''JUST THE MATHS''

by

A.J. Hobson

TEACHING SLIDES - TABLE OF CONTENTS

(Average number of pages = 1038 ¸140 = 7.4 per unit)

All units are in presented as .PDF files

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Page last changed: 9 September 2002
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FOREWORD

(Home) (About the Author) (Teaching Units) (Teaching Slides)

In 35 years of teaching mathematics to Engineers and Scientists, I have frequently been made aware (by students) of a common cry for help. "We're coping, generally, with our courses", they may say, "but it's Just the Maths". This is the title chosen for the package herein.

Traditional text-books and programmed learning texts can sometimes include a large amount of material which is not always needed for a particular course; and which can leave students feeling that there is too much to cope with. Many such texts are biased towards the mathematics required for specific engineering or scientific disciplines and emphasise the associated practical applications in their lists of tutorial examples. There can also be a higher degree of mathematical rigor than would be required by students who are not intending to follow a career in mathematics itself.

"Just the Maths" is a collection of separate units, in chronological topic-order, intended to service foundation level and first year degree level courses in higher education, especially those delivered in a modular style. Each unit represents, on average, the work to be covered in a typical two-hour session consisting of a lecture and a tutorial. However, since each unit attempts to deal with self-contained and, where possible, independent topics, it may sometimes require either more than or less than two hours spent on it.

"Just the Maths" does not have the format of a traditional text-book or a course of programmed learning; but it is written in a traditional pure-mathematics style with the minimum amount of formal rigor. By making use of the well-worn phrase, "it can be shown that", it is able to concentrate on the core mathematical techniques required by any scientist or engineer. The techniques are demonstrated by worked examples and reinforced by exercises that are few enough in number to allow completion, or near-completion, in a one-hour tutorial session. Answers to exercises are supplied at the end of each unit of work.

A.J. Hobson
January 2002

Page last changed: 9 September 2002
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ABOUT THE AUTHOR

Tony Hobson was, until retirement in November 2001, a Senior Lecturer in Mathematics of the School of Mathematical and Information Sciences at Coventry University. He graduated from the University College of Wales, Aberystwyth in 1964, with a BSc. Degree 2(i) in Pure Mathematics, and from Birmingham University in 1965, with an MSc. Degree in Pure Mathematics. His Dissertation for the MSc. Degree consisted of an investigation into the newer styles teaching Mathematics in the secondary schools of the 1960's with the advent of experiments such as the Midland Mathematics Experiment and the School Mathematics Project. His teaching career began in 1965 at the Rugby College of Engineering Technology where, as well as involvement with the teaching of Analysis and Projective Geometry to the London External Degree in Mathematics, he soon developed a particular interest in the teaching of Mathematics to Science and Engineering Students. This interest continued after the creation of the Polytechnics in 1971 and a subsequent move to the Coventry Polytechnic, later to become Coventry University. It was his main teaching interest throughout the thirty six years of his career; and it meant that much of the time he spent on research and personal development was in the area of curriculum development. In 1982 he became a Non-stipendiary Priest in the Church of England, an interest he maintained throughout his retirement. Tony Hobson died in December 2002.

The set of teaching units for "Just the Maths" has been the result of a pruning, honing and computer-processing exercise (over some four or five years) of **many** years' personal teaching materials, into a form which may be easily accessible to students of Science and Engineerng in the future.

Page last changed: 17 January 2003
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"JUST THE MATHS"

SLIDES NUMBER

1.1

ALGEBRA 1
(Introduction to algebra)

by

A.J. Hobson

UNIT 1.1 - ALGEBRA 1
INTRODUCTION TO ALGEBRA

DEFINITION

An "Algebra" uses Equality (=), Addition (+), Subtraction (-), Multiplication ($\times$ or .) and Division ($\div$).

The Algebra of Numbers = "ARITHMETIC"

1.1.1 THE LANGUAGE OF ALGEBRA

a, b and c denote **constant** numbers of arithmetic;
x, y and z denote **variable** numbers of arithmetic

(a) $a + b$ is the "sum of a and b".

$a + a$ is written $2a$, $\quad a + a + a$ is written $3a$.

(b) $a - b$ is the "difference of a and b".

(c) $a \times b$, $a.b$, ab is the "product of a and b".

$a.a$ is written a^2 $\quad a.a.a$ is written a^3

$-1 \times a$ is written $-a$ and is the "negation" of a.

(d) $a \div b$ or $\frac{a}{b}$ is the "quotient" or "ratio" of a and b.

(e) $\frac{1}{a}$, [also written a^{-1}], is the "reciprocal" of a.

(f) $\mid a \mid$ is the "modulus", "absolute value" or "numerical value" of a.

