks in Section 3. Be prepared to discuss them in class.

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**HW3.11** Convert between forms, evaluate some logarithms, and sketch and analyze graphs of some related functions in this exercise. In other words, practice.

1. Convert the following equations to logarithmic form.

a.  $6^3 = 216$

d.  $a^b = c$

b.  $(\frac{1}{3})^5 = \frac{1}{243}$

e.  $2.5^{-2} = 0.16$

c.  $7^p = q$

f.  $e^2 \approx 7.389$      *Note:  $e = 2.718281828 \dots$*

2. Convert the following equations to exponential form.

a.  $\log_4 64 = 3$

d.  $\log_{10} 10^5 = 5$

b.  $\log \frac{1}{8} 4096 = -4$

e.  $\log_m n = p$

c.  $\log_2 \frac{1}{8} = -3$

f.  $\ln(20.085) \approx 3$      *Note:  $\ln(x) = \log_e x$*

3. Evaluate the following logs.

a.  $\log_2 64$

d.  $\log_5 \sqrt[3]{5^2}$

b.  $\log_3 \frac{1}{9}$

e.  $\log_{0.5} (\frac{1}{2})^{-12}$

c.  $\log_{2.5} 1$

f.  $\log_4^1 16$

4. For each of the following (related) functions:

- Sketch the general shape of the function.
- Give the domain and range of the function.
- Describe the set of all  $x$  values where the function is increasing or decreasing.
- Describe the set of all  $x$  values where the function is concave up or down.

v. Give the set of all turning points.

a.  $f(x) = 4^x$

c.  $f(x) = \log_4 x$

b.  $f(x) = \frac{1}{4^x}$

d.  $f(x) = \log_4^{-1} x$

**HW3.12** For each of the three functions given below in parts 1 through 3, reflect the function

- a. through the  $y$ -axis.
- b. through the  $x$ -axis.
- c. through the line  $y = x$ .

For *each* reflection of the given function:

- (1) Represent the specified reflection by an expression.
  - (2) Sketch the general shape of the function and the specified reflection on a single pair of axes. Label the two graphs.
  - (3) Give the domain of the reflection.
1.  $y = e^x$ , where  $x$  is any real number.
  2.  $y = \log_{10} x$ , where  $x > 0$ .
  3.  $y = \frac{1}{3^x}$ , where  $x$  is any real number.

**HW3.13** Describe how the various exponential and logarithmic functions are related, where  $a$  is an arbitrary positive real number, not equal to 1.

1. Describe the relationship between  $y = a^x$  and  $y = a^{-x}$ .
2. Describe the relationship between  $y = a^x$  and  $y = \log_a x$ .
3. Describe the relationship between  $x = a^y$  and  $y = \log_a x$ .

**HW3.14** Properties of logarithms.

1. Prove the following properties, where  $a$  is a positive real number not equal to 1, by converting the given equation to exponential form.
  - a.  $\log_a 1 = 0$
  - b.  $\log_a a = 1$
  - c.  $\log_a a^x = x$ , for any real number  $x$

2. Suppose  $a$ ,  $r$  and  $s$  are positive real numbers, where  $a$  is not equal to 1. Some additional properties of logarithms are

- i.  $a^{\log_a r} = r$
- ii.  $\log_a rs = \log_a r + \log_a s$
- iii.  $\log_a \frac{r}{s} = \log_a r - \log_a s$
- iv.  $\log_a r^c = c \log_a r$ , for every real number  $c$

*Note: These properties can be proved using the laws of exponents and the fact that logarithmic and exponential functions are inverse functions.*

a. Use the properties of logs to express each of the following as a combination of  $\log_a x$ ,  $\log_a y$ , and  $\log_a z$ .

- (1)  $\log_a \frac{(x^2y)^2}{z}$
- (2)  $\log_a \sqrt{\left(\frac{x}{y}\right)^3}$
- (3)  $\log_a \frac{z\sqrt{x}}{(xy)^{\frac{1}{4}}}$
- (4)  $a^{\log_a(\log_a xy)}$

b. Use the properties of logs to express each of the following as the log of a single expression.

- (1)  $4 \log_a x - 2 \log_a y - 3 \log_a z$
- (2)  $\frac{1}{4} \log_a z + \frac{1}{2} \log_a y + \frac{3}{4} \log_a x$
- (3)  $6 \log_a (a^{\log_a x})$
- (4)  $\log_a (a^{2 \log_a x}) - \log_a (a^{3 \log_a y})$

c. Show that  $\log_a x = \frac{\ln x}{\ln a}$  by completing the following steps:

- Let  $y = \log_a x$ .
- Convert the equation from logarithmic to exponential form.
- Take the natural logarithm of both sides of the new equation.
- Solve for  $y$ . Hint: Use property iv.

**HW3.15** Compare the growth of an exponential function to a power function.

Consider the following salary negotiations between a king and a wanderer.

*King's offer:* I'll pay you 1¢ for the first day of work, and then the square of the number of the day for each day thereafter. That is, I'll pay you 1¢ for day 1, 4¢ for day 2, 9¢ for day 3, and so on.

*Wanderer's request:* Please pay me 2¢ for the first day and then twice as much (as the previous day's pay) for each day thereafter. In other words, pay me 2¢ for day 1, 4¢ for day 2, 8¢ for day 3, and so on.

Since the king was desperate for help, and the wanderer's request seemed reasonable, the king agreed.

1. Represent the king's offer by a function  $K$ , where  $K(t)$  is the amount of money (in cents) the king is willing to pay the wanderer for day  $t$ , where  $t = 1, 2, 3, 4, \dots$
2. Represent the wanderer's request by a function  $W$ , where  $W(t)$  is the amount of money (in cents) the wanderer wants to be paid for day  $t$ , where  $t = 1, 2, 3, 4, \dots$
3. Consider the wanderer's request.
  - a. On which day would the king pay the wanderer \$20.48?
  - b. What is the first day on which the king would pay the wanderer more than \$10,000?
  - c. How much would the king pay the wanderer on the 30th day?
4. Compare the king's offer and the wanderer's request.
  - a. On which day(s) would the wanderer receive the same amount with either plan?
  - b. Based on the two plans, what is the difference in the amount the wanderer would be paid on:
    - (1) day 10?
    - (2) day 15?
    - (3) day 20?
  - c. Graph the functions modeling the king's offer and the wanderer's request on the same pair of axes. Compare the growth rates of the two functions as the days go by.
  - d. Did the king make the right decision?

## SECTION 4

### Fitting Curves to Discrete Functions



Function classes enable you to fit a curve to—or find a model of—a set of real data. You can then use the model to analyze the data and make predictions. For example, suppose you are conducting a study of the number of HMOs (health maintenance organizations) in the United States between 1978 and 1990, and gather the following data, where  $t = 0$  corresponds to the baseline for your study, 1978, when there were 5,000 HMOs.

specified form. (As you look through the list, determine the degree of each polynomial.)

$$f(x) = 3x^3 - 4x^2 + 9x + 15$$

$$g(r) = r^{100}$$

$$k(x) = 2.5x^{10} - 3x^5 + 1.1$$

$$h(t) = 26.78$$

Some polynomials can be categorized based on the value of their degree. For instance,

Degree	Category	Example
0	Constant	$f(x) = -6.78$
1	Linear	$m(t) = 2t - 8.5$
2	Quadratic	$d(s) = 6s^2 - 4.5s + 99$
3	Cubic	$p(x) = x^3 - 2x + 1.4$

Polynomials are old friends. When you took algebra, you sketched their graphs by plotting points. You factored and simplified their expressions. You found their  $x$ - and  $y$ -intercepts. Given a verbal description of a polynomial, you represented it symbolically and vice versa. For instance, given information about a line, you found its corresponding expression, and given the equation for a line, you determined whether it rose or fell, described its steepness, and indicated where it intersected the vertical axis. In the next task, you will examine some of these ideas again. Later in this section, you will use your calculator to analyze the behavior of a rational function near a hole or a vertical asymptote.

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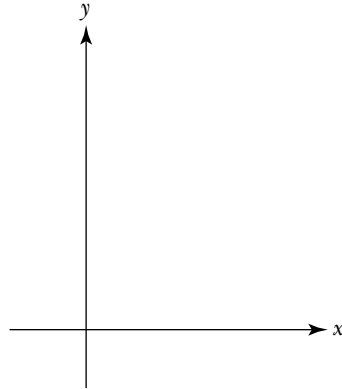
### Task 3-1: Examining Polynomial Functions

1. The domain of any polynomial is the set of all real numbers. Explain why this is the case by considering its general form.

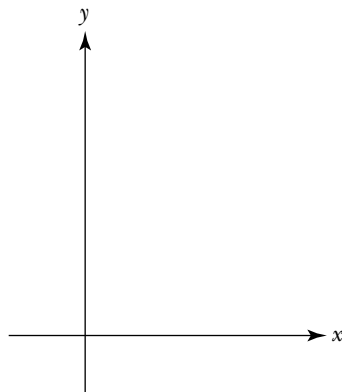
2. The  $y$ -intercept of a function is the value where the graph of the function crosses the vertical axis. If  $y_0$  is the  $y$ -intercept of a function  $f$ , then  $f(0) = y_0$ , and the graph of  $f$  passes through the point  $P(0, y_0)$ .

a. For each of the following polynomials find the value of its  $y$ -intercept. Mark its location on the vertical axis.

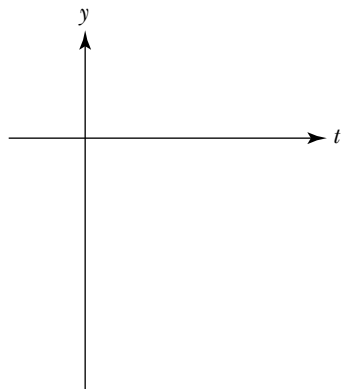
(1)  $f(x) = 3x^3 - 4x^2 + 9x + 15$



(2)  $g(x) = 2.5x^{10} - 3x^5 + 1.1$

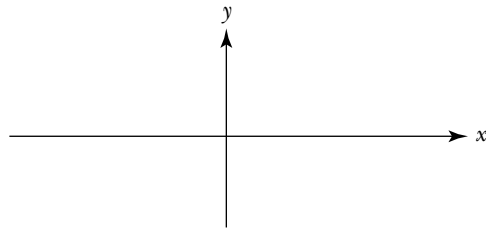


(3)  $h(t) = -6.78$

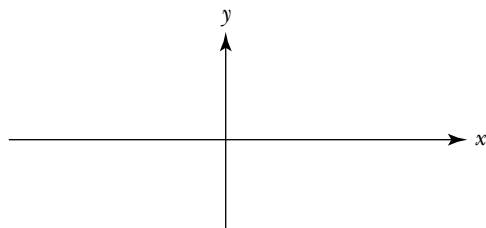


- b.** A polynomial has one, and only one,  $y$ -intercept. Explain why this is always the case. (First explain why a polynomial must have a  $y$ -intercept, and then explain why it cannot have more than one.)
- 3.** A polynomial crosses the vertical axis exactly one time. On the other hand, it might intersect—touch or cross—the horizontal axis once, twice, 10 times, or maybe not at all. Mathematically speaking, an  $x$ -intercept of a function is a value where the graph of the function intersects the horizontal axis. If  $x_0$  is an  $x$ -intercept of a function  $f$ , then  $f(x_0) = 0$ , and the graph of  $f$  passes through the point  $P(x_0, 0)$ . An  $x$ -intercept is also referred to as a *zero* or a *root* of the function.
- a.** If  $x_0$  is a root of a polynomial function, then  $(x - x_0)$  is a factor of the function. Consequently, one way to find the roots of the function is to:
- i.** Factor the polynomial expression.
  - ii.** Set each of the factors equal to 0 and solve for  $x$ .
- (1)** Find all the roots of the following polynomials by factoring. Show your work. Use this information to mark the  $x$ -intercepts on the given horizontal axis.

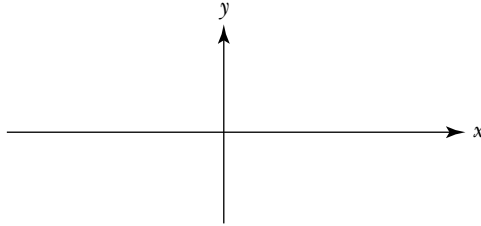
**(a)**  $g(x) = x^2 - 5x - 6$



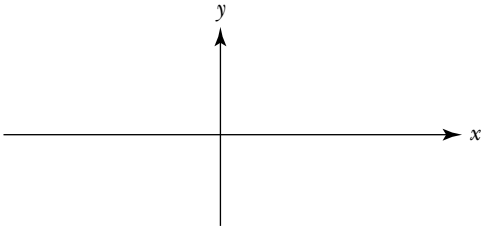
**(b)**  $p(x) = 3x^3 - 3x$



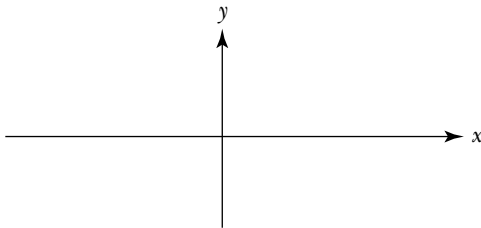
(c)  $h(x) = (x - 0.5)^5$



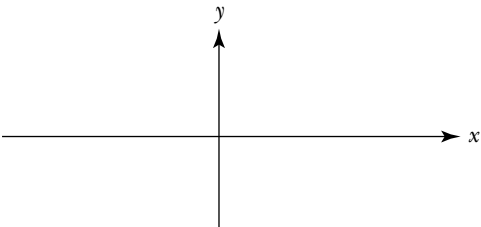
(d)  $m(x) = 26.78$



(e)  $y = \frac{5}{6}x - 10$



(f)  $q(x) = x^2 - 5x + 6$



- (2) Explain why the method of factoring method works; that is, explain why factoring a polynomial and setting its factors to 0 gives all the roots of the polynomial.



- b.** In the case of a second degree polynomial—that is, a polynomial which has the form

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$

—you can use the *quadratic formula* to find its roots where

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- (1)** Show that the quadratic formula yields the same results as factoring.

Consider  $f(x) = x^2 + 2x - 3$ .

- (a)** Use the quadratic formula to find the roots of  $f$ .

- (b)** Use factoring to find the roots of  $f$ . Your results should be the same as in part (a). If not, check your work.

- (2)** If the roots are not integers, it is almost impossible to factor a quadratic function by hand. The quadratic formula, on the other hand, always works.

- i.** Find the roots of the following polynomials by using the quadratic formula. Show your work.
- ii.** Use the roots to express each function as a product of its factors.

**(a)**  $q(x) = -x^2 + 5x - 4$

$$(b) f(x) = 3x^2 + 5x - 1$$

$$(c) r(x) = -4x^2 + 4x - 1$$

$$(d) s(x) = -2x^2 + 0.5x - 1$$

- (3) The graph of a quadratic function,  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ , is shaped like a “U” and is called a *parabola*. As a result, there are three possible scenarios: its graph does not intersect the horizontal axis, in which case the function has no real roots; its graph touches the horizontal axis at one point, in which case the function has one real root; or its graph intersects the axis at two distinct points, in which case it has two real roots.

The part of the quadratic formula under the square root sign,  $b^2 - 4ac$ , is called the *discriminant*. The sign of the discriminant indicates whether the quadratic function has no, one, or two real roots. For each of the cases given below:

- i. Give the sign of the discriminant: positive (+), negative (-), or zero (0).
- ii. Explain why this sign implies that a quadratic function has the specified number of roots.

iii. Sketch two quadratic functions that have the given number of roots.

(a) Case 1: no real roots.

(b) Case 2: one real root.

(c) Case 3: two real roots.

4. Examine the impact of the coefficients on the shape and location of the graphs of linear and quadratic functions.

a. A first degree polynomial is called a *linear function* and has the form  $f(x) = mx + b$ , where  $m \neq 0$ . Examine how the values of the coefficients  $m$  and  $b$  affect the graph of the function.

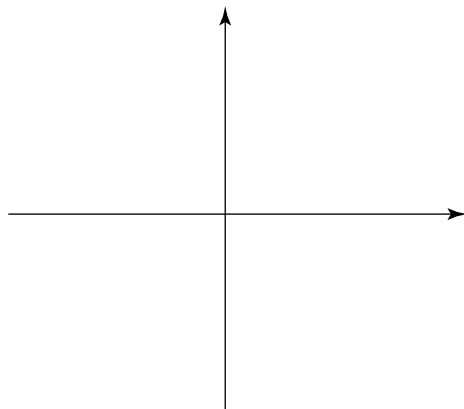
- (1) Describe the general shape of the graph of a first degree polynomial.
  
- (2) Describe the impact the sign of  $m$  has with respect to whether a linear function is increasing or decreasing. Support your claims with some sample sketches.
  
  
  
  
  
  
  
  
  
  
- (3) Describe what the size of the value of  $m$  (large positive versus small positive, more negative versus less negative) tells you about the steepness of the graph of the function. Support your description with some sample sketches.
  
  
  
  
  
  
  
  
  
  
- (4) Describe what the value of  $b$  tells you about the graph of the function. Support your description with a sketch.
  
  
  
  
  
  
  
  
  
  
- (5) Describe the relationship between two linear functions that have the same value for their leading coefficient,  $m$ . Support your description with a sketch.

(6) Find an equation for the  $x$ -intercept of  $f(x) = mx + b$ , where  $m \neq 0$ , in terms of  $m$  and  $b$ .

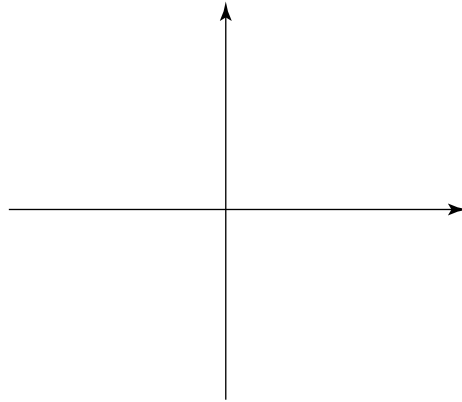
(7) What happens if  $m = 0$ ? Give the general form for the polynomial. Describe the shape and location of the graph. Support your description with some sample sketches.

b. Recall that the graph of a quadratic function,  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ , is always a parabola. As a result, the graph has exactly one turning point. The graph, however, can have a large variety of shapes—for instance, it can be concave up or down, be skinny or fat, and have zero, one, or two  $x$ -intercepts. Examine how the sign and the value of the coefficients  $a$  and  $c$  affect the shape and location of a parabola.

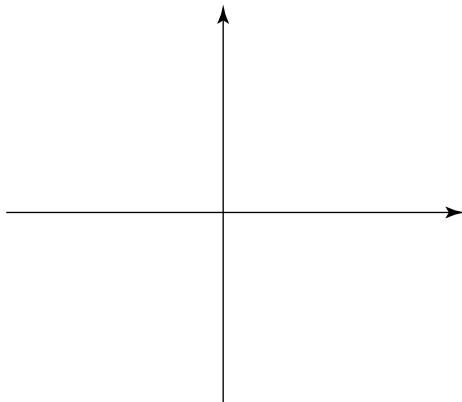
(1) On the axes given below, draw rough sketches of the graphs of  $y = x^2$  and  $y = -x^2$ . Describe in general how the sign of the leading coefficient,  $a$ , affects the concavity of a quadratic function.



(2) On the axes given below, draw rough sketches of the graphs of  $y = x^2 + 3$ ,  $y = x^2 - 1$ , and  $y = x^2 + 1.5$ . Describe how varying the value of  $c$  affects the location of the graph of  $f(x) = x^2 + c$ .



- (3) On the axes given below, draw rough sketches of the graphs of  $y = x^2$ ,  $y = 2x^2$ , and  $y = x^2/2$ . Describe how varying the value of  $a$  affects the shape of the graph of  $f(x) = ax^2$ .



A *rational function* is the quotient of two polynomials. An important question when analyzing the behavior of a rational function is:

What impact does a zero in the denominator of a rational function have on the shape of its graph?

For instance, neither of the following rational functions,  $r$  or  $s$ , is defined at  $x = 3$ , since you cannot divide by zero.

$$r(x) = \frac{x^2 - 2x - 3}{x - 3}$$

$$s(x) = \frac{x + 2}{x - 3}$$

The impact, however, of this zero in the denominator on the graphs of  $r$  and  $s$  is quite different; one graph has a *hole*, whereas the other has a *vertical asymptote*. In the next task, you will investigate why this is the case. You will also learn how to determine whether a zero in the denominator results in a hole or an asymptote by examining the expression for the rational function.

Before starting the task, read “Evaluating a Function Using a Table,” in the appendix for your calculator.

### Task 3-2: Analyzing Rational Functions

- Examine the behavior of a rational function near a hole.

Consider

$$r(x) = \frac{x^2 - 2x - 3}{x - 3}$$

for  $x$  near 3.

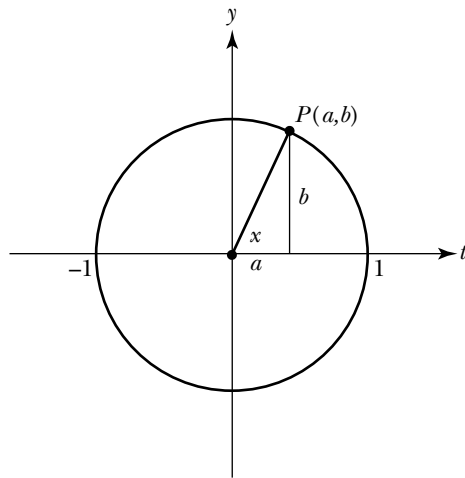
- Give the domain of  $r$ .
- Examine the limiting behavior of  $r$  as  $x$  approaches 3 from either side.
  - Find some values of  $r(x)$  as  $x$  approaches 3 from the left by using your calculator to fill in the following table. Consider at least four digits to the right of the decimal place.

$x$	$r(x)$
2.0	
2.5	
2.75	
2.9	
2.99	
2.999	
2.9999	
3.00	und

Note: “und” indicates that  $r$  is not defined at  $x = 3$ .

- Find some values of  $r(x)$  as  $x$  approaches 3 from the right by filling in the following chart.

the *sine* function returns the second coordinate, the *tangent* function returns the quotient of the coordinates, and the *secant*, *cosecant*, and *cotangent* functions are the reciprocals of these three basic functions; that is, the six fundamental trigonometric functions are defined as follows:



Sine	$\sin(x) = b$	Cosecant	$\csc(x) = \frac{1}{b}$
Cosine	$\cos(x) = a$	Secant	$\sec(x) = \frac{1}{a}$
Tangent	$\tan(x) = \frac{b}{a}$	Cotangent	$\cot(x) = \frac{a}{b}$

*Note: If you are accustomed to defining the trigonometric functions using a right triangle, observe that the length of the hypotenuse of the triangle is always equal to 1 with the unit circle approach. Consequently,  $\sin(x) = b$ , or  $\sin(x) = b/1$ , is equivalent to  $\sin(x) = b/c$ , where  $c$  is the length of the hypotenuse of the corresponding triangle.*

Since the terminal point of the vector repeatedly passes through the same points on the unit circle, and since the values of the trigonometric functions depend on the coordinates of these points, the values of each function are also repetitive. As a result, the trigonometric functions are said to be *periodic* functions, where the period of the function is the smallest interval containing one complete cycle through the sequence of values.

In the next task, you will investigate the trigonometric functions. You will evaluate them at various angles, find their domains, and analyze their graphs.

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### Task 3-5: Graphing Basic Trigonometric Functions

1. Describe the domains of the trigonometric functions. For each function, describe the values of  $x$  where the function is *not* defined.

*Note: Some trig functions are defined for any value of  $x$ . Others have restrictions on their domains, since you cannot divide by 0. Use the definitions of the trigonometric functions and the results from the end of Task 3.4 to determine which functions have restrictions on their domains and the values where these functions are not defined.*

- a.  $\sin(x)$



- b.  $\cos(x)$
  - c.  $\tan(x)$
  - d.  $\csc(x)$
  - e.  $\sec(x)$
  - f.  $\cot(x)$
2. Find the places where the graphs of the trigonometric functions intersect the horizontal axis; that is, for each of the trigonometric functions, describe the set of all values of  $x$  where the function is equal to zero.

*Note: Use the definitions of the trigonometric functions, the results from the end of Task 3-4, and the fact that the quotient of two numbers equals 0 if and only if the numerator of the quotient equals 0.*

- a.  $\sin(x)$
  - b.  $\cos(x)$
  - c.  $\tan(x)$
  - d.  $\csc(x)$
  - e.  $\sec(x)$
  - f.  $\cot(x)$
3. Sine, cosine and tangent are considered to be the basic trigonometric functions since the other three functions can be expressed in terms of them.

- a. Express  $\csc(x)$  in terms of  $\sin(x)$ .

$$\csc(x) =$$

- b. Express  $\sec(x)$  in terms of  $\cos(x)$ .

$$\sec(x) =$$

- c. Express  $\cot(x)$  in terms of  $\tan(x)$ .

$$\cot(x) =$$

- d. Actually you only need the sine and cosine functions, since the tangent function, and hence the cotangent function, can be expressed in terms of these functions.

- (1) Express  $\tan(x)$  in terms of  $\sin(x)$  and  $\cos(x)$ .

$$\tan(x) =$$



## 5. Graph the trigonometric functions.

Use your calculator to graph the six trigonometric functions, for  $-2\pi \leq x \leq 2\pi$ . Sketch the graphs below. Label the horizontal axis of each graph in terms of  $\pi$ .

*Note: Define the sine, cosine and tangent functions using the appropriate keys on your calculator. Define the cosecant, secant and cotangent functions in terms of these functions, recalling that  $\csc(x) = 1/\sin(x)$ ,  $\sec(x) = 1/\cos(x)$  and  $\cot(x) = 1/\tan(x)$ .*

**6.** Analyze the graphs of the three basic trigonometric functions.

Use your graphs from part 5 to fill in the chart given below, for  $-2\pi \leq x \leq 2\pi$ . Express your responses in terms of  $\pi$ . For example, to enter the zeros for  $\sin(x)$ , write  $-2\pi, -\pi, 0, \pi, 2\pi$ . To indicate the intervals where  $\sin(x)$  is increasing, use interval notation and write  $(-2\pi, -\frac{3\pi}{2}), (-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, 2\pi)$ .

Function	$\sin(x)$	$\cos(x)$	$\tan(x)$
Range			
Zeros			
Location of vertical asymptotes			
Intervals where increasing			
Intervals where decreasing			
Intervals where concave up			
Intervals where concave down			

Most real-life situations, such as the length of the days and nights throughout the year, cannot be modeled by the basic trigonometric functions. Usually the function has to be stretched or shrunk vertically or horizontally, and shifted left, right, up, or down to fit the data under consideration. In the case of the sine function, this can lead to messy looking functions such as

$$f(x) = 0.25 \sin(4x - 6) + 10.99$$

or

$$f(x) = 0.25 \sin(4(x - \frac{3}{2})) + 10.99$$

where the *general sine function* has the form

$$f(x) = a \sin(b(x - c)) + d$$

The values of  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers, which determine the shape of the function's graph. They indicate how the graph of the general sine function can be obtained by shrinking, stretching, and shifting the graph of  $\sin(x)$ .

In the next two tasks, you will use your calculator to determine how varying the values of  $a$ ,  $b$ ,  $c$  and  $d$  affects the shape and location of the basic sine function. To begin with, you will change one coefficient at a time, and then in the homework you will examine the impact of changing all of them at once.

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### Task 3-6: Stretching and Shrinking the Sine Function

1. Examine the impact of the value of  $a$  by comparing the graph of  $a \sin(x)$  to the graph of  $\sin(x)$ , when  $a > 0$ .
    - a. Consider some specific functions. For each of the functions given below in parts (1) through (4):
      - i. Use your calculator to graph the given function and  $\sin(x)$  on the same pair of axes for  $-2\pi \leq x \leq 2\pi$ . Use the TRACE feature to compare the shapes of the two graphs.
      - ii. Sketch the graphs of the two functions. Label the given function,  $\sin(x)$ , the horizontal axis (in terms of  $\pi$ ) and the vertical axis.
      - iii. State the range of the given function.
- (1)  $f(x) = 4\sin(x)$

Range of  $f(x) = a \sin(x)$ , when  $a = 4$ :

(2)  $f(x) = 2.5\sin(x)$

Range of  $f(x) = a \sin(x)$ , when  $a = 2.5$ :

The term “inverse function” is familiar. In Unit 2, Section 3 you learned how to develop new functions from existing ones by performing reflections. In particular, given a one-to-one function  $f$ , you constructed its *inverse function*  $f^{-1}$  by reflecting  $f$  through the line  $y = x$ . Finding the reflection of a graph through  $y = x$  involved flipping the graph over the diagonal line. Representing the reflection of a function through  $y = x$  by an expression involved interchanging the  $x$ 's and  $y$ 's in the expression for the given function and solving for  $y$ .

Since an exponential function is a one-to-one function, it makes sense to find its inverse. This is the approach which you will use in the next task to develop the concept of a logarithmic function.

### Task 3-10: Investigating the Relationship Between Exponential and Logarithmic Functions

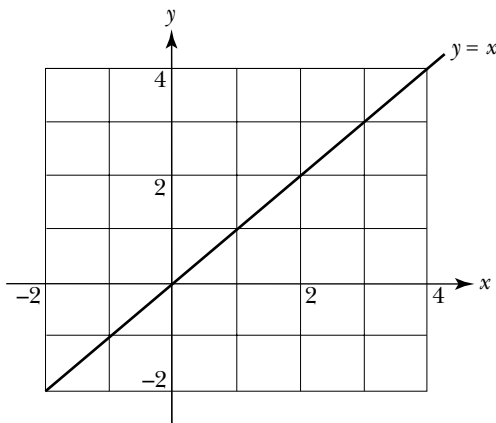
- Examine the reflection of a given exponential function through the line  $y = x$ .

Consider the exponential function with base 2,  $y = 2^x$ , where  $-\infty < x < +\infty$ .

- Find some values of  $y = 2^x$  by filling in the following table.

$x$	-2	-1	0	1	2
$y$					

- Graph  $y = 2^x$  and its reflection through  $y = x$  on the same pair of axes. Use the entries in the table for  $y = 2^x$ , recalling that you can graph the reflection by graphing the ordered pairs in reverse order.



c. The reflection of  $y = 2^x$  through the line  $y = x$  is called a *logarithmic function with base 2*. Try to represent the reflection by an expression.

(1) Interchange the  $x$ 's and  $y$ 's in  $y = 2^x$ .

(2) Try to solve for  $y$ .

*Observation: After interchanging the  $x$ 's and  $y$ 's in the given exponential expression,  $y = 2^x$ , it is impossible to solve  $x = 2^y$  for  $y$  algebraically. Instead, we write*

$$y = \log_2(x)$$

*which is read, "y equals log base 2 of x."*

2. Graph some given exponential functions and their corresponding logarithmic functions on the same pair of axes.

First, let's generalize the comments about logarithmic notation which were made above. If  $a$  is a positive real number, not equal to 1, you can find the inverse of exponential function  $y = a^x$  as follows:

$y = a^x$	Given
$x = a^y$	Interchange $x$ and $y$
??	Solve for $y$

Since there is no straightforward way of solving  $x = a^y$  for  $y$ , the notation

$$y = \log_a(x) \quad \text{or} \quad y = \log_a(x)$$

which is read, "y equals log base  $a$  of  $x$ ," is used to denote the inverse of  $y = a^x$ . It is called the *logarithmic function with base  $a$* . In other words:

- $y = \log_a(x)$  is equivalent to  $x = a^y$ , where
- $y = \log_a(x)$  is the reflection of  $y = a^x$  through the line  $y = x$ .

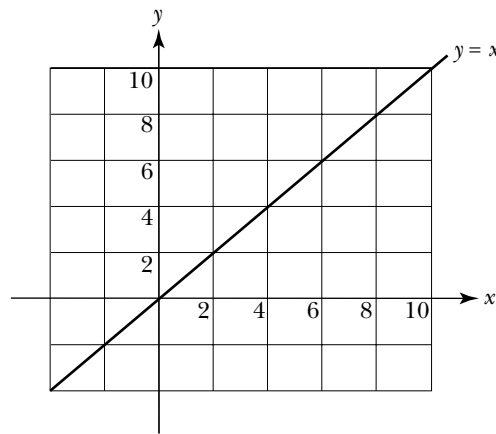
The important thing to realize is that  $x = a^y$  and  $y = \log_a(x)$  are simply two different ways of referring to the same function; the first is called the *exponential form* of the logarithmic function base  $a$  and the second is called the *logarithmic form*.

Now, consider some specific functions.

*Note: The following exponential functions are ones you considered in the last task.*

a. Find the logarithm associated with the exponential growth function  $y = 3^x$ .

- (1) Give the exponential form of the associated logarithm.
- (2) Give the logarithmic form of the associated logarithm.
- (3) Plot a few points on the graph of  $y = 3^x$ . Use this information to sketch a graph of  $y = 3^x$  and the graph of its associated logarithm. Label the two graphs.



- (4) Check your graphs using your calculator.

Note: The base of an exponential or logarithmic function can be any positive number not equal to 1. A special base, which you will examine more closely later on, is the number  $e$ , where  $e = 2.718281828\dots$ . The logarithm base  $e$ ,  $\log_e(x)$ , is called the *natural logarithm* function and is usually denoted by  $\ln(x)$ .

The LN key on your calculator computes the logarithm function base  $e$ , whereas the LOG key computes the logarithm base 10. To compute the logarithm with other bases, you may need to use the following equality (which you will prove in HW3.14):

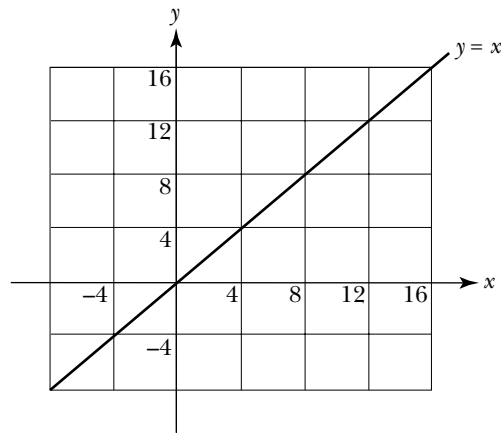
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

- (5) Recall that the domain of  $y = 3^x$  is the set of all real numbers, or  $-\infty < x < +\infty$ . Give the domain of its associated logarithmic function.

- b. Find the logarithm associated with the exponential decay function  $y = \left(\frac{1}{2}\right)^x$ .



- (1) Give the exponential form of the associated logarithm.
  
- (2) Give the logarithmic form of the associated logarithm.
  
- (3) Plot a few points on the graph of  $y = (\frac{1}{2})^x$ . Use this information to sketch graph of  $y = (\frac{1}{2})^x$  and the graph of its associated logarithm. Label the two graphs. Check your graphs using your calculator.



- (4) Give the domain of the associated logarithmic function.

3. Generalize your observations concerning logarithmic functions.

$y = \log_a x$	$0 < a < 1$	$1 < a$
Sketch of a typical graph		
Domain		
Range		
x-intercept(s)		
y-intercept		
Intervals where increasing		
Intervals where decreasing		

Intervals where concave up		
Intervals where concave down		
Intervals where one-to-one		

In the last task, you examined the relationship between a given exponential function and its associated logarithmic function.

The next task will give you some practice working directly with logarithmic functions. In particular, you will:

- Convert back and forth between the two representations for logs, namely the exponential and logarithmic forms.
- Examine ways to evaluate a logarithmic expression, such as  $\log_2 128$ .
- Examine ways to graph a logarithmic function directly, without first graphing its associated exponential function.

### Task 3-11: Evaluating and Graphing Log Functions

1. Practice using logarithmic notation by converting between exponential and logarithmic forms.

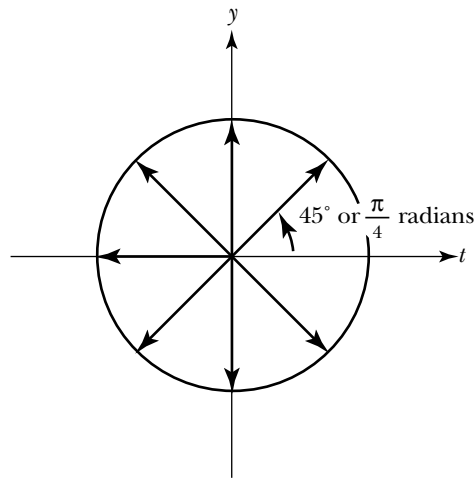
As stated above,  $5^y = x$  is equivalent to  $y = \log_5 x$ , where  $5^y = x$  is in exponential form, and  $y = \log_5 x$  is in logarithmic form. The following table gives some equations in exponential and/or logarithmic forms. Fill in the missing entries.

Exponential form	Logarithmic form
$2^4 = 16$	$\log_2 16 = 4$
$5^3 = 125$	
	$\log_5 1 = 0$
	$\log_7 7 = 1$
$\pi^2 = 9.86 \dots$	$\log_\pi 9.86 \dots = 2$
$2^{-3} = (\frac{1}{8})$	
$e = 2.718 \dots$	$\log_e 2.718 \dots = 1$
$10^4 = 10,000$	
	$\log_6(\frac{1}{6}) = -1$
$\pi^1 = 3.2415 \dots$	
	$\log_2 \sqrt[3]{2} = \frac{1}{3}$

### Task 3-4: Measuring Angles

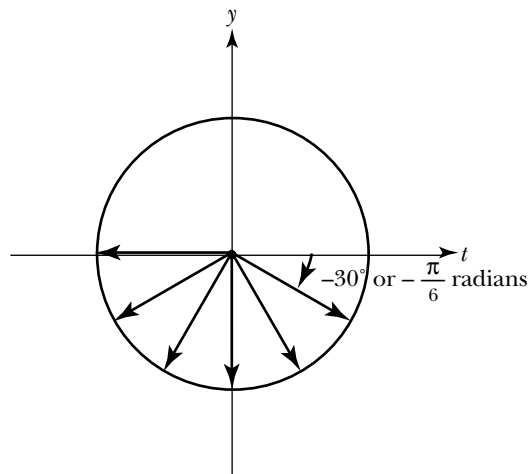
1. Convert between radian and degree measure.

- a. Consider some positive angles. Using increments of  $45^\circ$ , label the circle below in degrees and radians, for  $0^\circ < x \leq 360^\circ$ . Indicate each angle with an arrow. (To help you get started, the first angle is already labeled.)



- b. Consider some negative angles, by reversing direction and traveling clockwise around the unit circle. Using increments of  $-30^\circ$ , label the circle below in degrees and in the equivalent value in radians in terms of  $\pi$ , for  $-180^\circ \leq x < 0^\circ$ . Indicate each angle with an arrow.

*Note: The first angle is marked  $-30^\circ$  and with its equivalent value  $-\frac{\pi}{6}$  radians.*



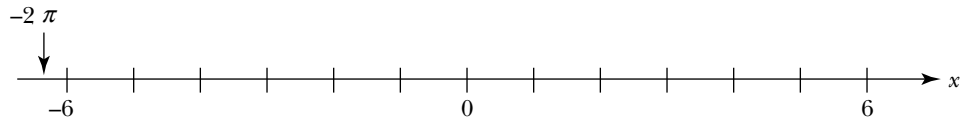
- c. Approximate the decimal value of some radian measures.

A graphing calculator expresses radian measure in terms of its decimal equivalent using the fact that

$$180^\circ = \pi \text{ radians} \approx 3.1416 \text{ radians}$$

Label all the  $\frac{1}{4}$  values of  $\pi$  between  $-2\pi$  and  $2\pi$ —that is, label the values of  $-2\pi$ ,  $-\frac{7}{4}\pi$ ,  $-3/2\pi$ , ...,  $2\pi$ —on the number line given below.

*Note:  $-2\pi$ , which is approximately equal to  $-6.28$ , is already labeled.*



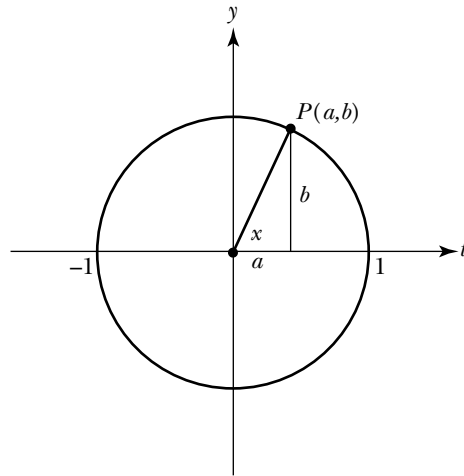
d. Practice converting back and forth.

Convert between degrees and radians (in terms of  $\pi$  and as a decimal value) as you fill in the missing values on the table given below.

Degrees	Radians ( $\pi$ )	$\approx$ Radians (#)
0		
	$\frac{\pi}{4}$	
		1.5708
135		
		3.1416
	$\frac{3\pi}{2}$	
360		
		9.4248
	$4\pi$	
-45		
	$-\frac{\pi}{2}$	
		-3.1416
	$-\frac{3\pi}{4}$	
-360		
		-12.5664

2. Examine the relationship between an angle and the terminal point of its associated vector.

As usual, suppose  $x$  is an angle created by rotating a vector of length 1 around the unit circle. Let  $P(a,b)$  be the terminal point of the vector.

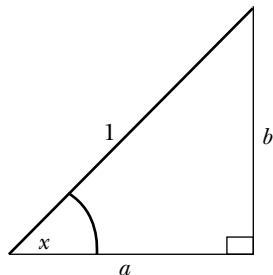


- a. Find the coordinates of the terminal point  $P(a,b)$  of the vector for given values of  $x$  by filling in the table below.

$x$	$-2\pi$	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	$0$	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$a$									
$b$									

- b. Find the values of the coordinates of the terminal point of the vector when  $x = \frac{\pi}{4}$  radians.

- (1) The Pythagorean Theorem gives a relationship between the lengths of the sides of a right triangle and its hypotenuse. State this relationship for the triangle given below.



- (2) If  $x = \frac{\pi}{4}$  radians, the lengths of the two sides of the right triangle given above are equal. Find the length of each side. (Express the values in terms of a radical, not a decimal.)

c. The unit vector can rotate any number of times around the circle. The size of the angle reflects the number of rotations. For example, if the vector rotates around twice in a counterclockwise direction it forms an angle of  $720^\circ$  or  $4\pi$  radians. If it rotates one and one-half times around in a clockwise direction, it creates an angle of size  $-540^\circ$  or  $-3\pi$  radians. In general, the size of the angle gets larger and larger as the vector rotates around the circle in the counterclockwise direction, and it gets smaller and smaller (more and more negative) as the vector rotates in the clockwise direction. As the vector repeatedly rotates around the circle (in either direction), the terminal point of the vector,  $P(a,b)$ , passes through the same points over and over again. As a result, many different angles—both positive and negative—are associated with the same values of  $a$  and  $b$ .

(1) Describe the set of all positive and negative values of  $x$  where the first coordinate of the terminal point of the associated vector is 0—that is, where  $a = 0$ .

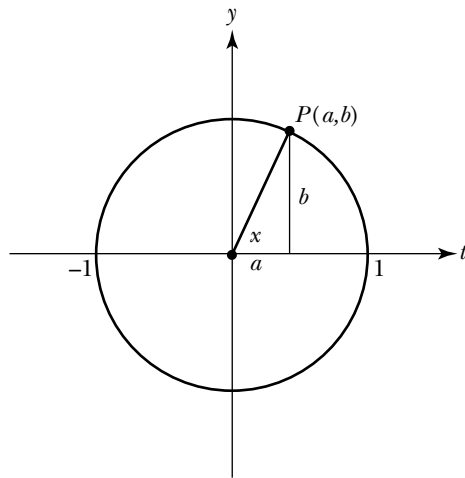
(2) Describe the set of all positive and negative values of  $x$  where the second coordinate of the terminal point of the associated vector is 0—that is, where  $b = 0$ .

(3) Describe the set of all positive and negative values of  $x$  where the first and second coordinates of the terminal point of the associated vector are equal—that is, where  $a = b$ .

So, you have a unit vector rotating around a unit circle. The vector forms an angle with the positive horizontal axis. When the vector rotates in a counterclockwise direction, the resulting angle is positive; otherwise it is negative. The angles can be measured in degrees or radians. Moreover, as the vector goes around and around the circle, its terminal point repeatedly passes through the same values, and the associated angle gets more and more positive or more and more negative.

The domains of the various trigonometric functions are collections of angles. The value of a particular trigonometric function at a given angle is determined by the coordinates of the terminal point of the associated vector. In other words, a trig function accepts an angle as input and then returns a number that is derived from the coordinates of the terminal point of the associated vector. In particular, if  $x$  is the angle formed by rotating a vector around the unit circle and  $P(a,b)$  is the point where the vector terminates on the circle, the *cosine* function returns the first coordinate of  $P$ ,

the *sine* function returns the second coordinate, the *tangent* function returns the quotient of the coordinates, and the *secant*, *cosecant*, and *cotangent* functions are the reciprocals of these three basic functions; that is, the six fundamental trigonometric functions are defined as follows:



Sine	$\sin(x) = b$	Cosecant	$\csc(x) = \frac{1}{b}$
Cosine	$\cos(x) = a$	Secant	$\sec(x) = \frac{1}{a}$
Tangent	$\tan(x) = \frac{b}{a}$	Cotangent	$\cot(x) = \frac{a}{b}$

*Note: If you are accustomed to defining the trigonometric functions using a right triangle, observe that the length of the hypotenuse of the triangle is always equal to 1 with the unit circle approach. Consequently,  $\sin(x) = b$ , or  $\sin(x) = b/1$ , is equivalent to  $\sin(x) = b/c$ , where  $c$  is the length of the hypotenuse of the corresponding triangle.*

Since the terminal point of the vector repeatedly passes through the same points on the unit circle, and since the values of the trigonometric functions depend on the coordinates of these points, the values of each function are also repetitive. As a result, the trigonometric functions are said to be *periodic* functions, where the period of the function is the smallest interval containing one complete cycle through the sequence of values.

In the next task, you will investigate the trigonometric functions. You will evaluate them at various angles, find their domains, and analyze their graphs.

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### Task 3-5: Graphing Basic Trigonometric Functions

1. Describe the domains of the trigonometric functions. For each function, describe the values of  $x$  where the function is *not* defined.

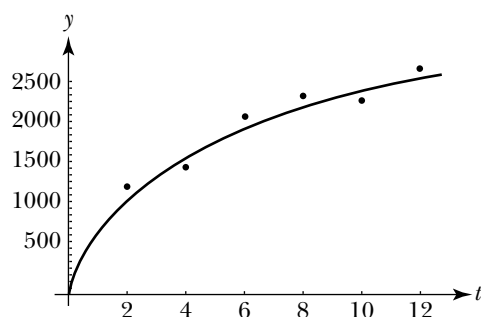
*Note: Some trig functions are defined for any value of  $x$ . Others have restrictions on their domains, since you cannot divide by 0. Use the definitions of the trigonometric functions and the results from the end of Task 3.4 to determine which functions have restrictions on their domains and the values where these functions are not defined.*

- a.  $\sin(x)$

$t$	0	2	4	6	8	10	12
Year	'78	'80	'82	'84	'86	'88	'90
# HMOs (100's)	50	1203	1499	2089	2306	2356	2650

Table 1.

The graph of this discrete data is represented by the scatter plot given below. Rather than working directly with a bunch of unconnected points, it is often much easier to try and find a continuous curve that “fits” or models the data—that is, an expression whose graph is very similar to the scatter plot representing the given data. In this instance, the data have a logarithmic shape, and the function  $f(t) = 10^3 \ln(t + 1) + 50$ , whose graph is shown below by the curved line, appears to fit the data fairly closely.


 Graph of Data in Table 1 and  $f(t) = 10^3 \ln(t + 1) + 50$ 

The new fit function can be used to predict what happens at points that are not in the original data set. For example, to predict how many HMOs there were in 1981, you could calculate  $f(3)$ . Be careful, however. If you use the fit function to predict what happens at values that lie above or below the endpoints of the original domain, scrutinize your results with care. The fit function may behave very differently outside the domain of the data set and thus give inaccurate approximations.

Of course, not all data can be fitted nicely with one curve. Sometimes you have to use several curves and model the data with a piecewise-defined function.

In the next task, you will use your calculator to fit curves to some discrete functions and make predictions about the data for values not in the domain of the data set. Since finding a curve which best fits the data typically involves plotting the data and guessing the general shape of its graph, you will start by reviewing the general shape of some fundamental function classes.

---

### Task 3-12: Modeling Data

1. First, recall the general shape of some basic functions.

Draw a rough sketch indicating the general shape of the following functions.



a.  $f(x) = ax + b$ , where  $a \neq 0$

b.  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$

c.  $f(x) = ax^3 + bx^2 + cx + d$ , where  $a \neq 0$

d.  $f(x) = a + b \ln(x)$ , where  $x > 0$

e.  $f(x) = ab^x$

f.  $f(x) = a \sin(bx + c) + d$

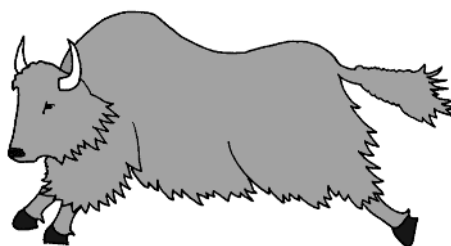
2. Review the section in the appendix for your calculator that describes how to fit a curve to a discrete function.

3. Model some situations with a function.

For each of the following discrete functions:

- i. Represent the discrete function by a scatter plot.
  - ii. Find a continuous curve that fits the data as closely as possible. Record your function in your activity guide.
  - iii. Plot the discrete function and the curve fit on the same pair of axes. Sketch a copy of the graphs in your activity guide.
  - iv. Use the fit function to answer any questions pertaining to the data set.
- a. Make your first pass at fitting a curve to some data while thinking about a yak named Charlie.

A mountain yak named Charlie is sprinting across a field moving at a nearly constant rate. By measuring Charlie's distance at 1-second intervals from a fixed point you have collected the following data:



Time (s)	Distance (m)
0	0.0
1	20.4
2	39.8
3	60.0
4	79.5
5	99.8
6	120.0
7	143.1
8	159.7
9	181.0
10	200.5

- (1) Represent Charlie's movement with a scatter plot.
- (2) Describe the shape of the scatter plot. What type of function might you use to fit the data?
- (3) Since Charlie is moving at nearly a constant rate, his distance versus time graph is almost linear. Fit a linear function to the data. Record your function below.

(4) Sketch graphs of the original data and the function modeling Charlie's movements on the same pair of axes.

(5) Use your fit function to estimate Charlie's distance at:

(a) 2.5 seconds

(b) 7.9 seconds

(c) 0.25 seconds

(6) Find the size of the error—that is, the difference between the actual data and the value of the fit function—at 7 seconds.

b. Fit a curve to data collected from moving in front of the motion detector.

Suppose for 13 seconds you walked back and forth in front of the motion detector and collected the following data:

Time (s)	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Distance (m)	2	3.26	3.36	2.21	0.86	0.57	1.58	2.99	3.48	2.62	1.18	0.5	1.19	2

(1) Plot the data that you collected on a scatter plot.

(2) Find a curve that fits your distance versus time graph. Record your function below.

(3) Sketch graphs of the data that you collected and the curve that models your movement on the same pair of axes.

- (4) Using the function that models your movement, give a detailed description of the way you moved in front of the detector. Include in your description approximations of
- the time intervals when you were increasing your distance from the detector.
  - the time intervals when you were decreasing your distance from the detector.
  - the times when you reversed direction.

c. Fit a function to a population growth situation.

Your scientist friend is at it again, growing little beasties in Petri dishes in her lab. At first it is fairly easy to keep track of how many there are, but then the total number starts increasing faster than she can count. Before she admits defeat, however, she manages to collect the following data:

Time (min.)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Number of beasties	6	10	15	27	42	75	125	200	325

- (1) Plot the data that your friend has collected.
- (2) Fit the data with a continuous function. Record the function in the space below.

(3) Sketch graphs of the data and the curve on the same pair of axes.

(4) Use the function that fits the data to estimate the number of beasties that your friend had at the following times. (Since fractional beasties don't make much sense, round your responses down.)

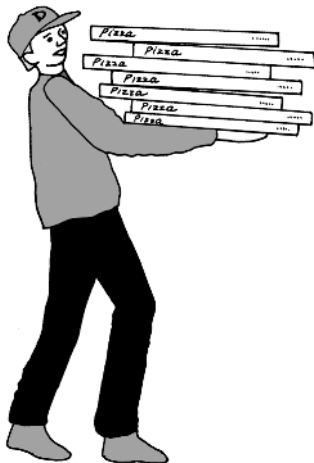
(a) At 20 seconds.

(b) At one minute and forty-five seconds.

(c) At 3 minutes and 15 seconds.

d. Model total sales in your pizza business.

Five years ago you bought a pizza place in the town of Collegeville. During the past year business has been phenomenal—both home deliveries and your take-out service have increased rapidly. Unfortunately, times haven't always been as good, as the following sale records show:



Year ( $t$ )	Total sales (\$1000's)
0.5	24
1.0	33
1.5	37
2.0	38
2.5	36
3.0	37
3.5	38
4.0	42
4.5	51
5.0	65



- (5) Based on your model, describe the way your business has grown during the past 5 years. During what time periods were sales increasing? Decreasing? During which 6-month period did you have your fastest growth?
- (6) Analyze the accuracy of your model at some specified times by filling in the following chart:

Year ( $t$ )	Total sales (\$1000's)	$S(t)$	Error
0			
2.0			
2.5			
3.0			
3.5			
5.0			

### Unit 3 Homework After Section 4

- Complete the tasks in Section 4. Be prepared to discuss them in class.

#### HW3.16 Analyze two supply and demand functions.

Supply Data

Price (\$)	Number of tickets
0	0
20	4,000
40	13,000
60	20,000
80	36,000
100	50,000

You have agreed to be the promoter of a major outdoor concert. There are an incredible number of little details to which you must attend, but one of the major decisions you must make is how much to charge for each ticket. You realize that the number of tickets you would be willing to sell increases as the price of the tickets increases, since your net profit will be greater and your ticket selling operation will be operating at increasingly efficient levels. After some back-of-the-envelope calculations you come up with the data (given in the table on the left) concerning the number of tickets you would be willing to supply at various prices.

Demand Data

Price (\$)	Number of tickets
15	50,000
35	30,000
55	10,000
75	5,000
95	2,500

Although you're willing to supply more tickets as the price increases, the potential ticket buyers feel exactly the opposite way. The number of tickets they are willing to buy goes down as the price per ticket increases. To try to determine the number of tickets that will be demanded by the customers at various prices, you hire a marketing firm to take a survey. The table given to the left summarizes their report.

1. Find a function that models your supply data. Name your function  $S$ .
2. Find a function  $D$  that models the consumers' demand data. Name your function  $D$ .
3. Graph the four functions—the supply data, the demand data, and the two fit functions,  $S$  and  $D$ —on the same pair of axes. Sketch the graphs. Label the four graphs.
4. Use  $S$  and  $D$  to approximate solutions to the following questions. Show your work.
  - a. If the tickets cost \$25 each, how many tickets would the public be willing to buy?
  - b. If the tickets cost \$75, how many tickets would you be willing to sell?
  - c. If the tickets cost \$65, what is the difference between the number of tickets you would be willing to promote and the number the public would be willing to buy?
  - d. An equilibrium point is the point where the supply and demand graphs intersect.
    - (1) Use  $S$  and  $D$  to find the price of tickets at the equilibrium point.
    - (2) Find the quantity of tickets at the equilibrium point.
    - (3) Explain what these values represent with regard to your interest as the supplier and the consumers' interests.



**HW3.17** Write your journal entry for Unit 3. As usual, before you begin to write, review the material in the unit. Think about how it all fits together. Try to identify what, if anything, is still causing you trouble.

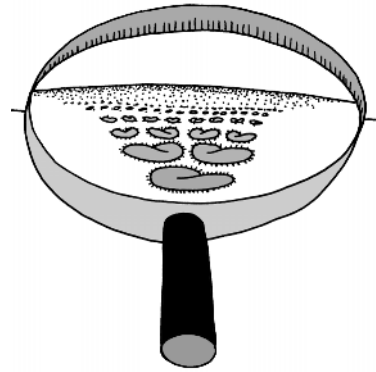
1. Reflect on what you have learned in this unit. Describe in your own words the concepts that you studied and what you learned about them. How do they fit together? What concepts were easy? Hard? What were the main (important) ideas? Give some examples of the main ideas.
2. Reflect on the learning environment for the course. Describe the aspects of this unit and the learning environment that helped you understand the concepts you studied. What activities did you like? Dislike?



**SECTION 3**

**Exponential and Logarithmic Functions**

Suppose your best friend is a biologist who studies the growth rate of paramecia. Currently, she is examining a variety where every hour the size of the total population doubles as each paramecium becomes two paramecia. In this case, the population is said to be “growing exponentially,” and with this rapid rate of growth, it will not take long for her to have a lot of paramecia. In the next task you will use an exponential function to model your friend’s growth situation and investigate the impact on the model of changing the initial condition. You will also model the decay situation that results when her paramecia suddenly begin to die.




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**Task 3-8: Modeling Situations Using Exponential Functions**

1. Model the exponential growth situation of your biologist friend’s paramecia, noting that each hour the population doubles.

Suppose  $t$  is the number of hours that have passed and  $A(t)$  is the number of paramecia at the end of  $t$  hours. If she begins her experiment with one paramecium, then  $A(0) = 1$ . Moreover, since the population doubles by the end of each hour,  $A(1) = 2$ ,  $A(2) = 4$ , and so on.

- a. Calculate the results of the experiment at the end of each of the first 5 hours by completing the following table.

$t$	0	1	2	3	4	5
$A(t)$	1	2				

- b. Since the value of  $A(t)$  can be expressed as a power of 2, a mathematical description of your friend’s situation is

$$A(t) = 2^t, \text{ where } t \geq 0$$

Use this function to calculate the number of paramecia at some particular times and also to determine the time when a given number exists.

- (1) Since the population is continually increasing, it makes sense not only to talk about the number of paramecia at the end of each hour but also at each moment in time. Use your calculator to approximate the number of paramecia at the end of the following

times. Be as accurate as possible. Round your answers down to the nearest whole paramecium.

(a) Find  $A(t)$  when  $t = 8$ —that is, find the number of paramecia after 8 hours.

(b) Find  $A(t)$  at the end of 10 hours, 15 minutes.

(c) Find  $A(t)$  when  $t = 17$ .

(d) Find  $A(t)$  after 20 hours, 30 minutes.

(e) Find  $A(t)$  at the end of 2 days.

(f) Find  $A(t)$  after 100 hours.

(2) Approximate as accurately as you can the number of hours which need to pass before your friend has the following number of paramecia:

(a) 1,000 paramecia

*Hint: Find the zero of  $f(t) = 2^t - 1000$ . Alternately, find the intersection of the graph of  $f(t) = 2^t$  and the graph of  $g(t) = 1000$ .*

(b) 100,000 paramecia

(c) 1 million paramecia

- c. After a short period of time, the function  $A(t) = 2^t$  grows very rapidly. Use your calculator to examine the graph of  $A$ . Sketch the graphs for the specified values of  $t$  in your activity guide. Label the axes.

(1)  $0 \leq t \leq 5$

(2)  $0 \leq t \leq 10$

2. Consider a different initial condition. Examine how this change affects the expression representing the situation and how it affects the graph.
- a. Suppose she starts her experiment with 10 paramecia (instead of 1). Model the new situation with a function called  $S$ .
- (1) Calculate the number of paramecia at the end of each hour, for the first 5 hours. Express the result in terms of a power of 2. Note the impact of the assumption that initially she has 10 paramecia.

$t$ (hrs)	$S(t)$ (value)	$S(t)$ (in terms of a power of 2)
0	10	$10 \cdot 2^0$
1	$2 \cdot 10$ or 20	$10 \cdot 2^1$
2		
3		
4		
5		

(2) Represent  $S$  by an expression.

$$S(t) = \quad , \text{ where } t \geq 0$$

(3) Since the population is doubling, the expression representing  $A$  (which you considered in part 1) and the one representing  $S$  both have the same base, namely 2. The only thing that differs is their initial condition, since by assumption  $A(0) = 1$  whereas  $S(0) = 10$ .

Graph  $A$  and  $S$  on the same pair of axes for  $0 \leq t \leq 5$ . Describe the relationship between the two graphs. Support your description with a sketch.

3. Consider another scenario. Suppose something goes wrong and your friend's paramecia start to die. When she initially notices the problem she has a million paramecia. One hour later she has half that many, and after each subsequent hour she has half again as much. Model the new situation with an exponential decay function called  $D$ .
- a. Calculate some values of  $D(t)$  and record the results in the following table.

$t$ (hrs)	$D(t)$ (value)	$D(t)$ (in terms of a power of $\frac{1}{2}$ )
0	1,000,000	$1,000,000 \cdot (\frac{1}{2})^0$
1	$\frac{1}{2} \cdot 1,000,000$ or 500,000	$1,000,000 \cdot (\frac{1}{2})^1$
2		
3		
4		
5		

b. Represent  $D$  by an expression.

$$D(t) = \quad \text{where } t \geq 0$$

c. Answer some questions about the situation.

(1) Find the population at the end of 4 hours, 30 minutes.

(2) During what 15-minute time interval does the population drop below 20,000?

d. Describe the shape and location of the graph of  $D$ . Support your description with a sketch.

In the last task you used exponential functions to model several situations. You analyzed two growth functions, namely  $A(t) = 2^t$  and  $S(t) = 10 \cdot 2^t$ , and a decay function,  $D(t) = 1,000,000 \cdot (\frac{1}{2})^t$ . You compared the shapes of their graphs and observed how changing the initial condition affects the location of the graph. The domains of the exponential functions which you con-

sidered do not contain any negative numbers, since in your friend's experiment it doesn't make sense to talk about negative time. Exponential functions, however, can be defined for all real numbers.

In general, if  $a$  is a fixed positive number, not equal to 1, then

$$f(x) = c \cdot a^x, \quad \text{where } -\infty < x < +\infty$$

is called an *exponential function* with base  $a$  and initial condition  $c$ . When  $0 < a < 1$ ,  $f$  is an *exponential decay* function. When  $a > 1$ ,  $f$  is an *exponential growth* function. Since  $f(0) = c$ , the value of  $c$  determines where the graph crosses the vertical axis.

In the next task, you will compare the behavior of exponential growth functions and the behavior of exponential decay functions for different values of the base. You will also compare the behavior of an exponential growth function, such as  $g(x) = 3^x$ , and its associated exponential decay function,  $d(x) = (\frac{1}{3})^x$ .

Before beginning Task 3-9, recall two rules for manipulating exponents:

$$\left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x}$$

and

$$a^{m/n} = \sqrt[n]{a^m}$$

According to these rules,  $(\frac{1}{8})^x$  and  $\frac{1}{8^x}$  and  $8^{-x}$  are three different ways of writing the same thing, and

$$8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

---

### Task 3-9: Comparing Exponential Functions

1. Analyze how the value of the base affects the shape and location of an exponential growth function.
  - a. Consider the behavior of three exponential growth functions whose bases are  $a = 2$ ,  $a = 3$ , and  $a = 5$ . Let

$$f(x) = 2^x$$

$$g(x) = 3^x$$

$$h(x) = 5^x$$

- (1) Compare some output values for  $f$ ,  $g$ , and  $h$ .

Evaluate the functions at a variety of inputs, negative and positive, large and small. Record your results in the table given below.

In the last task, you used your calculator to examine how the values of the coefficients  $a$  and  $b$  affect the shape of the graph of the general sine function  $f(x) = a \sin(b(x - c)) + d$ . In the next task, you will use a similar approach to investigate how the values of  $c$  and  $d$  affect the location of the graph.

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### Task 3-7: Shifting the Sine Function

1. Investigate the impact of the value of  $c$  by comparing the graph of  $\sin(x - c)$  to the graph of  $\sin(x)$ .
  - a. Use your calculator to explore how the value of  $c$  affects the location of the graph of  $f(x) = \sin(x - c)$ . Examine functions where  $c$  is positive and where  $c$  is negative by graphing functions such as

$$r(x) = \sin(x - \pi/2)$$

$$g(x) = \sin(x + 1)$$

$$h(x) = \sin(x + \pi)$$

for  $-2\pi \leq x \leq 2\pi$ , on the same pair of axes with  $\sin(x)$ . Sketch the graphs. Label the functions, the horizontal axis (in terms of  $\pi$ ) and the vertical axis.

**b.** Generalize your observations.

Describe the impact of the value of  $c$  by comparing the location of the graph of  $\sin(x - c)$  to the graph of  $\sin(x)$ . Describe what happens when  $c < 0$  and when  $c > 0$ .

**2.** Analyze the impact of the value of  $d$  by comparing the graph of  $\sin(x) + d$  to graph of  $\sin(x)$ , where  $d \neq 0$ .

**a.** Use your calculator to discover how the value of  $d$  affects the graph of  $f(x) = \sin(x) + d$ . Examine functions where  $d$  is positive and where  $d$  is negative by graphing functions such as

$$p(x) = \sin(x) + 4$$

$$q(x) = \sin(x) - 2$$

$$r(x) = \sin(x) + 0.5$$

for  $-2\pi \leq x \leq 2\pi$ , on the same pair of axes with  $\sin(x)$ . Sketch the graphs. Label the functions. Give the range of each function.



b. Generalize your observations.

(1) Describe the impact the value of  $d$  by comparing the location of the graph of  $\sin(x) + d$  to the graph of  $\sin(x)$ . State what happens when  $d < 0$  and when  $d > 0$ .

(2) Give the range of  $f(x) = \sin(x) + d$  in terms of  $d$ .

### Unit 3 Homework After Section 2

- Complete the tasks in Section 2. Be prepared to discuss them in class.

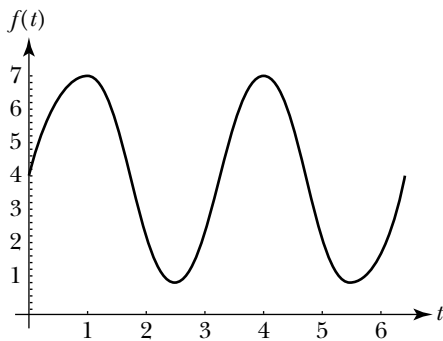
**HW3.6** Consider the general sine function  $f(x) = a \sin(b(x - c)) + d$ .

1. Summarize the results of Tasks 3.6 and 3.7.
2. For each of the following sinusoidal functions:

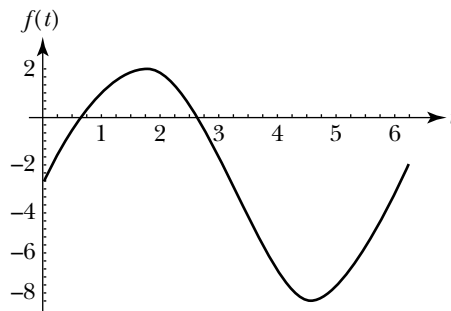
Value	Impact on the Graph of $\sin(x)$
$0 < a < 1$ $a > 1$	
$0 < b < 1$ $b > 1$	
$c < 0$ $c > 0$	
$d < 0$ $d > 0$	

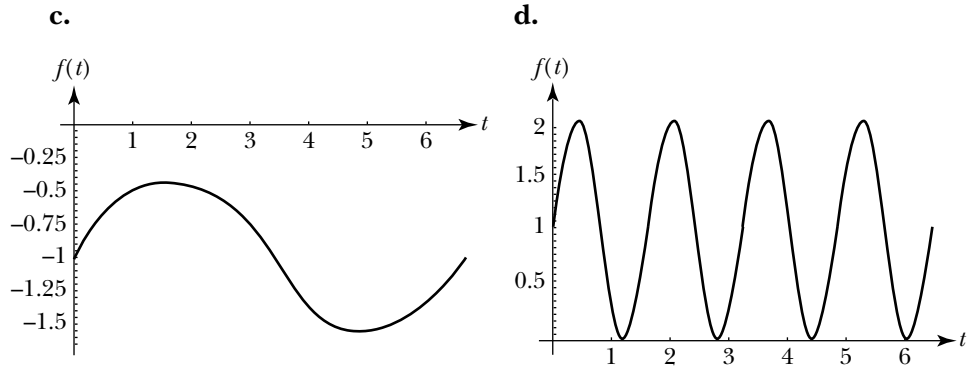
- i. Compare the shape of the graph of the function to the graph of  $\sin(t)$ . Analyze:
- horizontal stretch or shrink
  - vertical stretch or shrink
  - shift up or down
  - shift left or right
- ii. Sketch the graph by hand. Carefully label the axes on your graph.
- iii. Use your calculator to check that your sketch is correct.
- a.  $f(t) = \sin(2t) + 6$
  - b.  $f(t) = 2\sin(t - \pi)$
  - c.  $f(t) = \sin\left(\frac{\pi}{5}(t - 5)\right) - 3$
3. Represent each of the following graphs by an expression of the form  $a \sin(b(t - c)) + d$ .
- i. Find the size of the shift up or down. Use this information to determine the value  $d$ .
  - ii. Find the range of the function. Use this information to determine the value  $a$ .
  - iii. Find the period of the function. Use this information to determine the value  $b$ .
  - iv. Find the size of the shift left or right. Use this information to determine the value  $c$ .

a.



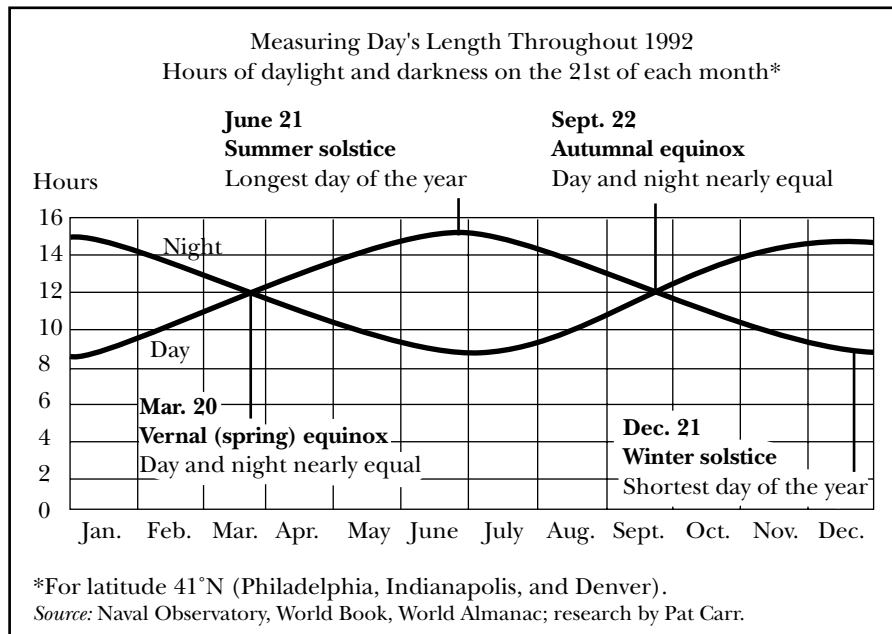
b.





**HW3.7** Express the lengths of days and nights as general sine functions.

The graph below displays the day’s length throughout 1992 for latitude  $41^\circ\text{N}$ . Note that an equinox—a day when the hours of daylight and darkness are nearly equal—occurs each spring and fall when the sun shines directly at the equator. The summer solstice—the longest day of the year—occurs when the earth tilts the northern hemisphere closest to the sun, and the winter solstice—the longest night of the year—occurs when the southern hemisphere is closest to the sun.



1. Write a paragraph detailing what the graph tells you about the relationship between the length of the days and nights over the course of a year.
2. Represent the length of the days throughout the year by a general sine function called *day*.
  - a. Assume  $day(t) = a \sin(b(t - c)) + d$ . Suppose the vernal equinox corresponds to the origin in the basic sine function. Find the values of  $d$  and  $a$ .

- b. Denote the first day of the successive months by 0 to 11—that is, denote January 1 by 0, February 1 by 1, and so on. How would you denote the end of December? What is the period of function *day*? Use this information to find the value of *b*.
- c. Assume each month has 30 days. Given that March 1 is denoted by 2, how would you denote the vernal equinox, which occurs on March 20? Determine the value of *c*.
- d. Use your graphing calculator to check that your function is correct.
3. Represent the length of the nights throughout the year by a general sine function called *night* by modifying the general sine function for *day*.
4. Plot *night* and *day* on the same pair of axes using your calculator. Sketch the graphs.

### HW3.8 Examine a concise way of expressing multiples of $\pi$ .

The six fundamental trigonometric functions are periodic, since their values keep repeating themselves. For example,  $\sin(t)$  is 0 when  $t$  is *any* integer multiple of  $\pi$ , such as  $-10\pi$ ,  $0\pi$ ,  $8\pi$ ,  $23\pi$ , and so on. A concise way to represent all the integer multiples of  $\pi$  is to write:

$$n\pi, \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

which stands for

$$\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$$

*Note: The sequence of three dots on either end of the list implies that the list continues on indefinitely using the same pattern.*

1. Consider the following sequences. In each case:
- Give a verbal description of all the multiples of  $\pi$  represented by the notation.
  - Give the sequence for which the notation stands.
    - $2n\pi$ , where  $n = 0, \pm 1, \pm 2, \pm 3, \dots$
    - $(2n + 1)\pi$ , where  $n = 0, \pm 1, \pm 2, \pm 3, \dots$
    - $\frac{n\pi}{2}$ , where  $n = 0, \pm 1, \pm 2, \pm 3, \dots$
    - $\frac{(2n + 1)\pi}{2}$ , where  $n = 0, \pm 1, \pm 2, \pm 3, \dots$
2. Express each of the following sequences as a multiple of  $\pi$ .
- $\dots, -8\pi, -4\pi, 0, 4\pi, 8\pi, \dots$
  - $\dots, -\pi, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \dots$

3. Express the answer to each of the following as a sequence or as a multiple of  $\pi$ . Support your answer with a sketch.
  - a. Find all the values of  $\theta$  such that  $\cos(\theta) = -1$ .
  - b. Find all the values of  $\theta$  such that  $\sin(\theta) = -1$ .
  - c. Find all the values of  $\theta$  such that  $\sin(\theta) = \cos(\theta)$ .
  - d. Find all the values of  $\theta$  such that  $\sin(\theta) + \cos(\theta) = 0$ .

**HW3.9** Show that some fundamental trigonometric identities hold.

1. Use the Pythagorean Theorem and the definitions of sine and cosine to show that

$$\sin^2 \theta + \cos^2 \theta = 1$$

*Note:  $\sin^2 \theta = (\sin \theta)^2$ ,  $\cos^2 \theta = (\cos \theta)^2$ , and so on.*

2. Use the Pythagorean Theorem and the definitions of tangent and secant to show that

$$\tan^2 \theta + 1 = \sec^2 \theta$$

3. Explain why  $\sin(\theta + 2\pi) = \sin(\theta)$ , for every value of  $\theta$ .
4. Explain why  $\cos(\theta + 2\pi) = \cos \theta$ , for every value of  $\theta$ .
5. Explain why  $\sin(\theta + \pi) = -\sin(\theta)$ , for every value of  $\theta$ .
6. Explain why  $\cos(\theta + \pi) = -\cos(\theta)$ , for every value of  $\theta$ .

**HW3.10** For each of the following combination of functions:

- i. Try to predict the shape of the graph of the combination.
  - ii. Check your prediction using your calculator.
  - iii. Sketch the graph.
1.  $f(t) = -\sin(t)$ , where  $-4\pi \leq t \leq 4\pi$ .
  2.  $f(t) = \sin(t) + t$ , where  $-4\pi \leq t \leq 4\pi$ .
  3.  $f(t) = \sin(t) - t$ , where  $-4\pi \leq t \leq 4\pi$ .
  4.  $f(t) = t \sin(t)$ , where  $-4\pi \leq t \leq 4\pi$ .
  5.  $f(t) = t^2 \sin(t)$ , where  $-4\pi \leq t \leq 4\pi$ .
  6.  $f(t) = \sin(t)/t$ , where  $-4\pi \leq t \leq 4\pi$  but  $t \neq 0$ . What type of discontinuity does  $f$  appear to have at  $t = 0$ ?

It is important to note that in all three of these cases the arrow notation indicates that the values of  $x$  get arbitrarily close to  $a$ , but do not reach  $a$ .

What you are really interested in, however, is the limiting behavior of a function. In particular, how do the output values of a function behave as the associated inputs approach a particular number? For example, what happens to the value of  $f(x) = x + 5$ , as the values of  $x$  get closer and closer to 6? In other words, does  $x + 5$  approach a particular value, as  $x$  approaches 6? If so, what? Or, as another example, suppose *DayLight* is a function that returns the length of each day during the year. What happens to the value of  $DayLight(t)$ , as  $t$  approaches the vernal equinox? One way to investigate the answers to questions such as these is to graph the given function and observe what's happening. Another way is to create an *input-output table*. You will use both of these approaches in the next task.

Caution! As you examine the limiting behavior of the following functions, keep in mind that there are two “approachings” going on: one by the inputs to the function and one by the corresponding output values of the function. These are different and they happen simultaneously.

### Task 4-2: Analyzing the Limiting Behavior of Functions

1. Examine the limiting behavior of  $f(x) = x + 5$  as  $x$  approaches 6.
  - a. Use an input–output table to examine the limiting behavior of the function.
    - (1) Consider the sequence of numbers approaching 6 from the right and the sequence approaching 6 from the left that you defined in Task 4-1, part 4. Calculate the value of  $f(x) = x + 5$  at each input and record the result in the *input–output table* given below.

Term # $i$	Left Sequence $x = 6 - 10^{-i}$	Output Values $f(x)$	Right Sequence $x = 6 + 10^{-i}$	Output Values $f(x)$
1	5.9	10.9		
2				
3				
4				
5				
	↓ $6^-$ (from the left)	↓ <input type="text"/>	↓ $6^+$ (from the right)	↓ <input type="text"/>

- (2) The output values appear to approach a particular value as  $x$  approaches 6 from the left—that is, as  $x \rightarrow 6^-$ . What value is this? Record your response in the first box in the last row of the input–output table.

This value is called the *left-hand limit* of the function  $f$  as  $x$  approaches 6. One way to express this mathematically is to write

$$f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6^-$$

or, instead of using the name of the function, you can use the expression that represents the function and write

$$x + 5 \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6^-$$

Fill in the boxes with the appropriate value.

- (3) Next note what happens when you approach 6 from the other side. What value does the function’s output values appear to approach as  $x$  approaches 6 from the right—that is, as  $x \rightarrow 6^+$ ? Record your response in the last row of the input–output table.

This value is called the *right-hand limit* of the function  $f$  as  $x$  approaches 6. In this case, you write

$$f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6^+ \quad \text{or} \quad x + 5 \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6^+$$

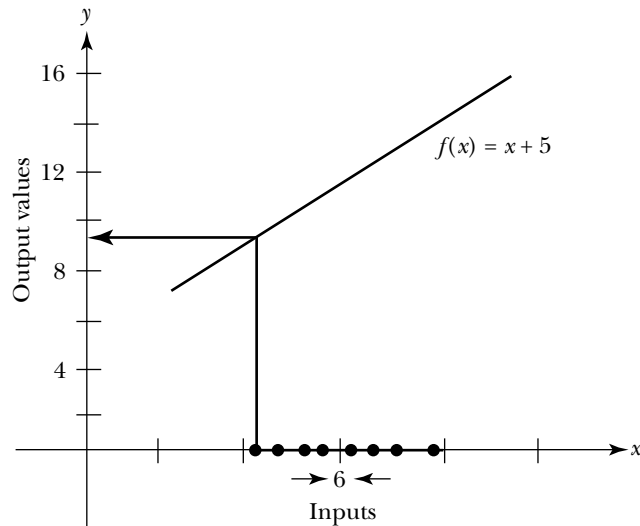
where the boxes are filled in with the appropriate value.

- (4) You should have concluded that the right- and left-hand limits are the same. If you didn’t, retrace your steps and check your calculations. Because the two one-sided limits are equal, this value is called the *limit* of the function  $f$  as  $x$  approaches 6, and you write:

$$f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6 \quad \text{or} \quad x + 5 \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 6$$

- b.** Investigate the limiting behavior of this function again, but this time use a graphic approach.

- (1) Consider the graph of  $f(x) = x + 5$  for  $2 \leq x \leq 10$  given below. Using a pencil and straightedge, indicate the output value corresponding to each of the specified inputs located to the left and right of  $a = 6$ . (The first association is already marked.)



- (2) Based on your graph, what can you conclude about the behavior of the output values of  $f$  as  $x$  approaches 6 from the left and from the right? Mark the value on the vertical axis. Indicate the limiting behavior of the function as  $x \rightarrow 6^-$  and as  $x \rightarrow 6^+$  by placing vertical arrows along the vertical axis in the appropriate directions.
- (3) You should have reached the same conclusion using the input–output table and the graph. If you did not, try again.
2. Examine the limiting behavior of a piecewise-defined function near a hole.

*Note: Keep in mind that you are interested in the limiting behavior of a function near the specified point, not at the point. Consequently, the function does not have to be defined at the point.*

Consider the following function which has a hole at  $x = 2$ :

$$h(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ x^2, & \text{if } x > 2 \end{cases}$$

- a. Use an input–output table to analyze the limiting behavior of  $h$  as  $x$  approaches 2.

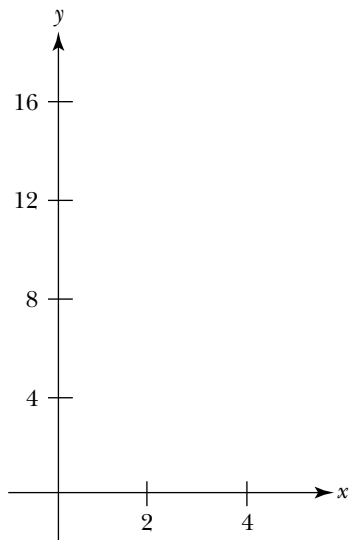
Fill in the table below. Define a sequence of numbers that approaches  $a = 2$  from the left, and a sequence that approaches  $a = 2$  from the right. Evaluate  $h$  at each of these inputs. Predict the limiting behavior of  $h$  as  $x \rightarrow 2^-$  and as  $x \rightarrow 2^+$ .



Term # $i$	Left Sequence $x = 2 - \square$	Output Values $h(x)$	Right Sequence $x = 2 + \square$	Output Values $h(x)$
1				
2				
3				
4				
5				
	↓ $2^-$	↓ $\square$	↓ $2^+$	↓ $\square$

- b. Use a graphic approach to analyze the limiting behavior of  $h$  as  $x$  approaches 2.

On the axes below, sketch the graph of  $h$  for  $0 \leq x \leq 4$ , where  $x \neq 2$ . Using a pencil and straightedge, indicate on the graph the output values corresponding to several input values immediately to the left of  $a = 2$  and several immediately to the right of  $a = 2$ . On the vertical axis, mark the value  $h$  approaches as  $x$  approaches 2. Use vertical arrows to indicate the limiting behavior of  $h$  as  $x \rightarrow 2^-$  and  $x \rightarrow 2^+$ .



- c. Summarize your observations from parts a and b. (The results should be the same.)

$$h(x) \rightarrow \square \text{ as } x \rightarrow 2^- \quad \text{and} \quad h(x) \rightarrow \square \text{ as } x \rightarrow 2^+$$

Consequently,

$$h(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 2$$

- d. Express the left- and right-hand limits, representing  $h$  by the appropriate expression to the left and to the right of  $a = 2$ .

$$\boxed{\phantom{000}} \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 2^-$$

and

$$\boxed{\phantom{000}} \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 2^+$$

- e. Explain why it is legal to consider the limit of a function at a hole.
3. In this task, you have considered two approaches for examining the limiting behavior of a function: using an input–output table and using the graph. Describe the relationship between these two approaches.

In general, a function  $f$  has a limit  $L$  as  $x$  approaches  $a$ , if  $f(x)$  can be made to get as close to  $L$  as you wish by restricting  $x$  to a small interval about  $a$ , but excluding  $a$ . In this case you write

$$f(x) \rightarrow L \text{ as } x \rightarrow a$$

Two important observations:

- When you refer to the limit of a function as  $x$  approaches  $a$ , you are referring to how  $f$  behaves near  $a$ , not at  $a$ . In fact, a function does not even have to be defined at  $x = a$  for the limit to exist.
- In order for  $L$  to be the limit of a function  $f$  as  $x$  approaches  $a$ , the values of  $f(x)$  must be getting closer and closer to  $L$  as  $x$  approaches  $a$  from both the left and from the right. The left-hand and right-hand limits must both exist and must both be equal to  $L$ .

In the last task, you used two methods to help make a reasonable guess for the value of  $L$ :

- You examined an input–output table for  $f$  as  $x \rightarrow a$  from both the left and the right.
- You examined the graph of  $f$  for  $x$  near  $a$ .

In the next task, you are going to use the same methods to estimate the value of the limit of a given function as  $x$  approaches a specified value, but this time instead of doing all your work by hand, you will use your calculator to construct an input–output table and to graph the function near the indicated point.

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### Task 4-3: Approximating Limits Using a Graphing Calculator

1. Construct some sequences using your calculator.

- a. Read the section on sequences and lists in the appendix for your graphing calculator. Determine how to:
- Construct a list in the home screen.
  - Generate a sequence using an expression.
  - Display a list or sequence on the home screen.
  - Store a list or sequence in a table.
  - Access the terms in a list or sequence.

Work through the section interactively. Try it, as you read along.

b. Practice what you have learned.

- (1) Store  $\{1, -6, 2.5, 23, 0\}$  in a list called  $L1$ . Place the list in the first column in a table on your calculator.
- (2) Store the first five terms of the sequence whose typical term is  $6 - 10^{-i}$  in a list named  $LX$ . Display at least five digits to the right of the decimal point.

(a) Record the terms in  $LX$ .

(b) Describe the behavior of the terms in  $LX$ .

- (3) Set  $a = -4$ ,  $b = 2$  and  $n = 6$ . Store the sequence  $\{a + b^{-i}, 1 \leq i \leq n\}$  in a sequence named  $RX$ .

(a) List the terms in  $RX$ .

- You examined an input–output table for  $f$  as  $x \rightarrow a$  from both the left and the right.
- You examined the graph of  $f$  for  $x$  near  $a$ .

In the next task, you are going to use the same methods to estimate the value of the limit of a given function as  $x$  approaches a specified value, but this time instead of doing all your work by hand, you will use your calculator to construct an input–output table and to graph the function near the indicated point.

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Work through the section interactively. Try it, as you read along.

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- (1) Store  $\{1, -6, 2.5, 23, 0\}$  in a list called  $L1$ . Place the list in the first column in a table on your calculator.
- (2) Store the first five terms of the sequence whose typical term is  $6 - 10^{-i}$  in a list named  $LX$ . Display at least five digits to the right of the decimal point.

(a) Record the terms in  $LX$ .

(b) Describe the behavior of the terms in  $LX$ .

- (3) Set  $a = -4$ ,  $b = 2$  and  $n = 6$ . Store the sequence  $\{a + b^{-i}, 1 \leq i \leq n\}$  in a sequence named  $RX$ .

(a) List the terms in  $RX$ .

(b) Describe the behavior of the terms in  $RX$ .

(4) Modify  $RX$  so that  $b = 10$ . Store the sequence in the second column in a table on your calculator. List the terms in the new sequence below.

2. Investigate how you can use your calculator to analyze the limiting behavior of a function.

Analyze  $f(x) = (x^3 + 2x^2 + x + 2)/(x + 2)$  as  $x$  approaches  $-2$ .

*Note: See "Examining the Limiting Behavior of a Function" in the appendix for your graphing calculator.*

a. Create an input–output table.

(1) Define the function  $f(x) = (x^3 + 2x^2 + x + 2)/(x + 2)$ .

(2) Set  $a = -2$ .

(3) Define two sequences of values, one approaching  $a = -2$  from the left and one approaching  $a = -2$  from the right. Store the left sequence in  $LX$  (which stands for left sequence of  $x$ -values), where

$$LX = \{a - b^{-i}, 1 \leq i \leq n\},$$

and the right sequence in  $RX$  (for right sequence of  $x$ -values), where

$$RX = \{a + b^{-i}, 1 \leq i \leq n\}.$$

Let the base  $b = 2$ , consider the first five terms in the sequence—that is, let  $n = 6$ . Display at least four digits to the right of the decimal point.

(4) Construct the associated sequences of output values of the function  $f$ .

(a) For each input value  $x$  in  $RX$ , calculate  $f(x)$ , and store the values in a list called  $LFX$  (for left output values of  $f(x)$ ).

(b) Similarly, for each  $x$  in  $RX$ , calculate  $f(x)$ , and store the values in a list called  $RFX$  (for right output values of  $f(x)$ ).

(5) Place the two list of input values,  $LX$  and  $RX$ , and the associated lists of output values,  $LFX$  and  $RFX$ , in an input-output table. Copy

the entries in the table on your calculator into the following table, and predict the limiting behavior of  $f$  as  $x$  approaches  $-2$ .

Term # $i$	Left Sequence (LX) $x = a - b^{-i}$	Output Values (LFX) $f(x)$	Right Sequence (RX) $x = a + b^{-i}$	Output Values (RFX) $f(x)$
	↓ $-2^-$	↓ <input type="text"/>	↓ $-2^+$	↓ <input type="text"/>

(6) Draw some conclusions about the limiting behavior of  $f(x) = (x^3 + 2x^2 + x + 2)/(x + 2)$  as  $x$  approaches  $-2$ .

b. Graph the function using your calculator.

(1) Graph  $f(x) = (x^3 + 2x^2 + x + 2)/(x + 2)$  for  $x$  near  $-2$ , for instance for  $-4 \leq x \leq 0$ . Sketch the graph below.

(2) Illustrate graphically that the value of the limit  $L$  (which you determined using your input–output table) is reasonable.

As usual, mark  $L$  on the vertical axis. Use a pencil and straight-edge to indicate the output values corresponding to several input values to the left and to the right of  $a = -2$ . Indicate the behavior of  $f$  as  $x \rightarrow -2^-$  and as  $x \rightarrow -2^+$  with vertical arrows.

- (3) Use the TRACE feature on your calculator to double-check your conclusion.

Zoom in on a region of the graph of  $f$  near  $a = -2$ . Analyze the behavior of  $f(x)$  as  $x \rightarrow -2^-$  and as  $x \rightarrow -2^+$  by moving the cursor along the graph and observing what happens to the values of  $f(x)$  as  $x$  approaches  $-2$  from the left and the right.

- c. Summarize your observations.

Based on your tables and your graph it appears that

$$f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow -2^- \quad \text{and} \quad f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow -2^+$$

Consequently,

$$f(x) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow -2$$

3. Analyze some other functions using your calculator.

If your input–output table does not provide enough information about the behavior of a function near  $x = a$ , then:

- Consider more terms in the left sequence of input values  $LX$  or in the right sequence  $RX$  by increasing the value of  $n$ .
- Consider sequences which approach  $a$  more rapidly by increasing the value of the base  $b$ , say from 2 to 10.
- Increase the precision of the values in the sequence by displaying more digits.

- a. Analyze the limiting behavior of

$$f(x) = \begin{cases} x + 2, & \text{if } x < 0 \\ \sin(x) + 2, & \text{if } x > 0 \end{cases}$$

as  $x \rightarrow 0$ .

- (1) Use your calculator to construct an input–output table for  $f$  as  $x$  approaches 0 from the left and the right. Copy the table below. Label the columns and indicate the limiting behavior of  $f$  as  $x \rightarrow 0^-$  and  $x \rightarrow 0^+$ .

- (2) Sketch a graph of  $f$  for  $x$  near 0. Use a pencil and straightedge to illustrate the limiting behavior of  $f$  as  $x \rightarrow 0^-$  and  $x \rightarrow 0^+$ .

- (3) Summarize your observations about the behavior of  $f$  for  $x$  near 0. Express your result using the arrow ( $\rightarrow$ ) notation for limits.

**b.** Analyze the limiting behavior of

$$f(x) = |x + 1|$$

as  $x \rightarrow -1$ .

- (1) Use your calculator to construct an input–output table for  $f$  for  $x$  near  $-1$ . Copy the table below. Label the columns and indicate the limiting behavior of  $f$  as  $x \rightarrow -1^-$  and  $x \rightarrow -1^+$ .



- (2) Sketch a graph of  $f$  for  $x$  near  $-1$ . Use a pencil and straightedge to illustrate the limiting behavior of  $f$  as  $x \rightarrow -1^-$  and  $x \rightarrow -1^+$ .

- (3) Represent  $f(x) = |x + 1|$  by a piecewise-defined function.

$$f(x) = \begin{cases} & \text{if } x < -1 \\ & \text{if } x \geq -1 \end{cases}$$

- (4) Summarize your observations about the limiting behavior of  $f$  for  $x$  near  $-1$ . Express your result using the arrow ( $\rightarrow$ ) notation for limits and the piecewise representation of  $f$ .

Each of the functions you have examined up to this point has had a limit at the indicated point. It is possible, however, that the limit does not exist.

- The value of a limit must be a finite real number. If the output values “explode” as  $x$  approach  $a$ —that is, if they go off to plus or minus infinity—then the limit does not exist at  $x = a$ . Consequently, a function does not have a limit at a vertical asymptote.
- The value of the left- and right-hand limits must be the same. If the output values approach one number as  $x$  approaches  $a$  from the right and a different number as  $x$  approaches  $a$  from the left, then the limit does not exist at  $x = a$ . Consequently, a function does not have a limit at a jump.

Consider these two situations in the next task.

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#### Task 4-4: Examining Situations Where the Limit Does Not Exist

1. Use input–output tables to show that the following limits do not exist. In each case:
  - i. Construct the associated input–output table using your calculator. Copy the table in your activity guide.

then the following rules hold:

- The limit of the sum, difference, or product of two functions is the sum, difference, or product of the limits.

$$\begin{aligned}\lim_{x \rightarrow a} (f(x) \pm g(x)) &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \\ \lim_{x \rightarrow a} (f(x) \cdot g(x)) &= \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M\end{aligned}$$

- The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not 0.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$

- The limit of the product of a constant and a function is the product of the constant and the function.

$$\lim_{x \rightarrow a} c \cdot f(x) = c \cdot \lim_{x \rightarrow a} f(x) = c \cdot L$$

Caution! Always interpret the result when you use substitution to calculate a limit. Keep in mind what the value of the limit tells you about the limiting behavior of the function. Remember that the output values of the function approach the value of the limit as the inputs get closer and closer to the specified number; that is, if  $\lim_{x \rightarrow a} f(x) = L$ , then  $f(x)$  approaches  $L$  as  $x$  approaches  $a$  from both sides.