$\mid a \mid = a$ when a is positive or zero;
$\mid a \mid = -a$ when a is negative or zero.

Rules for combining fractions.

1.
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

2.
$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

3.
$$\frac{a}{b} . \frac{c}{d} = \frac{a.c}{b.d}$$

4.
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a.d}{b.c}$$

EXAMPLES

1. How much more than the difference of 127 and 59 is the sum of 127 and 59 ?

   Difference of 127 and 59 is $127 - 59 = 68$.
   Sum of 127 and 59 is $127 + 59 = 186$.
   Sum exceeds the difference by $186 - 68 = 118$.

2. What is the reciprocal of the number which is 5 multiplied by the difference of 8 and 2 ?
   Reciprocal of $5.(8-2) = \frac{1}{30}$.
3. Calculate the value of $4\frac{2}{3} - 5\frac{1}{9}$ expressing the answer as a fraction.
   $\frac{14}{3} - \frac{46}{9} = \frac{126-138}{27} = -\frac{12}{27} = -\frac{4}{9}$ or $\frac{42}{9} - \frac{46}{9} = -\frac{4}{9}$
4. Remove the modulus signs from the expression $\mid a-2 \mid$ in the cases when (i) a is greater than (or equal to) 2 and (ii) a is less than 2.
   (i) If a is greater than or equal to 2,
   $$\mid a-2 \mid = a-2$$
   (ii) If a is less than 2,
   $$\mid a-2 \mid = -(a-2) = 2-a$$

1.1.2 THE LAWS OF ALGEBRA

(a) The Commutative Law of Addition

$$a + b = b + a$$

(b) The Associative Law of Addition

$$a + (b + c) = (a + b) + c$$

(c) The Commutative Law of Multiplication

$$a.b = b.a$$

(d) The Associative Law of Multipication

$$a.(b.c) = (a.b).c$$

(e) The Distributive Laws

$$a.(b + c) = a.b + a.c$$
$$(a + b).c = a.c + b.c$$

Note for later: $(a + b).(c + d) = a.c + b.c + a.d + b.d$

1.1.3 PRIORITIES IN CALCULATIONS

Problem: $5 \times 6 - 4 = 30$ - $4 = 26$ or $5 \times 2 = 10$????

B.O.D.M.A.S.

B	brackets	( )	First Priority
O	of	$\times$	Joint Second Priority
D	division	$\div$	Joint Second Priority
M	multiplication	$\times$	Joint Second Priority
A	addition	+	Joint Third Priority
S	subtraction	-	Joint Third Priority

Exs.

$5 \times (6 - 4) = 5 \times 2 = 10$
$5 \times 6 - 4 = 30 - 4 = 26.$

$12 \div 3 - 1 = 4 - 1 = 3$
$12 \div (3 - 1) = 12 \div 2 = 6.$

1.1.4 FACTORS

If a number can be expressed as a product of other numbers, each of those other numbers is called a "Factor" of the original number.

EXAMPLES

1.

$$70 = 2 \times 7 \times 5$$

These are "prime" factors

2. Show that the numbers 78 and 182 have two common factors which are prime numbers.

$$78 = 2 \times 3 \times 13$$

$$182 = 2 \times 7 \times 13$$

Common factors are 2 and 13 - both prime.

Highest Common Factor, h.c.f.

$$90 = 2 \times 3 \times 3 \times 5$$

and

$$108 = 2 \times 2 \times 3 \times 3 \times 3$$

$$\text{h.c.f} = 2 \times 3 \times 3 = 18$$

Lowest Common Multiple, l.c.m.

$$15 = 3 \times 5$$

and

$$20 = 2 \times 2 \times 5$$

$$\text{l.c.m.} = 2 \times 2 \times 3 \times 5 = 60$$

Lowest Terms

Common factors may be cancelled to leave the fraction in its "lowest terms".

$$\frac{15}{105} = \frac{3 \times 5}{3 \times 5 \times 7} = \frac{1}{7}$$

"JUST THE MATHS"

SLIDES NUMBER

1.2

ALGEBRA 2
(Numberwork)

by

A.J. Hobson

UNIT 1.2 - ALGEBRA 2
NUMBERWORK

1.2.1 TYPES OF NUMBER

(a) NATURAL NUMBERS

$$1, 2, 3, 4, ...$$

(b) INTEGERS

$$......-5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ ......$$

(c) RATIONALS

The ratio of two integers or a terminating decimal or a recurring decimal

$$\frac{2}{5} = 0.4 \ \text{ and } \ \frac{3}{7} = 0.428714287142871....$$

(d) IRRATIONALS

Neither the ratio of two integers nor a recurring decimal

$$\pi \simeq 3.1415926....., \quad e \simeq 2.71828..... \ \sqrt{2} \simeq 1.4142135....$$

$$\sqrt{5} \simeq 2.2360679....$$

The above four types of number form the system of **"Real Numbers".**

1.2.2 DECIMAL NUMBERS

(a) Rounding to a specified number of decimal places

When rounding to n decimal places, the digit in the n-th place is left as it is when the one after it is below 5; otherwise it is taken up by one digit.