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### Task 4-7: Calculating Limits Using Substitution

Evaluate some limits using substitution and interpret the results. In each exercise:

- Determine if the function is continuous at  $x = a$ . Justify your conclusion.
  - If it is continuous, find the limit using substitution.
  - If you can find the limit using substitution, interpret the result. Explain what the value of the limit tells you about the limiting behavior of the function.
- Consider  $\lim_{x \rightarrow 1.5} (6x^2 - 2x + 5)$ .
    - Is  $f(x) = 6x^2 - 2x + 5$  continuous at  $x = 1.5$ ? Justify your conclusion.

- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
2. Consider  $\lim_{t \rightarrow 0} \log_2 t$ .
- a. Is  $f(t) = \log_2 t$  continuous at  $t = 0$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
3. Consider  $\lim_{x \rightarrow 106} 16.78$ .
- a. Is  $f(x) = 16.78$  continuous at  $x = 106$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
4. Consider  $\lim_{r \rightarrow -2} \frac{r^3 - 1}{r + 6.5}$ .
- a. Is  $h(r) = (r^3 - 1)/(r + 6.5)$  continuous at  $r = -2$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.

- c. If you evaluated the limit using substitution, interpret the result.
5. Consider  $\lim_{x \rightarrow -4} \sqrt{x}$ .
- a. Is  $f(x) = \sqrt{x}$  continuous at  $x = -4$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
6. Consider  $\lim_{n \rightarrow -1} (n^{10} - 8n^8 + 6n^6 - 4n^4 + 2n^2 - 100)$ .
- a. Is  $f(n) = n^{10} - 8n^8 + 6n^6 - 4n^4 + 2n^2 - 100$  continuous at  $n = -1$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
7. Consider  $\lim_{t \rightarrow \pi} 6\left(\sin\left(\frac{t}{2}\right) + \cos(t)\right)$ .
- a. Is  $f(t) = 6(\sin(t/2) + \cos(t))$  continuous at  $t = \pi$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.

- c. If you evaluated the limit using substitution, interpret the result.
8. Consider  $\lim_{x \rightarrow 1} (e^{(x-1)} \ln(x))$ .
- a. Is  $g(x) = e^{(x-1)} \ln(x)$  continuous at  $x = 1$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.
9. Consider  $\lim_{x \rightarrow 4} \frac{1/2^x}{\log_4 x}$ .
- a. Is  $q(x) = (1/2^x)/\log_4 x$  continuous at  $x = 4$ ? Justify your conclusion.
- b. If the answer to part a is “yes,” use substitution to evaluate the limit.
- c. If you evaluated the limit using substitution, interpret the result.

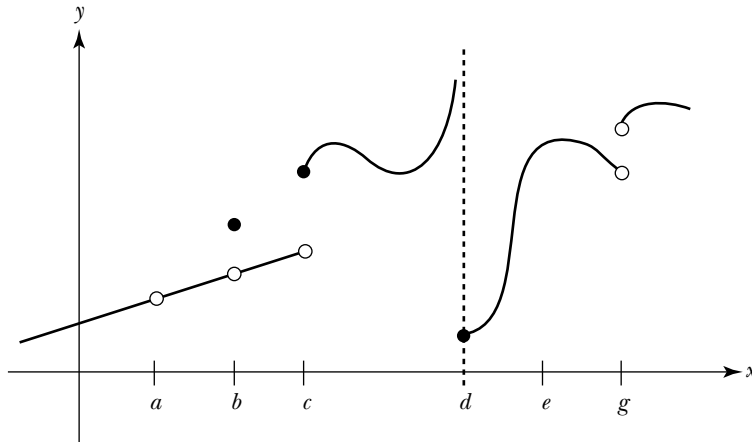
### Unit 4 Homework After Section 2

- Complete the tasks in Section 2. Be prepared to discuss them in class.

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#### HW4.6 Analyze some piecewise-defined functions.

1. Consider the piecewise-defined function  $f$  represented by the following graph:



- Identify the points on the horizontal axis where  $f$  is not continuous. For each point, determine which part of the limit-based definition for continuity  $f$  fails to satisfy: (i)  $f$  is not defined at the point, or (ii)  $f$  is defined, but the limit does not exist at the point, or (iii)  $f$  is defined and the limit exists, but the value of the limit is not equal to the value of the function at the point.
- Classify each of the function's points of discontinuity as a jump, blowup, or removable discontinuity.
- List all the intervals where  $f$  is continuous.
- Mark each of the following statements true or false. Justify each response.

$$(1) \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

$$(4) \lim_{x \rightarrow d} f(x) = f(d)$$

$$(2) \lim_{x \rightarrow b} f(x) = f(b)$$

$$(5) \lim_{x \rightarrow e} f(x) = f(e)$$

$$(3) \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$$

$$(6) \lim_{x \rightarrow g^-} f(x) < \lim_{x \rightarrow g^+} f(x)$$

- Analyze the following piecewise-defined functions.

$$a. \text{ Let } g(x) = \begin{cases} -x, & \text{if } x \leq 0 \\ x^2, & \text{if } 0 < x < 4 \\ 2x, & \text{if } x \geq 4 \end{cases}$$

- Sketch a graph of the function.
- Identify any points in the domain where the function is discontinuous. Classify each point of discontinuity.
- List all the intervals where the function is continuous.
- Evaluate each of the following limits, if possible. If the limit does not exist, explain why.

$$(a) \lim_{x \rightarrow -6} g(x)$$

$$(c) \lim_{x \rightarrow 4} g(x)$$

$$(b) \lim_{x \rightarrow 0} g(x)$$

$$(d) \lim_{x \rightarrow 6} g(x)$$

$$(e) \lim_{x \rightarrow 3.5} g(x) \qquad (f) \lim_{x \rightarrow 4^+} g(x)$$

$$b. \text{ Let } h(x) = \begin{cases} \sin(x), & \text{if } x \leq 0 \\ \ln(x), & \text{if } x > 0 \end{cases}$$

- (1) Sketch a graph of the function.
- (2) Identify any points in the domain where the function is discontinuous. Classify each point of discontinuity.
- (3) List all the intervals where the function is continuous.
- (4) Evaluate each of the following limits, if possible. If the limit does not exist, explain why.

$$(a) \lim_{x \rightarrow -\pi} h(x) \qquad (c) \lim_{x \rightarrow 1} h(x)$$

$$(b) \lim_{x \rightarrow 0} h(x) \qquad (d) \lim_{x \rightarrow 0^-} h(x)$$

$$c. h(x) = \begin{cases} 2, & \text{if } x < 4 \\ 3, & \text{if } x \geq 4 \end{cases}$$

- (1) Sketch a graph of the function.
- (2) Identify any points in the domain where the function is discontinuous. Classify each point of discontinuity.
- (3) List all the intervals where the function is continuous.
- (4) Evaluate each of the following limits, if possible. If the limit does not exist, explain why.

$$(a) \lim_{x \rightarrow -7} h(x) \qquad (c) \lim_{x \rightarrow 7} h(x)$$

$$(b) \lim_{x \rightarrow 4^+} h(x) \qquad (d) \lim_{x \rightarrow 4} h(x)$$

**HW4.7** In this exercise and the following two exercises, you will think about three important theorems concerning continuous functions and limits and explain why their statements are reasonable.

According to the *Intermediate Value Theorem for Continuous Functions*:

If  $f$  is continuous over the closed interval  $[a, b]$ , then  $f$  takes on every value between  $f(a)$  and  $f(b)$ . In other words, if  $y$  is any value between  $f(a)$  and  $f(b)$ , then there exists an  $x$ -value  $c$  between  $a$  and  $b$  such that  $f(c) = y$ .

1. Apply the Intermediate Value Theorem for Continuous Functions to a specific example. Consider  $f(x) = 2x + 1$ , where  $-2 \leq x \leq 4$ .
  - a. Sketch the graph of  $f$  for  $-2 \leq x \leq 4$ . Label  $-2$  and  $4$  on the horizontal axis and  $f(-2)$  and  $f(4)$  on the vertical axis.
  - b. In order to apply the theorem,  $f$  must be continuous over the given closed interval. Explain why  $f$  is continuous on the closed interval  $[-2, 4]$ .

- c. The theorem claims that if  $y$  is any value between  $f(-2)$  and  $f(4)$ , there exists an input value  $c$  between  $-2$  and  $4$  such that  $f(c) = y$ . Apply the theorem for  $y = 6$ ; that is, find a value for  $c$  between  $-2$  and  $4$  such that  $f(c) = 6$ . Label  $c$  and  $f(c)$  on the graph you sketched in part a and indicate the relationship between  $c$  and  $f(c)$ .
  - d. Apply the theorem for  $y = -2$ ; that is, find a value for  $c$  between  $-2$  and  $4$  such that  $f(c) = -2$ . Label  $c$  and  $f(c)$  on the graph you sketched in part a and indicate the relationship between  $c$  and  $f(c)$ .
2. Explain why the Intermediate Value Theorem for Continuous Functions makes sense. Support your explanation with an appropriate diagram; that is, on a pair of axes:
    - a. Label  $a$  and  $b$  on the horizontal axis, where  $a < b$ .
    - b. Sketch the graph of a squiggly function  $f$  that is continuous over the closed interval  $[a, b]$ .
    - c. Label  $f(a)$  and  $f(b)$  on the vertical axis.
    - d. Pick a value between  $f(a)$  and  $f(b)$  and label it  $y$ .
    - e. Show that the conclusion of the Intermediate Value Theorem for Continuous Functions holds; that is, show that there exists an  $x$ -value  $c$  between  $a$  and  $b$  such that  $f(c) = y$ . Label  $c$  on the horizontal axis.
  3. For a given value of  $y$  between  $f(a)$  and  $f(b)$ , is it possible for there to be more than one choice of  $c$  between  $a$  and  $b$  such that  $f(c) = y$ ? If so, draw a diagram supporting your conclusion. If not, explain why not.
  4. Apply the Intermediate Value Theorem to some real-life situations. For each of the following situations:
    - i. Model the scenario with a graph.
    - ii. Illustrate the conclusion on your graph.
    - a. *Scenario:* The light turns green, and you step on the gas pedal in your car. Fifteen seconds later, you level off your speed at 60 MPH.  
*Conclusion:* According to the Intermediate Value Theorem, there exists a time in the 15-second time interval when you are going 28 MPH.
    - b. *Scenario:* You walk back and forth in front of a motion detector for 20 seconds. You vary your velocity. Your minimum distance is 0.5 meters from the detector and your maximum distance is 8.5 meters.  
*Conclusion:* According to the Intermediate Value Theorem, there exists a time in the 20-second time interval when you are 5.25 meters from the detector.
    - c. *Scenario:* You have the flu. Over a three-day period, your temperature fluctuates between  $99.8^\circ$  and  $103.2^\circ$ .  
*Conclusion:* According to the Intermediate Value Theorem, there exists a time in the three-day period when your temperature is  $100^\circ$ .
  5. Use the Intermediate Value Theorem to show that the equation  $x^5 - 3x^4 - 2x^3 - x + 1 = 0$  has a solution between 0 and 1.



**HW4.8** According to the *Max-Min Theorem for Continuous Functions*:

If  $f$  is continuous over the closed interval  $[a,b]$ , then  $f$  takes on both an absolute minimum and an absolute maximum on  $[a,b]$ . In other words:

- There exists a value  $m$  between  $a$  and  $b$  such that  $f(m) \leq f(x)$  for all  $x$  between  $a$  and  $b$ ; that is,  $f$  has an absolute minimum at  $x = m$ .
  - There exists a value  $M$  between  $a$  and  $b$  such that  $f(x) \leq f(M)$  for all  $x$  between  $a$  and  $b$ ; that is,  $f$  has an absolute maximum at  $x = M$ .
1. Apply the Max-Min Theorem for Continuous Functions to a specific example. Consider  $f(x) = x^2 - 1$ , where  $-2 \leq x \leq 3$ .
    - a. Sketch the graph of  $f$  for  $-2 \leq x \leq 3$ . Label  $-2$  and  $3$  on the horizontal axis and  $f(-2)$  and  $f(3)$  on the vertical axis.
    - b. In order to apply the theorem,  $f$  must be continuous over the given closed interval. Explain why  $f$  is continuous on the closed interval  $[-2,3]$ .
    - c. The theorem claims that  $f$  has an absolute minimum on  $[-2,3]$ —that is, there exists a value  $m$  between  $-2$  and  $3$  such that  $f(m) \leq f(x)$  for all  $x$  between  $-2$  and  $3$ . Find  $m$ . Label  $m$  and  $f(m)$  on the graph you sketched in part a.
    - d. The theorem claims that  $f$  has an absolute maximum on  $[-2,3]$ —that is, there exists a value  $M$  between  $-2$  and  $3$  such that  $f(x) \leq f(M)$  for all  $x$  between  $-2$  and  $3$ . Find  $M$ . Label  $M$  and  $f(M)$  on the graph you sketched in part a.
  2. Explain why the Max-Min Theorem for Continuous Functions makes sense. Support your explanation with an appropriate diagram. On a pair of axes:
    - a. Label  $a$  and  $b$  on the horizontal axis, where  $a < b$ .
    - b. Sketch the graph of a (squiggly) function  $f$  which is continuous over the closed interval  $[a,b]$ .
    - c. Show that there exists an input value  $m$  between  $a$  and  $b$  such that  $f(m) \leq f(x)$  for all  $x$  between  $a$  and  $b$ . Label  $m$  and  $f(m)$  on your diagram.
    - d. Show that there exists an input value  $M$  between  $a$  and  $b$  such that  $f(x) \leq f(M)$  for all  $x$  between  $a$  and  $b$ . Label  $M$  and  $f(M)$  on your diagram.
  3. The Max-Min Theorem for Continuous Functions states that every function  $f$  which is continuous over a closed interval  $[a,b]$  takes on both an absolute minimum and an absolute maximum on  $[a,b]$ . If you want to determine where the absolute extrema exist, what values would you examine? In other words, which values in the interval  $[a,b]$  would be candidates for the absolute maximum and absolute minimum?
  4. Is it possible for there to be more than one choice of  $m$  between  $a$  and  $b$  such that  $f(m) \leq f(x)$  for all  $x$  between  $a$  and  $b$ ? If so, draw a diagram supporting your conclusion. If not, explain why not. Similarly, is it possible for there to be more than one choice of  $M$  between  $a$  and  $b$  such that  $f(x) \leq f(M)$  for all  $x$  between  $a$  and  $b$ ? If so, draw a diagram supporting your conclusion. If not, explain why not.

sible for there to be more than one choice of  $M$  between  $a$  and  $b$  such that  $f(x) \leq f(M)$  for all  $x$  between  $a$  and  $b$ ? If so, draw a diagram supporting your conclusion. If not, explain why not.

5. Apply the Max-Min Theorem to some real-life situations. For each of the following situations:
  - i. Model the scenario with a graph.
  - ii. Label the absolute extrema for your model.
- a. You work an 8-hour shift at a pretzel factory. At the start of your shift, your production rate is low, but it continues to increase as you settle into a routine. Two hours before the end of the shift, you start thinking about what you are going to do after work, and your production rate decreases until it's time to quit.
- b. You're home alone watching a scary movie, on a dreary night. Each time a scary part comes on, your heart rate increases dramatically and then returns to normal when the scary part is over. The movie is 117 minutes long, and there are 7 scenes that frighten you.
- c. You create a distance-versus-time graph using a motion detector. Starting 7.75 meters from the detector, you walk toward the detector for 6.5 seconds. You walk faster and faster for the first 4 seconds, and then slower and slower for the next 2.5 seconds. You stop at the half-meter mark.

**HW4.9** According to the *Sandwich Theorem*:

If  $f$ ,  $g$ , and  $h$  are three functions, where for all  $x$  near some real number  $c$

$$f(x) \leq g(x) \leq h(x)$$

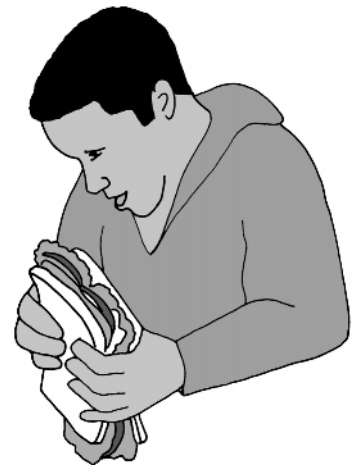
and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$$

then

$$\lim_{x \rightarrow c} g(x) = L$$

In other words, whenever the graph of  $g$  is “sandwiched” between the graphs of  $f$  and  $h$  for all  $x$  near some real number  $c$ , and  $f$  and  $h$  have the same limit as  $x$  approaches  $c$ , then  $g$  also has the same limit.



1. Begin by examining the two conditions for the Sandwich Theorem.
  - a. The first condition states that  $f$ ,  $g$ , and  $h$  are three functions, where  $f(x) \leq g(x) \leq h(x)$  for all  $x$  near  $c$ .

- (1) Illustrate what this means graphically as follows: Choose an arbitrary value for  $c$  on the horizontal axis. For  $x$  near  $c$ , sketch graphs of functions  $f$ ,  $g$ , and  $h$  which satisfy the necessary condition. Label the graphs  $f$ ,  $g$ , and  $h$ .
- (2) Describe the relationship among the graphs of  $f$ ,  $g$ , and  $h$  for  $x$  near  $c$ .
- b. The second condition states that  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ . Consequently, the graphs of  $f$  and  $h$  “merge” as  $x$  gets close to  $c$ . Explain why this is the case. Support your explanation with an appropriate diagram.
2. Apply the Sandwich Theorem to a specific example. In particular, use the Sandwich Theorem to find  $\lim_{x \rightarrow 0} x \sin(1/x)$ .
- a. Explain why you cannot evaluate  $\lim_{x \rightarrow 0} x \sin(1/x)$  directly.
- b. In order to use the Sandwich Theorem you need to show that you can “sandwich” the graph of  $g(x) = x \sin(1/x)$  for  $x$  near 0 between the graphs of two functions  $f$  and  $h$ , where  $f$  and  $h$  have the same limit as  $x$  approaches 0. Consider  $f(x) = -|x|$  and  $h(x) = |x|$ .
- (1) Show that the hypotheses of the theorem hold.
- (a) Show that  $f(x) \leq g(x) \leq h(x)$  for all  $x$  near 0 by using your calculator to sketch, on one pair of axes, the graphs of  $f$ ,  $g$ , and  $h$  for  $x$  near 0.
- (b) Use a graphic approach to show that  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$ .
- (2) Find  $\lim_{x \rightarrow 0} x \sin(1/x)$ .
3. Explain why the Sandwich Theorem makes sense in general. Support your explanation with an appropriate diagram. On a single pair of axes:
- a. Label  $c$  on the horizontal axis and  $L$  on the vertical axis.
- b. For  $x$  near  $c$ , sketch graphs of three arbitrary functions  $f$ ,  $g$ , and  $h$  where
- $f(x) \leq g(x) \leq h(x)$  for all  $x$  near  $c$
  - $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$
- c. Use the graph to show that the conclusion of the Sandwich Theorem makes sense; that is, use a graphic approach to show that in this case it is reasonable to assume that  $\lim_{x \rightarrow c} g(x)$  equals  $L$ .

## SECTION 3

## More Limits

At this point, you should have a solid conceptual understanding of what a limit is. You know how to analyze the limiting behavior of a function using an input–output table and a graphic approach. You can recognize when

happens if you need to find the area of a shape for which there is no formula, such as the area of an oddly shaped region bounded by two curves? Limits can help you do this and do much more. They are one of the most fundamental building blocks in calculus. You will use them frequently.

Your primary goal in this unit is to develop a conceptual understanding of what a limit is. You will examine the limiting behavior of a function by analyzing input–output tables and by scrutinizing the graph of the function. You will investigate what happens to the output values of a function as the function’s inputs get closer and closer to a specified point. You will use your understanding of limit to develop a definition for continuity.

Because forming conjectures concerning the value of a limit can involve doing lots of calculations, you will use your calculator to do many of the computations for you. You will use it to define sequences that approach a specified value from the left and the right, to create input–output tables, and to graph the functions.

As you use your calculator remember not to sit passively while the calculator does all the work. Think about what’s happening. As we mentioned in the Preface, each time you ask the computer to do something, keep in mind the following:

- What you have commanded the calculator to do
- Why you asked it to do whatever it is doing
- How it might be doing whatever you have told it to do
- What the results mean

Thinking this way will help you form a mental image associated with the process of a limit. It will help you understand what is happening and help you recognize when to use a limit to solve a problem.

## SECTION 1

### Limiting Behavior of Functions

Your study of limits begins by thinking about the meaning of the term *approach*. In the first task, you will examine questions such as: What is a sequence of numbers? What does it mean to say that a sequence of numbers gets closer and closer to or “approaches” another number? How can you represent a typical term in a sequence? How can you generate a sequence of numbers that approaches a given value?

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#### Task 4-1: Constructing Sequences of Numbers

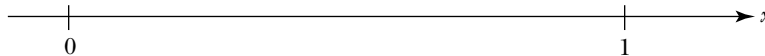
1. Examine some equivalent ways to represent a sequence of small numbers.
  - a. Consider the sequence of numbers in the first column of the chart given below. Complete the unfilled entries in the chart, expressing

each of the terms in the sequence as a fraction, as a fraction whose denominator is a power of 10, and as a negative power of 10.

Decimal representation	Fraction	Fraction with denominator power of 10	Negative power of 10
0.1	$\frac{1}{10}$		
0.01			
0.001			$10^{-3}$
0.0001			
0.00001		$\frac{1}{10^5}$	

b. Represent some of the terms in the sequence on the real number line.

(1) Mark the location of 0.1 and 0.01.



(2) Describe how you would determine the location of 0.001.

c. Suppose you use the same pattern to continue defining terms in this sequence. Consider the term that has 1 in the 42nd decimal place and zeros everywhere else.

(1) Represent this term as a fraction whose denominator is a power of 10.

(2) Represent this term as 10 raised to a negative exponent.

d. It is impossible to write down every member in the extended sequence, because the sequence goes on forever. You can, however, represent a “typical term” in the sequence, since the terms follow a specific pattern. In particular, the first term has a 1 in the first decimal place, the second term has a 1 in the second decimal place, the third term has a 1 in the third decimal place, and so on. In general, the  $i$ th term

has a 1 in the  $i$ th decimal place and zeros everywhere else, where  $i$  is any positive integer.

(1) Represent the  $i$ th term in the sequence as a fraction whose denominator is a power of 10.

(2) Represent the  $i$ th term in the sequence as 10 raised to a negative exponent.

e. If you continue to define members in this sequence using the same pattern, the terms get closer and closer to a particular value.

(1) What is this value?

(2) Will the terms in the sequence ever actually reach this value? Explain why or why not.

(3) The sequence is said to be approaching this value *from the right*. What does “from the right” mean in this case?

2. Construct another sequence of numbers.

a. Sequences can be defined using bases other than 10. For instance, the chart given below gives some members of a sequence defined using powers of 2. Complete this chart and then compare this sequence to the one you considered in part 1.

*Note: A series of three dots indicates that some terms in the sequence are missing. Do not fill in anything in this space.*

Decimal representation	Fraction	Fraction with denominator power of 2	Negative power of 2
	$\frac{1}{2}$		
	$\frac{1}{4}$		
	$\frac{1}{8}$		

	$\frac{1}{16}$		
	$\frac{1}{32}$		
· · ·	· · ·	· · ·	· · ·
	$\frac{1}{512}$		
· · ·	· · ·	· · ·	· · ·
	$\frac{1}{2048}$		

- b.** Extend this sequence using the same pattern.
- (1) What value is the sequence approaching?
  - (2) Will the sequence ever reach this value?
  - (3) Is the sequence approaching the value from the left or the right?
- c.** Represent a typical item in this new sequence by an expression—that is, represent the  $i$ th term in the sequence by an expression, where  $i$  is any positive integer.
- (1) Express the  $i$ th term in the sequence as a fraction whose denominator is a power of 2.
  - (2) Express the  $i$ th entry in the sequence as 2 raised to a negative exponent.
- d.** Compare the two sequences you considered above, one of which was defined in terms of powers of 10 and the other in terms of powers of 2.

One of the sequences approaches zero from the right “more

rapidly” than the other. Determine which sequence approaches zero more rapidly and explain why. Support your conclusion by locating some of the entries in each sequence on the number line given below. Mark the first three terms in the base 10 sequence with a small X and mark the first three terms in the base 2 sequence with a small O.



3. The two sequences that you considered above both approach zero from the right. Define two sequences that approach zero *from the left*.
  - a. Using powers of 10, construct a sequence that approaches zero from the left.
    - (1) List some of the terms in the sequence.
    - (2) Represent the  $i$ th term in this sequence by an expression, where  $i$  is any positive integer.
  - b. Using powers of 2, construct a sequence that approaches zero from the left.
    - (1) List some of the terms in the sequence.
    - (2) Represent the  $i$ th term in the sequence by an expression, where  $i$  is any positive integer.
4. Construct sequences that approach a value other than zero.
  - a. Consider the sequence
 
$$6.1, 6.01, 6.001, 6.0001, \dots$$
    - (1) Assuming the sequence continues on indefinitely, what value does the sequence appear to be approaching? Is the sequence approaching this value from the right or the left?



(2) Represent the  $i$ th term in the sequence by an expression.

*Hint:  $n.001$  can be rewritten as  $n + 10^{-3}$ , where  $n$  is an integer.*

b. Using powers of 10, construct a sequence that approaches 6 from the left.

(1) List some of the terms in this sequence.

(2) Represent the  $i$ th term in this new sequence by an expression, where  $i$  is any positive integer.

c. Using powers of 2, construct a sequence that approaches  $-4$  from the left. Represent the  $i$ th term in the sequence, where  $i$  is any positive integer.

d. Using powers of 2, construct a sequence that approaches  $-4$  from the right. Represent the  $i$ th term in the sequence, where  $i$  is any positive integer.

In the last task, you examined the limiting behavior of a sequence. You explored ways to construct a sequence of numbers that gets closer and closer to a particular value, such as 0, 6, or  $-4$ , without ever actually reaching the value. In general, if  $a$  is a fixed real number and the values of  $x$  are getting closer and closer to  $a$  from the right, you write

$$x \rightarrow a^+$$

where the arrow sign  $\rightarrow$  is read as “approaches” and the little  $^+$  sign indicates that the values of  $x$  are greater than—and hence to the right of— $a$ . Similarly, if the values of  $x$  are approaching  $a$  from the left, you write

$$x \rightarrow a^-$$

Finally, if they are approaching  $a$  from both sides—that is, if  $x \rightarrow a^+$  and  $x \rightarrow a^-$ —you write

$$x \rightarrow a$$

It is important to note that in all three of these cases the arrow notation indicates that the values of  $x$  get arbitrarily close to  $a$ , but do not reach  $a$ .

What you are really interested in, however, is the limiting behavior of a function. In particular, how do the output values of a function behave as the associated inputs approach a particular number? For example, what happens to the value of  $f(x) = x + 5$ , as the values of  $x$  get closer and closer to 6? In other words, does  $x + 5$  approach a particular value, as  $x$  approaches 6? If so, what? Or, as another example, suppose *DayLight* is a function that returns the length of each day during the year. What happens to the value of  $DayLight(t)$ , as  $t$  approaches the vernal equinox? One way to investigate the answers to questions such as these is to graph the given function and observe what's happening. Another way is to create an *input-output table*. You will use both of these approaches in the next task.

Caution! As you examine the limiting behavior of the following functions, keep in mind that there are two “approachings” going on: one by the inputs to the function and one by the corresponding output values of the function. These are different and they happen simultaneously.

### Task 4-2: Analyzing the Limiting Behavior of Functions

1. Examine the limiting behavior of  $f(x) = x + 5$  as  $x$  approaches 6.
  - a. Use an input–output table to examine the limiting behavior of the function.
    - (1) Consider the sequence of numbers approaching 6 from the right and the sequence approaching 6 from the left that you defined in Task 4-1, part 4. Calculate the value of  $f(x) = x + 5$  at each input and record the result in the *input–output table* given below.

Term # $i$	Left Sequence $x = 6 - 10^{-i}$	Output Values $f(x)$	Right Sequence $x = 6 + 10^{-i}$	Output Values $f(x)$
1	5.9	10.9		
2				
3				
4				
5				
	↓ $6^-$ (from the left)	↓ <input type="text"/>	↓ $6^+$ (from the right)	↓ <input type="text"/>

- (2) Sketch a graph of  $f$  for  $x$  near  $-1$ . Use a pencil and straightedge to illustrate the limiting behavior of  $f$  as  $x \rightarrow -1^-$  and  $x \rightarrow -1^+$ .

- (3) Represent  $f(x) = |x + 1|$  by a piecewise-defined function.

$$f(x) = \begin{cases} & \text{if } x < -1 \\ & \text{if } x \geq -1 \end{cases}$$

- (4) Summarize your observations about the limiting behavior of  $f$  for  $x$  near  $-1$ . Express your result using the arrow ( $\rightarrow$ ) notation for limits and the piecewise representation of  $f$ .

Each of the functions you have examined up to this point has had a limit at the indicated point. It is possible, however, that the limit does not exist.

- The value of a limit must be a finite real number. If the output values “explode” as  $x$  approach  $a$ —that is, if they go off to plus or minus infinity—then the limit does not exist at  $x = a$ . Consequently, a function does not have a limit at a vertical asymptote.
- The value of the left- and right-hand limits must be the same. If the output values approach one number as  $x$  approaches  $a$  from the right and a different number as  $x$  approaches  $a$  from the left, then the limit does not exist at  $x = a$ . Consequently, a function does not have a limit at a jump.

Consider these two situations in the next task.

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#### Task 4-4: Examining Situations Where the Limit Does Not Exist

1. Use input–output tables to show that the following limits do not exist. In each case:
  - i. Construct the associated input–output table using your calculator. Copy the table in your activity guide.

- ii. Explain why the information on your table supports the claim that the limit does not exist.
- a. Show that the limit of

$$p(x) = \frac{x + 6}{x - 1} \quad \text{as } x \rightarrow 1$$

does not exist.

- b. Show that the limit of

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ \cos(x), & \text{if } x \geq 0 \end{cases} \quad \text{as } x \rightarrow 0$$

does not exist.

2. Use a graphic approach to show that the following limits do not exist. In each case:
- i. Sketch a graph of the function.
- ii. Explain why the graph supports the claim that the limit does not exist.

a. Show that the limit of

$$g(x) = \begin{cases} x^2, & \text{if } x \leq 2 \\ 2x + 3, & \text{if } x > 2 \end{cases} \text{ as } x \rightarrow 2$$

does not exist.

b. Show that the limit of

$$h(x) = \frac{1}{x} \text{ as } x \rightarrow 0$$

does not exist.

### Unit 4 Homework After Section 1

- Complete the tasks in Section 1. Be prepared to discuss them in class.

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#### HW4.1 What is a limit?

Write a one-page essay explaining what a limit is. Include in your explanation a description of how one can find a reasonable estimate for a

limit using an input–output table and using a graph. Also include a description of how you can determine when a limit does not exist.

**HW4.2** Consider another notation for limits.

In this section, we used the notation  $f(x) \rightarrow L$  as  $x \rightarrow a$  to indicate that the limit of  $f(x)$  as  $x$  approaches  $a$  equals  $L$ . Another way to denote this is with the notation

$$\lim_{x \rightarrow a} f(x) = L$$

Similarly, writing  $f(x) \rightarrow LL$  as  $x \rightarrow a^-$  is equivalent to writing  $\lim_{x \rightarrow a^-} f(x) = LL$ , and  $f(x) \rightarrow RL$  as  $x \rightarrow a^+$  is equivalent to  $\lim_{x \rightarrow a^+} f(x) = RL$ , where  $LL$  and  $RL$  represent the left- and right-hand limits respectively.

1. Give a verbal description of each of the following limits.

a.  $\lim_{x \rightarrow 0} \frac{x}{x^3 - 6x^2 + 5x} = \frac{1}{5}$

b. If  $f(x) = \sqrt[3]{2x^2 - 4x + 8}$ , then  $\lim_{x \rightarrow 0} f(x) = 2$ .

c.  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$

d. Let  $g(t) = \begin{cases} 0, & \text{if } t < -10 \\ t + 6.5, & \text{if } t > -10 \end{cases}$

(1)  $\lim_{t \rightarrow -10^+} g(t) = -3.5$

(2)  $\lim_{t \rightarrow -10^-} g(t) = 0$

2. Express each of the following limits using the new lim notation.

a.  $-2\sin(\theta) \rightarrow -2$ , as  $\theta \rightarrow \frac{\pi}{2}$

b.  $\frac{t^2 - \frac{1}{4}}{t - \frac{1}{2}} \rightarrow 1$ , as  $t \rightarrow \frac{1}{2}$

c.  $100 \rightarrow 100$ , as  $x \rightarrow 5$

d. Let  $h(r) = \begin{cases} r^2 + 4r + 8, & \text{if } r < -2 \\ -r^2 + 4r + 4, & \text{if } r \geq -2 \end{cases}$

(1)  $h(r) \rightarrow 4$ , as  $r \rightarrow -2^-$

(2)  $h(r) \rightarrow -8$ , as  $r \rightarrow -2^+$

e.  $g(x) \rightarrow M$ , as  $x \rightarrow b^-$

3. Model the responses to each of the following questions as a limit using the lim notation.



- a. A patient receives a 200-milligram dose of a drug. Suppose  $a(t)$  represents the amount of the drug present in her bloodstream after  $t$  hours. What will be the amount of the drug in her bloodstream as the time approaches  $2\frac{1}{2}$  hours after the initial dose?
- b. A company makes footballs. When  $n$  balls are made the average cost per ball is  $C(n)$ . As the number of footballs produced increases, the average cost per ball drops. What will be the average cost per ball as the number of balls produced nears 1,000,000?
- c. Suppose  $h(t) = \log(t + 1) + 0.75$ , where  $t \geq 0$ , gives the average height in feet of a particular animal after  $t$  years. What will the average height of the animal be as it approaches age 10?
- d. Suppose  $m(x)$  is the slope of the secant line determined by the points  $Q(x, f(x))$  and  $P(2, 10)$  which lie on the graph of  $f$ . What value does the slope of the secant line approach as  $x$  gets closer and closer to 2?

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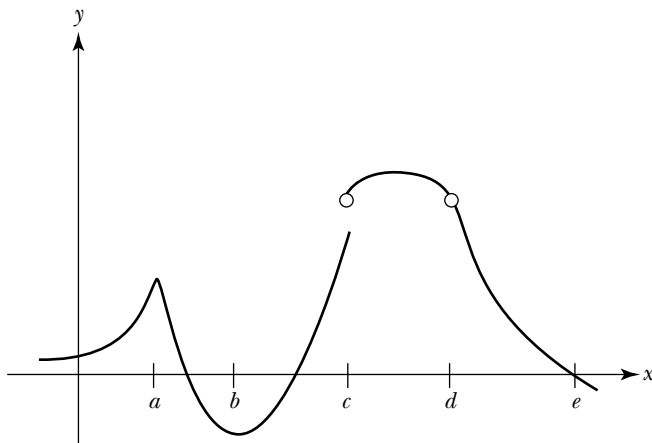
**HW4.3** Examine the limiting behavior of some functions.

1. Use a graphic approach to show that the value for each of the following limits is reasonable. Label the graph carefully.
  - a.  $2x - 3 \rightarrow -7$  as  $x \rightarrow -2$
  - b.  $\lim_{x \rightarrow 10} 5 = 5$
  - c.  $x^2 + 1 \rightarrow 5$  as  $x \rightarrow 2^-$
  - d.  $\lim_{x \rightarrow \pi} \sin(x) = 0$
2. Use an input–output table to show that the value of each of the following limits is reasonable.
  - a.  $\lim_{x \rightarrow 3} 2^x = 8$
  - b.  $4.5 \rightarrow 4.5$  as  $x \rightarrow -3$
  - c.  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$
  - d. Let  $p(x) = \begin{cases} -x + 1, & \text{if } x \leq 0 \\ x^2 + 1, & \text{if } x > 0 \end{cases}$ 
    - (1)  $\lim_{x \rightarrow 0} p(x) = 1$
    - (2)  $\lim_{x \rightarrow -2} p(x) = 3$
    - (3)  $p(x) \rightarrow 5$  as  $x \rightarrow 2$
3. Show that each of the following limits does not exist. Use:
  - i. a graphic approach
  - ii. an input–output table
  - a.  $\lim_{x \rightarrow 0} \frac{1}{x^2}$

- b. The limit of  $h(x)$  as  $x \rightarrow -5$ , where  $h(x) = \begin{cases} -3, & \text{if } x \leq -5 \\ 4.5, & \text{if } x > -5 \end{cases}$

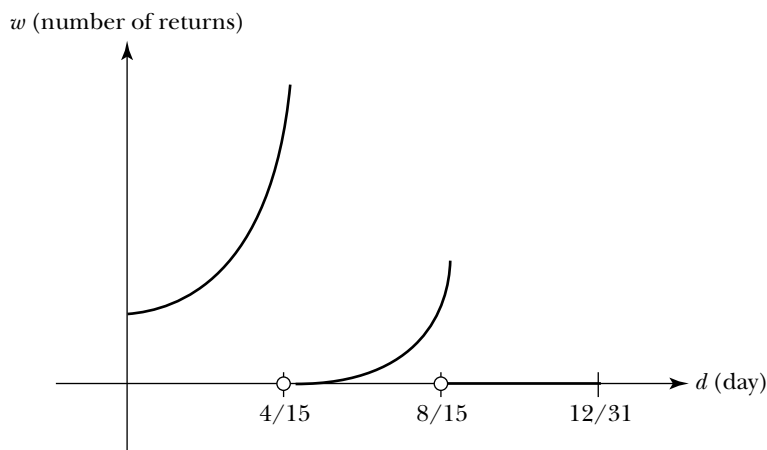
**HW4.4** Find reasonable values for some limits by considering the graph of a function or a table of values.

1. Consider the following graph of function  $h$ :



Mark the values of  $L, M, N, P, Q,$  and  $R$  on the vertical axis of the graph of  $h$  where:

- |  |  |
|--|--|
| a. $h(x) \rightarrow L$ as $x \rightarrow a$   | d. $\lim_{x \rightarrow c^+} h(x) = P$       |
| b. $\lim_{x \rightarrow b} h(x) = M$           | e. $\lim_{x \rightarrow d} h(x) = Q$         |
| c. $h(x) \rightarrow N$ as $x \rightarrow c^-$ | f. $h(x) \rightarrow R$ as $x \rightarrow e$ |
2. Suppose you help people prepare their federal income tax returns. Most of your clients manage to file their returns on time by April 15. Those who don't request a 4-month extension. The following graph shows your average workload over the course of a year.







a. Analyze your workload near April 15. Indicate the values of the limits on the vertical axis.

(1) Label the value of  $\lim_{d \rightarrow 4/15^-} w(d)$ .

(2) Label the value of  $\lim_{d \rightarrow 4/15^+} w(d)$ .

(3) Interpret your answers to parts (1) and (2). What do they say about your workload at this time of year?

b. Find and interpret each of the following limits. If the limit does not exist, explain why not.

(1)  $\lim_{d \rightarrow 12/31^-} w(d)$

(2)  $\lim_{d \rightarrow 8/15} w(d)$

(3)  $\lim_{d \rightarrow 6/15} w(d)$

3. Suppose you use your calculator to examine the behavior of a function  $f$  for  $x$  near  $-2$  and discover that:

$x$	$f(x)$	$x$	$f(x)$
-2.5	7.8701	-1.5	10.7500
-2.05	8.0600	-1.95	10.5678
-2.005	8.3456	-1.995	10.2345
-2.0005	8.4567	-1.9995	10.0034
-2.00005	8.4999	-1.99995	10.0001

a. From the information that your calculator gives you, what might you conclude about:

(1) the left-hand limit of  $f(x)$  as  $x$  approaches  $-2$ ?

(2)  $\lim_{x \rightarrow -2^+} f(x)$ ?

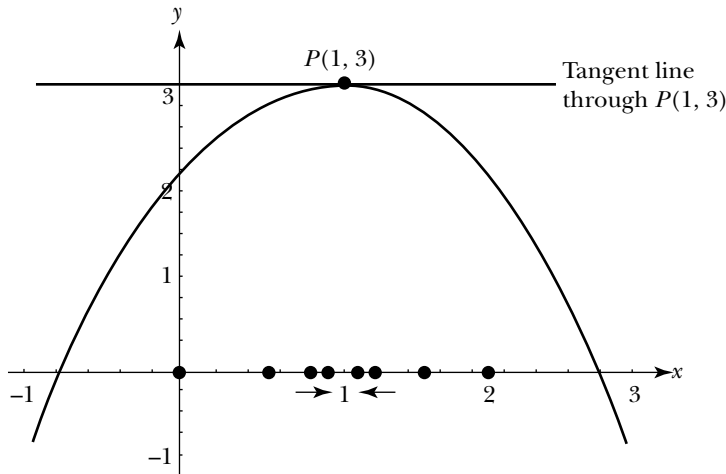
(3) the limit of  $f(x)$  as  $x$  approaches  $-2$ ?

b. Assume  $f$  is defined at  $x = -2$ . Describe the behavior of the graph of  $f$  for  $x$  near  $-2$ .

**HW4.5** Use limit-based argument to find the slope of a tangent line to a curve. This exercise is a preview of things to come.

The slope of the tangent line to the graph of  $f(x) = -x^2 + 2x + 2$  at the point  $P(1,3)$  is equal to 0. Show that this is reasonable, and then consider the general case.

1. Consider the following graph of  $f(x) = -x^2 + 2x + 2$ .



The items in the sequence of points indicated on the horizontal axis approach 1 from the left and from the right. Corresponding to each item in the sequence, there is a point on the graph of  $f$  and there is a secant line which passes through the point on the graph and  $P$ . Use a graphic approach to examine the limiting behavior of the slopes of these secant lines as  $x \rightarrow 1$ .

- a. Consider the sequence of  $x$ -values approaching 1 from the left. Use a pencil and straightedge to draw the secant lines determined by  $P$  and each point on the graph that corresponds to an item in the sequence. Describe the behavior of the slopes of these secant lines as  $x \rightarrow 1^-$ .
  - b. Similarly, consider the sequence of  $x$ -values approaching 1 from the right. Draw the secant lines determined by  $P$  and each point on the graph that corresponds to an item in the sequence. Describe the behavior of the slopes of these secant lines as  $x \rightarrow 1^+$ .
  - c. Based on your observations, does it appear that the slopes of the secant lines are approaching 0 as  $x \rightarrow 1$ ?
2. Use an input–output table to examine the behavior of the slopes of the secant lines as  $x \rightarrow 1$ .
- a. Fill in the entries in the following input–output table. Indicate the values that the slopes of the secant lines appear to be approaching as  $x$  approaches 1 from the left and the right.

LeftSeq $x$	$f(x)$	Slope of Secant Line Determined by $P(1, 3)$ and $Q(x, f(x))$	RightSeq $x$	$f(x)$	Slope of Secant Line Determined by $P(1, 3)$ and $Q(x, f(x))$
0	2	$\frac{3-2}{1-0} = 1$	2		
0.5			1.5		

0.75			1.25		
0.9			1.1		
↓ 1 <sup>-</sup>		↓ <input type="text"/>	↓ 1 <sup>+</sup>		↓ <input type="text"/>

- b. Based on the input–output table, does it appear that the slopes of the secant lines are approaching 0 as  $x \rightarrow 1$ ?
3. Consider the general case. Examine the limiting behavior of the slopes of the secant lines as  $x \rightarrow a$ , where  $-1 < a < 3$ .
- Make a clean sketch of the graph of  $f(x) = -x^2 + 2x + 2$  for  $-1 \leq x \leq 3$ .
  - Let  $a$  be a number between  $-1$  and  $3$ .
    - Mark  $a$  on the horizontal axis and  $f(a)$  on the vertical axis.
    - Mark the point  $P(a, f(a))$  on the graph of  $f$ .
  - Consider a point close to  $x = a$ .
    - Let  $h$  be a small number. Mark  $a + h$  on the horizontal axis and  $f(a + h)$  on the vertical axis.
    - Mark the point  $Q(a + h, f(a + h))$  on the graph of  $f$ .
  - Using a straightedge, draw the secant line determined by the points  $P$  and  $Q$ .
  - Find an expression, in terms of  $a$  and  $h$ , for the slope of the secant line determined by  $P$  and  $Q$ .
  - Note that as  $h$  gets closer and closer to zero,  $a + h$  approaches  $a$ . Describe the behavior of the slopes of the secant lines determined by  $P$  and  $Q$  as  $h \rightarrow 0$ .

## SECTION 2

## Continuity, Limits, and Substitution

The word “continuous” means to go on without interruption. If you can trace through the graph of a function without lifting your pencil, the function is *continuous*; if it is not continuous, you have to lift your pencil at the place on the graph corresponding to the point of discontinuity.

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### Task 4-6: Identifying Continuous Functions

1. Use the limit-based definition of continuity to examine the continuity of some functions at a point.

For each function ask yourself:

- i. Does  $f(a)$  exist? If it does, find its value.
- ii. Does  $\lim_{x \rightarrow a} f(x)$  exist? If it does, find a reasonable value for the limit by considering the graph of the function near  $x = a$ . Sketch a graph of the function by hand or use your calculator.
- iii. Assuming the answers to i and ii are “yes,” does  $\lim_{x \rightarrow a} f(x) = f(a)$ ?

If the answer to all three questions is “yes,” then the function is continuous at  $x = a$ . If the answer to any one of them is “no,” then  $f$  is not continuous at  $x = a$ . Quit as soon as you encounter the first “no” answer.

- a. Determine if  $f(x) = \sin(x)$  is continuous at  $x = \pi$ .

(1) Responses to questions.

(2) Conclusion.

b.  $f(x) = \frac{1}{\cos(x)}$  at  $x = \frac{\pi}{2}$

(1) Responses to questions.

(2) Conclusion.

$$\text{c. } f(x) = \begin{cases} x - 2, & \text{if } x < 4 \\ \sqrt{x}, & \text{if } x \geq 4 \end{cases} \quad \text{at } x = 4$$

(1) Responses to questions.

(2) Conclusion.

$$\text{d. } f(x) = \begin{cases} e^x, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases} \quad \text{at } x = 0$$

(1) Responses to questions.

(2) Conclusion.

$$\mathbf{e.} \quad f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

(1) Responses to questions.

(2) Conclusion.

$$\mathbf{f.} \quad f(x) = \begin{cases} x^2, & \text{if } x \neq -2 \\ 5, & \text{if } x = -2 \end{cases} \quad \text{at } x = -2$$

(1) Responses to questions.

(2) Conclusion.

2. A function is *continuous over an interval*  $I$  if it is continuous at each point in the interval. Consider the basic function classes. Identify the intervals where functions in these classes are continuous.

a. The class of constant functions.

Consider an arbitrary constant function  $f(x) = c$ , where  $c$  is a constant. Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where a constant function is continuous.

**b.** The class of linear functions.

Consider an arbitrary linear function  $f(x) = mx + b$ , where  $m \neq 0$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where a linear function is continuous.

**c.** The class of parabolic functions.

Consider an arbitrary parabolic function  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where a parabolic function is continuous.

**d.** The class of power functions.

(1) Let  $n$  be an even integer, where  $n > 0$ . Consider an arbitrary even power function  $f(x) = x^n$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where an even power function is continuous.

(2) Let  $n$  be a odd integer, where  $n > 0$ . Consider an arbitrary odd power function  $f(x) = x^n$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where an odd power function is continuous.

**e.** The class of exponential functions.

Consider an arbitrary exponential function  $f(x) = ab^x$ , where  $b > 0$  and  $b \neq 1$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where an exponential function is continuous.

f. The class of logarithmic functions.

Consider an arbitrary logarithmic function  $f(x) = \log_b x$ , where  $b > 0$  and  $b \neq 1$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where a logarithmic function is continuous.

g. The general sine function.

Consider an arbitrary sine function  $f(x) = a \sin(b(x - c)) + d$ , where  $a \neq 0$  and  $b \neq 0$ . Sketch the general shape of the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where a sinusoidal function is continuous.

h. The cosine function.

Consider the cosine function  $f(x) = \cos(x)$ . Sketch the graph of  $f$ . Based on the graph, identify the set of all  $x$ -values where the cosine function is continuous.

3. In the last part, you identified where functions in the basic function classes are continuous. In addition, any combination of continuous functions is continuous on its domain. In particular, the sum, difference, product, quotient, or composition of continuous functions is continuous on its domain. This enables you to identify where numerous other functions are continuous. For example,
- Every polynomial is continuous for all real numbers, since a polynomial is the sum of power functions, which, in turn, are continuous.
  - Every rational function is continuous at all points where the denominator is not equal to zero, since a rational function is the quotient of two polynomials, which, in turn, are continuous.
  - Every extended power function ( $x^{m/n}$ ) is continuous on its domain, since it is the composition of two continuous functions.



- A complicated function, such as  $f(x) = (3\sin(0.5x) + \log_2(x))/3^x$  is continuous for all  $x > 0$ , since  $f$  is the quotient of two continuous functions. In particular, the numerator of  $f$  is continuous for all  $x > 0$ , since the numerator is a combination of a sinusoidal function (which is continuous for all real numbers) and a logarithmic function (which is continuous for all positive numbers). The denominator is continuous for all real numbers, since every exponential function is continuous for all real numbers. Moreover, the denominator is never 0, since an exponential function is never 0.

Use similar arguments to explain why each of the following functions is continuous on its specified domain.

a.  $g(x) = \frac{x^2 - 2.4x + 9.5}{\sqrt{x}}$ , where  $x > 0$ .

b.  $h(r) = 6 \log_{10}(r - 2) - 3r^{10}$ , where  $r > 2$ .

c.  $s(t) = \pi + \cos(t)$ , where  $t$  is any real number.

If a function  $f$  is continuous at  $x = a$ , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

This enables you to use *substitution* to find the exact value of the limit of a continuous function by simply evaluating the function at  $x = a$ . This powerful result makes calculating limits much easier—and certainly more precise—than using input–output tables or graphs.

Because a combination of continuous functions is continuous, you can find the limit of a combination by uncombining the components, finding the limit of each component, and then combining the results. In particular, if the limit of  $f$  and the limit of  $g$  as  $x$  approaches  $a$  exist, where

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M$$

then the following rules hold:

- The limit of the sum, difference, or product of two functions is the sum, difference, or product of the limits.

$$\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M$$

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M$$

- The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not 0.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$

- The limit of the product of a constant and a function is the product of the constant and the function.

$$\lim_{x \rightarrow a} c \cdot f(x) = c \cdot \lim_{x \rightarrow a} f(x) = c \cdot L$$

Caution! Always interpret the result when you use substitution to calculate a limit. Keep in mind what the value of the limit tells you about the limiting behavior of the function. Remember that the output values of the function approach the value of the limit as the inputs get closer and closer to the specified number; that is, if  $\lim_{x \rightarrow a} f(x) = L$ , then  $f(x)$  approaches  $L$  as  $x$  approaches  $a$  from both sides.

### Task 4-7: Calculating Limits Using Substitution

Evaluate some limits using substitution and interpret the results. In each exercise:

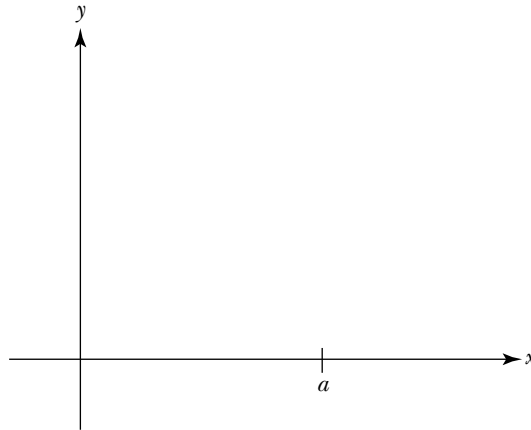
- Determine if the function is continuous at  $x = a$ . Justify your conclusion.
  - If it is continuous, find the limit using substitution.
  - If you can find the limit using substitution, interpret the result. Explain what the value of the limit tells you about the limiting behavior of the function.
- Consider  $\lim_{x \rightarrow 1.5} (6x^2 - 2x + 5)$ .
    - Is  $f(x) = 6x^2 - 2x + 5$  continuous at  $x = 1.5$ ? Justify your conclusion.

### Task 4-5: Inspecting Points of Discontinuity

Examine the various types of discontinuities and determine limit-based conditions guaranteeing that these types of discontinuities cannot occur.

*Note: When discussing the continuity or discontinuity of a function at a point, the function is assumed to be defined at that point.*

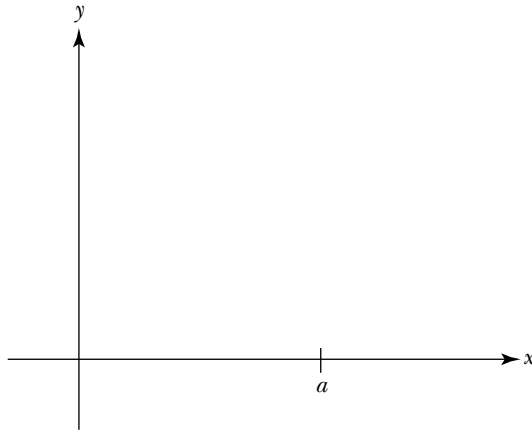
1. Examine the limiting behavior of a function near a jump discontinuity.
  - a. On the axes below, sketch the graph of a function that is defined at  $x = a$  and has a jump at  $x = a$ .



- b. On the vertical axis of your graph:
  - (1) Label  $f(a)$ .
  - (2) Label  $LL$ , where  $LL$  is the left-hand limit of  $f$  as  $x$  approaches  $a$ ; that is,  $LL = \lim_{x \rightarrow a^-} f(x)$ .
  - (3) Label  $RL$ , where  $RL$  is the right-hand limit of  $f$  as  $x$  approaches  $a$ ; that is,  $RL = \lim_{x \rightarrow a^+} f(x)$ .
- c. Does the limit of  $f$  as  $x$  approaches  $a$  (from both sides) exist if  $f$  has a jump discontinuity at  $x = a$ ? Justify your response.
- d. Clearly, a function that has a vertical jump at  $x = a$  is not continuous at  $x = a$ . The goal, however, is to find conditions that guarantee that a function *is* continuous. Based on your observations, what condition on the existence of the limit, and in particular on  $LL$  and  $RL$ , will eliminate the possibility of a function having a vertical gap at  $x = a$ ?

2. Examine the limiting behavior of a function near a blowup discontinuity.

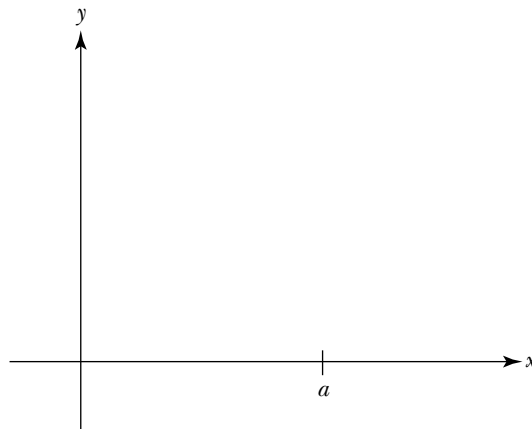
- a. On the axes below, sketch the graph of a function that is defined at  $x = a$  and has a vertical asymptote at  $x = a$ .



- b. Label  $f(a)$  on the vertical axis of your graph.
- c. Does the limit of  $f$  as  $x$  approaches  $a$  exist if  $f$  has a blowup discontinuity at  $x = a$ ? Justify your response.
- d. Based on your observations, what condition on  $\lim_{x \rightarrow a} f(x)$  will guarantee that a function does not have a vertical asymptote at  $x = a$ ?

3. Examine the limiting behavior of a function near a removable discontinuity.

- a. On the axes below, sketch the graph of a function that is defined at  $x = a$  and has a hole at  $x = a$ .



- b. On the vertical axis of your graph:
- (1) Label  $f(a)$ .
  - (2) Label  $L$ , where  $L = \lim_{x \rightarrow a} f(x)$ .
- c. In the case of a function that is defined at  $x = a$  and has a hole at  $x = a$ , what can you conclude about the relationship between the value of the limit as  $x$  approaches  $a$  and the value of  $f(a)$ ?
- d. Based on your observations, what limit-based condition guarantees that the point  $P(a, f(a))$  is connected to the graph? In other words, what must be true about the values of  $L$  and  $f(a)$  for a function not to have a hole at  $x = a$ ?

What conditions guarantee that a function is continuous at a point? According to your observations in Task 4-5, a function is continuous at a point if and only if:

- i. The function is defined at the point. (It doesn't make sense to talk about a function being continuous at a point where it is undefined.)
- ii. The limit of the function exists as  $x$  approaches the point. (This condition eliminates the possibility of the function having a jump or a blow-up discontinuity.)
- iii. The value of the limit equals the value of the function at the point. (This condition eliminates the possibility of the function having a removable discontinuity.)

In other words,  $f$  is continuous at  $x = a$  if and only if:

- i.  $f(a)$  exists
- ii.  $\lim_{x \rightarrow a} f(x)$  exists
- iii.  $\lim_{x \rightarrow a} f(x) = f(a)$

The third condition provides you with a quick way to calculate the limit of a continuous function. In particular, it says that if a function is continuous at  $x = a$ , you can find the value of  $\lim_{x \rightarrow a} f(x)$  by simply evaluating  $f(a)$ . This enables you to find the exact value of a limit instead of using an input-output table or a graph to find a reasonable approximation. To use this approach you need to be able to identify if a function is continuous. This is the goal of the next task.

calculate the right-hand limit by substituting into the expression that defines the function to the right of the end-point. There are several possible outcomes: (1) If the left- and right-hand limits are equal, but the function is not defined, then it has a hole. (2) If the left- and right-hand limits are equal and the function is defined, then it is continuous. (3) If the left- and right-hand limits are not equal, the function has a jump. (4) If either the left- and/or the right-hand limit does not exist, the function has a vertical asymptote. These four possibilities are based on ideas you considered in earlier sections. They should seem reasonable to you.

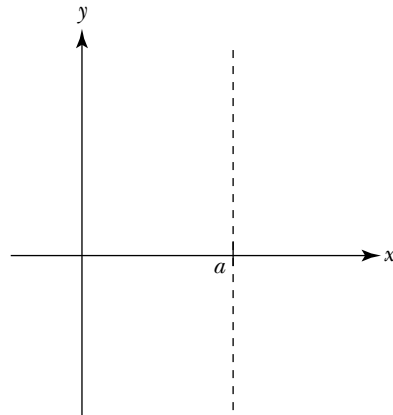
Before beginning the next task, read through this introduction again and then use the ideas as you do the task.

### Task 4-8: Using Limits to Investigate Functions

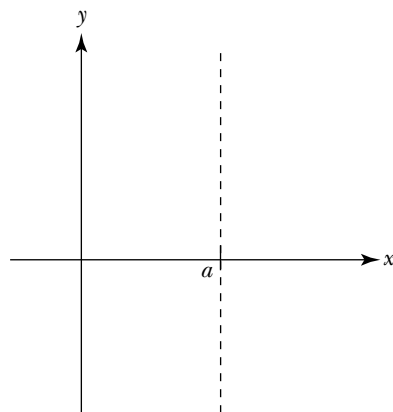
1. First consider some general situations.

a. Suppose  $f$  has a vertical asymptote at  $x = a$ . For each of the following cases, sketch the graph of  $f$  for  $x$  near  $a$ .

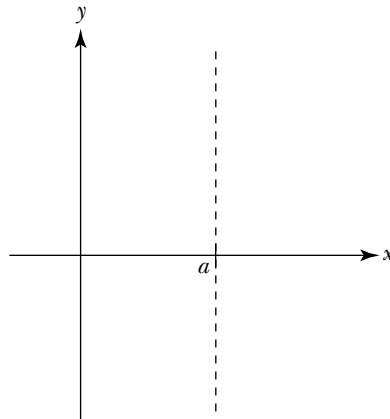
(1)  $\lim_{x \rightarrow a^-} f(x) = +\infty$  and  $\lim_{x \rightarrow a^+} f(x) = +\infty$



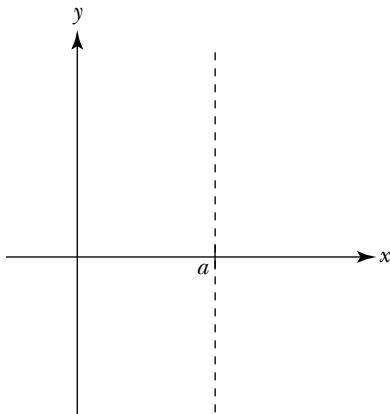
(2)  $\lim_{x \rightarrow a^-} f(x) = -\infty$  and  $\lim_{x \rightarrow a^+} f(x) = -\infty$



$$(3) \lim_{x \rightarrow a^-} f(x) = +\infty \text{ and } \lim_{x \rightarrow a^+} f(x) = -\infty$$

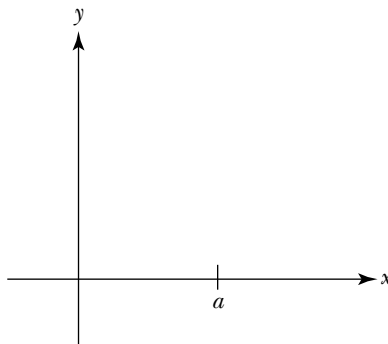


$$(4) \lim_{x \rightarrow a^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow a^+} f(x) = +\infty$$

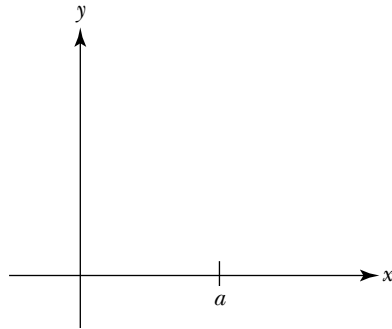


- b.** Suppose  $f$  is a piecewise-defined function where  $x = a$  is the common end-point of two adjacent subdomains. For each of the following lists of conditions, sketch the graph of a function satisfying the conditions for  $x$  near  $a$ .

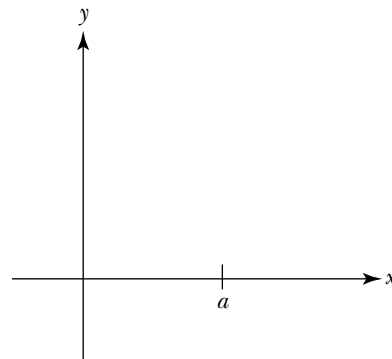
$$(1) \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x), \text{ but } f(a) \text{ does not exist.}$$



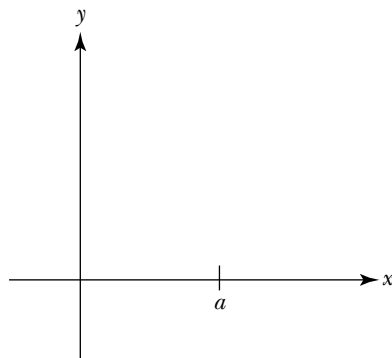
(2)  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$  and  $f(a)$  exists.



(3)  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$  and  $f(a)$  exists.



(4)  $\lim_{x \rightarrow a^-} f(x)$  does not exist,  $\lim_{x \rightarrow a^+} f(x)$  does exist, and  $f$  is defined at  $x = a$ .



2. Use limits to analyze the behavior of some rational functions (without sketching the graph).

For each value of  $a$ , determine if the given function  $f$  has a hole, vertical asymptote, or is continuous at  $x = a$ .

- If  $f$  has a hole at  $x = a$ , evaluate the limit of  $f(x)$  as  $x \rightarrow a$ . Find the  $(x,y)$ -coordinates of the hole.



- If  $f$  has a vertical asymptote, describe the behavior of the function near the asymptote using one-sided limits.
- If  $f$  is continuous, evaluate the limit of  $f(x)$  as  $x \rightarrow a$ .

a. Consider  $f(x) = \frac{x^2 + x - 6}{x - 2}$ .

(1) Analyze the behavior of  $f$  for  $x$  near 2.

(2) Analyze the behavior of  $f$  for  $x$  near  $-2$ .

b. Consider  $f(x) = \frac{x^2 + 6x + 5}{x^2 + 4x - 5}$ .

(1) Analyze the behavior of  $f$  for  $x$  near 1.

(2) Analyze the behavior of  $f$  for  $x$  near  $-5$ .

(3) Analyze the behavior of  $f$  for  $x$  near 0.

c. Consider  $f(x) = \frac{-x^3 + x}{x^5 - x}$ .

(1) Analyze the behavior of  $f$  for  $x$  near 2.

(2) Analyze the behavior of  $f$  for  $x$  near  $-1$ .

(3) Analyze the behavior of  $f$  for  $x$  near 0.

(4) Analyze the behavior of  $f$  for  $x$  near 1.

3. Use limits to analyze the behavior of some piecewise-defined functions (without sketching the graph).

For each value of  $a$ , find the left- and right-hand limits as  $x \rightarrow a$ . Interpret the results by determining if the given function has a hole, a jump, or is continuous at  $x = a$ .

a. Consider  $f(x) = \begin{cases} \sin(x), & \text{if } x < 0 \\ 2^x, & \text{if } 0 \leq x < 2 \\ x^2, & \text{if } x > 2 \end{cases}$

(1) Analyze the behavior of  $f$  for  $x$  near 0.

(a) Find  $\lim_{x \rightarrow 0^-} f(x)$ .

(b) Find  $\lim_{x \rightarrow 0^+} f(x)$ .

(c) Interpret your results.

(2) Analyze the behavior of  $f$  for  $x$  near 2.

(a) Find  $\lim_{x \rightarrow 2^-} f(x)$ .

(b) Find  $\lim_{x \rightarrow 2^+} f(x)$ .

(c) Interpret your results.

b. Consider  $f(x) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x < -1 \\ \frac{1}{3}x + \frac{1}{3}, & \text{if } -1 \leq x < 2 \\ (x - 3)^2, & \text{if } x > 2 \end{cases}$

(1) Analyze the behavior of  $f$  for  $x$  near  $-1$ .

(a) Find  $\lim_{x \rightarrow -1^-} f(x)$ .

(b) Find  $\lim_{x \rightarrow -1^+} f(x)$ .

(c) Interpret your results.

(2) Analyze the behavior of  $f$  for  $x$  near 2.

(a) Find  $\lim_{x \rightarrow 2^-} f(x)$ .

(b) Find  $\lim_{x \rightarrow 2^+} f(x)$ .

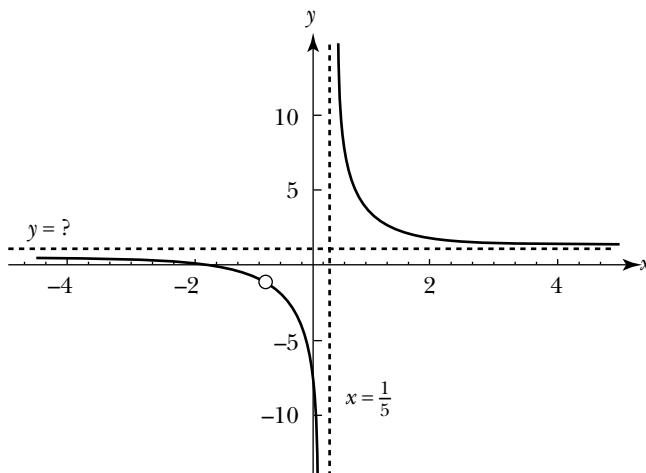
(c) Interpret your results.

Thus far you have restricted your attention to thinking about how a function's output values behave as its inputs approach a finite number, such as 3, or  $-100$ , or  $\pi$ . Another possibility is to examine how  $f(x)$  behaves as the  $x$ -values explode; that is, what happens as  $x \rightarrow -\infty$  or as  $x \rightarrow +\infty$ ? If  $f(x)$  approaches a finite value, then the function has a *horizontal asymptote*.

For example, consider once again the rational function

$$f(x) = \frac{3x^2 + 9x + 6}{5x^2 + 4x - 1} = \frac{3(x+1)(x+2)}{(x+1)(5x-1)}, \quad \text{where } x \neq -1 \text{ and } x \neq \frac{1}{5}$$

You know that  $f$  has a hole at  $P(-1, -\frac{1}{2})$  and a vertical asymptote at  $x = \frac{1}{5}$ . Moreover, based on the graph of  $f$ , it appears that the graph levels off as  $x$  get larger and larger in both the positive and negative directions, and hence that the output values are getting closer and closer to a particular number as the value of  $x$  explodes. In other words, it appears that  $f$  has a horizontal asymptote. The question is: where?



You could use an input–output table to find a reasonable approximation for this horizontal asymptote. However, if you could evaluate

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 9x + 6}{5x^2 + 4x - 1} \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{3x^2 + 9x + 6}{5x^2 + 4x - 1}$$

you would know the exact value. The next task will guide you through the process of how to do this.

---

**Task 4-9: Using Limits to Locate Horizontal Asymptotes**

Obviously, you cannot evaluate a limit such as

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 9x + 6}{5x^2 + 4x - 1}$$

by substituting  $-\infty$  for  $x$ . One way around this is to find an equivalent expression for the rational function, where each term in the new expression is either a constant or has the form  $c/x^n$ , where  $n$  is a positive integer and  $c$  is a constant. For instance, multiplying the numerator and the denominator of the function given above by  $1/x^2$ , gives you an equivalent expression where each term has the desired form:

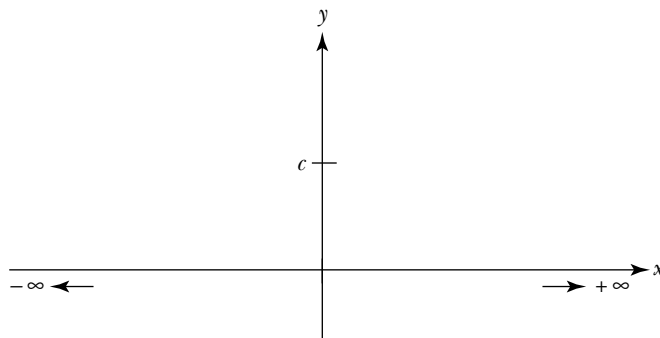
$$\frac{3x^2 + 9x + 6}{5x^2 + 4x - 1} = \frac{(3x^2 + 9x + 6) \cdot \frac{1}{x^2}}{(5x^2 + 4x - 1) \cdot \frac{1}{x^2}} = \frac{3 + (9/x) + (6/x^2)}{5 + (4/x) - (1/x^2)}$$

You can then take the limit of this equivalent expression as  $x$  approaches  $\pm\infty$ . To do so, you need to understand how the constant function  $c$  and the function  $c/x^n$  behave as  $x$  approaches  $\pm\infty$ .

1. Suppose  $c$  is a constant. Examine the limiting behavior of the constant function  $c$  as  $x$  approaches  $+\infty$ . In particular, use a graphic approach to show that

$$\lim_{x \rightarrow -\infty} c = c \quad \text{and} \quad \lim_{x \rightarrow +\infty} c = c$$

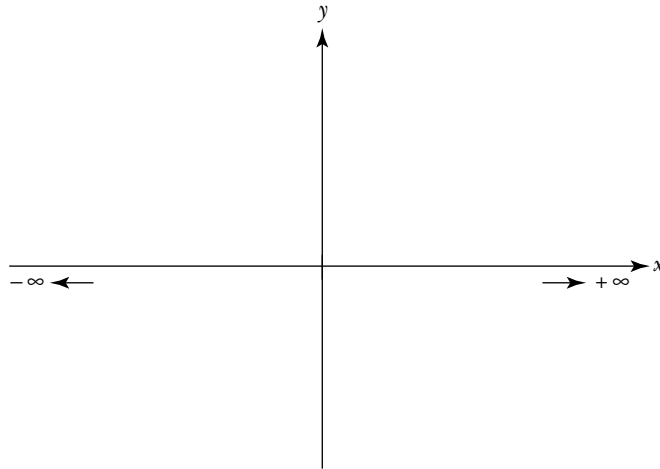
- a. Sketch the graph of a constant function where  $c$  is an arbitrary constant.



- b. Indicate some output values of the constant function as  $x$  gets larger and larger in the negative direction, and as  $x$  gets larger and larger in the positive direction. Show that the output value equals  $c$ , no matter how big or small  $x$  gets.
2. Suppose  $n$  is a positive integer and  $c$  is a constant. Examine the limiting behavior of  $c/x^n$  as  $x$  approaches  $+\infty$ .

- a. Consider  $n = 1$ . Use a graphic approach to show that it is reasonable to conclude that

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$



- b. Consider  $n = 3$ . Use input–output tables to show that it is reasonable to conclude that

$$\lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x^3} = 0$$

$x$	$\frac{1}{x^3}$
-1	
-10	
-100	
-1,000	
-10,000	

$x$	$\frac{1}{x^3}$
1	
10	
100	
1,000	
10,000	

- c. Generalize your observations. Explain why it is reasonable to assume that

$$\lim_{x \rightarrow \infty} \frac{c}{x^n} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{c}{x^n} = 0$$

where  $n$  is a positive integer and  $c$  is a constant.

3. Work through an example. Show that

$$\lim_{x \rightarrow \pm\infty} \frac{3x^2 + 9x + 6}{5x^2 + 4x - 1} = \frac{3}{5}$$

and consequently that  $f(x) = (3x^2 + 9x + 6)/(5x^2 + 4x - 1)$  has a horizontal asymptote at  $y = \frac{3}{5}$ .

- a. First, find an equivalent expression for

$$\frac{3x^2 + 9x + 6}{5x^2 + 4x - 1}$$

where each term is a constant or has the form  $c/x^n$ . One way to do this is to divide each term in the numerator and in the denominator by the highest power of  $x$  that appears in the given function, which in this example is  $x^2$ . Try it.

- b. Next, take the limit of the equivalent expression as  $x \rightarrow \pm\infty$ , noting that  $c \rightarrow c$  and  $c/x^n \rightarrow 0$  as  $x \rightarrow \pm\infty$ , where  $n$  is a positive integer and  $c$  is a constant.

4. Evaluate the limits of some rational functions as  $x \rightarrow \pm\infty$  and locate the horizontal asymptotes. Use the approach developed in the previous part.

a.  $\lim_{x \rightarrow \infty} \frac{2x^2 - 5}{3x^2 - x + 7}$

$$\mathbf{b.} \quad \lim_{x \rightarrow -\infty} \frac{-0.5 + 1.25x - x^3}{0.75 + 5x^4}$$

$$\mathbf{c.} \quad \lim_{x \rightarrow \infty} \frac{\frac{1}{2}x + \frac{2}{3}x^3}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

$$\mathbf{d.} \quad \lim_{x \rightarrow \infty} \left( 7 + \frac{3}{x} \right)$$

$$\mathbf{e.} \quad \lim_{x \rightarrow \pm \infty} \frac{\frac{3}{x^{10}} - \frac{7}{x^6} + 90}{\frac{4}{x^7} - 10}$$



5. The limit of a rational function as  $x \rightarrow +\infty$  may not exist and, hence, a rational function does not necessarily have a horizontal asymptote.
- a. Consider a specific example

$$f(x) = \frac{x^3 - 6x^2 + 3}{2x^2 - 9x - 13}$$

Show that  $f$  does not have a horizontal asymptote by showing that  $\lim_{x \rightarrow \pm\infty} f(x)$  does not exist.

- b. Give two other examples of rational functions which do not have horizontal asymptotes.
- c. Describe how can you recognize when a rational function does not have a horizontal asymptote by looking at the expression representing the function.

### Unit 4 Homework After Section 3

- Complete the tasks in Section 3. Be prepared to discuss them in class.

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**HW4.10** Use pencil and paper to find each of the following limits, if it exists. If it does not exist, explain why.

1. Let  $f(x) = \frac{2x^2 + 6x + 4}{x^2 - x - 2}$ .

- a.  $\lim_{x \rightarrow -1} f(x)$                       c.  $\lim_{x \rightarrow 0} f(x)$   
 b.  $\lim_{x \rightarrow 2} f(x)$                         d.  $\lim_{x \rightarrow \infty} f(x)$
2. Let  $f(x) = \frac{10x^6 + 3x^3 - x^2 + 7}{5x^6 + 8x^5 - 1}$ .  
 a.  $\lim_{x \rightarrow 1} f(x)$                       c.  $\lim_{x \rightarrow -\infty} f(x)$   
 b.  $\lim_{x \rightarrow 0} f(x)$
3. Let  $g(x) = \frac{2}{x^2 - 1}$ .  
 a.  $\lim_{x \rightarrow 0} g(x)$                       c.  $\lim_{x \rightarrow -1^-} g(x)$   
 b.  $\lim_{x \rightarrow 1^+} g(x)$                       d.  $\lim_{x \rightarrow \infty} g(x)$
4. Let  $f(x) = \begin{cases} x^3, & \text{if } x < 2 \\ x + 6, & \text{if } 2 \leq x < 4 \\ \frac{x^3}{3x^3 + 4}, & \text{if } x \geq 4. \end{cases}$   
 a.  $\lim_{x \rightarrow 2} f(x)$                       c.  $\lim_{x \rightarrow 4^-} f(x)$   
 b.  $\lim_{x \rightarrow 3} f(x)$                       d.  $\lim_{x \rightarrow 4^+} f(x)$
5. Let  $g(x) = \begin{cases} 2^x, & \text{if } x < 0 \\ \frac{2x^3 - 4x^2 + x - 1}{-8x^3 - 6x - 1}, & \text{if } x \geq 0. \end{cases}$   
 a.  $\lim_{x \rightarrow -2} g(x)$                       c.  $\lim_{x \rightarrow 1} g(x)$   
 b.  $\lim_{x \rightarrow 0} g(x)$                       d.  $\lim_{x \rightarrow \infty} g(x)$
6. Let  $h(x) = \begin{cases} \frac{x^2 + x - 12}{x + 4}, & \text{if } x < 2 \\ \frac{9x^4 - 3x^3 + 6}{-3x^4 - 8x^2 + 7}, & \text{if } x \geq 2. \end{cases}$   
 a.  $\lim_{x \rightarrow -4} h(x)$                       c.  $\lim_{x \rightarrow \infty} h(x)$   
 b.  $\lim_{x \rightarrow 1} h(x)$
7. Let  $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & \text{if } x < 2 \\ \frac{x^4 - x^3 + 7}{2x^4 - 4x^3 + 5}, & \text{if } x \geq 2. \end{cases}$

- a.  $\lim_{x \rightarrow 1} f(x)$                       c.  $\lim_{x \rightarrow 2^+} f(x)$   
 b.  $\lim_{x \rightarrow 2^-} f(x)$                       d.  $\lim_{x \rightarrow -\infty} f(x)$

**HW4.11** (CAS activity) Limits can be evaluated using a CAS. Before beginning this activity, determine how to use your CAS to evaluate the limit of a function represented by an expression and by a piecewise-defined function. Determine how to evaluate one- and two-sided limits as  $x$  approaches a finite value and as  $x$  approaches  $\pm\infty$ .

1. For each of the following functions:
  - i. Use your CAS to evaluate the specified limits. Record your results.
  - ii. Interpret the results. What does the value of the limit tell you about the behavior of the function near  $x = a$ ?
  - iii. Use your CAS to graph the function. Check that the values of the limits returned by your CAS are reasonable and that your interpretation of the results makes sense.

a. Let  $f(x) = \sqrt[3]{x^3 - 3x^2 + 6.5}$ .

(1)  $\lim_{x \rightarrow 1} f(x)$

(2)  $\lim_{x \rightarrow 4} f(x)$

b. Let  $g(x) = \sin(x)/x$ .

(1)  $\lim_{x \rightarrow 0^1} \sin(x)/x$  and  $\lim_{x \rightarrow 0^+} \sin(x)/x$ .

(2)  $\lim_{x \rightarrow -\infty} \sin(x)/x$  and  $\lim_{x \rightarrow +\infty} \sin(x)/x$ .

c. Let  $h(t) = \begin{cases} 4e^{-2t}, & \text{if } t < 0 \\ 4\cos(t), & \text{if } 0 \leq t < \pi \\ \sin(2t), & \text{if } t \geq \pi. \end{cases}$

(1)  $\lim_{t \rightarrow -\infty} h(t)$

(2)  $\lim_{t \rightarrow 0^-} h(t)$  and  $\lim_{t \rightarrow 0^+} h(t)$ .

(3)  $\lim_{t \rightarrow \pi^-} h(t)$  and  $\lim_{t \rightarrow \pi^+} h(t)$ .

2. Consider  $g(r) = \frac{-7.5 - 5r}{-10.14 - 6.76r + 1.5r^2 + r^3}$

Analyze  $g$  using your CAS. Record your results.

- a. Use limits to find any horizontal asymptotes of  $g$ .
- b. Find any vertical asymptotes of  $g$ . Use limits to determine the shape of the graph of  $g$  near each vertical asymptote.
- c. Find the  $x$ -coordinate of any holes in the graph of  $g$ . Use limits to find the  $y$ -coordinate of each hole.

**HW4.12** Use limits to analyze some situations.

- 1. Suppose  $l(t) = 5 \log_2(t + 1)$  gives the length in feet of a certain poisonous snake after  $t$  years. Snakes of this type live 8 years. Find the length a typical snake is approaching toward the end of its lifetime.
- 2. Tests have indicated that the amount of a drug in milligrams remaining in a person's bloodstream after  $h$  hours is given by

$$a(h) = \frac{23 - h}{4 + h^2}$$

Find and interpret the following limits:

- a.  $\lim_{h \rightarrow 3.5^-} a(h)$
- b.  $\lim_{h \rightarrow \infty} a(h)$

- 3. Joe is always trying new software packages. He has kept track of how long it takes him to feel comfortable with each one. Based on these data, he



knows that average number of hours it will take him to learn the  $n$ th new package is given by  $c(n) = 4 + 6/n$ .

- a. What is the longest amount of time it has taken him to become familiar with a new package?
  - b. Joe acquires more and more software packages. As the years pass, what is the average time it takes him to learn a package?
4. Suppose the population in thousands of Hometown, USA  $t$  years from now is given by

$$P(t) = \frac{25t^2 + 125t + 200}{2t^2 + 5t + 40}$$

- a. What is the current population?
  - b. What will the population be 13 years from now?
  - c. What will the population be in the long run?
  - d. Describe how the population of Hometown changes as time goes by.
5. Suppose you write a best-selling book and your royalties in millions of dollars after  $x$  years total

$$P(x) = \frac{4x^2}{x^2 + 4}$$

- a. How much will you earn after  $2\frac{1}{2}$  years?
- b. How much will you earn in the long run?

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**HW4.13** Write your journal entry for Unit 4. As usual, before you begin to write, review the material in the unit. Think about how it all fits together. Try to identify what, if anything, is still causing you trouble.

1. Reflect on what you have learned in this unit. Describe in your own words the concepts that you studied and what you learned about them. How do they fit together? What concepts were easy? Hard? What were the main (important) ideas? Give some examples of the main ideas.
2. Reflect on the learning environment for the course. Describe the aspects of this unit and the learning environment that helped you understand the concepts you studied. What activities did you like? Dislike?

There are a number of ways to approximate the area of a region under a curve and over a given interval. In the next few tasks, you will apply a rectangular approach, which involves covering the region with rectangles, formed by using the right endpoint of each subinterval. To use this approach:

- Partition the given interval into equal size pieces or subintervals.
- For each subinterval, form a rectangle whose base is the subinterval and whose height is the value of the function at the right endpoint of the subinterval.
- Calculate the area of each rectangle.
- Approximate the area under the curve by summing the areas of the rectangles.

In Task 5-9, you will apply this approach to approximate the area of the region bounded by the Susquehanna River which you considered in Task 5-7, part 6.

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### Task 5-9: Applying a Rectangular Approach

*Note: A blowup of the region under consideration appears on the next page.*

1. Place the region on a coordinate system, with the southwest corner of the property at the origin and the horizontal and vertical axes determined by the highway and the western boundary respectively. Carefully label the axes in terms of feet.
2. Suppose  $h$  is the function whose value at points along the southern boundary is determined by measuring the distance from the highway to the riverbank. For example, if you look carefully at the diagram,  $h(0)$  is approximately 131 feet. Approximate the value of  $h$  at each of the following points along the southern boundary:

$$h(0) = 131'$$

$$h(25) =$$

$$h(100) =$$

$$h(275) =$$

$$h(350) =$$

3. Suppose you wanted to approximate of the area of the property by dividing the property into 11 rectangular strips of equal widths, running north to south. A mathematician refers to the base of each strip as a *subinterval*. Find the size of each subinterval.
4. Assume the height of each rectangular strip is determined by the value of  $h$  at the *right* endpoint of the associated subinterval.
  - a. Sketch the 11 rectangles on the blowup of the region.



- b. First make sure that the column headings on the chart given below make sense to you, then fill in the chart.

Subinterval # $i$	Right End-Point of $i$ th Subinterval $x$ (ft)	Height of $i$ th Rectangle $h(x)$ (ft)	Width of $i$ th Rectangle $\Delta x$ (ft)	Area of $i$ th Rectangle $h(x) \cdot \Delta x$ (ft <sup>2</sup> )
1	50			
2	100			
3	150			
4	200			
5	250			
6	300			
7	350			
8	400			
9	450			
10	500			
11	550			

- c. Use the information on the chart to approximate the area of the region. How does this result compare to the approximation you found in Task 5-7, part 6?
- d. Describe how you can make your approximation more precise.

In the last task, you approximated the area of a region by using a small number of rectangles. Your goal is to find the actual value. Intuitively, it makes sense that the more rectangles you consider, the more accurate your approximation will be. In the next task, you will develop the mathematical notation for the general situation. You will represent the approximate area of a region by an expression and then pass to the limit to represent the exact area.



The approximate area is represented by a Riemann sum. In Task 5-9, you calculated a Riemann sum by hand. Obviously, when you do a calculation by hand, you need to keep the number of rectangles—that is, value of  $n$ —very small. Larger values of  $n$ , however, give more precise approximations. This sounds like a perfect task to do on your calculator. In the next task, you will use your graphing calculator to calculate some Riemann sums and observe the limiting behavior of the Riemann sums as the size of  $n$  increases. Using your calculator will not only help you do the calculations, it should also help you develop a better understanding of the mathematical notation.

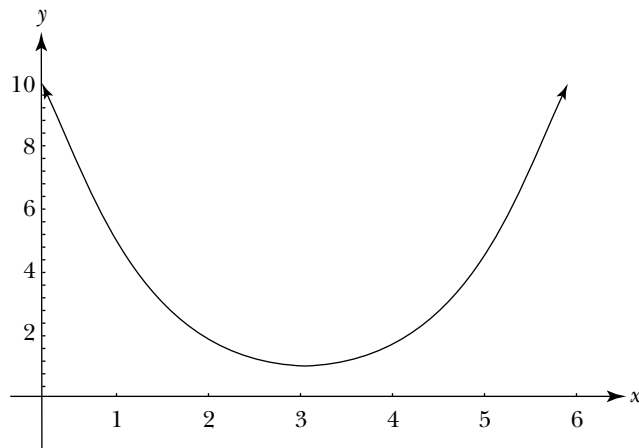
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### Task 5-11: Calculating Riemann Sums

1. Approximate the area of the region bounded by the  $x$ -axis and the graph of

$$f(x) = x^2 - 6x + 10, \quad \text{where } 2 \leq x \leq 5$$

- a. Consider the following graph of  $f$ .



- (1) Shade the area of the region under the graph of  $f$  and over the closed interval  $[2, 5]$ .
- (2) Suppose you approximate the area of this region using six rectangles, which all have the same width and where the height of each rectangle is determined by the value of  $f$  at the right endpoint of its base. Draw the six rectangles, using a pencil and a straightedge.
- (3) If you use six rectangles to estimate the area, will the approximation be greater than, equal to, or less than the exact value? Justify your response.

- (4) Suppose you use 10 rectangles, or 100, or 1000 to estimate the area. How will the approximate value compare to the exact value as the number of rectangles increases?
- (5) The program RECT in the appendix for your graphing calculator illustrates using a sequence of rectangles determined by the right endpoint of the subintervals. Load the program and run it for  $f(x) = x^2 - 6x + 10$ , where  $2 \leq x \leq 5$ , and  $n = 6$ . Observe what happens as the value of  $n$  increases. Does the demonstration support the conclusion you reached using a straightedge?
- b. Use your graphing calculator to approximate the area of the region under the graph of  $f(x) = x^2 - 6x + 10$  and over the closed interval  $[2,5]$ , using six rectangles. As you enter the code in your calculator, think carefully about what is going on here. Try to connect the code with the associated mathematical notation which you developed in Task 5-10, noting that in this example,  $a = 2$ ,  $b = 5$ , and  $n = 6$ .

*Note: See "Constructing a Riemann Sum" in the appendix for your graphing calculator.*

- (1) Define the function  $f(x) = x^2 - 6x + 10$ .
- (2) Set  $a = 2$ ,  $b = 5$ , and  $n = 6$ . Define  $\Delta x$  to be  $(b - a)/n$ .
- (3) Define the right endpoints of the  $n$  subintervals. Store the values in a sequence called  $X$ , where

$$X = \{a + i\Delta x, 1 \leq i \leq n\}.$$

Note that  $X$  contains  $n$  values, where the first item is  $x_1$ , the second is  $x_2$ , and the last is  $x_n$  or  $b$ .

- (4) Calculate the Riemann sum  $\sum_{i=1}^n f(x_i)\Delta x$ —that is, find the approximate area of the region using a rectangular approach with the right endpoint of each subinterval, where  $n = 6$ . Record your result below.

- c. Use your calculator to make your approximation more precise.

*Note: To calculate the Riemann sum for a new value of  $n$ :*

- i. Edit and reexecute the assignment statement for  $n$ .
  - ii. Reexecute the assignment statements for  $\Delta x$  and  $x$  and the expression for the Riemann sum (in this order), which all depend on the value of  $n$ .
- (1) Use your calculator to complete the following chart. Display at least four digits to the right of the decimal point.

$n$	$\sum_{i=1}^n f(x_i)\Delta x$
6	
10	
25	
75	
100	
250	

- (2) As  $n$  gets larger and larger, the Riemann sum gets closer and closer to the exact value of the area. Based on the entries in your chart, what might you conclude about the limiting behavior of the associated Riemann sum?

$$\sum_{i=1}^n f(x_i)\Delta x \rightarrow \boxed{\phantom{000}} \quad \text{as } n \rightarrow \infty$$

2. Consider the region bounded by the graph of  $g(x) = e^x$  and the  $x$ -axis, where  $-1 \leq x \leq 1$ .
- a. Use your calculator to plot the region.
    - (1) Sketch the graph in the space below.

- (2) On the graph, shade the region under consideration.
- (3) Suppose  $n = 8$ . Sketch the eight rectangles you would use to approximate the area of the region using the rectangular approach with the right endpoint of each subinterval.
- b. Use your calculator to find the value of the Riemann sum for  $n = 8$ . Enter your result below.
- c. Explain why using the right endpoint of each subinterval to determine the height of the associated rectangle will always result in an approximation that is larger than the exact value.
- d. The area of the region under the graph of  $g(x) = e^x$  and over the closed interval  $-1 \leq x \leq 1$  is very close to 2.3504. Use your calculator to determine the minimum number of rectangles you need to consider so that the difference between the Riemann sum and 2.3504 is less than or equal to 0.05. Record the results of your trials in the table below.

Trial Number	$n$	$\sum_{i=1}^n f(x_i)\Delta x$	Error
1			
2			
3			

If the limit of the Riemann sum  $\sum_{i=1}^n f(x_i)\Delta x$  exists as  $n \rightarrow \infty$ ,  $f$  is said to be *integrable* over the closed interval  $[a, b]$ . A shortcut for denoting this limit is the definite integral notation

$$\int_a^b f(x) \, dx$$

which is read, “the integral from  $a$  to  $b$  of  $f(x) \, dx$ .”



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### Task 5-13: Checking the Connection Between Derivatives and Definite Integrals

Consider the function  $f(x) = 5$ , where  $2 \leq x \leq 9$ . Define the associated cumulative area function and show that its derivative equals 5.

1. Sketch a graph of  $f$  for  $2 \leq x \leq 9$ .



2. Find the cumulative area function  $A$  defined by  $f$ .

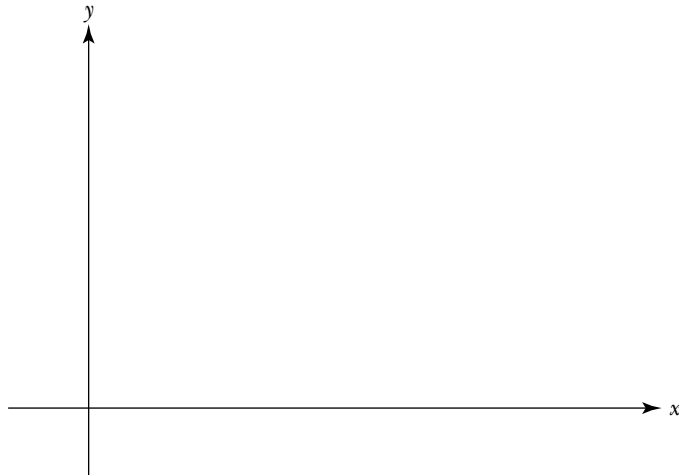
Note that in this case,  $A(x)$  equals the area of the region under the graph of  $f$  and over the closed interval  $[2, x]$ , where  $x$  is any value between 2 and 9 inclusive.

- a. First, interpret some values of  $A(x)$ .
  - (1) For each of the following, make a small sketch of  $f$  and shade the associated region.
    - (a)  $A(4)$
    - (b)  $A(6.5)$
    - (c)  $A(9)$
  - (2) Explain why  $A(2) = 0$ .
  - (3) Give the domain of  $A$ .

- b. Use the graph of  $f$  to calculate some numerical values of  $A(x)$ . Represent each value as a definite integral.

	$x$				
	2	3	7.5	8	9
$A(x)$ (numerical value)					
$A(x)$ (represented by a definite integral)					

- c. Sketch a graph of  $A$  for  $2 \leq x \leq 9$ .



- d. Use your graph to find an expression for  $A$ .
3. You now have *two* functions, the original function  $f(x) = 5$  and a new function  $A(x) = 5x - 10$ .  $A$  is the accumulation function associated with  $f$ , which is defined by integrating  $f$ . The Fundamental Theorem of Calculus claims that the derivative of the accumulation function equals the original function—that is,  $A'(x) = f(x)$ . So, in this instance,  $A'(x)$  should equal 5. Show that this is true, using both a graphic approach and the definition of derivative.
- a. Use a graphic approach to find  $A'(x)$ .