362.5863 = 362.586 to 3 decimal places;
362.5863 = 362.59 to 2 decimal places;
362.5863 = 362.6 to 1 decimal place;
362.5863 = 363 to the nearest whole number.

(b) Rounding to a specified number of significant figures

The first significant figure of a decimal quantity is the first non-zero digit from the left, whether it be before or after the decimal point.

0.02158 = 0.0216 to 3 significant figures;
0.02158 = 0.022 to 2 significant figures;
0.02158 = 0.02 to 1 significant figure.

1.2.3 USE OF ELECTRONIC CALCULATORS

(a) B.O.D.M.A.S.

$7.25 + 3.75 \times 8.32 = 38.45$, not 91.52.

$6.95 \div [2.43 - 1.62] = 8.58$, not 1.24

(b) Other Useful Numerical Functions

$\sqrt{x}$, x^2, x^y, $x^{\frac{1}{y}}$, using, where necessary, the "shift" control

$\sqrt{173} \simeq 13.153, \;\; 173^2 = 29929, \;\; 23^3 = 12167, \;\; 23^{\frac{1}{3}} \simeq 2.844$

(c) The Calculator Memory

For $(1.4)^3 - 2(1.4)^2 + 5(1.4) - 3 \simeq 2.824$, store each of the four terms in the calculation (positively or negatively) then recall their total sum at the end.

1.2.4 SCIENTIFIC NOTATION

(a) Very large numbers written as

$$a \times 10^n$$

n is a positive integer
a lies between 1 and 10.

$$521983677.103 = 5.21983677103 \times 10^8$$

(b) Very small numbers written as

$$a \times 10^{-n}$$

n is a positive integer
a lies between 1 and 10.

$$0.00045938 = 4.5938 \times 10^{-4}$$

Note:
Enter numbers by using the **EXP** or **EE** buttons

EXAMPLES

1. Key in the number 3.90816×10^{57} on a calculator.
   Press **3.90816** **[EXP]** **57**
2. Key in the number 1.5×10^{-27} on a calculator
   Press **1.5** **[EXP]** **27** [+/-]

Notes:
(i) Scientific notation is also called
floating point notation.

(ii) Accuracy = one significant figure more than the least number of significant figures in any measurement.

1.2.5 PERCENTAGES

Definition

A percentage is a fraction whose denominator is 100.

$\frac{17}{100} = 17\%$

EXAMPLES

1.
$$\frac{2}{5} = \frac{2}{5} \times \frac{20}{20} = \frac{40}{100} = 40\%$$

2.
$$27\% \text{ of } 90 = \frac{27}{100} \times 90 = \frac{27}{10} \times 9 = 24.3$$

3.

$$30\% = \frac{30}{100} = 0.3$$

1.2.6 RATIO

We may use a colon (:)

eg. 7:3 instead of either $7 \div 3$ or $\frac{7}{3}$.

7:3 could also be written $\frac{7}{3}$:1 or 1:$\frac{3}{7}$.

EXAMPLES

1. Divide 170 in the ratio 3:2

   170 is made up of $3 + 2 = 5$ parts, each of value $\frac{170}{5} = 34$.

   $3 \times 34 = 102$ and $2 \times 34 = 68$.

   $170 = 102 + 68$

2. Divide 250 in the ratio 1:3:4

   250 is made up of $1 + 3 + 4 = 8$ parts, each of value $\frac{250}{8} = 31.25$.

   $3 \times 31.25 = 93.75$ and $4 \times 31.25 = 125$.

   $250 = 31.25 + 93.75 + 125$.

"JUST THE MATHS"

SLIDES NUMBER

1.3

ALGEBRA 3
(Indices and radicals (or surds))

by

A.J.Hobson

UNIT 1.3 - ALGEBRA 3
INDICES AND RADICALS (or Surds)

1.3.1 INDICES

(a) Positive Integer Indices

Let a and b be arbitrary numbers
Let m and n be natural numbers

Law No. 1

$$a^m \times a^n = a^{m+n}$$

Law No. 2

$$a^m \div a^n = a^{m-n}$$

assuming m greater than n.