Consider the graph of  $A(x) = 5x - 10$  in part 2c. Show that the slope of the tangent line to any point on the graph of  $A$  is 5.

- b. Use the definition of derivative to find  $A'(x)$ .

Show that

$$\lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = 5, \quad \text{when } A(x) = 5x - 10.$$

## Unit 5 Homework After Section 2

- Complete the tasks in Section 2. Be prepared to discuss them in class.

---

**HW5.10** Consider, once again, the region bounded by the  $x$ -axis and the graph of

$$f(x) = x^2 - 6x + 10, \quad \text{where } 2 \leq x \leq 5$$

In Task 5-11 you estimated the area of this region using a rectangular approach, where the height of each rectangle was determined by the value of the function at the right endpoint of the associated subinterval. Consider three other approaches for estimating the area.

1. In parts a–c below, estimate the area of the region under the graph of  $f$  and over the closed intervals  $[2,5]$  using the indicated approach. In each case:
  - i. Sketch the graph of  $f$  for  $2 \leq x \leq 5$ .
  - ii. Subdivide the interval  $[2,5]$  into six equal subintervals. Cover the region with the specified shapes. For example, in part a, cover the region with rectangles formed by using the value of  $f$  at the left endpoint of each subinterval.
  - iii. Approximate the area of the region by finding the areas of the specified shapes and summing the results.
    - a. Rectangular approach using left endpoint: Form rectangles where the height of each rectangle is determined by the value of  $f$  at the left endpoint of each subinterval.
    - b. Midpoint approach: Form rectangles where the height of each rectangle is determined by the value of  $f$  at the midpoint of each subinterval.
    - c. Trapezoidal approach: Form trapezoids where the heights of each trapezoid are determined by the value of  $f$  at the left and right endpoints of each subinterval.
2. Which approach (rectangular approach using the right endpoint, rectangular approach using the left endpoint, midpoint approach, or trapezoidal approach) appears to give the most accurate result for this problem. Why?



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**HW5.11** Investigate how the value of an approximation relates to the exact value when using different approaches.

Suppose you want to approximate the area of a region under a given curve and over a closed interval. For each of the statements 1–9 below:

- i. Sketch a graph of a function for which the statement is true.
  - ii. Shade the region.
  - iii. Sketch a typical rectangle or trapezoid. (Your example should work for any number of rectangles or trapezoids.)
1. You approximate the area using the rectangular approach with the left endpoint of each subinterval. The approximate area is greater than the exact area.
  2. You approximate the area using the rectangular approach with the left endpoint of each subinterval. The approximate area is less than the exact area.
  3. You approximate the area using the rectangular approach with the left endpoint of each subinterval. The approximate area equals the exact area.
  4. You approximate the area using the rectangular approach with the right endpoint of each subinterval. The approximate area is greater than the exact area.
  5. You approximate the area using the rectangular approach with the right endpoint of each subinterval. The approximate area is less than the exact area.
  6. You approximate the area using the rectangular approach with the right endpoint of each subinterval. The approximate area equals the exact area.
  7. You approximate the area using the trapezoidal approach. The approximate area is greater than the exact area.
  8. You approximate the area using the trapezoidal approach. The approximate area is less than the exact area.
  9. You approximate the area using the trapezoidal approach. The approximate area equals the exact area.

---

**HW5.12** Interpret some definite integrals.

1. Model some familiar situations and represent the regions by definite integrals.

Look again at the dock and the boat house which you considered in Task 5-7, parts 1 and 4. In each case:

- i. Model the situation by placing the region on a coordinate system and finding an expression for the function that defines the shape of the region.

- ii. Express the exact area of the region in terms of a definite integral using the expression from part i.
- iii. Evaluate the integral using a geometric approach.

- a. The deck of the dock.
- b. The back of the boat house. *Note:* The upper boundary of the back of the boat house is modeled by two expressions. Consequently, its area can be represented by the sum of two definite integrals. It is also possible to represent the area as a multiple of a single definite integral. Try to do it both ways.

2. Consider

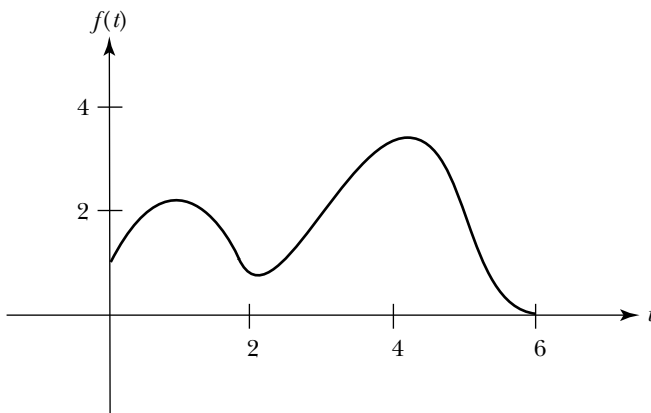
$$h(r) = \begin{cases} r + 1, & \text{if } r \leq 3 \\ 4, & \text{if } 3 < r < 5 \\ -4r + 24, & \text{if } r \geq 5 \end{cases}$$

Use a geometric approach to evaluate  $\int_0^6 h(r) dr$ .

- 3. Sketch the region whose area is given by  $\int_0^2 x^2 dx + \int_2^4 (x - 4)^2 dx$ .
- 4. Suppose  $f$  is a non-negative continuous function with domain  $1 \leq x \leq 3$ , where the area of the region under the graph of  $f$  over the closed interval  $[1,3]$  is greater than or equal to  $\frac{1}{2}f(1.5) + \frac{1}{2}f(2) + \frac{1}{2}f(2.5) + \frac{1}{2}f(3)$ . Sketch a graph of  $f$ .
- 5. Sketch the graph of a function  $g$ , where  $\int_{-4}^{-1} g(x) dx = 15/2$ .
- 6. Explain why the following inequalities are true.

a.  $\int_0^\pi \sin(\theta) d\theta < \pi$

b.  $\int_0^6 f(t) dt < 24$ , where



---

**HW5.13** Show that the Fundamental Theorem of Calculus holds for a linear function.

Consider  $f(x) = 2x$ , where  $0 \leq x \leq 4$ .

1. Graph  $f$  for  $0 \leq x \leq 4$ .
2. Let  $A$  be the accumulation function defined by  $f$ .
  - a. Interpret some values of  $A(x)$ . For each of the values of  $x$  in parts (1)–(6):
    - i. Draw a small sketch of  $f$  and shade the region whose area equals  $A(x)$ .
    - ii. Find the numerical value of  $A(x)$ .
    - iii. Represent  $A(x)$  by a definite integral.

(1) $x = 0$	(4) $x = 2.5$
(2) $x = 1$	(5) $x = 3$
(3) $x = 2$	(6) $x = 4$
  - b. Sketch a graph of  $A$  for  $0 \leq x \leq 4$ .
  - c. Find an expression that fits the graph of  $A$ .
3. Show that the Fundamental Theorem of Calculus holds. Use the definition of the derivative to show that  $A'(x) = f(x)$ .

---

**HW5.14** What is a definite integral? Write a short essay explaining what a definite integral is. In your essay:

- i. Give the definition of a definite integral as the limit of a Riemann sum.
- ii. Support your definition by sketching and labeling an appropriate diagram.

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**HW5.15** Write your journal entry for unit 5. As usual, before you begin to write, review the material in the unit. Think about how it all fits together. Try to identify what, if anything, is still causing you trouble.

1. Reflect on what you have learned in this unit. Describe in your own words the concepts that you studied and what you learned about them. How do they fit together? What concepts were easy? Hard? What were the important ideas? Give some examples of the main ideas.
2. Reflect on the learning environment for the course. Describe the aspects of this unit and the learning environment that helped you understand the concepts you studied. What activities did you like? Dislike?

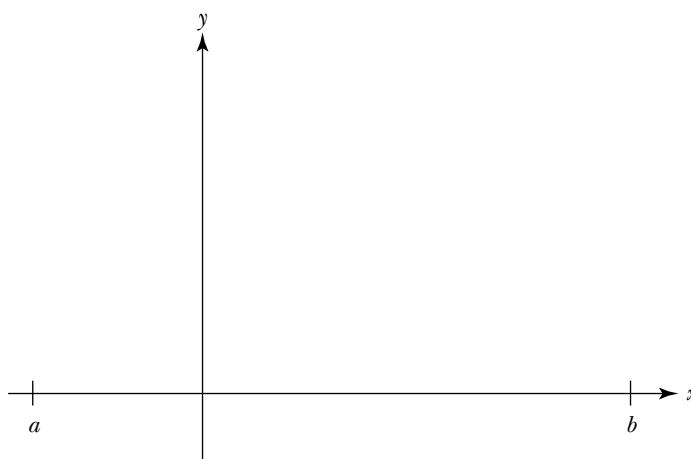
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### Task 5-10: Considering the General Situation

Suppose  $f$  is a non-negative, continuous function, where  $a \leq x \leq b$ . Consider the region under the graph of  $f$  and over the closed interval  $[a, b]$ . In this task, you will approximate the area of this region by covering it with  $n$  rectangles and then summing the areas of the rectangles. Finally, you will express the exact area in terms of a limit.

*Note: Do not label the graph or sketch the rectangles until you are instructed to do so.*

1. Sketch the graph of  $f$ , for  $a \leq x \leq b$ . Make the graph of  $f$  curvaceous and make it lie above the  $x$ -axis.



2. Partition the closed interval  $[a, b]$  into  $n$  equal subintervals and find a formula for the right endpoint of each subinterval. These subintervals will form the bases of the rectangles that you will use to approximate the area of the region.
  - a. Observe that the length of the entire interval is  $b - a$ . Find a formula for the width of each of the  $n$  subintervals in terms of  $a$ ,  $b$ , and  $n$ . Call this width  $\Delta x$ —which is read “delta  $x$ ” and denotes a small change in  $x$ .

$$\Delta x =$$

- b. Find formulas for the endpoints of the  $n$  subintervals and give each endpoint a name.

The initial endpoint is  $a$ . Call this  $x_0$ ; that is,

$$x_0 = a$$

- (1) The right endpoint of the first subinterval is  $\Delta x$  away from  $a$ . Find a formula for the endpoint of this subinterval in terms of  $a$  and  $\Delta x$ . Call this endpoint  $x_1$ .

$$x_1 =$$

- (2) The endpoints of the successive subintervals are  $\Delta x$  apart. Find formulas for the right endpoints of the second and third subin-

tervals in terms of  $a$  and a multiple of  $\Delta x$ . Call the endpoints of these subintervals  $x_2$  and  $x_3$ , respectively.

$$x_2 =$$

$$x_3 =$$

- (3) Recall that  $n$  represents the total number of subintervals. You know that  $n$  is a positive integer, but you don't know which one. Consequently, it is impossible to list all the values of the right endpoints of the subintervals. What might you do in this case? One approach is to consider an arbitrary subinterval—that is, instead of referring to the first, second, or third subinterval, refer to the  $i$ th subinterval where  $1 \leq i \leq n$ , as there are  $n$  subintervals.

Use the pattern that emerged when you found formulas for  $x_1$ ,  $x_2$ , and  $x_3$  to find a formula for the right endpoint of the  $i$ th subinterval. As before, express the formula in terms of  $a$  and a multiple of  $\Delta x$ . Call the endpoint of this subinterval  $x_i$ .

$$x_i = \quad , \quad \text{where } 1 \leq i \leq n$$

- (4) The right endpoint of the last, or  $n$ th, subinterval is  $b$ . Find a formula for the right endpoint of this subinterval in terms of  $a$  and a multiple of  $\Delta x$ .

$$x_n = b =$$

- c. Mark the values of  $x_i$ , where  $0 \leq i \leq n$ , on the  $x$ -axis on your diagram in part 1 as you complete the following instructions.

- (1) Relabel  $a$  as  $x_0$  and  $b$  as  $x_n$ .
- (2) Starting at the value for  $a$ , make three, equally spaced tick marks on the  $x$ -axis, representing the right endpoints of the first three subintervals. Label them  $x_1$ ,  $x_2$ , and  $x_3$ .
- (3) Because it's impossible to label the endpoint of every subinterval, after the tick mark for  $x_3$ , put three dots on the  $x$ -axis to indicate that some points are missing.
- (4) Make tick marks for  $x_{i-1}$  and  $x_i$ , which represent the left and right endpoints of the  $i$ th subinterval, and label them appropriately.
- (5) After the tick mark for  $x_i$ , put three more dots on the  $x$ -axis to indicate that more points are missing.
- (6) Finally, noting that  $b$  is  $x_n$ , make a tick mark for  $x_{n-1}$ , which is the left endpoint of the last—or  $n$ th—subinterval.

3. Finally, you are ready to cover the region with rectangles and then approximate the area under the graph of  $f$  over  $[a, b]$  by summing the areas of these rectangles.

- a. Sketch a few rectangles. On your diagram in part 1, sketch the first three rectangles, the  $i$ th rectangle, and the  $n$ th rectangle, using the value  $f$  at the right endpoint of each subinterval to determine each rectangle's height.

- b. Find a formula for the height of each rectangle. Express the height in terms of  $f$  and the right endpoint of the subinterval.
- (1) Express the height of the first rectangle in terms of  $f$  and  $x_1$ .
  
  - (2) Express the height of the second rectangle in terms of  $f$  and  $x_2$ .
  
  - (3) Express the height of the  $i$ th rectangle in terms of  $f$  and  $x_i$ .
  
  - (4) Express the height of the  $n$ th rectangle in terms of  $f$  and  $x_n$ .
- c. Find a formula for the area of each rectangle. Recall that the width of each rectangle is  $\Delta x$ .
- (1) Find a formula for the area of the first rectangle.
  
  - (2) Find a formula for the area of the second rectangle.
  
  - (3) Find a formula for the area of the  $i$ th rectangle.
  
  - (4) Find a formula for the area of the  $n$ th rectangle.
4. Approximate the area of the region by summing the areas of the rectangles. Express the sum using sigma notation (see HW5.9). This sum is called a *Riemann sum*.

5. What happens to the value of the Riemann sum if the number of rectangles is allowed to increase indefinitely? How might you express this using limit notation?

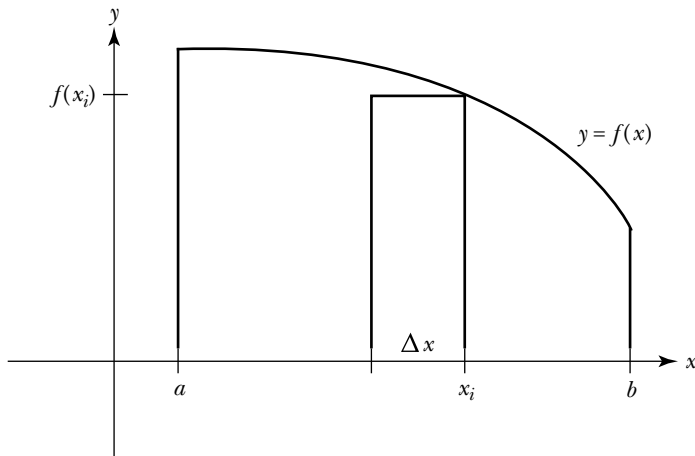


Congratulations! You have just developed the notation for what mathematicians call the *definite integral*, another extremely important calculus concept.

In the last task, you took a concrete idea—adding up the areas of rectangles—and represented it in an abstract way using mathematical notation. In particular, you considered a continuous, non-negative function  $f$  defined over a closed interval  $[a, b]$ . You let

$$\Delta x = \frac{b - a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{for } 0 \leq i \leq n$$

where  $n$  is a positive integer.



You discovered that

$$\lim_{n \rightarrow \infty} \underbrace{\sum_{i=1}^n \underbrace{f(x_i) \Delta x}_{\text{area of } i\text{th rectangle}}}_{\text{approximate area of region under curve}}$$

exact area of region under the graph of  $f$  over the interval  $[a, b]$

The approximate area is represented by a Riemann sum. In Task 5-9, you calculated a Riemann sum by hand. Obviously, when you do a calculation by hand, you need to keep the number of rectangles—that is, value of  $n$ —very small. Larger values of  $n$ , however, give more precise approximations. This sounds like a perfect task to do on your calculator. In the next task, you will use your graphing calculator to calculate some Riemann sums and observe the limiting behavior of the Riemann sums as the size of  $n$  increases. Using your calculator will not only help you do the calculations, it should also help you develop a better understanding of the mathematical notation.

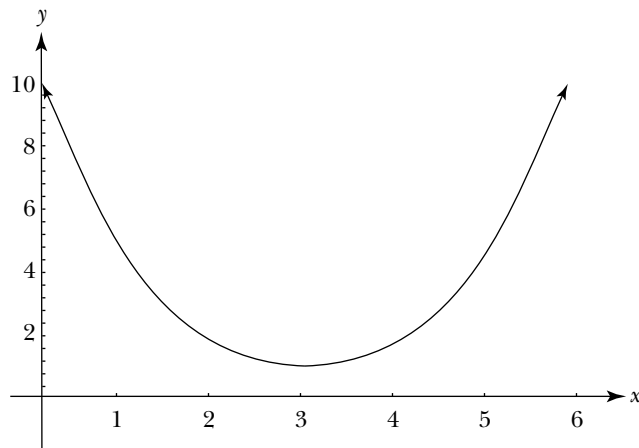
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### Task 5-11: Calculating Riemann Sums

1. Approximate the area of the region bounded by the  $x$ -axis and the graph of

$$f(x) = x^2 - 6x + 10, \quad \text{where } 2 \leq x \leq 5$$

- a. Consider the following graph of  $f$ .



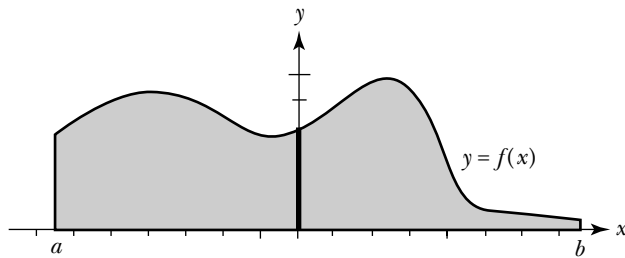
- (1) Shade the area of the region under the graph of  $f$  and over the closed interval  $[2, 5]$ .
- (2) Suppose you approximate the area of this region using six rectangles, which all have the same width and where the height of each rectangle is determined by the value of  $f$  at the right endpoint of its base. Draw the six rectangles, using a pencil and a straightedge.
- (3) If you use six rectangles to estimate the area, will the approximation be greater than, equal to, or less than the exact value? Justify your response.



---

**Task 5-8: Describing Some Possible Approaches**

1. You know how to find the areas of a variety of regular shapes. List the shapes whose areas you can calculate, and give the associated area formulas.
2. Suppose you want to find the area of an oddly shaped region lying under a given curve and over a closed interval, such as the shaded region in Figure 1.

**Figure 1.**

For each regular shape, which you listed in part 1, make a rough sketch of the region in Figure 1. Cover the region with pieces having the desired shape. For instance, cover the region with circles. The pieces can touch, but cannot overlap. Observe that you can approximate the area of the region by summing the areas of the pieces.

There are a number of ways to approximate the area of a region under a curve and over a given interval. In the next few tasks, you will apply a rectangular approach, which involves covering the region with rectangles, formed by using the right endpoint of each subinterval. To use this approach:

- Partition the given interval into equal size pieces or subintervals.
- For each subinterval, form a rectangle whose base is the subinterval and whose height is the value of the function at the right endpoint of the subinterval.
- Calculate the area of each rectangle.
- Approximate the area under the curve by summing the areas of the rectangles.

In Task 5-9, you will apply this approach to approximate the area of the region bounded by the Susquehanna River which you considered in Task 5-7, part 6.

---

### Task 5-9: Applying a Rectangular Approach

*Note: A blowup of the region under consideration appears on the next page.*

1. Place the region on a coordinate system, with the southwest corner of the property at the origin and the horizontal and vertical axes determined by the highway and the western boundary respectively. Carefully label the axes in terms of feet.
2. Suppose  $h$  is the function whose value at points along the southern boundary is determined by measuring the distance from the highway to the riverbank. For example, if you look carefully at the diagram,  $h(0)$  is approximately 131 feet. Approximate the value of  $h$  at each of the following points along the southern boundary:

$$h(0) = 131'$$

$$h(25) =$$

$$h(100) =$$

$$h(275) =$$

$$h(350) =$$

3. Suppose you wanted to approximate of the area of the property by dividing the property into 11 rectangular strips of equal widths, running north to south. A mathematician refers to the base of each strip as a *subinterval*. Find the size of each subinterval.
4. Assume the height of each rectangular strip is determined by the value of  $h$  at the *right* endpoint of the associated subinterval.
  - a. Sketch the 11 rectangles on the blowup of the region.

---

### Task 5-2: Discovering a Definition for the Derivative

1. First, construct a general sequence that approaches  $x = a$  from both sides.

a. Consider the sequence

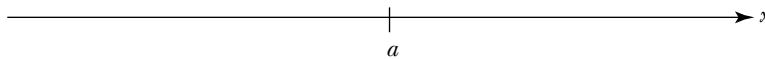
$$RX = \{a + 10^{-i}, 1 \leq i \leq n\}$$

The items in  $RX$  approach  $x = a$  from the right. Each item in  $RX$  is formed by adding a small positive number  $h$  to  $a$ , that is, each item has the form

$$a + h, \text{ where in this case } h = 10^{-i}$$

Analyze the limiting behavior of  $a + h$  as  $h$  approaches 0 from the right.

(1) Let  $h$  be a small positive number. Mark  $a + h$  on the axis below.



(2) Describe what happens to the value of  $a + h$  as  $h \rightarrow 0^+$ . Support your description by drawing an arrow on the diagram in part (1).

b. Now look to the left of  $a$ . A sequence such as

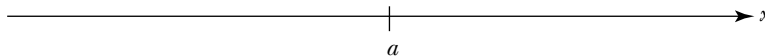
$$LX = \{a - 10^{-i}, 1 \leq i \leq n\}$$

approaches  $x = a$  from the left, where the items in  $LX$  have the form

$$a + h, \text{ where } h = -(10^{-i})$$

Each item in the sequence is formed by adding a negative number  $h$ , which is close to 0, to  $a$ . Analyze the limiting behavior of  $a + h$  as  $h$  approaches 0 from the left.

(1) Let  $h$  be a negative number close to 0. Mark  $a + h$  on the axis below.



(2) Describe what happens to the value of  $a + h$ , as  $h \rightarrow 0^-$ . Support your description by drawing an arrow on the diagram in part (1).

c. Describe, in general, the limiting behavior of  $a + h$  as  $h \rightarrow 0$ —that is, as  $h$  approaches 0 from both sides.

2. Find a general expression for the slope of the secant line determined by  $x = a$  and a point close to  $a$ .

a. Consider  $a + h$ , which is near  $a$  but to the right of  $a$ .

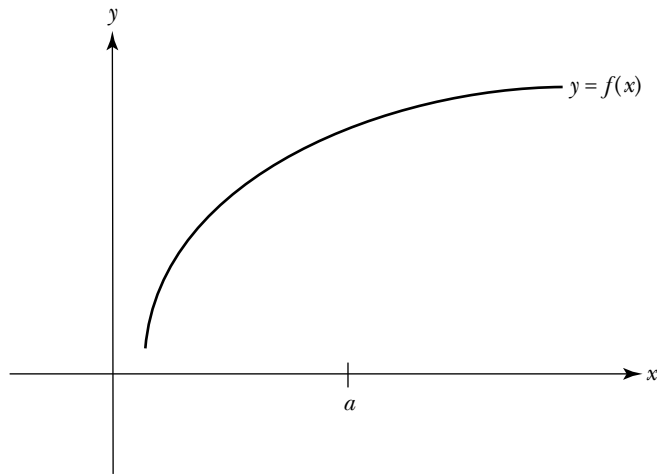
(1) Sketch the secant line corresponding to  $a$  and  $a + h$ . Label the diagram given below as follows:

(a) Mark  $a + h$  on the  $x$ -axis, where  $h$  is a small positive number.

(b) Mark  $f(a)$  and  $f(a + h)$  on the  $y$ -axis.

(c) Label the points  $P(a, f(a))$  and  $Q(a + h, f(a + h))$  on the graph of the function.

(d) Sketch the secant line determined by  $P$  and  $Q$ .



(2) Find an expression for the slope of the secant line in terms of the coordinates of  $P$  and  $Q$ —that is, in terms of  $a$ ,  $f(a)$ ,  $a + h$ , and  $f(a + h)$ . Simplify your expression.

(3) Describe the relationship between the values of the slopes of the secant lines as  $h \rightarrow 0^+$  and the value of the slope of the tangent line at the point  $P(a, f(a))$ .

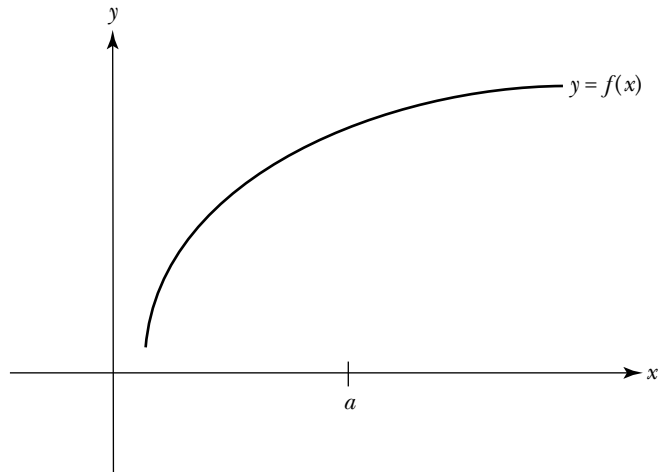
b. Do the same thing that you did above, but this time approach  $a$  from the left; that is, consider  $a + h$ , which is near  $a$  but to the left of  $a$ .

(1) Sketch the secant line corresponding to  $a$  and  $a + h$ . Label the diagram given below as follows:

(a) Mark  $a + h$  on the  $x$ -axis, where  $h$  is a negative number close to 0.

(b) Mark  $f(a)$  and  $f(a + h)$  on the  $y$ -axis.

- (c) Label the points  $P(a, f(a))$  and  $R(a + h, f(a + h))$  on the graph of the function.
- (d) Sketch the secant line determined by  $P$  and  $R$ .



- (2) Find the slope of the secant line in terms of the coordinates of  $P$  and  $R$ . Simplify your expression.
  - (3) Describe the relationship between the values of the slopes of the secant lines as  $h \rightarrow 0^-$  and the value of the slope of the tangent line at the point  $P(a, f(a))$ .
3. Find the slope of the tangent line by taking the appropriate limit of the expression for the slope of the secant line determined by  $x = a$  and  $x = a + h$ .

Congratulations! You have just developed the notation for what mathematicians call the *derivative*, one of the most important and fundamental concepts in calculus.

In particular, the derivative of a function  $y = f(x)$  at  $x = a$  is given by

$$\lim_{h \rightarrow 0} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\substack{\text{slope of secant line determined by } P(a, f(a)) \text{ and } Q(a+h, f(a+h))}} = \underbrace{\hspace{10em}}_{\substack{\text{slope of tangent line to graph of } f \text{ at } P(a, f(a))}}$$



provided the limit exists. If the limit exists,  $f$  is said to be *differentiable* at  $x = a$  and the limit can be denoted by  $f'(a)$ , which is read “ $f$  prime at  $a$ ” or “the derivative of  $f$  at  $a$ .” Evaluating a limit of this form gives a number, such as 10 or  $-1.5$  or 23.6. Although there has been a lot of talk about tangent lines, keep in mind that there are several ways to interpret this number. It equals the slope of the tangent line when  $x = a$ . It represents the rate of change of the function when  $x = a$ . And, more generally, it is the value of the derivative at  $x = a$ .

The derivative takes an  $x$ -value in the domain of  $f$  and, if the limit exists, returns the value of the slope of the tangent line to the graph of  $f$  at that point or the value of the rate of change at  $f$  at that point. Consequently, the *derivative of a function is itself a function*, where the domain of the derivative is contained in the domain of the underlying function. If  $f$  is the name of the underlying function, then the derivative of  $f$  is denoted by  $f'$ , and you can find the *general derivative* of  $f$ ,  $f'(x)$ , by evaluating the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If  $f$  is a polynomial, evaluating a limit of this form leads to an expression, such as  $2x$  or  $x^2 + 6$ . You can use the expression representing  $f'$  to find the value of the derivative at a particular input, such as  $x = 2$ .

In general, to find  $f'(x)$  using the definition of derivative, you do the following:

- Write down the *difference quotient*

$$\frac{f(x+h) - f(x)}{h}$$

- Simplify the difference quotient. Try to eliminate the  $h$  in the denominator so you will be able to evaluate the limit by substituting zero for  $h$ .
- Take the limit as  $h \rightarrow 0$ .

In the next task, you will represent the derivatives of some functions by expressions and then use the expressions to analyze the functions.

---

### Task 5-3: Representing a Derivative by an Expression

Find the derivative of a function using the definition. Use the derivative to calculate some tangent lines and analyze the behavior of the function. Before you begin, recall how to evaluate  $f(x+h)$ .

1. If  $f(x) = -2x^2 + x - \frac{1}{2}$ , you find  $f(x+h)$  by evaluating  $f$  at  $x+h$ ; that is, by substituting  $x+h$  in for  $x$ .

$$\begin{aligned} f(x+h) &= -2(x+h)^2 + (x+h) - \frac{1}{2} \\ &= -2(x^2 + 2xh + h^2) + (x+h) - \frac{1}{2} \\ f(x+h) &= -2x^2 - 4xh - 2h^2 + x + h - \frac{1}{2} \end{aligned}$$

Evaluate each of the following functions at  $x+h$ . Simplify the result, if possible.

You can then find the exact value for the area by making the approximations become more and more precise as you take the limit. This is the same approach you will use to find the slope of the tangent line: First approximate the value you want by applying a formula you already know, and then find the exact value by making the approximations become more exact as you take the limit. You will use this approach again and again throughout your study of calculus.

The objectives of this unit are to develop a conceptual understanding of the fundamental concepts, derivative and definite integral, and to make a first pass at examining how these two important ideas are related. Understanding what these ideas mean will help you recognize where they apply. Exploring ways to evaluate them directly and how to use them in a variety of situations is the focus of subsequent units.

## SECTION 1

### The Derivative

In Unit 1 you developed a conceptual understanding of the notion of the tangent line to a smooth curve at a point  $P$ . You observed that if you repeatedly zoom in and magnify the portion of the curve containing  $P$ , the curve “straightens out” and merges with the tangent line to the curve at  $P$ . You noted that the tangent line does not exist at a point where a function has a sharp peak or dip—that is, it does not exist at a *cusp*—because no matter how close you zoom in to a point of this type, the cusp is always there and the graph of the curve never straightens. In HW1.14 and HW4.5, you approximated the value of the slope of the tangent line at a given point by finding the slope of the secant line determined by the given point and a point nearby.

What does all this have to do with derivatives? The value of the derivative of a function at a given point represents the value of the slope of the tangent line to the curve at that point. It also gives the rate of change of the function at the point. In other words,

$$\begin{aligned} \text{derivative of } f \text{ when } x \text{ equals } a & \\ &= \text{slope of the tangent line to the graph of } f \text{ at } P(a, f(a)) \\ &= \text{rate of change of } f \text{ when } x \text{ equals } a \end{aligned}$$

The question, as you know, is: How can you find the exact value? In the next task, you will use limits to find the slope of the tangent line for a specific function. In the following task, you will develop a definition for derivative by formalizing the approach that you develop in Task 5-1. Note: Before beginning the next task, review the ideas developed in homework problems HW1.14 and HW4.5.

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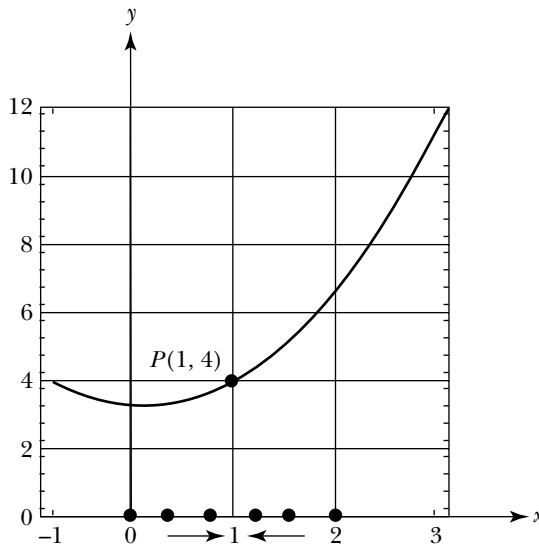
#### Task 5-1: Examining an Example

Consider  $f(x) = x^2 + 3$  at  $a = 1$ . As you complete this task you will: use a graphic and a numeric approach to examine the limiting behavior of the slopes of the secant lines determined by  $P(1,4)$ ; based on your observations,

find a reasonable value for the slope of the tangent line when  $a = 1$ ; use a symbolic approach to find the actual value of the slope; and show that the three approaches—graphic, numeric, and symbolic—yield the same result.

1. Use a graphic approach to examine the limiting behavior of the slopes of the secant lines through  $P(1,4)$ .
  - a. Consider the graph of  $f$  given below. The sequence of points indicated on the horizontal axis approach 1 from the left and from the right. Mark the points on the graph that correspond to the items in the sequence. Use a pencil and straightedge to draw the secant lines determined by  $P(1,4)$  and each of these points.

*Note: Secant lines are not line segments. They go on indefinitely. Extend each line through the two points which determine it.*



- b. Based on the behavior of the secant lines, find a reasonable value for the slope of the tangent line to the graph of  $f(x) = x^2 + 3$  at  $a = 1$ .  
*Caution:* The two axes have different scales.
  - c. If you have a program for your graphing calculator that illustrates the limiting behavior of the secant lines as they approach a tangent line, load the program and run it for  $f(x) = x^2 + 3$  at  $a = 1$ . Observe what happens to the slopes of the secant lines as the lines approach  $a = 1$  from the left and from the right. Does the demonstration support the conclusion you reached using a straightedge in parts a and b?
2. Try this again, only this time use a numeric approach to examine the limiting behavior of the slopes of the secant lines determined by  $P(1,4)$ , as  $x \rightarrow 1$  from both sides. As you enter the code in your graphing calculator, think carefully about what is going on here.

*Note: See "Approximating a Rate of Change," in the appendix for your graphing calculator.*



- a. Define the function  $f(x) = x^2 + 3$ .
- b. Set  $a = 1$ .
- c. Define two sequences of values, one approaching  $a = 1$  from the left and one approaching  $a = 1$  from the right. Store the left sequence in  $LX$ , where

$$LX = \{a - b^{-i}, 1 \leq i \leq n\},$$

and the right one in  $RX$ , where

$$RX = \{a + b^{-i}, 1 \leq i \leq n\}.$$

Let the base  $b = 2$ , consider the first five terms in the sequence—that is, let  $n = 5$ . Display at least five digits to the right of the decimal point.

- d. Calculate the slopes of the secant line determined by  $P(a, f(a))$  and  $Q(x, f(x))$ , for each  $x$  in the two sequences.

- (1) For each  $x$  in the left sequence  $LX$ , calculate the slope of the secant line,

$$\frac{f(x) - f(a)}{x - a},$$

and store the values in a list called  $LDQ$ , which stands for “left difference quotient.”

- (2) Similarly, for each  $x$  in the right sequence  $RX$ , calculate the slope of the secant line and store the values in a list called  $RDQ$ , for “right difference quotient.”

- e. Place the lists of input values,  $LX$  and  $RX$ , and the lists of slopes of the associated secant lines,  $LDQ$  and  $RDQ$ , in a table. Copy the entries in the table on your calculator into the table below, and predict the limiting behavior of the slopes of the secant lines determined by  $P(a, f(a))$  and  $Q(x, f(x))$  when  $a = 1$ .

Term # $i$	Left Seq (LX) $x = a - b^{-i}$	Left Difference Quotient (LDQ) $m = \frac{f(x) - f(a)}{x - a}$	Right Seq (RX) $x = a + b^{-i}$	Right Difference Quotient (RDQ) $m = \frac{f(x) - f(a)}{x - a}$
	↓ $1^-$	↓ <input type="text"/>	↓ $1^+$	↓ <input type="text"/>

- f. Summarize your observations.

Slope of secant line determined by  $P(1,4) \rightarrow \boxed{\phantom{000}} \text{ as } x \rightarrow 1^-$

and

Slope of secant line  
determined by  $P(1,4)$   $\rightarrow$   as  $x \rightarrow 1^+$

Therefore,

Slope of secant line  
determined by  $P(1,4)$   $\rightarrow$   as  $x \rightarrow 1$

- g. Does the conclusion you reached using a numeric approach support the conclusion you reached in part 1 using a graphic approach? If not, look over your work again.
3. One more time. Use a symbolic approach to find the exact value of the slope of the tangent line to the  $f(x) = x^2 + 3$  at  $a = 1$ .
- a. Find a general expression for the slope of the secant line through  $P(1,4)$  and  $Q(x, f(x))$ , where  $f(x) = x^2 + 3$ ; that is, find an expression (in terms of  $x$ ) for
- $$\frac{f(x) - f(1)}{x - 1}$$
- b. Calculate the actual value of the slope of the tangent line to the graph of  $f(x) = x^2 + 3$  at  $x = 1$  by finding the limit of the general expression for the slope of the secant lines (which you developed above in part a) as  $x \rightarrow 1$ .
- c. This result should agree with the conjectures you made based on the graph in part 1 and your tables in part 2. Does it? If not, try to find where you went astray....

In the specific example considered in the last task, you analyzed the behavior of the slopes of the secant lines as they approached the tangent line to the graph at a particular point. You estimated the slope of the tangent line using graphic and numeric approaches, and then found the exact value by taking the appropriate limit. In the next task, you will generalize your observations and, in the process, discover a definition for the derivative.

The underlying idea is as follows: As long as the graph of  $f$  is smooth at  $x = a$ —that is, as long as  $f$  is continuous and its graph does not have a sharp peak, dip, or vertical tangent at  $x = a$ —you can:

- Construct two sequences, one approaching  $a$  from the left and one from the right.
- Find an expression for the slope of the secant line determined by  $P(a, f(a))$  and the point on the graph corresponding to an item in the sequence.
- Find the slope of the tangent line by taking the limit of the slopes of the secant lines as the items in the sequence move closer and closer to  $a$ .

This is the same approach that you used in the last task to find the slope of the tangent line to the graph of  $f(x) = x^2 + 3$  at  $x = 1$ . There will be one notational change, however. The items in the sequence that get closer and closer to  $a$  will be defined in terms of  $a + h$ , where  $h \rightarrow 0$ , instead of in terms of  $x$  where  $x \rightarrow a$ . This change in notation reflects the way the left and right sequences are constructed.

Although this initial encounter with the concept of a definite integral will involve using a definite integral to represent the area or a region, it can be used to represent numerous quantities that accumulate. In the following units, you will return to your study of integrals and explore other situations where the idea is applicable. In addition, you will develop a straightforward way to find exact solutions to these types of problems (without evaluating a limit).

Your study of definite integrals begins as you approximate some areas in the next three tasks.

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### Task 5-7: Finding Some Areas

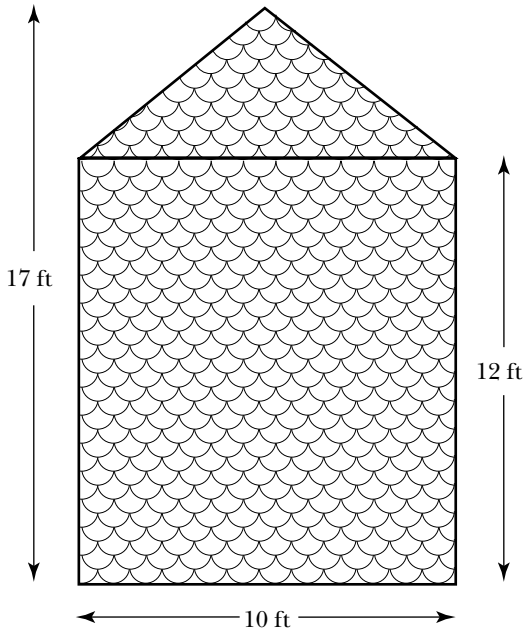
Should your client buy this property? You have graduated from college after distinguishing yourself in this course and are working for a team of real estate consultants. One of your clients is considering buying some vacation property that borders the Susquehanna River. The property comes equipped with a boat house and dock, but there are a few problems with them. The dock needs to be rebuilt, the back of the boat house that faces the river is in desperate need of paint, and the survey information has been lost. Your client needs to know the areas of the dock and the back of the boathouse (so she can calculate the cost of refurbishing them), and she needs to determine the area of the property (so she can determine if she can afford to buy it).



1. The dock has good supports, but the wood on the deck part is rotten. The dock deck is 4 feet wide and 6 feet long. Do a scale drawing of the deck and find its total area.
  
2. Suppose the local lumber mill sells leftover treated outdoor boards which are exactly 4" wide, 2" deep, and 4' long. How many boards will be needed to rebuild the deck? Sketch the boards needed on your drawing above.

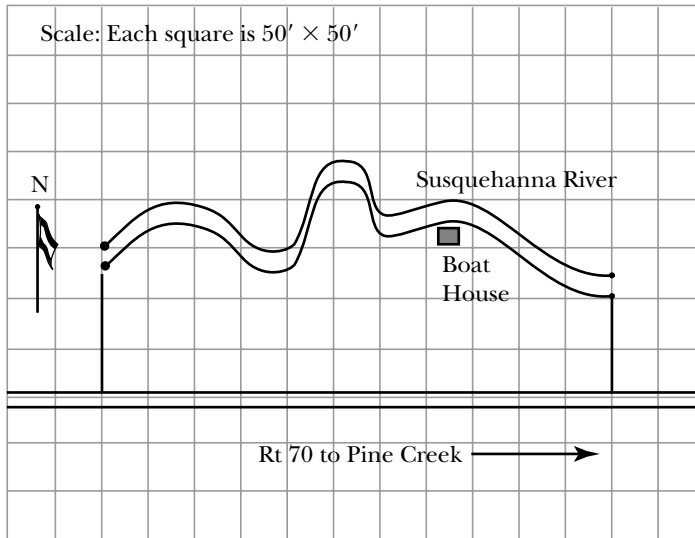
3. If the treated outdoor boards cost \$9.50 each and your labor is free, how much will it cost to refurbish the deck on the dock?

4. The back of the boat house is depicted in the diagram below. How many square feet need to be painted? Explain how you figured this out.



5. If a quart of special preservative costs \$12.25 and covers 50 square feet, how much will the preservative cost to refurbish the back face of the boat house?

6. The seller has made a careful sketch of the site boundaries on graph paper, where each square in the grid is  $50' \times 50'$ , as shown below. The property is bounded on the north by the Susquehanna River and on the south by Rt. 70. Approximately how many square feet are included in the property? Describe briefly the techniques you thought about using to find the answer, and then explain why you chose the one you used.

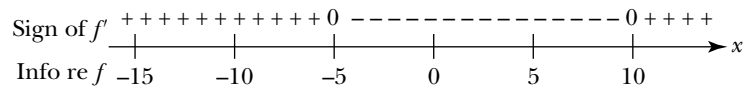


7. If the property is being sold for 1 penny per square foot, approximately how much is the seller asking for this plot of land?
  
8. After paying your consulting fee, your client can only afford to pay a total of \$1200 for the property and the lumber and paint to fix things up. Can she afford to buy it?

In part 6 of the last task, you approximated the area of the region bounded by the Susquehanna River and Route 70. Chances are that you did this by counting the number of squares in the grid covering the region and then summing the areas of the squares. Explore some other possible ways to do this in Task 5-8.

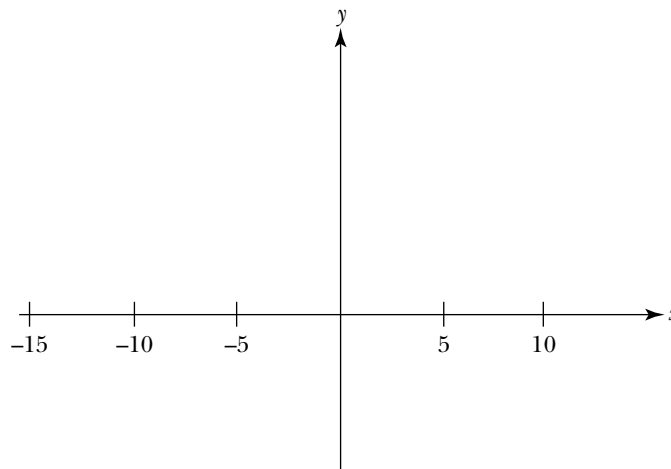
### Task 5-6: Gleaning Information About the Graph of a Function from Its Derivative

1. Consider the following sign chart:

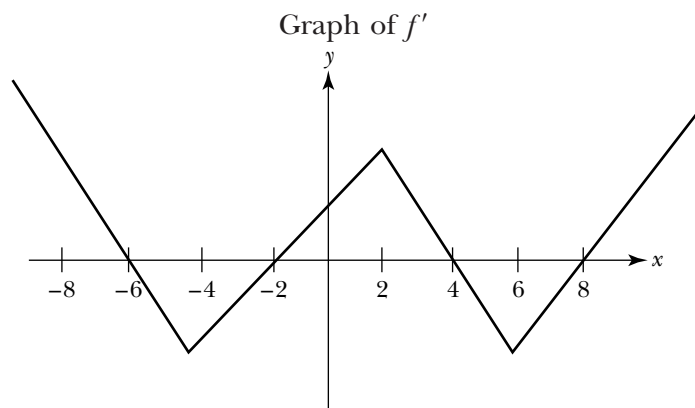


Assume  $f$  is defined for all real numbers. In addition, assume that the only zeros of  $f'$  are at  $x = -5$  and  $x = 10$ . Use the sign chart for  $f'$  to answer the following questions about the shape of the graph of  $f$ .

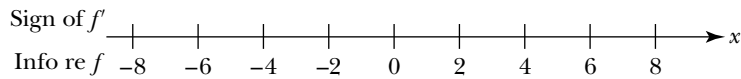
- Determine the intervals where  $f$  is increasing (inc). Record this information regarding  $f$  on the sign chart given above.
- Determine the intervals where  $f$  is decreasing (dec). Record this information regarding  $f$  on the sign chart given above.
- Find all the  $x$ -values where  $f$  has a local maximum (LM) or a local minimum (lm) and where the graph of  $f$  levels off (LO). Record this information regarding  $f$  on the sign chart given above.
- Sketch the graph of a function  $f$  whose derivative has the given sign chart.



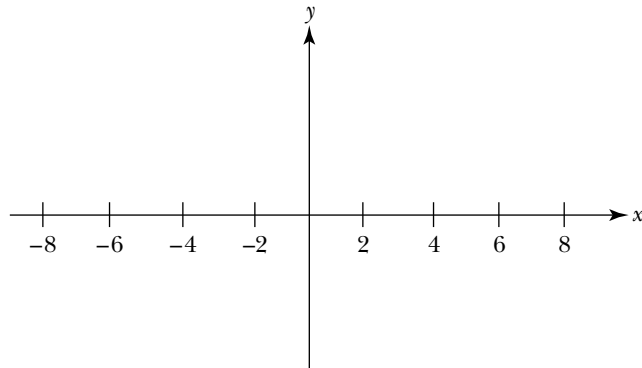
2. Consider the following graph of the derivative  $f'$ :



- a. Find the associated sign chart for  $f'$ .



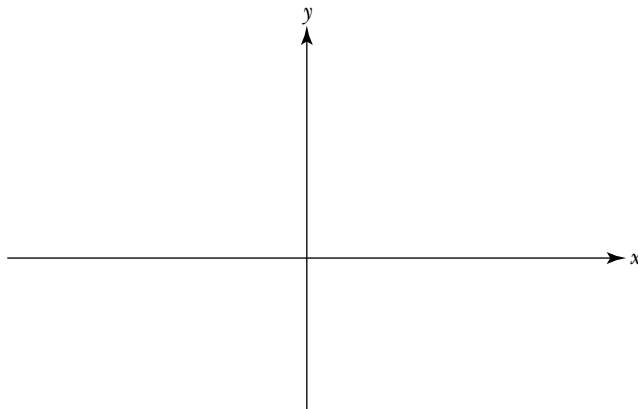
- b. Describe the shape of the graph of  $f$  by filling in information regarding  $f$ ; that is, indicate the intervals where  $f$  is increasing and decreasing, and mark the  $x$ -values of all the extrema of  $f$ .
- c. Suppose the graph of  $f$  lies above the horizontal axis for  $-8 \leq x \leq 8$ ; that is,  $f(x) > 0$  for  $-8 \leq x \leq 8$ . Sketch a graph of a function  $f$ , where the graph of  $f'$  is given above.



3. On the axes given below, sketch the general shape of the graph of a function  $f$  that satisfies the following conditions:

- $f$  defined for all real numbers
- $f'(x) = x^2$
- $f(0) = 3$

*Note: First, find the sign chart for the derivative  $f'$  and fill in the “info re  $f$ .”*



## Unit 5 Homework After Section 1

- Complete the tasks in Section 1. Be prepared to discuss them in class.

**HW5.1** Calculate some derivatives using the definition. Use the derivative to analyze the function. Compare the graphs of a function and its derivative.

Recall that, by definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

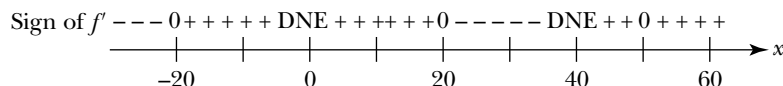
- For each of the following functions:
  - Use the definition to find its general derivative.
  - Sketch the graphs of  $f$  and  $f'$  on the same pair of axes. Label the graphs.
    - $f(x) = 6$
    - $f(x) = 6.5x - 10$
    - $f(x) = x^3$ . Note:  $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
- Suppose  $f(x) = x^2 - 2x + 5$ .
  - Use the definition to find  $f'(x)$ .
  - Sketch the graphs of  $f$  and  $f'$  on the same pair of axes. Label the graphs.
  - Use  $f'(x)$  to find the equation of the tangent line to the graph of  $f$  at  $P(2, f(2))$ .
  - Use the expression for  $f'(x)$  to find the  $(x, y)$ -coordinates of the point on the graph of  $f$  where the tangent line is horizontal to the graph.
  - Use the expression for  $f'(x)$  to find the intervals where  $f$  is increasing and the intervals where  $f$  is decreasing.
- Use the definition of derivative to show that each of the following statements is true. Support the conclusion of the statement by graphing a typical  $f$  and its derivative  $f'$  on the same pair of axes. Label the graphs.
  - The derivative of a constant function is always 0; that is, if  $f(x) = c$ , where  $c$  is a fixed constant, then  $f'(x) = 0$  for all  $x$ .
  - The derivative of a linear function is the slope its graph; that is, if  $f(x) = mx + b$ , where  $m \neq 0$ , then  $f'(x) = m$  for all  $x$ .



**HW5.2** Sketch the graph of a function that satisfies the given conditions.

1. Sketch the graph of a function  $f$  satisfying the following conditions:

- $f$  is defined for all real numbers.
- The sign chart of the derivative of  $f$  is



where DNE indicates that the derivative does not exist.

2. Sketch the graph of a function  $s$  satisfying the following conditions:

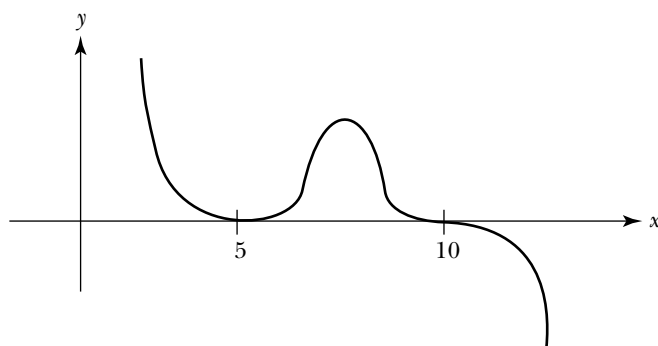
- $s$  is continuous for all real numbers.
- $s$  is differentiable everywhere, except at  $t = 6$ —that is  $s'(6)$  DNE.
- $s'(t) \leq 0$  for all  $t$ , except  $t = 6$ .
- $s'(3) = s'(9) = 0$ .
- $s(6) = 0$ .

3. Sketch the graph of a function  $h$  satisfying the following conditions:

- $h$  is continuous for all real numbers.
- $h'(x) = x^2 - x$ .
- The graph of  $h$  passes through the point  $P(\frac{1}{2}, -4)$ .

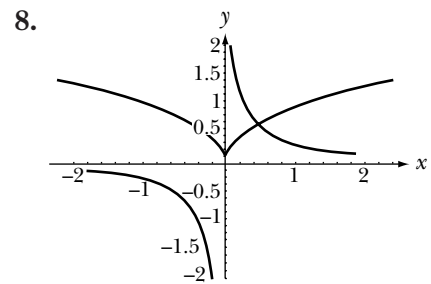
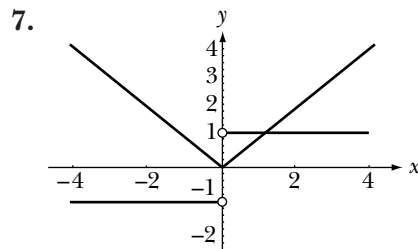
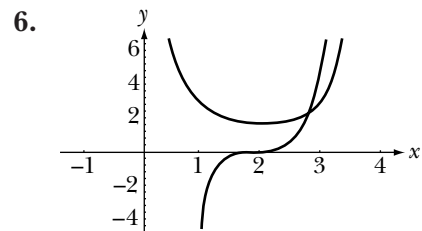
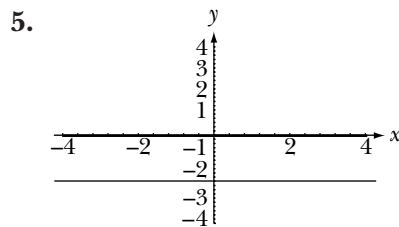
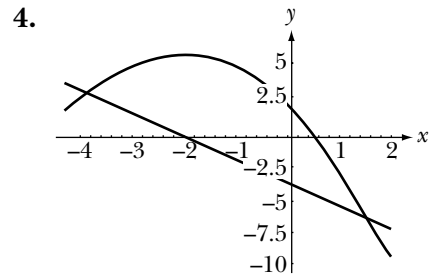
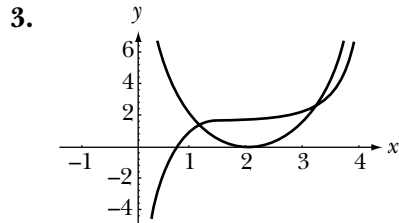
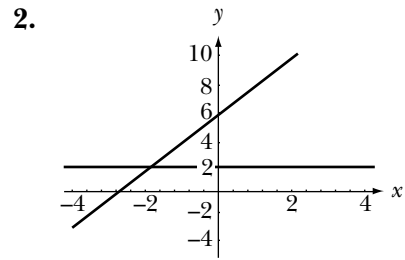
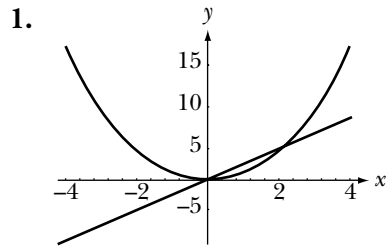
4. Sketch the graph of a function  $f$  satisfying the following conditions:

- $f$  is continuous for all real numbers.
- $f(x) \leq 0$ , for all  $x$ .
- $f(10) = 0$ .
- The graph of  $f'$  has the following shape:



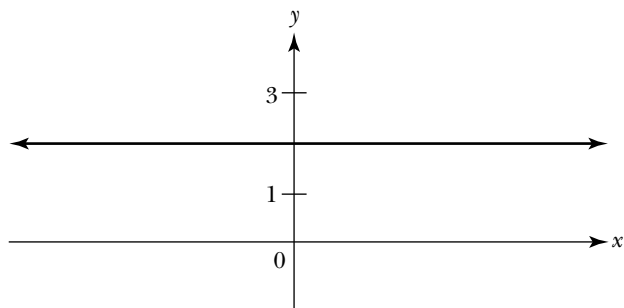
**HW5.3** Compare the graphs of a function and its derivative.

Each of the following graphs displays the graph of a function  $f$  and its derivative  $f'$ . Indicate which graph is which. Justify your choices.

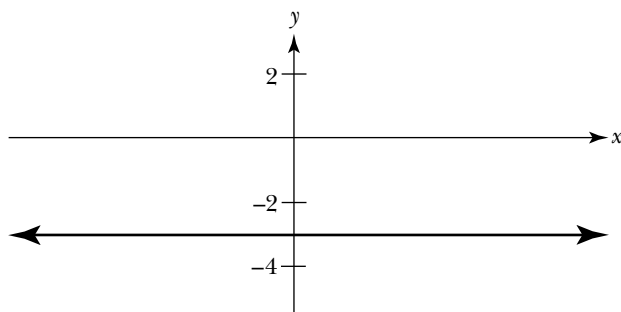


**HW5.4** Many functions have the same derivative. Make a first pass at determining the relationship between two functions that have the same derivative in this exercise.

1. Suppose the graph of  $f'$  is given below:



- a. Based on the shape of the graph of  $f'$ , explain why the graph of  $f$  is a line.
  - b. Give the slope of the graph of  $f$ .
  - c. Assume  $f$  passes through the point  $P(0,0)$ .
    - (1) Represent  $f$  by a graph. Label your axes.
    - (2) Represent  $f$  by an expression.
  - d. Instead of assuming the graph of  $f$  passes through the point  $P(0,0)$ , assume the graph of  $f$  passes through the point  $P(0,-3)$ .
    - (1) Represent  $f$  by a graph. Label your axes.
    - (2) Represent  $f$  by an expression.
2. Suppose the graph of  $f'$  is given below:



- a. Assume  $f(0) = 2$ . Represent  $f$  by a graph and an expression.
  - b. Assume  $f(2) = 0$ . Represent  $f$  by a graph and an expression.
3. What's going on here? In each instance, there are at least two functions that have the same derivative. Describe the relationship between the graphs of linear functions that have the same derivative.

**HW5.5** The value of  $f'(a)$  equals the slope of the tangent line to the graph of  $f$  when  $x = a$ . It also gives the rate of change of  $f$  when  $x = a$ . With this interpretation, instead of finding the limit of the slopes of secant lines, you think about taking the limit of average rates of change. Calculate some average rates of change in this exercise.

Suppose  $b$  is a point near  $a$  in the domain of  $f$ . Then

average rate of change of  $f$  between  $x = a$  and  $x = b$

$$= \text{slope of secant line determined by } P(a, f(a)) \text{ and } Q(b, f(b))$$

$$= \frac{f(b) - f(a)}{b - a}$$

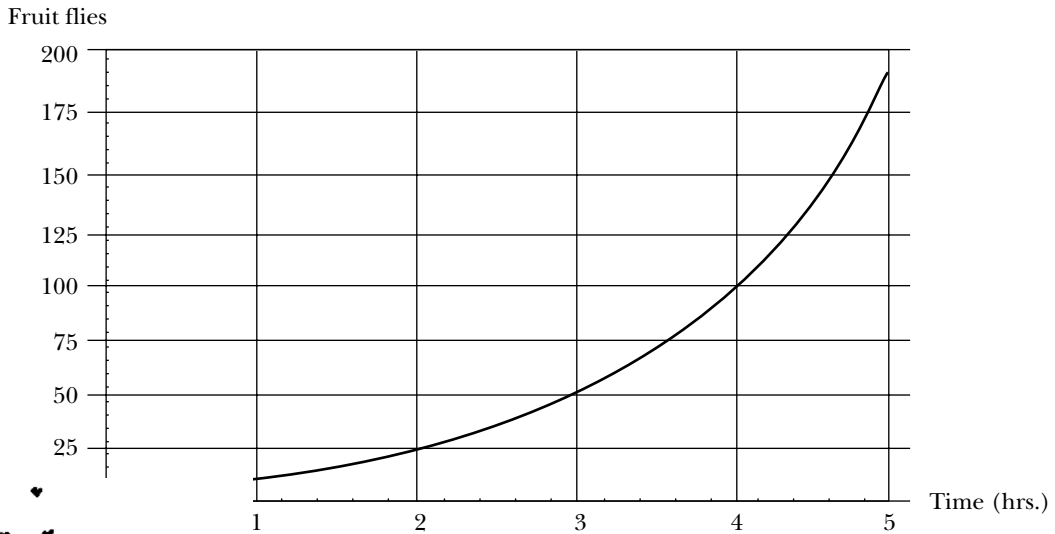
$$= \frac{\text{change in output values}}{\text{change in input values}}$$

whereas

$$\begin{aligned}
 & \text{(instantaneous) rate of change of } f \text{ when } x = a \\
 &= \text{slope of the tangent line to the graph of } f \text{ at } P(a, f(a)) \\
 &= \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a} \\
 &= f'(a)
 \end{aligned}$$

Calculate some average rates of change.

1. Fruit flies multiply fast. The following graph of the function  $N$  shows the growth of a fruit fly population during a 5-hour period.

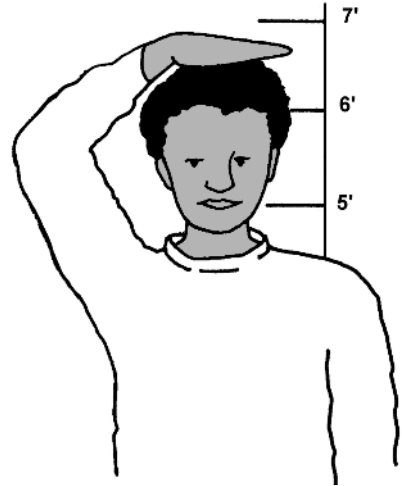


- a. Approximate the numerical value for the average rate of change in the fruit fly population over the following time intervals:
  - (1) Between 3 and 5 hours
  - (2) Between 3 and 4 hours
  - (3) Between 1 hour, 30 minutes and 3 hours
- b. Give a graphic representation of each of the numerical values which you found in part a by drawing the associated secant line on the graph of  $N$ . Label the lines (1), (2), and (3) to indicate which time interval the line represents.

2. The following table shows the growth chart for Mark:

Age (yr.)	0	2	4	6	8	10	12	14	16
Height (in.)	19	23	28	36	46	55	66	70	72

- a. Graph Mark's height versus time.
- b. Describe Mark's growth rate during the 16-year period.
- c. Find Mark's average rate of growth in inches per year from the time he was born until he was 4 years old.
- d. Find Mark's average rate of growth in inches per year from age 12 to age 16.
- e. Find the 2-year period when Mark had his largest average rate of growth.
- f. Find the four year period when he had his slowest average rate of growth.

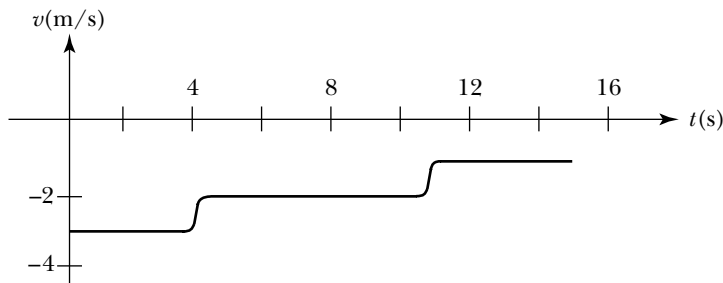


3. Suppose  $f(x) = x^2 + 3$ .

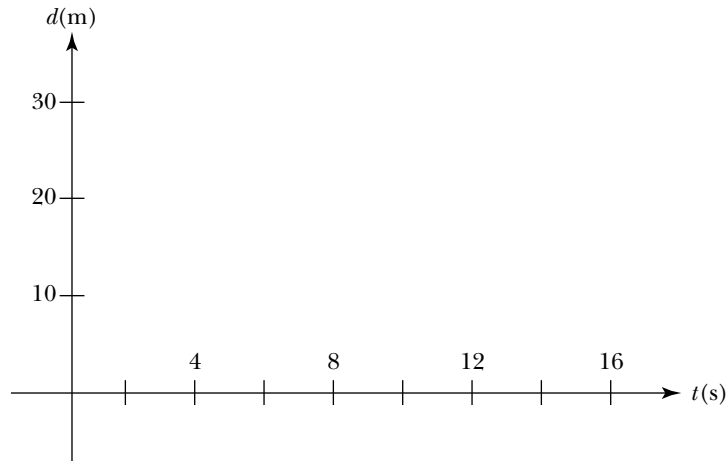
- a. Find the average rate of change of  $f$ :
  - (1) Between  $x = 1$  and  $x = -1$
  - (2) Between  $x = 1$  and  $x = 0$
  - (3) Between  $x = 1$  and  $x = 2$
  - (4) Between  $x = 1$  and  $x = 3$
- b. Graph the function  $f$  for  $-1 \leq x \leq 3$ .
- c. On your graph, carefully sketch the secant lines determined by the pairs of input values given in part a. Label each secant line on your graph and indicate its slope.

**HW5.6** In the case of the distance function, the derivative gives the velocity of an object at any given time. Apply this interpretation of derivative as you do this exercise.

Time is running out. Suppose you have 15 seconds to get to your destination 30 meters away. You start off at an acceptable pace, but because you are loaded down with books, you keep slowing down. The graph below shows your velocity function for the next 15 seconds.



1. Give a verbal description of your movement. For instance, you might say, “During the first 4 seconds my velocity is  $-3$  meters/second. Therefore, during the first 4 seconds, I am decreasing my distance from my destination, moving at a rate of 3 meters each second,” and so on.
2. Approximate the total distance you move during the 15-second time period. Recalling that initially you are 30 meters from your destination, do you make it?
3. Sketch your underlying distance function.




---

**HW5.7** Think about the meaning of the *Mean Value Theorem for Derivatives*, which says:

If  $y = f(x)$  is continuous on the closed interval  $[a, b]$  and is differentiable at every value in the open interval  $(a, b)$ , then there is at least one point  $c$  between  $a$  and  $b$  at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

1. Explain why the Mean Value Theorem for Derivatives makes sense. Develop your explanation as follows:
  - a. Sketch an appropriate diagram.
    - (1) Sketch a pair of axes, and label  $a$  and  $b$  on the horizontal axis where  $a < b$ .
    - (2) Sketch the graph of a “smooth” function  $f$  over  $[a, b]$ —that is, a function which is continuous on the closed interval  $[a, b]$  and is differentiable at every value in the open interval  $(a, b)$ .
    - (3) Label  $f(a)$  and  $f(b)$  on the vertical axis.
    - (4) Label the points  $A(a, f(a))$  and  $B(b, f(b))$ . Draw a line through  $A$  and  $B$ .

b. Answer some questions.

(1) What information does the value of

$$\frac{f(b) - f(a)}{b - a}$$

give you about the line determined by the points  $A$  and  $B$ ?

(2) If  $c$  is *any* point between  $a$  and  $b$ , what information does the value  $f'(c)$  give you about the tangent line to the graph of  $f$ ?

c. Show that the conclusion to the Mean Value Theorem for Derivatives holds.

(1) Find  $c$  between  $a$  and  $b$  so that

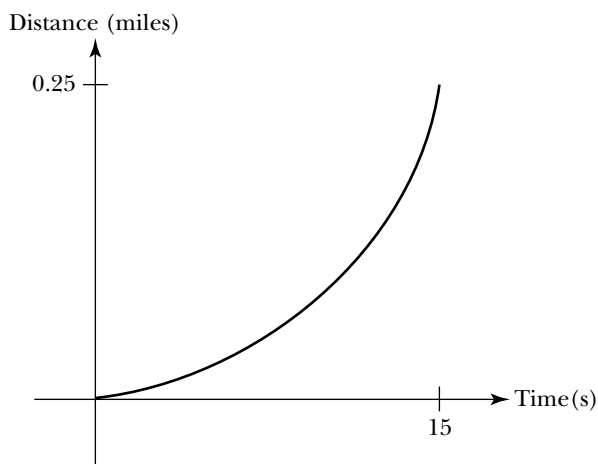
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

(2) Label  $f(c)$  on the vertical axis. Sketch the tangent line at  $C(c, f(c))$ .

(3) Describe the relationship between the line connecting the points  $A$  and  $B$  and the tangent line at  $x = c$ .

2. Apply the Mean Value Theorem for Derivatives to some real-life situations.

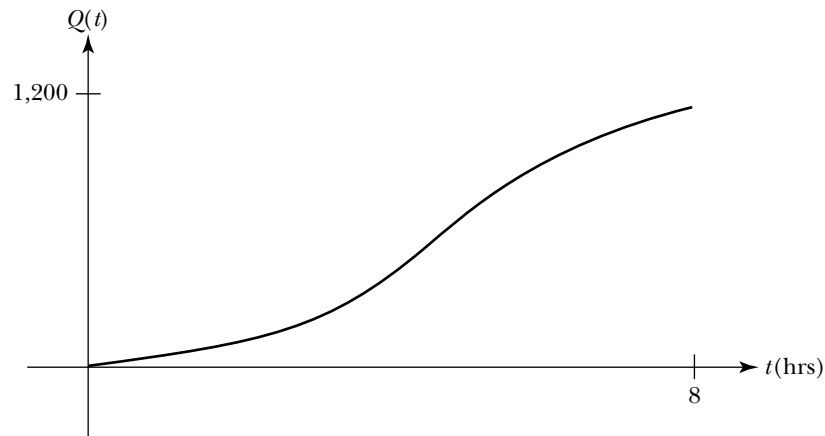
a. You stop at a red light. When the light turns green, you start off slowly, continually increasing your speed. After 15 seconds you have traveled a quarter of a mile, where your motion is depicted in the following distance versus time graph:



(1) Show that between 0 and 15 seconds your average speed is 60 MPH. *Hint:* 15 seconds =  $\frac{1}{240}$  hours.

(2) On the graph, sketch the secant line whose slope corresponds to your average speed during the 15-second time interval.

- (3) Note that your speed at any given moment corresponds to the slope of the tangent line at that point. According to the Mean Value Theorem for Derivatives, at some time  $c$  between 0 and 15 seconds your speed was 60 MPH. Label  $c$  on the  $t$ -axis. Sketch the tangent line whose slope is 60.
- (4) The posted speed limit is 60 MPH. A state trooper pulls you over at 15 seconds. Do you deserve a ticket? Justify your response.
- b. You work at a pretzel factory, where you can make 1,200 pretzels during your 8-hour shift. The number of pretzels you produce in  $t$  hours is given by the production function  $Q(t)$ , whose graph is given below:



- (1) Find your average rate of productivity between 0 and 8 hours.
- (2) On the graph, sketch the secant line whose slope corresponds to your average rate of productivity during an 8-hour shift.
- (3) According to the Mean Value Theorem for Derivatives, there exists at least one time  $c$  during your shift when your productivity rate equals your average rate of production. In your case, there are two choices for  $c$ , say,  $c_1$  and  $c_2$ . Label  $c_1$  and  $c_2$  on the  $t$ -axis. Sketch the associated tangent lines.
- (4) Label the time on the  $t$ -axis when your rate of productivity is the greatest. Sketch the associated tangent line.
- (5) The moment of maximum efficiency, which you labeled in part (4), is called your *point of diminishing returns*. Explain why this terminology makes sense.

---

**HW5.8** What is a derivative? Write a short essay explaining what a derivative is. In your essay:

- i. Describe the various ways to interpret the value of a derivative.
- ii. Give a formal definition of “derivative.”
- iii. Explain why  $f'(a)$  gives the slope of the tangent line to the graph of  $f$  at  $P(a, f(a))$ . Support your explanation with a diagram. Label your diagram.



- iv. Describe conditions under which the derivative does not exist.
- v. Summarize how you can use the derivative to gain information about the behavior of its underlying function.

**HW5.9** Explore the use of sigma notation in this exercise. You will use this notation in the next section.

Many times throughout your study of mathematics, you will be asked to find the sum of a sequence of numbers. For example, suppose your credit card company has a strict policy against late payments. For each day your payment is late, the company adds to your outstanding bill the dollar amount corresponding to the number of days your payment is overdue. In other words, if you are three days late, \$6 is added to your bill—that is, \$3 for the third day plus \$2 for the second day and \$1 for the first day.

When there are only three terms to be added together, it is easy to write out  $1 + 2 + 3$ , but what if you are 100 days late? It's tedious to write out  $1 + 2 + 3 + 4 + 5 + \cdots + 100$ . Consequently, a special notation called *sigma notation* has been developed to represent sums of numbers. Using sigma notation, you can represent

$$1 + 2 + 3 + 4 + 5 + \cdots + 100 \text{ as } \sum_{i=1}^{100} i$$

The capital Greek letter sigma  $\Sigma$  represents a sum and the variable  $i$  is the *index* of summation. In this case, the values of  $i$  range through the integers from 1 to 100. The limits on the index will vary depending on your problem, as will the expression to be summed. For instance,

$$\sum_{i=12}^{16} i^2 \text{ represents } 12^2 + 13^2 + 14^2 + 15^2 + 16^2.$$

1. Rewrite each of the following summations as the sum of a sequence of numbers. Do not calculate the actual value of the sum.

a. $\sum_{i=3}^7 (2i + 1)$	c. $\sum_{i=-3}^2 (-i + 3)$
b. $\sum_{i=1}^4 i^3$	d. $\sum_{i=0}^5 \frac{i}{i + 1}$

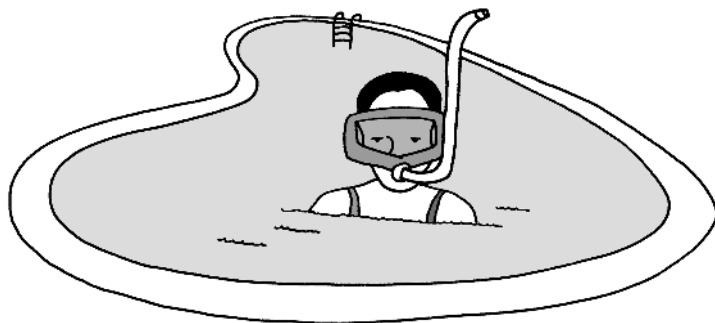
2. Represent the following sums using sigma notation.

a. $2 + 3 + 4 + \cdots + 12$	c. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{12}$
b. $1 + 4 + 9 + 16 + \cdots + 81$	d. $15 + 20 + 25 + \cdots + 65$

3. Because calculating these long sums can be very tedious, it is helpful to let your calculator sum the items in a sequence for you.
  - a. Read about how to sum the items in a sequence in the appendix for your graphing calculator.
  - b. Look again at the sums you considered in parts 1 and 2. In each case,
    - i. Translate the mathematical notation for the sum into an expression that you can enter into your calculator.
    - ii. Use your calculator to find the value of the sum.
    - iii. Record your result.

## SECTION 2

### The Definite Integral



The definite integral is one of the most important and fundamental concepts in calculus. It is extremely useful for measuring quantities and determining how quantities accumulate. For example, definite integrals can be used to help a maintenance worker (who has taken calculus!) figure out how much rubber sealer he will need to coat the bottom of a kidney

shaped swimming pool; or the president of Chemical Bank can use a definite integral to determine the accumulated net assets on a particular day, based on the rate at which money has been deposited or withdrawn from the bank up to that time. Surprising as it may seem, these two types of calculations—finding the area of an oddly shaped region and determining an accumulation resulting from a variable rate of change—are very closely connected mathematically.

In order to be able to identify situations where it is appropriate to use a definite integral, you need to develop a conceptual understanding of what a definite integral represents. This is the primary objective of this section. The tasks have been designed to help you discover its definition and develop a mental image associated with the concept of definite integral.

In this section, you will examine the relationship between definite integrals and areas. You will use your graphing calculator to do some of the calculations and to help you understand the mathematical notation underlying the definition of definite integral. You will explore ways to approximate areas and accumulations and think about how to make your approximations more exact. You will develop a general method which applies in all cases. Finally, you will make a first pass at examining the connection between definite integrals and derivatives.

limits, which are called the *left-* and *right-hand derivatives* exist, but have different values; that is, if

$$\lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h} \neq \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

- A limit does not exist if its output values explode as the input values get closer and closer to the designated number. Consequently,  $f'(a)$  does not exist if

$$\frac{f(a+h) - f(a)}{h}$$

blows up as  $h$  approaches 0 from the left and/or the right; that is, if

$$\lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h} = \pm\infty \quad \text{and/or} \quad \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} = \pm\infty$$

In the next task, you will examine what goes wrong when you try to evaluate the limit at a place where the derivative does not exist. As you do the task, keep two things in mind: First, the following are equivalent ways of saying that  $x = a$  is not in the domain of  $f'$ :

- $f$  is not differentiable at  $x = a$ .
- $f'(a)$  does not exist.
- $f'$  is undefined at  $x = a$ .

Second, as a line becomes more and more vertical, the slope of the line approaches  $+\infty$  or  $-\infty$ . Consequently, the slope of a vertical line is undefined.

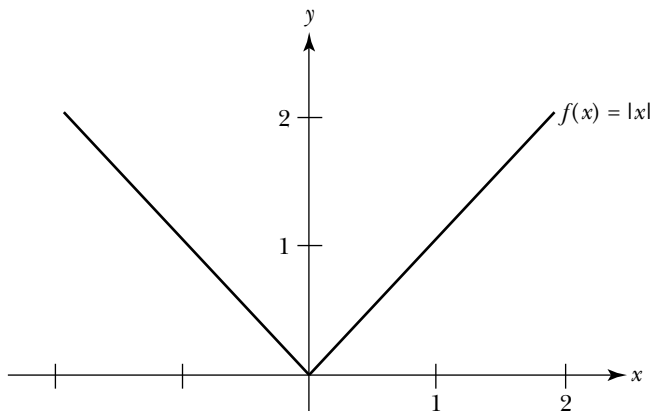
### Task 5-4: Inspecting the Domain of a Derivative

The domain of  $f'$  is contained in the domain of  $f$ . What points are left out?

1. If the graph of a function  $f$  has a sharp “peak” or “dip” at  $x = a$ , then  $f'(a)$  does not exist. Examine why this is true.

Actually, this is not a new idea. Remember the dude with the pointy hair in Unit 1? There were no tangent lines on the tips of his hair...

- a. Consider the graph of  $f(x) = |x|$ , which has a sharp dip at  $x = 0$ .



- (1) Use a graphic approach to find the left-hand derivative of  $f$  at  $x = 0$ .

The *left-hand derivative* of  $f$  at  $x = 0$  is the limit of the slopes of the secant lines determined by  $P(0, f(0))$  and  $Q$ , where  $Q$  approaches  $P$  from the left. Consequently, one way to determine the left-hand derivative of  $f$  at  $x = 0$  is to find the slope of the secant line determined by  $P(0, f(0))$  and say  $Q(-1, f(-1))$ . Next, find the slope of the secant line for another value of  $Q$  which is closer to  $P$ , but still to its left, for instance,  $Q(-0.5, f(-0.5))$ . Find the value of the left-hand derivative by generalizing your observations. Record your conclusion below.

The left-hand derivative of  $f$  at  $x = 0$  is \_\_\_\_\_ .

- (2) Use a graphic approach to find the right-hand derivative of  $f$  at  $x = 0$ .

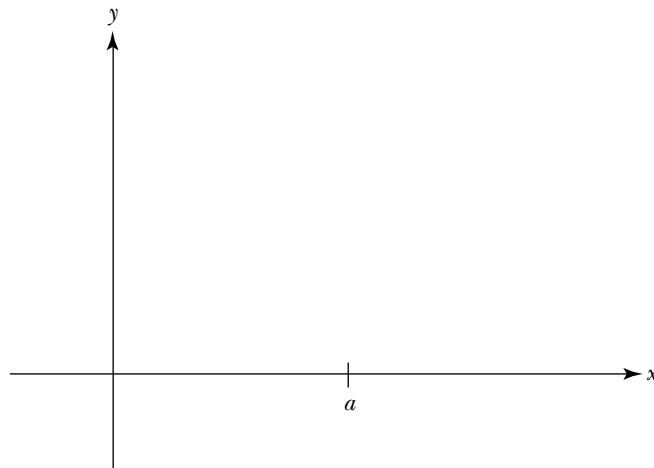
The *right-hand derivative* of  $f$  at  $x = 0$  is the limit of the slopes of the secant lines determined by  $P(0, f(0))$  and  $Q$ , where  $Q$  approaches  $P$  from the right. Repeat the process you used to find the left-hand derivative in part (1), but this time consider a couple of values of  $Q$  which are close to  $P(0, f(0))$  but to the right of  $P$ . Record your conclusion below.

The right-hand derivative of  $f$  at  $x = 0$  is \_\_\_\_\_ .

- (3) Explain why  $f(x) = |x|$  is not differentiable at  $x = 0$ .

**b.** Generalize your observations.

- (1) Sketch an arbitrary function  $f$  that is defined at  $x = a$  and has a sharp peak or dip at  $x = a$ .



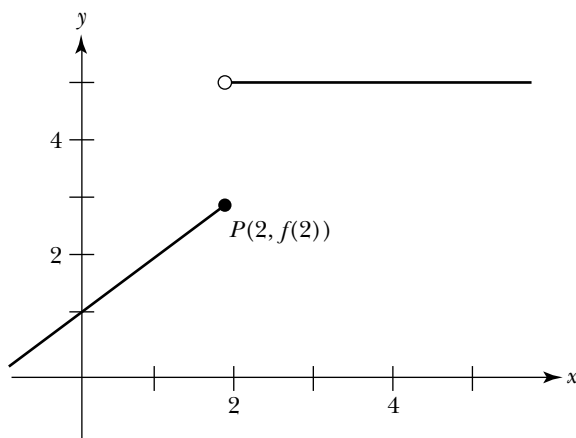
(2) Explain why  $f$  is not differentiable at  $x = a$ .

2. If the graph of a function  $f$  has a jump discontinuity at  $x = a$ , then  $f'(a)$  does not exist. Examine why this is true.

a. Consider the function

$$f(x) = \begin{cases} x + 1, & \text{if } x \leq 2 \\ 5, & \text{if } x > 2 \end{cases}$$

The graph of  $f$  has a jump discontinuity at  $x = 2$ .



(1) Find the value of the left-hand derivative of  $f$  at  $x = 2$  by calculating the slope of several secant lines determined by  $P(2, f(2))$  and some points on the graph of  $f$  to the left of  $P$ .

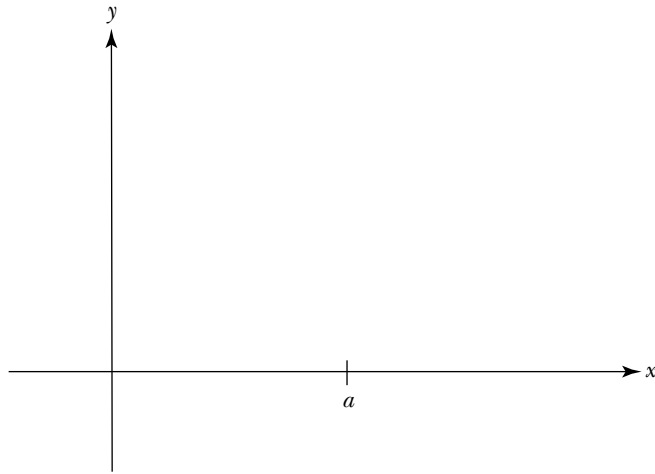
The left-hand derivative of  $f$  at  $x = 2$  is \_\_\_\_\_.

(2) Explain why the right-hand derivative of  $f$  at  $x = 2$  does not exist by sketching several secant lines determined by  $P(2, f(2))$  and some points on the graph of  $f$  to the right of  $P$ .

(3) Explain why  $f$  is not differentiable at  $x = 2$ .

b. Generalize your observations.

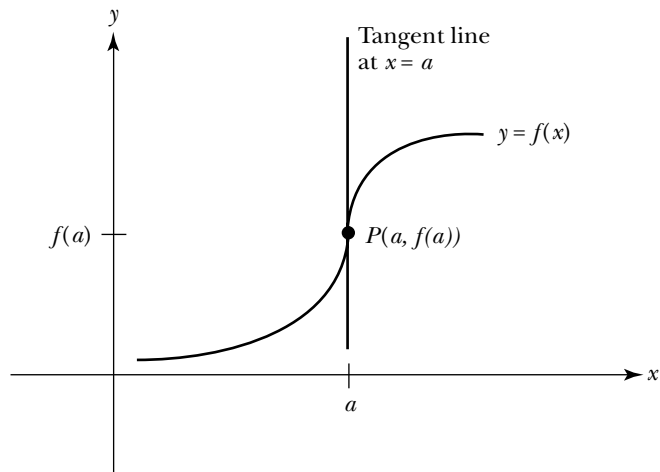
- (1) Sketch an arbitrary function  $f$  that has a jump discontinuity at  $x = a$ . Label the point  $P(a, f(a))$  on your graph.



- (2) Explain why  $f$  is not differentiable at  $x = a$ . In the case of your function, which one-sided derivative does not exist?

3. If the graph of a function  $f$  has a vertical tangent at  $x = a$ , then  $f'(a)$  does not exist. Examine why this is true.

Consider the graph of the following function which has a vertical tangent at  $x = a$ .



- a. Explain why the left-hand derivative of  $f$  at  $x = a$  does not exist. Sketch the secant lines determined by  $P(a, f(a))$  and several points on the graph of the function to the left of  $P$ . Describe what happens to the value of the slope of the secant line as the points get closer and closer to  $P$ .
  
- b. Using a similar approach as in part a, explain why the right-hand derivative of  $f$  at  $x = a$  does not exist.
  
- c. Explain why  $f$  is not differentiable at  $x = a$ .

In Unit 1, you used the motion detector to create distance versus time graphs, and you analyzed the behavior of the tangent line as it traveled along the graphs. You discovered that the value and sign of the slope of the tangent line provide a lot of information about the shape and behavior of the graph, including the following facts:

- The value of the slope of the tangent line corresponds to your velocity or instantaneous rate of change at a given time.
- The magnitude of the slope of the tangent line corresponds to the speed you are moving at a particular time.
- The sign of the slope indicates whether the function modeling your motion rises or falls from left to right. The sign is positive when the graph is increasing and negative when it is decreasing.
- If the value of the slope is zero, the graph of the function has a horizontal tangent.
- A change in the sign of the slope as the tangent line travels along the graph from left to right indicates the location of the function's local extrema. If the sign of the slope changes from positive to zero to negative, the function has a local maximum. If it changes from negative to zero to positive the function has a local minimum.
- If the graph lies above the tangent line, the function is concave up. If it lies below, it is concave down.

In the next task, you will reexamine these ideas and express them in terms of the derivative. You will also investigate how the graph of the derivative is related to the graph of the given function. In particular, based on behavior of  $f$ , you will determine the following:

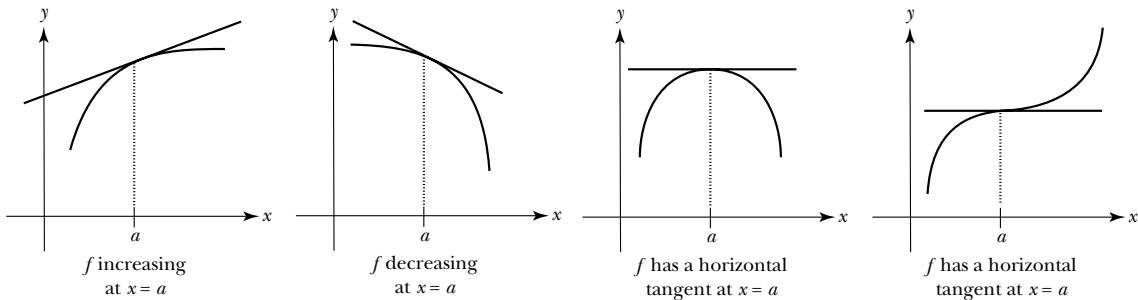
- The places where  $f'$  is undefined
- The places where the graph of  $f'$  intersects the horizontal axis

- The intervals where the graph of  $f'$  lies above the horizontal axis and the intervals where the graph lies below

As you explore the relationship between  $f$  and  $f'$ , you will develop a statement of the First Derivative Test.

### Task 5-5: Investigating the Relationship Between a Function and Its Derivative

1. Summarize what you know about the relationship between a function and the tangent line to its graph. Express these ideas in terms of the derivative and describe their impact on the graph of the derivative.
  - a. Consider the cases where a function  $f$  is increasing, decreasing, or has a horizontal tangent at  $x = a$ .



Summarize what's happening in each of these situations by filling in the following chart. The first row has been completed for you. Look it over carefully before completing the other rows.

*Note: TL stands for "tangent line."*

Description of $f$ at $x = a$	Shape of Graph of $f$ near $x = a$	Sign of Slope of TL at $x = a$ . i.e., Sign of $f'(a)$	Location of Graph of $f'$ at $x = a$
$f$ increasing	Rises from left to right	+	Above horizontal axis
$f$ decreasing			
$f$ has a horizontal tangent			





*Appendix:*  
**USING THE TI-83 GRAPHING CALCULATOR  
IN WORKSHOP CALCULUS**

**Christa Fratto, Nancy Baxter Hastings,  
and Barbara E. Reynolds**

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## 1. GENERAL INFORMATION

The TI-83 is a graphing calculator with a variety of scientific, statistical and financial applications and features. This appendix will focus on the features of the TI-83 that you will use in *Workshop Calculus: Guided Exploration with Review*. For additional information on these and other features, please consult the *TI-83 Guidebook*.

Throughout this appendix, calculator keys will be indicated in boldface type, with second function keys enclosed in square brackets. For example, **ENTER** and **2nd [QUIT]** are calculator keys that you will use frequently.

## 2. CALCULATOR BASICS

### 2.1. TURNING THE CALCULATOR ON/OFF

To turn the calculator on, press the **ON** key. To turn the calculator off, press **2nd [OFF]**. The **2nd** key is the yellow key in the upper left corner of the calculator. The **[OFF]** key is the **2nd** function of the **ON** key.

If the calculator is left on for several minutes without any activity, it will turn off automatically. When the calculator is turned off, either manually or automatically, the settings and memory are retained. This means that when the calculator is turned on again, the home screen will be displayed exactly as it was when the calculator was turned off.

### 2.2. THE KEYBOARD

The keyboard on the TI-83 calculator is organized into various clusters of keys. The *graphing keys* are in the top row, just under the display screen. These keys are used to access the calculator's interactive graphing features and include the **Y=** and the **GRAPH** keys. The *cursor keys* are the blue keys with the white arrows in the upper right corner of the keyboard. They are used to move the cursor on the screen and to highlight choices in a list or a menu. The *editing keys* are to the left of the cursor keys and are used to edit expressions and values. They include the **MODE** and the **DEL** keys. The *advanced function keys* are in the following row and they are used to display the menus which access the calculator's advanced functions. They include the **MATH** and the **PRGM** keys. Finally, the *scientific calculator keys* are located in the bottom half of the keyboard. They are used to access the capabilities of a standard scientific calculator.

Most of the keys on the keyboard can perform 2 or more functions, depending on whether you first press a modifier key. There are two modifier keys, **2nd** and **ALPHA**. The **2nd** key is yellow and can be used to access those functions which are printed in yellow above many of the keys.

Similarly, the **ALPHA** key is green, and can be used to access the green letters or symbols. If you press **2nd [A-LOCK]**, then you will activate the *alpha lock* function. In this mode, the subsequent keystrokes will automatically be the green letters above each key. The **A** in the upper right corner of the

screen indicates that the *alpha lock* feature is turned on. To exit the *alpha lock* mode, press the **ALPHA** key again.

---

### Example

To use the square root key, press **2nd** and then press the  $\sqrt{\quad}$  key which appears in yellow above the  $x^2$  key.

The **ENTER** key evaluates an expression or executes an instruction. The **X,T, $\theta$ ,n** key pastes an X, T,  $\theta$ , or  $n$  at the cursor location in just one keystroke, as opposed to pressing **ALPHA [X]**, for instance. In *Workshop Calculus*, you will be working almost exclusively in *function* mode; therefore, pressing the **X,T, $\theta$ ,n** key will paste an X at the cursor location. The T,  $\theta$ , and  $n$  are pasted when the calculator is in *parametric*, *polar* or *sequential* mode, respectively.

### 2.3. PREDEFINED FUNCTIONS AND CONSTANTS

The calculator has several predefined functions and constants stored in its memory. The trigonometric functions sine, cosine and tangent, along with their corresponding inverse functions, are located in the fourth row of the keyboard. The mathematical constants  $\pi$  and  $e$  are the second functions of the  $\wedge$  and  $\div$  keys, respectively. The common logarithm ( $\log_{10}$ ) and the natural logarithm keys are located in the first column of the keyboard with  $10^x$  and  $e^x$  as their corresponding second functions.

Predefined functions are also located under various menus. For example, the absolute value function is located under the **MATH[NUM]** menu. All of the predefined functions are listed in the **CATALOG** (see Section 2.7).

### 2.4. THE HOME SCREEN

The home screen is where you execute instructions, evaluate expressions, and view results. To return to the home screen from other screens such as a graph or a table, press **2nd [QUIT]**.

To clear the home screen, press **CLEAR**. If you have begun to type a few characters, and not yet pressed **ENTER**, pressing **CLEAR** will only clear the entry line; pressing **CLEAR** again will erase the entire home screen.

### 2.5. ADJUSTING THE CONTRAST ON THE SCREEN

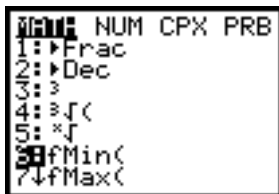
To adjust the contrast on the screen, press and release the **2nd** key and then press and hold the up or down arrows to make the screen lighter or darker. Each time you release the arrow keys, it is necessary to press **2nd** again to further adjust the contrast. As the contrast changes, a number ranging from 0 (lightest) to 9 (darkest) will appear in the upper right corner to indicate the current contrast level. You may not be able to see the number if the screen is too light or too dark.

### 2.6. SELECTING ITEMS FROM A LIST OR A MENU

To select an item from a list or a menu, which can be accessed from one of the *advanced function* keys, press the number of the appropriate item, or use the arrow keys to scroll down until the number of the item is highlighted, then press **ENTER**. The selected item will then be pasted at the current cursor location.

## Example

To select option **6:fMin(** from the **MATH** menu, either press **6** or use the arrow keys to highlight the **6:** and press **ENTER**.



## 2.7. THE CATALOG FEATURE

The **CATALOG** is an alphabetical list of all functions, symbols and instructions on the TI-83. An item selected from the **CATALOG** is pasted at the current cursor location. It is convenient to use this feature if you forget which menu contains a particular item. Each item in the **CATALOG** that you will use in *Workshop Calculus* can also be accessed from a menu or the keyboard.

To use the catalog, press **2nd [CATALOG]** and use the arrows to scroll to the appropriate item. Press **ENTER** to paste the item at the current cursor location.

To move directly to the first item that begins with a particular letter, press the desired letter, as the alpha-lock feature is automatically turned on when you are in the **CATALOG**. The symbols are located at the end of the list.



## 2.8. INSERTING AND DELETING CHARACTERS

To delete a character, use the left or right arrow keys to move the cursor so that it's on top of the character to be deleted and press **DEL**.

By default, if you type over existing code, the new characters will replace the existing characters. To insert a new character without deleting the existing character, use the cursor keys to position the cursor at the point where the new characters are to be inserted and press **2nd [INS]**. The new characters will be inserted to the *left* of the cursor. To exit *insert mode*, press **2nd [INS]** again.

## 2.9. EDITING AND RE-EXECUTING LINES OF CODE

You can reenter the previous entry or answer by pressing **2nd [ENTRY]** or **2nd [ANS]**, respectively. This feature is particularly useful to avoid retyping complicated lines of code.

2*17	34
Ans-6	28
2*17+Ans	62

## 2.10. INTERPRETING ERROR MESSAGES

The TI-83 displays a message when it encounters an error. This message gives two options, **1:Quit** which takes you back to the home screen, and **2:Goto** which takes you to the previous screen with the cursor at or near the error location. If you are unable to recognize an error, please consult *Appendix B* of the *TI-83 Guidebook*.

ERR:SYNTAX
1:Quit
2:Goto

## 2.11. TRANSFERRING PROGRAMS BETWEEN TI-83s

To transfer programs from one calculator to another, connect the two calculators with the black unit-to-unit link cable that came with your calculator. Firmly insert an end of the cable into the input/output port on the bottom edge of each calculator.

On the calculator which will *receive* the programs, press **2nd [LINK]**, scroll over to **RECEIVE** and choose **1:Receive**. Press **ENTER**.

SEND	RECEIVE
1:Receive	

On the calculator which will *transmit* the programs, press **2nd [LINK]** and choose **3:Prgm...**. Select the programs you wish to transmit by scrolling down to the desired name and pressing **ENTER**, at each name.

3:SEND	TRANSMIT
DIFFTEST	PRGM
DISMATCH	PRGM
LDECAY	PRGM
LGAMMA	PRGM
LINES	PRGM
LOGLOG	PRGM
MOTION	PRGM



Scroll over to **TRANSMIT** and choose **1:Transmit**. Press **ENTER**.



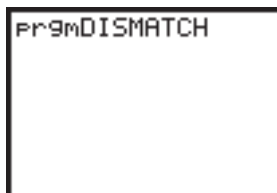
The calculator receiving the programs will display the message **Receiving...** When the programs have been transmitted, both calculators will display the names of the programs and the message **Done**.

### 2.12. EXECUTING A PROGRAM

To execute a program on the TI-83 calculator, press the **PRGM** key and select the appropriate program.



The name of the program will appear on the home screen. To begin the program, press **ENTER**.



When a program is completed, the word **Done** will appear on the screen. To execute the program again, press **ENTER**. To stop a program during execution, press **ON** and select **1:Quit**.

## 3. USING THE CBL SYSTEM

The Calculator-Based Laboratory System (CBL) is a portable data collection device for collecting real-world data. The data collected by the CBL can be transferred to the TI-83 graphing calculator for further analysis. In *Workshop Calculus*, you will use an ultrasonic motion detector to collect data. For further information, please consult the *CBL System Guidebook*.

The motion detector is a black box which enables you to plot distance (position) versus time graphs of your motion. As you move, the graph on

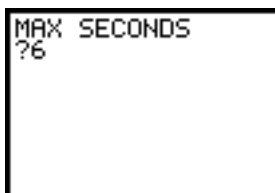
the calculator screen displays your distance (in meters) from the motion detector in real time. The motion detector is the origin from which distances are measured, and the term “distance” is short for “distance from the motion detector.”

When you create your graphs, you must be at least one-half meter away from the motion detector as it will not correctly measure anything closer. The motion detector sends out a series of ultrasonic pulses which bounce off the closest object in front of it; if you swing your arms or allow your clothing to flop, the motion detector will detect them instead of you. The motion detector sends out pulses in a cone-shaped form; it may detect objects such as the edge of a table or a chair which appear to be out of its range.

To connect a CBL and a motion detector to your calculator insert the adapter cord into the **SONIC** port on the CBL and insert the cord from the motion detector into the other end of the adapter cord. Connect your calculator and the CBL with the long (7 feet) version of the unit-to-unit link cable. Turn on both your calculator and the CBL.

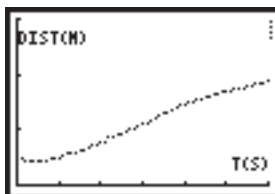
### 3.1. CREATING A GRAPH USING A MOTION DETECTOR

The program **MOTION** enables you to generate graphs using a motion detector. Load the program **MOTION** (see Section 31.1) and connect a motion detector to your calculator. Execute the program and press **ENTER**. Follow the directions that appear on the screen. In particular, type the number of seconds you wish to collect data and press **ENTER**.



A rectangular box representing a calculator screen. The text 'MAX SECONDS' is displayed on the top line, and '?6' is displayed on the second line.

The CBL will collect data every tenth of a second, from .1 seconds to **MAX SECONDS**. Press **ENTER** to start the graph. Begin moving when you hear a clicking sound from the detector. The graph representing your movement will appear on the screen. Note that the data for distance is in meters, and the data for time is in seconds.



After you complete your graph, press **ENTER**. You will be given two options. You can try again or you can quit program **MOTION** and analyze your data (see Section 4). If you choose to try again, you will be asked if

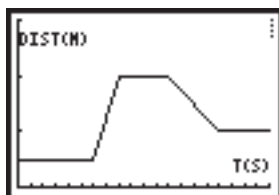
you want to save or erase the data for the current graph. If you choose to save the data, the old graph will remain on the screen.



When data is collected using program **MOTION**, the time data is stored in the list **L2**, the corresponding distance data for the most recently generated graph is stored in **L3**, and the corresponding distance data for an old graph (if saved) is stored in **L4**.

### 3.2. MATCHING A GRAPH USING A MOTION DETECTOR

The program **DISMATCH** generates a predefined graph for users to try to duplicate. Load the program **DISMATCH** (see Section 3.1.2) and connect a motion detector to your calculator. When you run the program, the following graph will appear on your screen. Note that the time interval for this function goes from 0 to 20 seconds. To try to duplicate the graph, press **ENTER**, wait for the clicking sound from the detector, and then start to move.



After you complete your graph, press **ENTER**. You will be given two options. You can try again or you can quit program **DISMATCH** and analyze your data (see Section 4). If you choose to try again, the data for your previous try will be erased.

When data is collected using program **DISMATCH**, the time data is stored in the list **L2**, the corresponding distance data for the graph which you created is stored in **L3**, and the corresponding distance data for the given distance match function is stored in **L4**.

## 4. ANALYZING DATA COLLECTED BY THE CBL SYSTEM

### 4.1. EXAMINING THE TABLE OF DATA

To examine the data collected by the CBL in tabular form, press the **STAT** key and choose **1:Edit**. Note that the columns in the table contain the following information:

- **L1** contains the data for the CBL system (you can ignore this).
- **L2** contains the time data (in seconds), .10, .20, ....
- **L3** contains the corresponding distances (in meters) for the most recently generated graph.
- With program **MOTION**, **L4** contains the corresponding distances for the old graph (if saved). With program **DISMATCH**, **L4** contains the corresponding distance data for the given distance match function.

Use the left and right arrow keys to scroll across a row in the table, observing, for instance, that **L3(4)** is the distance in meters corresponding to .40 seconds or **L2(4)**. Use the up and down arrow keys to scroll through a column of values.

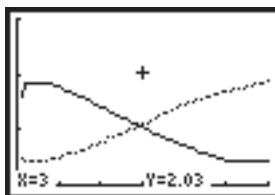
L2	L3	L4	3
.10	1.57	.46	
.20	1.85	.46	
.30	1.85	.45	
.40	1.85	.45	
.50	1.85	.44	
.60	1.84	.45	
.70	1.83	.45	
L3(4) = 1.85274			

To return to the home screen, press **2nd [QUIT]**. To view the graphs of the data, press **GRAPH**.

#### 4.2. VIEWING THE GRAPHS

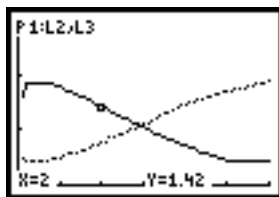
To view the graphs, press **GRAPH**. If two graphs are displayed, the most recently generated graph is displayed in connected mode, while the previous one, or in the case of **DISMATCH**, the distance match function, is in dot mode.

To display the coordinates of a point on the screen, use the arrow keys to move around the screen. The  $(x,y)$ -coordinates for the location of the cursor will be displayed at the bottom of the screen.



#### 4.3. TRACING THE GRAPHS

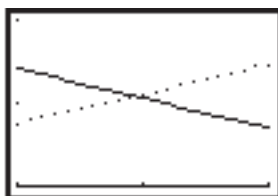
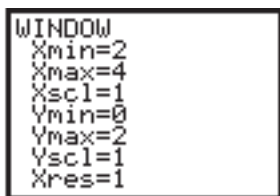
To trace the graphs, press the **TRACE** key. Use the up and down arrow keys to move between the graphs and the left and right arrows to move along a graph. The location of the cursor will be displayed at the bottom of the screen and the graph being traced will appear in the upper left corner, where **P1:L2,L3** indicates the most recent graph and **P2:L2,L4** indicates a previous graph, since the data for the graphs are stored in columns **L2** and **L3** or in **L2** and **L4** in the data table.



#### 4.4. ZOOMING IN

One way to examine a particular region of a graph more closely is to adjust the viewing window for the graph. To do this, press the **WINDOW** key and enter the desired values for **Xmin**, **Xmax**, **Ymin** and/or **Ymax**. To view the new portion of the graph, press the **GRAPH** key.

An alternate approach is to use the **ZOOM** feature (see Section 14).



### 5. ARITHMETIC OPERATIONS AND EXPRESSIONS

#### 5.1. ORDER OF OPERATIONS

The TI-83 calculator follows the order of operations described in the chart below. It is often helpful to use parentheses for grouping purposes.

1	Functions that are entered after the argument such as <b>2</b>
2	Powers and roots, such as <b>2^5</b>
3	Single argument functions that precede the argument, such as <b>sin(</b> or <b>log(</b>
4	Multiplication, implied multiplication and division
5	Addition and subtraction
6	Relational functions, such as <b>&gt;</b> or <b>≤</b>
7	Logic operator <b>and</b>
8	Logic operators <b>or</b> or <b>xor</b>

Calculations within parentheses are evaluated first. Multi-argument functions such as **nDeriv(X<sup>3</sup>,X,5)** are evaluated as they are encountered

## 5.2. ENTERING AND EVALUATING AN EXPRESSION

To evaluate an expression, type the expression and press **ENTER**.

The most recent entry will appear on the left at the bottom of the home screen, and the corresponding result will appear on the right. As you enter new expressions, new entry-result pairs will be added to the bottom of the screen.

## 5.3. SUBTRACTION AND NEGATION

The calculator distinguishes between subtraction and negation. To enter a negative number, use the gray (−) key; to subtract, use the blue — key.

## 5.4. IMPLIED MULTIPLICATION

The calculator recognizes implied multiplication. That is, the calculator recognizes  $2\pi$  as  $2 \times \pi$ ,  $4\sin(x)$  as  $4 \times \sin(x)$  and  $3(2x + e^x)$  as  $3 \times (2 \times x + e^x)$ . Note that parentheses should be used to avoid confusion when squaring a negative number. For example,  $-2^2 = -4$ , whereas  $(-2)^2 = 4$ .

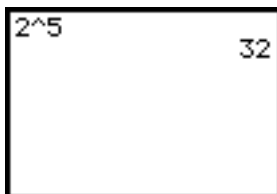
## 5.5. EXPONENTIATION

Use the ^ key to indicate that a number is raised to a power.

---

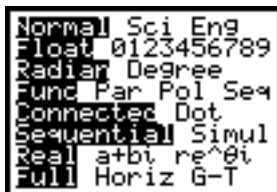
### Example

To enter  $2^5$ , type **2 ^ 5** and press **ENTER**. The calculator will return 32.



## 6. THE MODE KEY

The mode settings control how the TI-83 displays and interprets numbers and graphs. To change the mode settings, use the arrow keys to move the cursor to the appropriate setting and press **ENTER**. The features under the **MODE** key that you will use most often in *Workshop Calculus* are described below and the default settings are highlighted in the following screen.



### 6.1. FLOAT/0123456789

The *Float/0123456789* mode controls how many decimal places are displayed in a result. The default setting, **Float**, displays up to ten digits in a result. The calculator may also be set in *fixed decimal mode* where the number of digits to the right of the decimal point is specified. To specify the number of digits, move the cursor to the desired number and press **ENTER**.

### 6.2. Radian/DEGREE

The *angle mode* controls how the angle values in trigonometric functions are interpreted. **Radian** mode interprets angle values as radians and **Degree** mode interprets angle values as degrees.

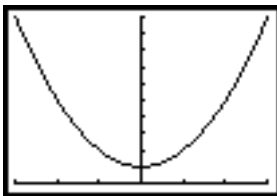
### 6.3. CONNECTED/DOT

The **Connected** plotting mode draws a graph by connecting each point calculated for the given function. The **Dot** plotting mode draws a graph by only plotting the calculated points. (See also Section 11.1., Choosing the Display Styles.)

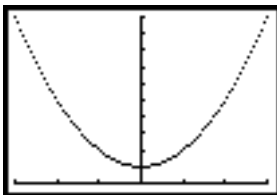
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#### Example

The graph of a function in **Connected** mode.

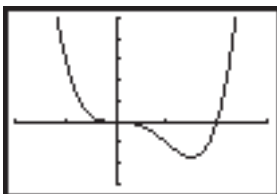


The graph of a function in **Dot** mode.

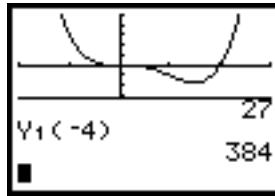


### 6.4. FULL/HORIZONTAL/GRAPH-TABLE SCREENS

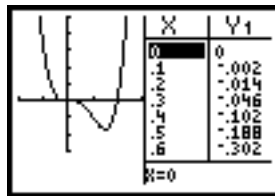
The *full screen mode* (**Full**) uses the entire screen to display a graph, table or home screen. An example of *full screen mode* appears below.



The *horizontal mode* (**Horiz**) splits the screen so that the current graph is on the top half of the screen and the home screen is on the bottom half. An example of *horizontal mode* appears below.



The *graph-table mode* (**G-T**) splits the screen so that the current graph is on the left half of the screen and its corresponding table is on the right half. An example of *graph-table mode* appears below.



## 7. USING VARIABLES

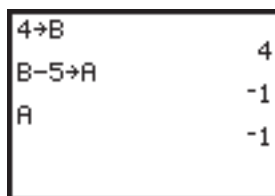
### 7.1. REAL VARIABLE NAMES

Variables which store real number values are given a single letter name. A variable which has not been assigned a value will initially be set to zero.

### 7.2. ASSIGNING A VALUE TO A VARIABLE

To assign a value to a variable, enter an expression or a number representing the value to be stored. Press the **STO->** key and type the name of the variable.

To display the value of a variable, type the name of the variable and press **ENTER**.



### 7.3. CLEARING A VARIABLE

To clear the value assigned to a variable, press **2nd** [**MEM**] and choose **2:Delete**.

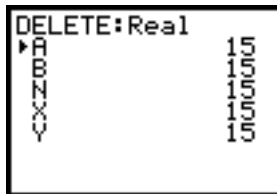




From the next menu, select **2:Real**.



Use the arrow keys to select the variable to be cleared and press **ENTER**.



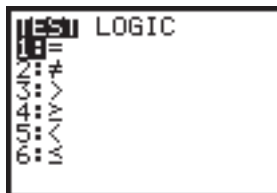
To return to the home screen, press **2nd [QUIT]**.

## 8. THE TEST MENU

The **TEST** menu is the second function of the **MATH** menu. The **TEST** menu has two submenus, **TEST** and **LOGIC**. The features that you will use most often in *Workshop Calculus* are described below.

### 8.1. RELATIONAL OPERATIONS

The relational operators are located in the **TEST** submenu. To access a relational operator, press **2nd [TEST]** and select the desired operator.

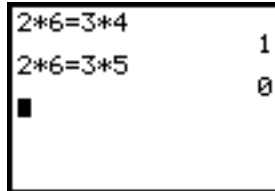


The calculator returns 0 if an expression is false and 1 if an expression is true.

---

### Example

If you enter  $2*6 = 3*4$ , the calculator will return 1 for true.  
 If you enter  $2*6 = 3*5$ , the calculator will return 0 for false.



```

2*6=3*4      1
2*6=3*5      0
  
```

## 8.2. BOOLEAN OPERATORS

The Boolean operators are located in the **LOGIC** submenu of the **2nd [TEST]** menu. To access a Boolean operator, press **2nd [TEST]**, scroll over to the **LOGIC** menu, and select the desired operator.



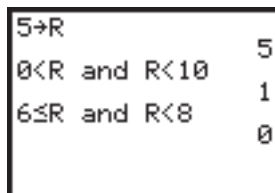
```

TEST LOGIC
1:and
2:or
3:xor
4:not(
  
```

---

### Example

Assign the value 5 to the variable  $R$ . To check if  $0 \leq R < 10$ , you must enter  $0 \leq R$  **and**  $R < 10$ . The calculator will return 1 for true. To check if  $6 \leq R < 8$ , you must enter  $6 \leq R$  **and**  $R < 8$ . The calculator will return 0 for false.



```

5→R
0<R and R<10  1
6≤R and R<8   0
  
```

## 9. REPRESENTING A FUNCTION BY AN EXPRESSION

### 9.1. STORING A FUNCTION IN THE Y= EDITOR

To represent a function by an expression, press the blue **Y=** key and enter the expression. Up to ten functions can be stored in the **Y=** Editor. **X** is the only valid independent variable in the **Y=** Editor when you are in **Func** mode.

```

Plot1 Plot2 Plot3
\Y1=X^4-2X^3
\Y2=e^(2X)
\Y3=sin(X-π)
\Y4=
\Y5=
\Y6=
\Y7=
    
```

### 9.2. SPECIFYING THE DOMAIN OF A FUNCTION

To specify the domain for a function entered in the **Y=** Editor, put parentheses around the entire expression. Then put the conditions on the domain in parentheses following the expression, separating the expression and the domain by the / symbol (use the ÷ key). Do not forget the / symbol. If there is more than one condition on the domain, for example,  $-3 < x \leq 2$ , you must separate it into two distinct conditions,  $-3 < x$  **and**  $x \leq 2$ .

```

Plot1 Plot2 Plot3
\Y1=(X^2-3)/(X>4)
\Y2=(2X-1)/( -3<X
and X≤2)
\Y3=
\Y4=
\Y5=
    
```

### 9.3. SELECTING A FUNCTION USING THE VARS KEY

To select a function from the list, press the **VAR**S key and select the **Y-VARS** submenu. Choose **1:Function** and the appropriate **Yn**. The **Yn** will appear at the current cursor location.

<pre> VAR S Y-VARS 1:Function... 2:Parametric... 3:Polar... 4:On/Off...                 </pre>	<pre> FUNCTION 1:Y1 2:Y2 3:Y3 4:Y4 5:Y5 6:Y6 7:Y7                 </pre>
--	--

This feature is used to evaluate a function at an input value, or a list of values. It is also used when combining functions.

### 9.4. EVALUATING A FUNCTION STORED IN THE Y= EDITOR

After a function has been defined in the **Y=** Editor, it can be evaluated at a specific input value by selecting the function using the **VAR**S key. The selected function will appear at the current cursor location. Enter the input value, enclosed in parentheses, after the function name.

To evaluate a function at several points in the homescreen, use the **2nd** **[ENTRY]** feature to avoid retyping.

If a function is evaluated at a point not in its domain, the calculator will return an error message.

$Y_2(-2.51)$	-6.020
$Y_2(0)$	-1.000
$Y_2(\sqrt{3})$	2.464

### 9.5. COMBINING FUNCTIONS

After a function has been defined in the **Y=** Editor, the function can be used in subsequent function definitions without retyping the expression. Select the name of the function using the **VARS** key.

Plot1	Plot2	Plot3
$Y_1 = X^2$		
$Y_2 = 2X$		
$Y_3 = -Y_1 + 4$		
$Y_4 = 2Y_2 + Y_2^2 + 6$		
$Y_5 = \text{abs}(Y_1)$		
$Y_6 =$		
$Y_7 =$		

This feature is particularly useful in constructing combinations of functions, including composition.

Plot1	Plot2	Plot3
$Y_1 = X^2$		
$Y_2 = 2X$		
$Y_3 = Y_1 + Y_2$		
$Y_4 = Y_2 / Y_1$		
$Y_5 = Y_1(Y_2)$		
$Y_6 =$		
$Y_7 =$		

### 9.6. SELECTING AND DE-SELECTING FUNCTIONS IN THE Y= EDITOR

A function defined in the **Y=** Editor may be selected by darkening the equal sign. This feature is used to determine which functions should be graphed or included in a table. When the equal sign = is not darkened in the **Y=** Editor, the calculator will not construct a graph or a table for that function. To select or de-select a function, move the cursor over the appropriate equal sign and press **ENTER**. If several equal signs are darkened, then multiple functions will be graphed on the same pair of axes.

#### Example

Note that the equal sign is darkened for **Y1** and **Y3** only.

Plot1	Plot2	Plot3
$Y_1 = X^2 + 1$		
$Y_2 = 2X - 41$		
$Y_3 = X * \sin(X)$		
$Y_4 = \ln(X)$		
$Y_5 =$		
$Y_6 =$		
$Y_7 =$		

## 10. REPRESENTING A PIECEWISE-DEFINED FUNCTION

To represent a piecewise defined function, enter each part of the function and its domain in a separate **Yn**. Each piece of the function and its corresponding domain is separated by the / symbol. Follow the process described in Section 9.2 to indicate the domain for each piece of the function.

### Example

Consider the function

$$f(x) = \begin{cases} -2x, & \text{if } x < -3 \\ x - 1, & \text{if } -3 \leq x < 2. \\ -3x + 10, & \text{if } x \geq 2 \end{cases}$$

You can store  $f$  in **Y1** through **Y3** as shown in the screen below.

The image shows a TI-83 calculator screen with the following text:

```

Plot1 Plot2 Plot3
\Y1[-2X)/(X<-3)
\Y2[(X-1)/(-3≤X
and X<2)
\Y3[-3X+10)/(X≥
2)
\Y4=
    
```

## 11. REPRESENTING A FUNCTION BY A GRAPH

Select the functions you wish to graph in the **Y=** Editor, (see Section 9.6). Press **GRAPH** to examine the graphs.

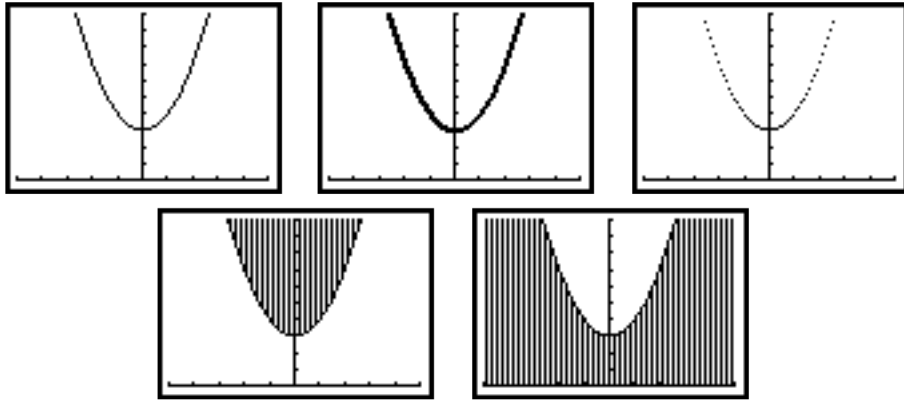
### 11.1. CHOOSING THE DISPLAY STYLES

When displaying several different graphs on the same screen, it may be helpful to use different kinds of graph styles to distinguish among the graphs. When the **Y=** Editor screen is displayed, the symbol to the left of each **Yn** indicates how the graph is to be drawn. When the cursor is over this symbol, pressing **ENTER** allows you to cycle through the choices.

There are seven different graphing styles: a *solid line*, a *thick solid line*, *shading above* or *below* the graph, a *moving cursor* tracing along the path of the graph, either *drawing* or *not drawing* the path, and a *small dot* at each plotted point. By default, the style is a solid line. You may wish to experiment with several different plotting styles.

### Example

The graphs below illustrate some of the different ways to plot  $y = x^2 + 3$ .



### 11.2. ADJUSTING THE VIEWING WINDOW

To adjust the viewing window, press the **WINDOW** key. Enter the minimum and maximum values for the  $x$  and  $y$  variables. When selecting the viewing window, be sure that you include the most “interesting” features of the graph, such as roots, turning points and inflection points.

The **Xscl** and **Yscl** variables set the distance between the tick marks on the  $x$  and  $y$  axes.

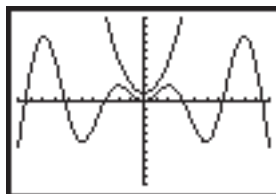
```

WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
Xres=1

```

### 11.3. VIEWING THE GRAPH

To view the graph of a function defined in the **Y=** Editor, press the **GRAPH** key. The calculator will graph the functions which are selected in the **Y=** Editor. Consider the screen in the example in Section 9.6. Note that the equal sign is darkened for **Y1** and **Y3** only.



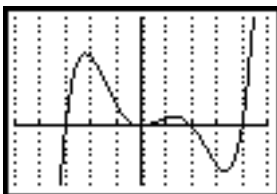
## 12. THE FORMAT MENU

The options under the **FORMAT** menu determine a graph's appearance on the display screen. Common features under the **FORMAT** menu are listed below with their default settings highlighted. The features used most often in *Workshop Calculus* are described below.

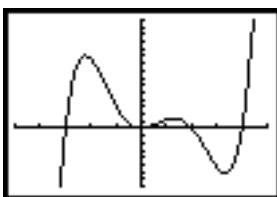


### 12.1. GRIDON/GRIDOFF

**GridOn** and **GridOff** determine whether grid points cover the viewing window in rows that correspond to the tick marks on each axis. This feature is useful when sketching graphs in your activity guide. An example of a graph in **GridOn** setting:

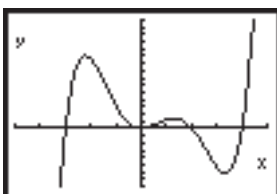


An example of a graph in **GridOff** setting:

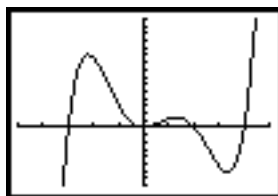


### 12.2. LABELON/LABELOFF

**LabelOn** and **LabelOff** determine whether to display labels for the axes,  $x$  and  $y$ . An example of a graph in **LabelOn** setting:



An example of a graph in **LabelOff** setting:

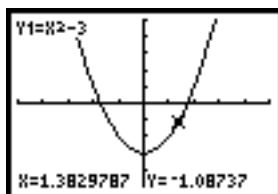


### 13. THE TRACE FEATURE

The **TRACE** feature allows you to move the cursor along the path of a function. The coordinates of the cursor location are displayed at the bottom of the screen. The **TRACE** feature is useful for distinguishing among several graphs on the screen and for determining points on the graphs when sketching them in your activity guide. This feature is also useful for analyzing the relationships between the graphs of combinations of functions.

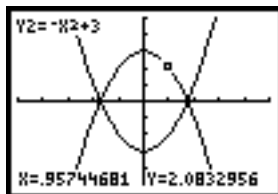
#### 13.1. TRACING A CURVE

To access the **TRACE** feature, press the **TRACE** key. A blinking cursor will appear on the graph of the function. Use the left and right arrow keys to move the cursor along the graph. The coordinates of the location of the cursor will appear at the bottom of the screen.



#### 13.2. MOVING AMONG CURVES

If several functions are graphed on the same pair of axes, use the up and down arrows to move the cursor among the graphs. The expression for the function being traced will appear in the upper left hand corner of the screen.

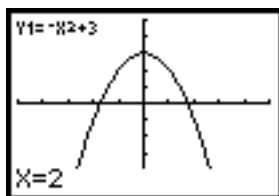


#### 13.3. EVALUATING A FUNCTION USING THE TRACE FEATURE

To evaluate a function at an input value and to move the cursor to the corresponding point on the graph, type the desired  $x$  value and press **ENTER**.

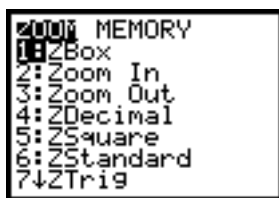


The associated  $y$  value will appear at the bottom of the screen. This procedure can be repeated as many times as necessary by entering more input values.



## 14. THE ZOOM FEATURE

The **ZOOM** menu provides alternate ways to adjust the viewing window. The features of this menu that you will use most frequently are described below. As always, for more information on these or the other **ZOOM** features, please consult the *TI-83 Guidebook*.

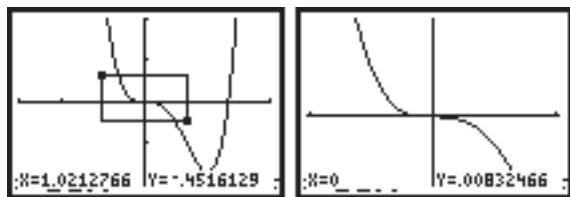


### 14.1. ZBOX

The **ZBox** feature draws a box to define the viewing window. To use this feature, press the **ZOOM** key and select **1:ZBox**. A blinking cursor will appear in the center of the screen.

Move the cursor to a point which you want to define as the corner of the box and press **ENTER**. Then use the arrow keys to draw the rest of the box.

When the box on the screen encloses the area to be enlarged, press **ENTER** to view this portion of the graph.



### 14.2. ZOOM IN AND ZOOM OUT

To zoom in or out around a particular point, press the **ZOOM** key and select **2:Zoom In** or **3:Zoom Out**. Move the cursor to the point which is to be the center of the new viewing window and press **ENTER** to view the new graph.

### 14.3. ZSTANDARD AND ZDECIMAL

The **ZStandard** feature adjusts the viewing window so that the  $x$  and  $y$  values range from  $[-10, 10]$  and replots the graph. To access this feature, press the **ZOOM** key and select **6:ZStandard**. The **ZDecimal** feature is similar to the **ZStandard** feature; it uses a smaller viewing window, but makes it easy to read decimal places of the coordinates when using the **TRACE** feature.

### 14.4. ZTRIG

The **ZTrig** feature adjusts the viewing window to preset values which are appropriate for trigonometric functions and replots the graph. To access this feature, press the **ZOOM** key and select **7:ZTrig**.

### 14.5. ZOOMSTAT

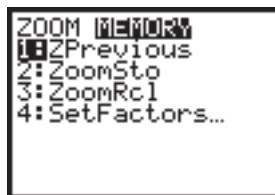
The **ZoomStat** feature is used when plotting discrete functions. This feature adjusts the viewing window so that all of the data points are displayed on the graph. To access this feature, press the **ZOOM** key and select **9:ZoomStat**.

### 14.6. ZOOMFIT

The **ZoomFit** feature adjusts the values for **Ymin** and **Ymax** so that the viewing window includes the minimum and maximum  $y$  values of the selected functions for the given values of **Xmin** and **Xmax**. The **ZFit** feature does not alter the values of **Xmin** and **Xmax**. To access this feature, press the **ZOOM** key and select **0:ZoomFit**.

### 14.7. ZPREVIOUS

The **ZPrevious** feature replots the graph using the window variables of the graph that was displayed before the execution of the last **ZOOM** command. To access this feature, press the **ZOOM** key and scroll over to the **MEMORY** submenu. Select the first option, **1:ZPrevious**.



## 15. REPRESENTING A FUNCTION BY A TABLE

A table of values can be created for a function defined in the **Y=** Editor. Begin by indicating the functions to be included in the table by darkening their equal signs (=) in the **Y=** Editor.

### 15.1. LISTING THE VALUES OF A FUNCTION IN A TABLE

Set the parameters of the table before viewing it. To do this, press **2nd** [**TBL SET**]. You will be prompted for an initial value in the table (**TblStart**), and

also for an increment by which the independent variable in the table is to increase ( $\Delta Tbl$ ).

Both **Indpnt** and **Depend** should be set to **Auto**. Note that by choosing **Auto** after **Indpnt** and **Depend**, the calculator will automatically calculate all values in the table. If you choose **Ask** after **Indpnt**, the calculator will let you select up to seven values manually. Always leave **Depend** set on **Auto**.



To view the table for the function, press **2nd** [TABLE]. Use the arrow keys to scroll up and down to see the values in the table. It is possible to scroll to values less than the value for **TblStart**. There is a column in the table for each function selected in the **Y=** Editor. However, you may need to scroll to the right to see them.

You may “zoom in” around a particular value in the table by adjusting the values for **TblStart** and  $\Delta Tbl$ . To do this, set **TblStart** to be the value and  $\Delta Tbl$  to be a very small increment. Use the up and down arrows to scroll through the table and to examine the behavior of the function near the value.

X	Y1
0	0
.1	.09983
.2	.19867
.3	.29552
.4	.38942
.5	.47943
.6	.56464

X=0

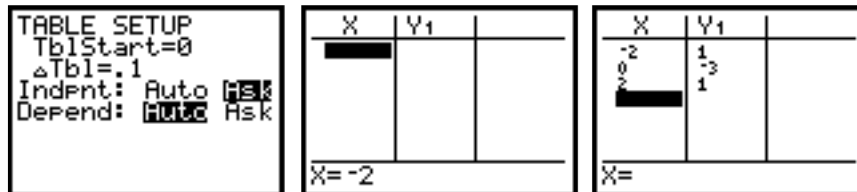
## 15.2. EVALUATING A FUNCTION USING A TABLE

To evaluate a function using a table, first define or select the function in the **Y=** Editor. Then, set the parameters for the table. Press **2nd** [TBLSET], choose **Ask** under **Indpnt** (leave **Depend** set to **Auto**). It does not matter what values are assigned to **TblStart** and  $\Delta Tbl$ . Press **2nd** [TABLE] to set up the table.

At the bottom of the screen, you will see **X=**. Enter a value for the independent variable—that is, the value at which you wish to evaluate the function. The value for the dependent variable will be calculated automatically and the result will appear in the table.

### Example

If the function  $y = x^2 - 3$  is stored in **Y1**, and you wish to evaluate it using a table at the values  $x = -2, 0, 2$ , you should see the screens below.

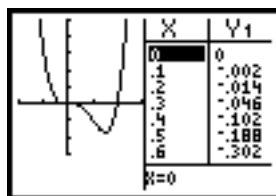


### 15.3. VIEWING A TABLE AND A GRAPH TOGETHER

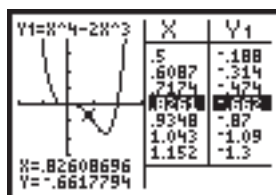
To view the table and graph of a function on the same screen, press the **MODE** key and highlight **G-T** (Graph–Table) on the bottom line of the screen. Press **ENTER**.



Set the parameters of the table by pressing **2nd** [TBLSET] and entering values for **TblStart** and **ΔTbl**. Establish the viewing window for the graph by pressing **WINDOW** and entering the appropriate values. To see the graph and table, press **GRAPH** or **2nd** [TABLE].



To examine the graph or table in more detail, that is, by zooming/tracing on the graph or by scrolling on the table, activate the appropriate part of the screen by pressing **GRAPH** or **[2nd] TABLE**. When the **TRACE** feature is used, as the cursor moves along the graph, the table automatically scrolls and highlights the current cursor values.



To revert back to a full screen, press the **MODE** key again and select **Full** on the bottom line.

## 16. REPRESENTING A DISCRETE FUNCTION

### 16.1. REPRESENTING A DISCRETE FUNCTION BY A TABLE

To represent a discrete function by a table, enter the set of ordered pairs into a list. For example, consider the following discrete function.

$$f = \{(0,3.11), (0.1,4.05), (0.3,6.89), (-0.5,11.78), (1.2,19.21), (-1.2,28.05)\}$$

To enter the data into a list, press the **STAT** key and choose **1:Edit...** Enter the first coordinates into **L1** and, in the same order, the associated second coordinates into **L2**.

L1	L2	L3	Z
0	3.11	-----	
.1	4.05		
.3	6.89		
-.5	11.78		
1.2	19.21		
-1.2	28.05		
-----			
L2(6) = 28.05			

An alternate way to enter data into a list is to enter the values into **L1** and **L2** on the home screen. To do this, enclose all the first coordinates in curly brackets, separated by commas, and store this new list in **L1**. Repeat the procedure with the associated second coordinates being stored into **L2**.

To view the table, press the **STAT** key and choose **1:Edit...**

```

{0,.1,.3,-.5,1.2
,-1.2}→L1
{0.1,3,-.5,1,...
{3.11,4.05,6.89,
11.78,19.21,28.0
5}→L2
{3.11 4.05 6.89...
    
```

### 16.2. REPRESENTING A DISCRETE FUNCTION BY A SCATTER PLOT

To view a discrete function represented by a scatter plot, press **2nd** [**STAT PLOT**] and select **1:Plot1** by pressing **ENTER**. When the next screen appears, select **On** and press **ENTER**. Be certain that the scatter plot is highlighted under **Type:** and that the appropriate lists (in this case, **L1** and **L2**) are entered after **Xlist:** and **Ylist:**. This information will be stored in **Plot1**. Press **2nd** [**QUIT**] to exit the **STAT PLOT** Editor.

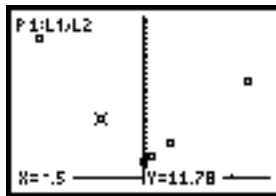
STAT PLOTS	
1:Plot1...Off	□
2:Plot2...Off	□
3:Plot3...Off	□
4:PlotsOff	□

1:Plot1	Plot2	Plot3
On	Off	Off
Type:	Scatter	Line
Xlist:	L1	
Ylist:	L2	
Mark:	+	.

To view the scatter plot, press the **ZOOM** key and select option 9, **9:ZoomStat**. Any function which is selected in the **Y=** Editor will also be graphed. If necessary, de-select the functions before graphing the data.



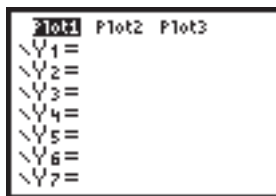
Use the **TRACE** feature to display the coordinates of the points on the scatter plot. Note that the cursor travels through the points in the order that they were entered into the lists.



To revert to the normal function graphing mode, go back into **STAT PLOT** and turn **Plot1 Off**.



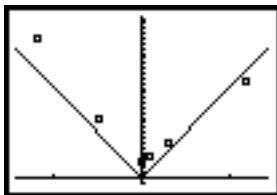
An alternate way to revert to the normal graphing mode is to go into the **Y=** Editor and move the cursor to the darkened **Plot1** at the top of the screen. Press **ENTER** so that the word **Plot1** is no longer highlighted. Once information has been stored in **Plot1** through the **STAT PLOT** Editor, **Plot1** may be turned on and off by selecting (or de-selecting) **Plot1** in the **Y=** Editor.



Additional scatter plots can be stored in **Plot2** and **Plot3**.

### 16.3. GRAPHING A CONTINUOUS AND A DISCRETE FUNCTION ON THE SAME AXES

To graph a continuous function and a discrete function on the same pair of axes, create a scatter plot in the way described in Section 16.2. Select the desired continuous functions in the **Y=** Editor. Press **GRAPH** to view the graphs.



### 17. FITTING A CURVE TO A DISCRETE FUNCTION

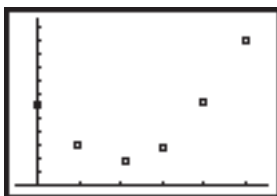
To fit a curve to a discrete function, follow the steps described below.

1. Enter the Data into a List and Examine the Scatter Plot
2. Fit a Function to the Data
3. Store the Fit Function in the **Y=** Editor
4. View the Data and the Fit Function on the Same Pair of Axes
5. Repeat if Necessary

#### 1. Enter the Data into a List and Examine the Scatter Plot

First enter the data into a list as described in Section 16. Examine the scatter plot to determine the type of curve to be used for the fit function. For example, consider the following data and the corresponding scatter plot:

$$\{(-0.04, 6.11), (0.96, 3.05), (2.10, 1.89), (3.02, 2.78), (3.97, 6.21), (5.03, 11.05)\}$$



#### 2. Fit a Function to the Data

After viewing the scatter plot for the data, decide what type of function you would like to use for the fit function. In this example, you may decide to fit a quadratic function to the data. Your other choices are:

Calculator Command	Equation Type
4:LinReg(ax + b)	linear functions of the form $y = ax + b$
5:QuadReg	quadratic functions of the form $y = ax^2 + bx + c$
6:CubicReg	cubic functions of the form $y = ax^3 + bx^2 + cx + d$
7:QuartReg	quartic functions of the form $y = ax^4 + bx^3 + cx^2 + dx + e$
8:LinReg(a + bx)	linear functions of the form $y = a + bx$
9:LnReg	logarithmic functions of the form $y = a + b \ln x$
0:ExpReg	exponential functions of the form $y = ab^x$
A:PwrReg	power functions of the form $y = ax^b$
B:Logistic	logistic functions of the form $y = \frac{c}{1 + ae^{-bx}}$
C:SinReg	sinusoidal functions of the form $y = a \sin(bx + c) + d$

To fit a quadratic function to the data, press the **STAT** key and scroll over to the **CALC** menu. In this example, select **5:QuadReg** and press **ENTER**.

```

EDIT  [2nd] [DEL] TESTS
4↑LinReg(ax+b)
5↓QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9:LnReg
0↓ExpReg

```

The word **QuadReg** will appear on the home screen. The default list names for each **STAT**[**CALC**] menu item are **L1** and **L2**. If the data are stored in **L1** and **L2**, simply press **ENTER**. If the data are stored in different lists, say **L3** and **L4**, you must specify the names of the lists by entering **QuadReg L3, L4**. The coefficients for the fit function will appear on the screen.

```

QuadReg
y=ax^2+bx+c
a=1.004335713
b=-4.007658424
c=5.93484731
R^2=.9966019161

```

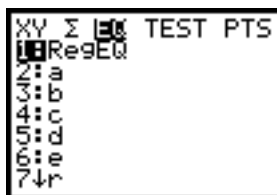
### 3. Store the Fit Function in the Y= Editor

To store the fit function in the **Y=** Editor without retyping the coefficients, go to the desired **Yn** and then press the **VARS** key.

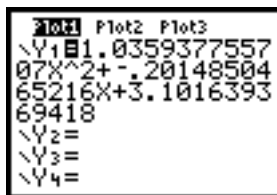




Choose **5:Statistics** and then scroll over to **EQ** and select **1:RegEQ**.



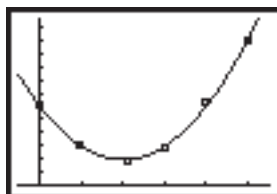
The fit function will be pasted into the **Y=** Editor at the appropriate **Yn**.



Check that the coefficients in **Yn** match the coefficients returned from the regression procedure. If a coefficient is very small, the **Y=** Editor may round it to zero. If this happens, enter the coefficient manually.

**4. View the Data and the Fit Function on the Same Pair of Axes**

To view the fit function along with the data, press the **GRAPH** key. Be sure that no other functions are selected in the **Y=** Editor.



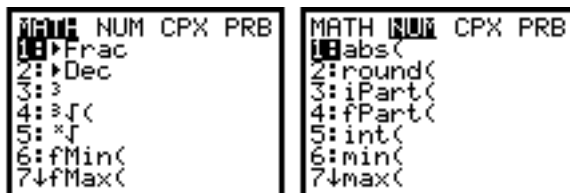
**5. Repeat if Necessary**

If the fit function does not appear to be a good fit for the data, repeat steps 2–4 with another possible fit function.

To revert to the normal function graphing mode, turn **Plot1 Off**.

## 18. THE MATH MENU

The **MATH** menu has four submenus. The features that you will use in *Workshop Calculus* are located in the **MATH** and **NUM** submenus and are described below.



### 18.1. REPRESENTING A VALUE AS A FRACTION

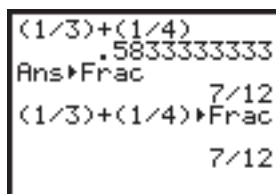
To convert a number to a fraction when the default representation is a decimal, enter the value to be converted and then press the **MATH** key and choose **1:->Frac**.

This feature can also be used to specify the form (fraction or decimal) for a result. Frequently, this feature will be used to convert the result of a previous calculation to a different form.

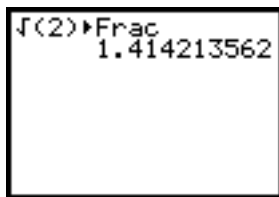
If your answer cannot be converted to a fraction, or if the resulting denominator is more than three digits, then the decimal value will be returned.

#### Example

Evaluate the expression  $\frac{1}{3} + \frac{1}{4}$ . By default, the result will be displayed as the decimal .5833333333. To convert the result to a fraction, press **2nd [ANS]** and then press the **MATH** key and select **1:->Frac**. Press **ENTER**. The calculator will return the value  $\frac{7}{12}$ . You may also use the **1:->Frac** option before evaluating the expression.



The number  $\sqrt{2}$  is an irrational number, therefore, by definition, it cannot be converted to a fraction. Thus, using the **1:->Frac** option on this value will return a decimal.

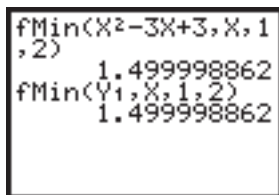


## 18.2. APPROXIMATING THE MINIMUM OR MAXIMUM VALUE OF A FUNCTION

To approximate the minimum or maximum value of a function over a particular interval, press the **MATH** key and choose **6:fMin(** or **7:fMax(**. Then enter the function, the independent variable, the lower bound and the upper bound of the interval, all separated by commas. The function can be represented by an expression or by name. Press **ENTER**.

### Example

To approximate the minimum of  $x^2 - 3x + 3$  over the interval  $[1,2]$ , press the **MATH** key and select **6:fMin(**. Enter **fMin( $x^2 - 3x + 3, x, 1, 2$ )**. If the function is stored in **Y1**, enter **fMin(Y1,  $x, 1, 2$ )** by using the **VAR**S key to access **Y1**.



## 18.3. EVALUATING A NUMERICAL DERIVATIVE

To find the numerical derivative of a function at a point, press the **MATH** key and choose **8:nDeriv(**. Enter your function, the variable of differentiation, and the value at which you would like to find the derivative, all separated by commas. The function can be represented by an expression or by name. Press **ENTER**.

### Example

To find the numerical derivative of  $x^2 - 2x$  with respect to  $x$  at  $x = 3$ , press the **MATH** key, choose **8:nDeriv(** and enter **nDeriv( $x^2 - 2x, x, 3$ )**. If the function is stored in **Y1**, enter **nDeriv(Y1,  $x, 3$ )**.

```

nDeriv(X^2-2X,X,3
)
nDeriv(Y1,X,3)
4
4

```

#### 18.4. EVALUATING A NUMERICAL INTEGRAL

To approximate a definite integral, press the **MATH** key and choose **9:fnInt()**. Enter the function, the variable of integration, and the lower and upper limits of integration, all separated by commas. The function can be represented by an expression or by name. Press **ENTER**.

---

##### Example

To approximate the definite integral of the function  $y = x^2$  over the interval  $[0,4]$ , press the **MATH** key, choose **9:fnInt()**, enter **fnInt** ( $x^2$ ,  $x$ , 0,4) and press **ENTER**. If the function is stored in **Y1**, enter **fnInt** (**Y1**,  $x$ , 0,4). The calculator will return 21.33333333 or  $64/3$ , if the result is represented by a fraction.

```

fnInt(X^2,X,0,4)
21.33333333
fnInt(X^2,X,0,4)▶
Frac
64/3
fnInt(Y1,X,0,4)
21.33333333

```

#### 18.5. THE ABSOLUTE VALUE FUNCTION

To access the absolute value function, press the **MATH** key, scroll over to the **NUM** submenu, choose **1:abs()** and enter the desired value or expression. Press **ENTER**. The absolute value function may also be used to define functions in the **Y=** Editor.

---

##### Example

To find the absolute value of  $-56$ , press the **MATH** key, scroll over to the **NUM** submenu, choose **1:abs()** and enter **abs**( $-56$ ). Press **ENTER**. The calculator will return 56.

If the variable  $N$  is assigned the value 10, to calculate the absolute value of  $2N - 33$ , select **1:abs()** from the **NUM** submenu and enter **abs**( $2N - 33$ ). Press **ENTER**. The calculator will return 13.

```
abs(-56)      56
10→N         10
abs(2N-33)    13
■
```

## 18.6. THE GREATEST INTEGER FUNCTION

To return the greatest integer less than or equal to a value, access the greatest integer function by pressing the **MATH** key, scrolling over to the **NUM** submenu, choosing **5:int(** and entering the desired value. Press **ENTER**.

### Example

To find the greatest integer less than or equal to 5.8 and the greatest integer less than or equal to  $-5.8$ , press the **MATH** key, scroll over to the **NUM** submenu, choose **5:int(** and enter **int(5.8)** and **int(-5.8)**.

```
int(5.8)      5
int(-5.8)     -6
```

When plotting the greatest integer function, it is best to set the display style to small dots (see Section 11.1).

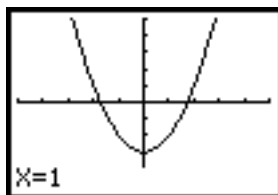
## 19. THE CALC MENU

There are several features under the **2nd [CALC]** (calculate) menu which can be used to analyze a function by examining its graph. The **2nd [CALC]** menu provides a graphical interpretation of many of the features found in the **MATH** menu. To use each of the following features, first select the functions you wish to consider in the **Y=** Editor.

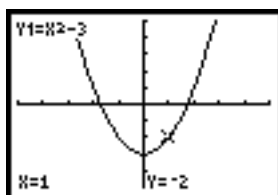
```
2nd [CALC]
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

### 19.1. EVALUATING A FUNCTION USING THE CALC MENU

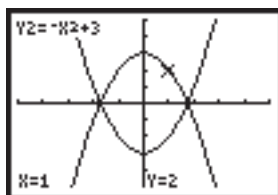
To evaluate the functions selected in the **Y=** Editor at a point, press **2nd** [**CALC**] and select **1:value**. At the bottom of the screen, you will see **X=**. Enter the value for the independent variable—that is, the input value where the functions are to be evaluated.



The calculator will compute the corresponding output value and the result will appear at the bottom of the screen.



If more than one function is selected, use the up and down arrows to move between the various graphs. The expression for the function will be shown at the top of the screen and the coordinates of the point will be shown at the bottom of the screen.



This procedure can be repeated as many times as necessary by entering additional input values.

### 19.2. FINDING A ZERO OF A FUNCTION

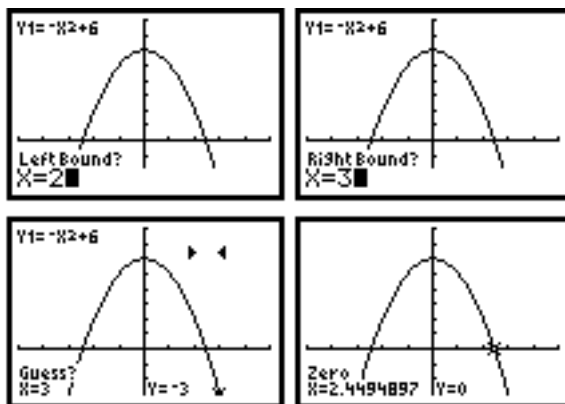
To approximate the zero of a function, press **2nd** [**CALC**] and select **2:zero**. Use the up and down arrows to select the appropriate function if there is more than one function on the screen.

You will be asked to select a region for the calculator to examine for a zero. The words **Left Bound?** will appear on the screen. Move the cursor to a point just left of the desired root and press **ENTER**. Repeat the process when the words **Right Bound?** appear, this time moving the cursor to a point just to the right of the root in question. You may also enter the values for the left and right bounds of the interval containing the root by typing the values

when prompted. Press **ENTER** again when the word **Guess?** appears. The coordinates of the zero of the function will appear at the bottom of the screen.

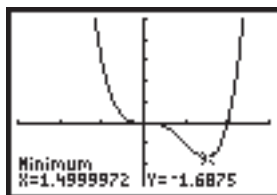
### Example

Consider the function  $y = -x^2 + 6$ . Based on the graph, it appears that there is a root between  $x = 2$  and  $x = 3$ . Using these values for the left and right bounds of the interval, you should see the screens below.



### 19.3. APPROXIMATING THE MINIMUM OR MAXIMUM VALUE OF A FUNCTION

To approximate the minimum or maximum value of a function over a specific interval, press **2nd [CALC]** and select the desired option (**3:minimum** or **4:maximum**). Use the up and down arrows to select the appropriate function. As when finding a zero of a function, the calculator will prompt you to select an interval for it to examine for a minimum or maximum. Follow the same procedure described in Section 19.2, and the approximate coordinates of the minimum or maximum of the function will appear at the bottom of the screen.

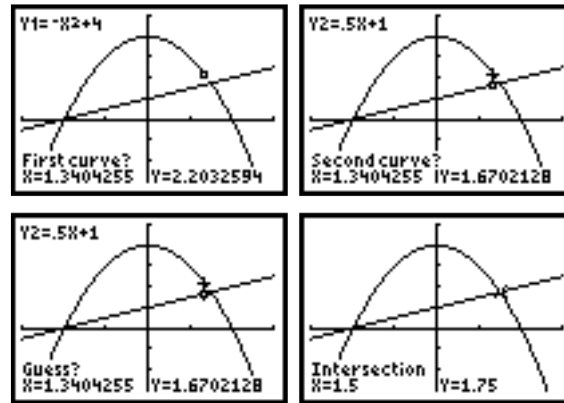


### 19.4. APPROXIMATING AN INTERSECTION OF TWO FUNCTIONS

To approximate an intersection of two functions, press **2nd [CALC]** and select **5:intersect**. When prompted, use the up and down arrows to select the first and second curves near a point of intersection. The approximate coordinates of the intersection of the two functions will appear at the bottom of the screen.

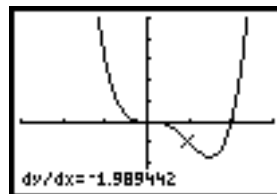
### Example

Consider the functions  $y = -x^2 + 4$  and  $y = \frac{1}{2}x + 1$ . To find their intersection in the first quadrant, you should see the screens below.



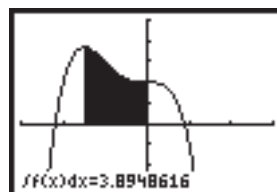
### 19.5. EVALUATING A NUMERICAL DERIVATIVE

To find the numerical derivative of a function at an input value, press **2nd** [CALC] and select **6:dy/dx**. Use the up and down arrows to select the appropriate function. Move the cursor to the point whose numerical derivative you wish to find, or enter the value directly. Press **ENTER**. The derivative of the function at that point will appear at the bottom of the screen.



### 19.6. EVALUATING A NUMERICAL INTEGRAL

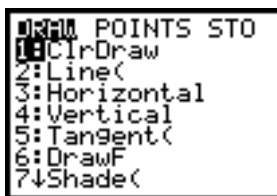
To approximate the value of a definite integral, press **2nd** [CALC] and select **7:∫f(x)dx**. Move the cursor and press **ENTER** to indicate the lower and upper limits of integration, or enter these values directly. The region will be shaded and the value of the definite integral will appear at the bottom of the screen.





## 20. THE DRAW MENU

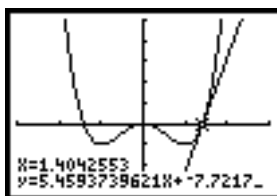
To access the **DRAW** menu, press **2nd [DRAW]**. Some features of the **DRAW** menu that you will use in *Workshop Calculus* are described below.



### 20.1. DRAWING A TANGENT LINE ON A GRAPH

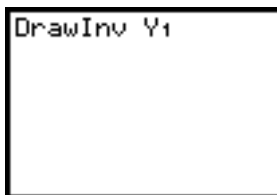
To draw a tangent line on a graph, first plot the function. Press **2nd [DRAW]** and choose **5:Tangent(**. The graph will appear. Move the cursor to the point on the graph where the tangent line is to be drawn or enter the  $x$  value directly. Press **ENTER**. The equation of the tangent line and the  $x$  coordinate will appear at the bottom of the screen.

Repeat the process to draw additional tangent lines.

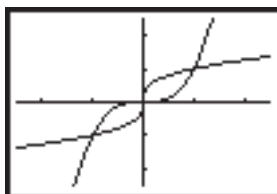


### 20.2. DRAWING THE REFLECTION THROUGH THE LINE $y = x$

To draw the reflection of a function through the line  $y = x$ , press **2nd [DRAW]** and select **8:DrawInv**. When the command appears on the home screen, enter the function by expression or by name. To set the viewing window to include the images of the function and its reflection, set **Xmin** equal to **Ymin** and **Xmax** equal to **Ymax**.

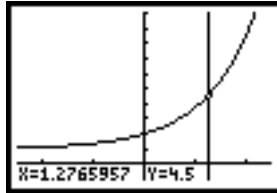


Press **ENTER** to see the graphs.

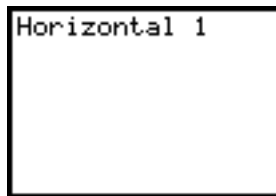


### 20.3. DRAWING A VERTICAL AND A HORIZONTAL LINE

To draw a horizontal or a vertical line when the graph is displayed, press **2nd [DRAW]** and choose **3:Horizontal** or **4:Vertical**. A line is displayed that moves along with the cursor. When the line is in the desired location, press **ENTER**. To draw additional lines, move the cursor and press **ENTER**.



To draw a horizontal or a vertical line from the home screen, press **2nd [DRAW]** and select **3:Horizontal** or **4:Vertical**. Then type the desired  $y$  value for a horizontal line or  $x$  value for a vertical line and press **ENTER**.



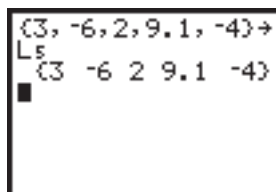
### 20.4. CLEARING A DRAWING

To clear a drawing created through the **2nd [DRAW]** menu, press **2nd [DRAW]** and select **1:ClrDraw**.

## 21. CONSTRUCTING LISTS AND SEQUENCES

### 21.1. CREATING A LIST IN THE HOME SCREEN

To enter data into a list on the home screen, enclose the data in curly brackets, separating the items by commas, and store the data into a list. The TI-83 has six predefined list names, **L1** through **L6**. These lists may be accessed as the second functions of the numbers 1 through 6.





```
seq(5-2^(-I),I,1
,5)→LX
(4.5 4.75 4.875...
```

A sequence can also be defined using values previously assigned to variables. For instance, you can assign values to the variables  $a = 7$ ,  $b = 2$  and  $n = 4$ .

```
7→A          7
2→B          2
4→N          4
█
```

And then store the sequence  $a - b^{-i}$  where  $i = 1, 2, \dots, n$  in the list **LX**.

```
seq(A-B^(-I),I,1
,N)→LX
(6.5 6.75 6.875...
█
```

### 21.3. MODIFYING A LIST OR A SEQUENCE

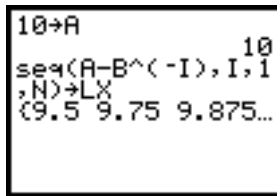
To avoid retyping the code, when modifying a previously defined list or sequence, use the **2nd [ENTRY]** feature to edit the previously executed code.

---

#### Example

To redefine the sequence **LX** defined in the last example so that it approaches the value 10 instead of the value 7, press **2nd [ENTRY]** as many times as necessary until the **7->A** statement appears on the screen. Edit the code, replacing the 7 with a 10 and press **ENTER**. You will need to use the **2nd [INS]** feature to insert the 0 after the 1.

Because the definition of **LX** depends on the value of **A**, you must reenter the definition of **LX** so that the value of **A** is updated. Use the **2nd [ENTRY]** feature again until the definition of **LX** appears on the screen. Press **ENTER** to re-execute the code.

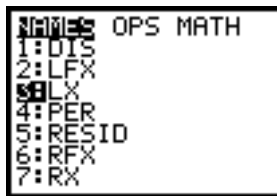


## 22. DISPLAYING A LIST

### 22.1. SELECTING THE NAME OF A LIST

A list with a predefined name, **L1** through **L6**, can be accessed by pressing **2nd** and the appropriate list name, for example **2nd [L2]**.

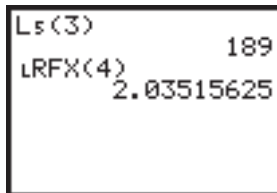
When a list with a user-defined name is referred to in a command or a calculation, the symbol **L** must appear before the list name to indicate that the variable is a list. To access a list with a user-defined name, press **2nd [LIST]** and select the appropriate list.



This feature will be used when accessing a particular term in a list, defining a function at the values in a list, finding the sum of a list, or entering a list into a table.

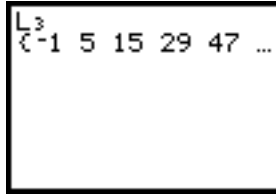
### 22.2. ACCESSING A TERM IN A LIST

To access a term in the list or a sequence, select the name of the list and enclose the number of the desired term in parentheses. Press **ENTER**.

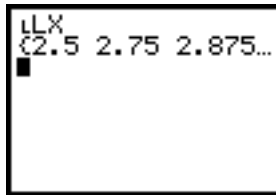


### 22.3. DISPLAYING A LIST ON THE HOME SCREEN

To display a list with a predefined name (**L1** through **L6**) on the home screen, enter the name of the list, for example **2nd [L3]**, and press **ENTER**. If the entire list does not fit on the screen, as indicated by three dots, use the left and right arrow keys to scroll through the items in the list.



To display a list with a user-defined name, recall the name of the list as described in section 22.1—that is, press **2nd** [**List**] and choose the appropriate list. Press **ENTER**.

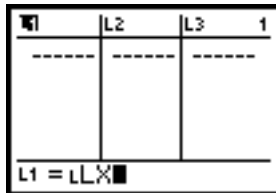


#### 22.4. STORING A LIST IN A TABLE

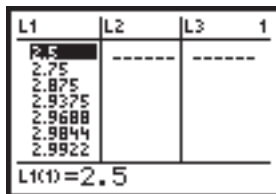
To store a list or a sequence in a table, press the **STAT** key and choose **1:Edit...**



Move the cursor to the top of the column where the list is to be stored. Enter the name of the list.



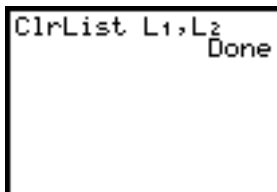
Press **ENTER** to store the list.



**Caution:** The TI-83 displays at most five digits for each entry in a table. If an entry contains more digits, the calculator rounds the entry, which may be misleading. To view a more precise value, highlight the entry using the up and down arrow keys and consider the value displayed at the bottom of the screen.

## 22.5. CLEARING A LIST

To clear a list, press the **STAT** key and select **4:ClrList**. Indicate the lists to be cleared by selecting the names of the appropriate lists, separated by commas. Press **ENTER**.



To clear all of the lists in the calculator's memory, press **2nd [MEM]** and choose **4:ClrAllLists**. Press **ENTER**.



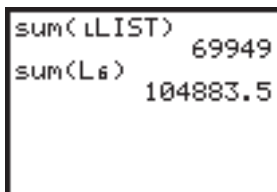
## 23. CALCULATING A SUM

### 23.1. SUMMING THE ITEMS IN A LIST

To find the sum of the items in a list, press **2nd [LIST]**, scroll over to the **MATH** menu and choose **5:sum(**.



Select the appropriate list name and press **ENTER**.



### 23.2. SUMMING THE ITEMS IN A SEQUENCE

To find the sum of the terms in a sequence that has not been stored in a list, use the format **sum(seq(expression, variable, begin, end))**.

---

**Example**

To calculate  $\sum_{i=4}^{10} 3i^2$ , enter the following code.

```
sum(seq(3I^2,I,4,
10))
1113
```

**23.3. CALCULATING A PARTIAL SUM**

To find the sum of a portion of a list—that is, a sum of the  $j^{\text{th}}$  through  $k^{\text{th}}$  items in a list, use the format **sum**(list name,  $j$ ,  $k$ ).

---

**Example**

To find the sum of the 3<sup>rd</sup> through 7<sup>th</sup> items in **L5**, enter the following code.

```
sum(L5,3,7)
```

**24. EXAMINING THE LIMITING BEHAVIOR OF A FUNCTION**

You can use the list feature to examine the limiting behavior of a function near a point.

Suppose  $f$  is a function defined near  $x = a$ . To examine the limiting behavior of  $f$  as  $x$  approaches  $a$  from both sides, create four lists—two containing sequences of input values (one that approaches  $a$  from the left and one that approaches  $a$  from the right) and two containing the associated lists of output values for  $f$ .

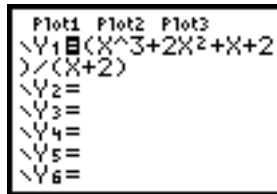
To illustrate the process, consider the function  $f(x) = \frac{x^3 + 2x^2 + x + 2}{x + 2}$  as  $x \rightarrow -2$  and use the following procedure.

1. Define the Function
2. Assign a Value to A
3. Assign Values to B and N
4. Define Two Sequences of Input Values
5. Construct the Associated Lists of Output Values
6. Store the Lists in an Input–Output Table



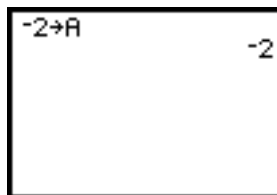
**1. Define the Function**

Define the function  $f$  using the Y= Editor.



**2. Assign a Value to A**

Store the value of  $a$  in A. In this case,  $a = -2$ .



**3. Assign Values to B and N**

Define two sequences of values, one approaching  $x = a$  from the left and one approaching  $x = a$  from the right. Store the left sequence in a list called **LX**, for “left  $x$  values” and store the right sequence in a list called **RX**, for “right  $x$  values.”

The general form for constructing a sequence of  $n$  values using powers of  $b > 1$  that approaches  $x = a$  from the left is  $a - b^{-i}$  where  $i = 1, 2, \dots, n$ . Similarly, the general form for constructing a sequence that approaches  $x = a$  from the right is  $a + b^{-i}$  where  $i = 1, 2, \dots, n$ . On the TI-83, you can use the **seq()** command to do this.

To prepare to define the left and right sequences of values, **LX** and **RX**, store the value of the base  $b$  in the variable  $B$  and the desired number of items in the sequence in the variable  $N$ . In this example, if  $b = 2$  and  $n = 6$ , then you would enter:



**4. Define Two Sequences of Input Values**

Type in the expression defining the left sequence and store it in **LX**. Similarly, type in the expression for the right sequence, this time storing it in **RX**. To save keystrokes, use the **2nd [ENTRY]** feature and edit the code for **LX**.

<pre>seq(A-B^(-I),I,1 ,N)→LX (-2.5 -2.25 -2....</pre>	<pre>seq(A+B^(-I),I,1 ,N)→RX (-1.5 -1.75 -1....</pre>
---	---

To construct a new sequence, repeatedly use the **2nd** [ENTRY] feature to edit the values of  $A$ ,  $B$  and  $N$  and to re-execute the code that defines **LX** and **RX**.

### 5. Construct the Associated Lists of Output Values

Construct the associated lists of output values of the function  $f$ . That is, for each input value  $x$  in **LX**, calculate  $f(x)$ . Store these values in a list called **LFX**. Similarly, for each  $x$  in **RX**, calculate  $f(x)$  and store these values in a list called **RFX**.

To evaluate a function stored in the **Y=** Editor at the values in **LX**, follow the usual process for evaluating a function at an input value. For example, consider the function  $f(x) = \frac{x^3 + 2x^2 + x + 2}{x + 2}$  and assume that it is stored in **Y1**. For the input value, you will select the list **LX** by using the process described in Section 22.1. Store this new list in **LFX**.

```
Y1(LX)→LFX
(7.25 6.0625 5....
```

To store the output values associated with the right sequence, repeat the process, this time evaluating the function at the values in **RX** and storing the results in **RFX**.

```
Y1(RX)→RFX
(3.25 4.0625 4....
```

### 6. Store the Lists in an Input–Output Table

Store the information in an input–output table. Store **LX**, **LFX**, **RX**, and **RFX** in a table in columns **L1**, **L2**, **L3**, and **L4**, respectively (see Section 22.4).

L1	L2	L3	1
-2.5	7.25	-1.5	
-2.25	6.0625	-1.75	
-2.125	5.5156	-1.875	
-2.063	5.2539	-1.938	
-2.031	5.126	-1.969	
-2.016	5.0627	-1.984	
-----			
L1(1)=-2.5			

An alternate and less complicated way to create an input–output table is to use the table feature on your calculator (see Section 15.2). With this approach, you enter the values in the left and right input sequences directly, instead of having the calculator generate the values. The procedure is as follows:

1. Define the function.
2. In 2nd[TBLSET], set INDPNT to Ask.
3. Press 2nd[TABLE] to set up the table.
4. Enter a sequence of input values that approaches  $x = a$  from the left (or from the right). Note that a table holds at most seven values.

## 25. APPROXIMATING A RATE OF CHANGE

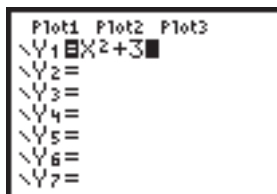
You can approximate a rate of change of a function on the TI-83 using the list feature. Create four separate lists, one for a sequence of  $x$  values approaching a value  $a$  from the left, one for the corresponding left difference quotients determined by  $P(x, f(x))$  and  $Q(a, f(a))$ , one for a sequence of  $x$  values approaching a value  $a$  from the right, and one for the corresponding right difference quotients determined by  $P(x, f(x))$  and  $Q(a, f(a))$ .

Suppose  $f$  is a continuous function defined near  $x = a$ . To examine the limiting behavior of the slopes of the secant lines as  $x$  approaches  $a$  from both sides, complete the following steps. To illustrate the process, consider  $f(x) = x^2 + 3$  and  $a = 1$  and use the following procedure.

1. Define the Function
2. Assign a Value to A
3. Assign Values to B and N
4. Define Two Sequences of Input Values
5. Calculate the Associated Difference Quotients
6. Store the Lists in a Table

### 1. Define the Function

Define the function in the Y= Editor.



**2. Assign a Value to A**

Store the value of  $a$  in **A**. In this case,  $a = 1$ .



**3. Assign Values to B and N**

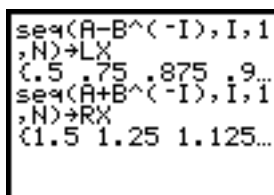
Store the value of  $b$  in **B** and  $n$  in **N**, where **N** represents the number of terms in each sequence and **B** is the base of the sequence. In this case, suppose  $b = 2$  and  $n = 5$ .



**4. Define Two Sequences of Input Values**

Construct two sequences of values, one, called **LX**, which approaches  $a$  from the left, and another, **RX**, which approaches  $a$  from the right.

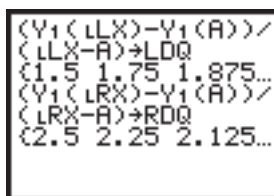
Use the **seq**( command and follow the procedure outlined in section 21.2 to construct the left and right sequences.



**5. Calculate the Associated Difference Quotients**

Calculate the slopes of the secant lines determined by  $P(a, f(a))$  and  $Q(x, f(x))$ . That is, for each  $x$  in **LX** and **RX**, find  $\frac{f(x) - f(a)}{x - a}$ . Store these

values in **LDQ**, for “left difference quotient,” and **RDQ**, for “right difference quotient.”



**6. Store the Lists in a Table**

Store the lists **LX**, **RX**, **LDQ** and **RDQ** in a table, using the process described in Section 22.4.

L1	L2	L3	1
1.5	1.5	1.5	
.75	1.75	1.25	
.875	1.875	1.125	
.9375	1.9375	1.0625	
.96875	1.9688	1.0313	
-----			
L1(1) =			

**26. CONSTRUCTING A RIEMANN SUM**

**26.1. CALCULATING A RIEMANN SUM**

You can calculate a Riemann sum for a function on the TI-83 using the list feature. First, divide the region into  $n$  subintervals and construct a sequence of the right endpoints of each subinterval. For each subinterval, multiply the height of the function at the right endpoint by the width of the subinterval and add the areas to find the Riemann Sum.

Suppose  $f$  is a continuous function defined over the interval  $[a,b]$ .

Complete the following steps to calculate the Riemann sum  $\sum_{i=1}^n f(x_i)\Delta x$ , where  $n$  is the number of subintervals,  $\Delta x = \frac{b-a}{n}$  is the width of each subinterval, and  $x_i = a + i\Delta x$  is the right endpoint of the  $i$ th subinterval. To illustrate the process, consider  $f(x) = x^2 - 6x + 10$  where  $2 \leq x \leq 5$  and  $n = 6$  and use the following procedure.

1. Define the Function
2. Assing Values to A, B, and N
3. Define D
4. Define the Sequence of Right Endpoints
5. Calculate the Riemann Sum

**1. Define the Function**

Define the function in the **Y=** Editor.

Plot1	Plot2	Plot3
Y1	X <sup>2</sup> -6X+10	
Y2		
Y3		
Y4		
Y5		
Y6		
Y7		

**2. Assign Values to A, B, and N**

Store the values of  $a$ ,  $b$ , and  $n$  in **A**, **B** and **N**. In this case,  $a = 2$ ,  $b = 5$  and  $n = 6$ .

```
2→A      2
5→B      5
6→N      6
```

**3. Define D**

Store the value of  $\Delta x$  in **D**, recalling that  $\Delta x = \frac{b-a}{n}$ .

```
(B-A)/N→D      .5
■
```

**4. Define the Sequence of Right Endpoints**

Define the sequence of the right endpoints of the  $n$  subintervals,  $\{a + i\Delta x, i = 1, 2, \dots, n\}$ . Store the values in a sequence called **X**. Note that **X** contains  $n$  values where the first item in **X** is  $x_1$ , the second is  $x_2$ , and the last is  $x_n$ .

```
seq(A+I*D, I, 1, N)
→X
(2.5 3 3.5 4 4.5)
```

**5. Calculate the Riemann Sum**

Calculate the Riemann sum  $\sum_{i=1}^n f(x_i)\Delta x$  by evaluating the function stored in the **Y=** Editor at each point in the sequence **X**, multiplying the result by  $\Delta x$ , which is stored in **D**, and summing the results (see Section 23).

```
sum(Y1(LX)*D)
6.875
```

To calculate the Riemann sum for a different value of  $n$ , assign a new value to  $N$ . Then use the **2nd [ENTRY]** feature to re-execute the code which depends on  $n$ : the definition of  $\Delta x$ ,  $(B-A)/N \rightarrow D$ ; the code for the sequence of right endpoints,  $\text{seq}(A + I * D, I, 1, N) \rightarrow X$ ; and the code for calculating the sum,  $\text{sum}(Y1(LX) * D)$ , in this order.

## 26.2. CONSTRUCTING A TABLE

To construct a table displaying the values of the right endpoints of each subinterval, the associated values of  $f$  at the right endpoints,  $f(x_i)$ , and the areas of the associated rectangles,  $f(x_i)\Delta x$ , follow steps 1–4 as described in the previous section, Section 26.1, and the three additional steps described below.

1. Define the Function
2. Assign Values to A, B, and N
3. Define D
4. Define the Sequence of Right Endpoints
5. Evaluate the Function at the Right Endpoints
6. Calculate the Areas of the Associated Rectangles
7. Store the Lists in a Table

### 5. Evaluate the Function at the Right Endpoints

Create a new list of values by evaluating the function at each point in the sequence  $X$ —that is, at each right endpoint of the subintervals. Call this list  $FX$ .

```
Y1(LX)→FX
{1.25 1 1.25 2 ...
█
```

### 6. Calculate the Areas of the Associated Rectangles

Calculate the areas of the associated rectangles,  $f(x_i)\Delta x$ , by multiplying the height of each rectangle,  $f(x_i)$ , by the width of the subinterval,  $\Delta x$ . Store the results in a list called  $FXD$ .

```
LFX*D→FXD
{.625 .5 .625 1...
```

7. Store the Lists in a Table

Store the lists containing the values of the right endpoints, **X**, the corresponding value of the function, **FX**, and the area of the associated rectangle, **FXD**, in a table.

<b>X</b>	<b>L2</b>	<b>L3</b>	<b>1</b>
2.5	1.25	.625	
3	1	.5	
3.5	1.25	.625	
4	2	1	
4.5	3.25	1.625	
5	5	2.5	

L1 = {2.5, 3, 3.5, 4...

26.3. SKETCHING THE RECTANGLES ASSOCIATED WITH A RIEMANN SUM

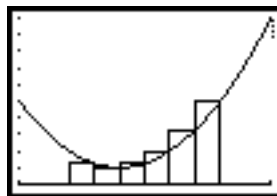
To sketch the rectangles whose heights are determined by the value of the function at the right endpoint of each subinterval, load the program **RECT** (see Section 31.3) into your calculator or transfer it from another calculator using the **LINK** feature.

Store the function in **Y1** and adjust the viewing window for the graph. To execute the program, press **PRGM** and select **RECT**. You will be asked to enter values for *a*, *b*, and *n*.

```

PRGMRECT
ENTER A
?2
ENTER B
?5
ENTER N
?6
    
```

The calculator will graph the function and the associated rectangles.



Press **ENTER** to see the associated Riemann sum.

```

RIGHT SUM
6.875
Done
    
```



To clear the rectangles from the graph, press **2nd [DRAW]** and choose **1:ClrDraw**.

## 27. USING THE TRAPEZOIDAL RULE

### 27.1. CALCULATING A TRAPEZOIDAL APPROXIMATION

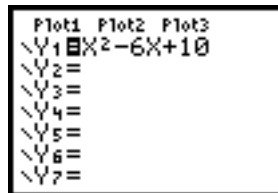
Suppose  $f$  is a continuous function over the closed interval  $[a, b]$ . Complete the steps listed below to approximate  $\int_a^b f(x) dx$  using the Trapezoidal Rule,

$\left( \frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(b) \right) \Delta x$ , where  $n$  is the number of subintervals,  $\Delta x = \frac{b-a}{n}$  and  $x_i = a + i\Delta x$ . To illustrate the process, consider  $f(x) = x^2 - 6x + 10$  where  $2 \leq x \leq 5$  and  $n = 6$  and use the following procedure.

1. Define the Function
2. Assign Values to A, B, and N
3. Define D
4. Define the Sequence of Right Endpoints
5. Calculate the Sum

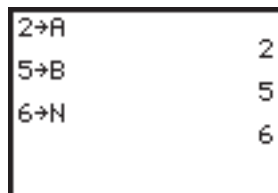
#### 1. Define the Function

Define the function in the **Y= Editor**.



#### 2. Assign Values to A, B, and N

Store the values of  $a$ ,  $b$ , and  $n$  in **A**, **B**, and **N**. In this case,  $a = 2$ ,  $b = 5$  and  $n = 6$ .



**3. Define D**

Store the value of  $\Delta x$  in **D**, recalling that  $\Delta x = \frac{b-a}{n}$ .

```
(B-A)/N→D
.5
```

**4. Define the Sequence of Right Endpoints**

Define a sequence containing the right endpoints of the first  $n - 1$  subintervals,  $\{a + i\Delta x, i = 1, 2, \dots, n - 1\}$ . Store the values in a sequence called **X**. Note that **X** contains  $n - 1$  values where the first item in **X** is  $x_1$ , the second is  $x_2$ , and the last is  $x_{n-1}$ .

```
seq(A+I*D,I,1,N-
1)→X
{2.5 3 3.5 4 4...
```

**5. Calculate the Sum**

Calculate the desired sum  $\left(\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(b)\right) \Delta x$ .

```
(.5Y1(A)+sum(Y1(
LX))+.5Y1(B))*D
6.125
```

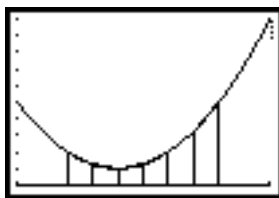
**27.2. SKETCHING THE TRAPEZOIDS ASSOCIATED WITH THE TRAPEZOIDAL RULE**

To sketch the curve along with the trapezoids associated with the Trapezoidal Rule, enter the program **TRAP** (see Section 31.4) into your calculator or transfer it from another calculator using the **LINK** feature.

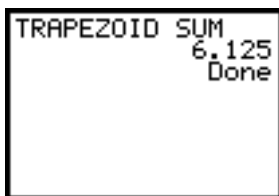
Store the function in **Y1** and adjust the viewing window. To execute the program, press **PRGM** and select **TRAP**. You will be asked to enter values for  $a$ ,  $b$ , and  $n$ .

```
PrgmTRAP
ENTER A
??
ENTER B
??
ENTER N
??
```

The calculator will graph the function and the associated trapezoids.



Press **ENTER** to see the associated trapezoidal sum.



To clear the trapezoids from the graph, press **2nd [DRAW]** and choose **1:ClrDraw**.

## 28. DEFINING AN ACCUMULATION FUNCTION

Suppose  $A$  is the accumulation function generated by  $f$ , where

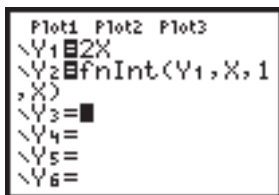
$$A(x) = \int_a^x f(t) dt \quad \text{for } a \leq x \leq b$$

Note that  $A$  is a function of  $x$  and that  $t$  is the (dummy) variable of integration (which in your calculator will also be denoted by  $x$ ).

To define  $A$  using the **Y=** Editor, first define  $f$ , and then define  $A$  by applying the **MATH[9:fnInt()]** operation to  $f$  (see Section 18.4).

### Example

Consider  $f(x) = 2x$  for  $1 \leq x \leq 3$ . First store  $f$  in **Y1**. To store  $A$  in **Y2**, press the **MATH** key and choose **9:fnInt()**, use the **VARS** key to select **Y1**, enter the variable of integration  $x$ , enter the lower limit of integration (which in this example is 1), and enter the upper limit of integration (which in the case of  $A$  is the independent variable  $x$ ).



Since  $A$  is a function of  $x$ , you can graph and evaluate  $A$  in the usual ways.

## 29. EULER'S METHOD

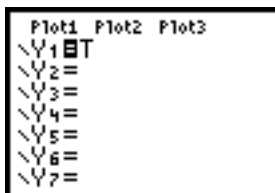
To use Euler's Method to find an approximate solution to the initial value problem,  $\frac{dy}{dt} = f(t,y)$ ,  $y(t_0) = y_0$  with step size  $\Delta t$ , complete the steps listed below. To illustrate the process, consider the initial value problem  $\frac{dy}{dt} = t$  with  $y(0) = 1$  and  $\Delta t = 0.5$  and use the following procedure.

Note that in this example, you are given the value of  $\Delta t$ . However, if you are given a value for  $n$  and asked to approximate a solution to an initial value problem over an interval  $[t_0, b]$ , then you will have to calculate  $\Delta t$ , where  $\Delta t = \frac{b-t_0}{n}$ .

1. Define  $\frac{dy}{dt}$
2. Assign Values to D, T and Y
3. Calculate S
4. Update the Values for T, Y and S
5. Further Update the Values for T, Y and S

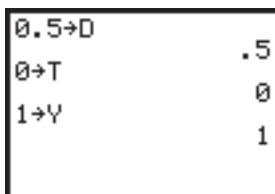
### 1. Define $\frac{dy}{dt}$

Define  $\frac{dy}{dt}$  in the **Y=** Editor.



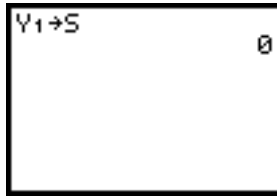
### 2. Assign Values to D, T and Y

Store the value of  $\Delta t$  in **D** and the initial values,  $t_0$  in **T** and  $y_0$  in **Y**. In this case,  $\Delta t = 0.5$ ,  $t_0 = 0$  and  $y_0 = 1$ .



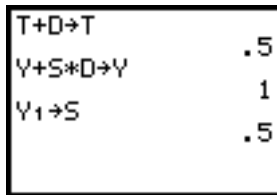
### 3. Calculate S

Calculate  $y'_0$  by evaluating the function **Y1**. Store the result in **S** (for "slope").



**4. Update the Values for T, Y and S**

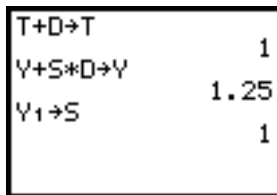
Calculate the values of  $t_1 = t_0 + \Delta t$  and  $y_1 = y_0 + y'_0 \Delta t$ . Store in **T** and **Y**, respectively. Calculate  $y'_1$  by evaluating the function **Y1**; store the result in **S**.



**5. Further Update the Values for T, Y and S**

Calculate  $t_i = t_{i-1} + \Delta t$ ,  $y_i = y_{i-1} + y'_{i-1} \Delta t$  and  $y'_i$  for  $i = 2, 3, \dots, n$ , by repeating Step 4.

Use the **2nd** [ENTRY] feature to repeatedly execute **T + D -> T**, **Y + S \* D -> Y** and **Y1 -> S**. Continuing the example, find  $t_2$ ,  $y_2$  and  $y'_2$ .

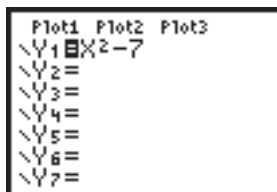


**30. NEWTON'S METHOD**

To use Newton's Method to approximate a real root to an equation  $f(x) = 0$ , complete the steps listed below. To illustrate the process, use Newton's Method to approximate  $\sqrt{7}$  to three decimal places by considering  $f(x) = x^2 - 7$  with  $x_0 = 3$ .

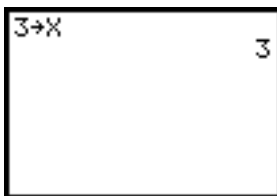
**1. Define the Function**

Define the function using the **Y=** Editor.

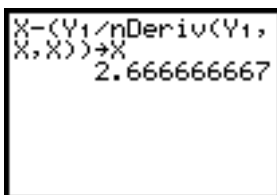


**2. Choose an Initial Guess**

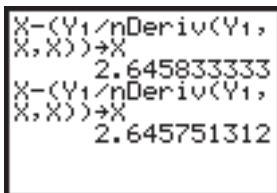
Choose an initial guess,  $x_0$ , for the root and store the value in **X**. In this example,  $x_0 = 3$ .

**3. Calculate  $x_1$** 

Calculate  $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ . Store the result in **X**.

**4. Calculate  $x_i$** 

Calculate  $x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$  for  $i = 1, 2, 3, \dots$  by using the **2nd [ENTRY]** feature to repeatedly process step 3.



Thus,  $\sqrt{7} \approx 2.645$ .

**31. PROGRAMS****31.1. PROGRAM MOTION<sup>1</sup>**

GridOff	Text(1,16,"TEXAS
AxesOff	INSTRUMENTS")
LabelOff	Text(8,30,"CBL SYSTEM")
PlotsOff	Text(15,16,"WORKSHOP
FnOff	CALCULUS")
ClrDraw	Text(29,26,"MOTION V1.0")

<sup>1</sup>Program MOTION is an adaptation of program HIKER, from Experiment M1 in the *Texas Instruments CBL System Experiment Workbook*, which is distributed with the CBL system.

```

Text(36,30,“(TASK 1.4)”)
Text(50,6,“PRESS [ENTER] ON
TI-83”)
Pause
ClrHome
Disp “TURN ON THE CBL”
Output(4,10,“[ENTER]”)
Pause
Full
ClrHome
Disp “NOW CHECKING THE”
Disp “CALCULATOR-CBL”
Disp “LINK CONNECTION”
Disp “PLEASE WAIT”
{1,0} → L1
Send(L1)
{0} → L2
Lbl M
{7} → L1
Send(L1)
Get(L2)
If dim(L2)=1 and L2(1)=0
Then
ClrHome
Disp “***LINK ERROR***”
Disp “PUSH IN THE LINK”
Disp “CORD CONNECTORS”
Disp “FIRMLY THEN HIT”
Disp “[ENTER]”
Pause
Goto M
End
Disp “ “
Output(6,1,“ STATUS:OK”)
Output(8,10,“[ENTER]”)
Pause
Func
ClrHome
ClrDraw
AxesOn
ClrList L2,L3
Disp “MAX SECONDS”
Input T
0 → Xmin
T → Xmax
1 → Xscl
0 → Ymin
4 → Ymax
1 → Yscl
T*10 → dim(L2)
T*10 → dim(L3)
seq(I,I,1,T,.1) → L2
{6,0} → L1
Send(L1)
{1,0} → L1
Send(L1)
{1,11,2} → L1
Send(L1)
Lbl 0
ClrHome
Disp “PRESS ENTER”
Disp “TO START GRAPH”
Pause
ClrDraw
Text(4,1,“DIST(M)”)
Text(51,81,“T(S)”)
{3,.1,-1,0} → L1
Send(L1)
For(I,1,T*10,1)
Get(L3(I))
Pt-On(L2(I),L3(I))
End
Pause
Menu(“*****OPTIONS*****”,“TRY
AGAIN”,1,“QUIT”,2)
Lbl 1
ClrHome
Menu(“*****OPTIONS*****”,“SAVE
DATA”,3,“ERASE DATA”,4)
Lbl 3
L3 → L4
ClrDraw
0 → Xmin
T → Xmax
1 → Xscl
Text(4,1,“DIST(M)”)
Text(51,81,“T(S)”)
Plot2(Scatter,L2,L4,·)
Goto 0
Lbl 4
Goto 0
Lbl 2
ClrHome
Plot1(xyLine,L2,L3,·)
Stop

```

31.2. PROGRAM DISMATCH<sup>2</sup>

GridOff	End
AxesOff	Disp ""
LabelOff	Output(6,1," STATUS: O.K.")
PlotsOff	Output(8,10,"[ENTER]")
FnOff	Pause
ClrDraw	Func
Text(1,16,"TEXAS INSTRUMENTS")	ClrHome
Text(8,30,"CBL SYSTEM")	ClrDraw
Text(15,16,"WORKSHOP CALCULUS")	AxesOn
Text(29,26,"DISMATCH V1.0")	ClrList L2,L4,L3
Text(36,32,"(TASK 1.5)")	AxesOn
Text(50,6,"PRESS [ENTER] ON TI-83")	0 → Xmin
Pause	20 → Xmax
ClrHome	0 → Ymin
Disp "TURN ON THE CBL."	3 → Ymax
Output(4,10,"[ENTER]")	1 → Xscl
Pause	1 → Yscl
Full	200 → dim(L2)
ClrHome	200 → dim(L3)
Disp "NOW CHECKING THE"	seq(I,I,1,20,.1) → L2
Disp "CALCULATOR-CBL"	{6,0} → L1
Disp "LINK CONNECTION."	Send(L1)
Disp "PLEASE WAIT..."	{1,0} → L1
{1,0} → L1	Send(L1)
Send(L1)	{1,11,2} → L1
{0} → L2	Send(L1)
Lbl M	Lbl 0
{7} → L1	ClrHome
Send(L1)	ClrDraw
Get(L2)	Text(4,1,"DIST(M)")
If dim(L2)=1 and L2(1)=0	Text(51,81,"T(S)")
Then	Line(0,.5,6,.5)
ClrHome	Line(6,.5,8,2)
Disp "***LINK ERROR***"	Line(8,2,12,2)
Disp "PUSH IN THE LINK"	Line(12,2,16,1)
Disp "CORD CONNECTORS"	Line(16,1,20,1)
Disp "FIRMLY THEN HIT"	Pause
Disp "[ENTER]."	{3,.1,-1,0} → L1
Pause	Send(L1)
Goto M	For(I,1,200,1)
	Get(L3(I))
	Pt-On(L2(I),L3(I))
	End

<sup>2</sup>Program DISMATCH is an adaptation of program DTMATCH, from Experiment M7 in the *Texas Instruments CBL System Experiment Workbook*, which is distributed with the CBL system.



Pause	$(.075*I-4) \rightarrow L4(I)$
Menu("*****OPTIONS*****", "TRY AGAIN", 1, "QUIT", 2)	End
Lbl 1	For(I, 81, 120, 1)
Goto 0	$2 \rightarrow L4(I)$
Lbl 2	End
ClrHome	For(I, 121, 160, 1)
Plot1(xyLine, L2, L3, ·)	$(-.025*I+5) \rightarrow L4(I)$
$200 \rightarrow \dim(L4)$	End
For(I, 1, 60, 1)	For(I, 161, 200, 1)
$.5 \rightarrow L4(I)$	$1 \rightarrow L4(I)$
End	End
For(I, 61, 80, 1)	Plot2(Scatter, L2, L4, ·)
	Stop

### 31.3. PROGRAM RECT

Disp "ENTER A"	$Y1+S \rightarrow S$
Input A	Line(X-H, 0, X-H, Y1)
Disp "ENTER B"	Line(X-H, Y1, X, Y1)
Input B	Line(X, 0, X, Y1)
Disp "ENTER N"	$X+H \rightarrow X$
Input N	$J+1 \rightarrow J$
$(B-A)/N \rightarrow H$	If $J < N$
$0 \rightarrow S$	Goto 1
ClrDraw	Pause
DispGraph	$HS \rightarrow S$
$0 \rightarrow J$	ClrHome
$A+H \rightarrow X$	Disp "RIGHT SUM"
Lbl 1	Disp S

### 31.4. PROGRAM TRAP

Disp "ENTER A"	$S+.5*(Y+Y1(X))*H \rightarrow S$
Input A	Line(X-H, 0, X-H, Y)
Disp "ENTER B"	Line(X-H, Y, X, Y1)
Input B	Line(X, 0, X, Y1)
Disp "ENTER N"	$X+H \rightarrow X$
Input N	$Y1(X-H) \rightarrow Y$
$(B-A)/N \rightarrow H$	$J+1 \rightarrow J$
ClrDraw	If $J < N$
DispGraph	Goto 1
$0 \rightarrow S$	Pause
$0 \rightarrow J$	ClrHome
$A+H \rightarrow X$	Disp "TRAPEZOID SUM"
$Y1(X-H) \rightarrow Y$	Disp S
Lbl 1	

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