Note:

$\frac{a^m}{a^m} = 1$ and $\frac{a^m}{a^m} = a^{m-m} = a^0$.

Hence, we **define** a^0 to be equal to 1.

Law No. 3

$$(a^m)^n = a^{mn}$$

$$a^m b^m = (ab)^m$$

EXAMPLE

Simplify the expression,

$$\frac{x^2y^3}{z} \div \frac{xy}{z^5}.$$

Solution

The expression becomes

$$\frac{x^2y^3}{z} \times \frac{z^5}{xy} = xy^2z^4.$$

(b) Negative Integer Indices

Law No. 4

$$a^{-1} = \frac{1}{a}$$

Note:

$\frac{a^m}{a^{m+1}} = \frac{1}{a}$ and $a^{m-[m+1]} = a^{-1}$.

Law No. 5

$$a^{-n} = \frac{1}{a^n}$$

Note:

$\frac{a^m}{a^{m+n}} = \frac{1}{a^n}$ and $a^{m-[m+n]} = a^{-n}$

Law No. 6

$$a^{-\infty} = 0$$

EXAMPLE

Simplify the expression,

$$\frac{x^5y^2z^{-3}}{x^{-1}y^4z^5} \div \frac{z^2x^2}{y^{-1}}.$$

Solution

The expression becomes

$$x^5y^2z^{-3}xy^{-4}z^{-5}y^{-1}z^{-2}x^{-2} = x^4y^{-3}z^{-10}.$$

(c) Rational Indices

(i) Indices of the form $\frac{1}{n}$ where n is a natural number.

$a^{\frac{1}{n}}$ means a number which gives a when it is raised to the power n. It is called an "n-th Root of a" and, sometimes there is more than one value.

ILLUSTRATION

$81^{\frac{1}{4}} = \pm 3$ but $(-27)^{\frac{1}{3}} = -3$ only

(ii) Indices of the form $\frac{m}{n}$ where m and n are natural numbers with no common factor.

$y^{\frac{m}{n}} = (y^m)^{\frac{1}{n}}$ or $(y^{\frac{1}{n}})^m$.

ILLUSTRATION

$27^{\frac{2}{3}} = 3^2 = 9$ or $27^{\frac{2}{3}} = 729^{\frac{1}{3}} = 9$

Note:
It may be shown that all of the standard laws of indices may be used for fractional indices.

1.3.2 RADICALS (or Surds)

"$\sqrt{}$" denotes the positive or **principal** square root of a number.

eg. $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

The number under the radical is called the **RADICAND**

The **principal n-th root** of a number a is $\sqrt[n]{a}$

n is the **index** of the radical.

ILLUSTRATIONS

1. $\sqrt[3]{64} = 4$ since $4^3 = 64$
2. $\sqrt[3]{-64} = -4$ since $(-4)^3 = -64$

3. $\sqrt[4]{81} = 3$ since $3^4 = 81$
4. $\sqrt[5]{32} = 2$ since $2^5 = 32$
5. $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$

Note:
If the index of the radical is an even number, then the radicand may not be negative.

(d) Rules for Square Roots

(i) $(\sqrt{a})^2 = a$

(ii) $\sqrt{a^2} = |a|$

(iii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(iv) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

assuming that all the radicals can be evaluated

ILLUSTRATIONS

1. $\sqrt{9 \times 4} = \sqrt{36} = 6$ and $\sqrt{9} \times \sqrt{4} = 3 \times 2 = 6$
2. $\sqrt{\frac{144}{36}} = \sqrt{4} = 2$ and $\frac{\sqrt{144}}{\sqrt{36}} = \frac{12}{6} = 2.$

(e) Rationalisation of Radical (or Surd) Expressions.

EXAMPLES

1. Rationalise the surd form $\frac{5}{4\sqrt{3}}$

   Solution

$$\frac{5}{4\sqrt{3}} = \frac{5}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{12}$$

2. Rationalise the surd form $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

   Solution

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{\sqrt[3]{ab^2}}{\sqrt[3]{b^3}} = \frac{\sqrt[3]{ab^2}}{b}$$

3. Rationalise the surd form $\frac{4}{\sqrt{5}+\sqrt{2}}$

   Solution

   We use $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$

$$\frac{4}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{4\sqrt{5}-4\sqrt{2}}{3}$$

4. Rationalise the surd form $\frac{1}{\sqrt{3}-1}$

   Solution

$$\frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}$$

(f) Changing numbers to and from radical form

$$| a^{\frac{m}{n}} | = \sqrt[n]{a^m}$$

EXAMPLES

1. Express the number $x^{\frac{2}{5}}$ in radical form

   Solution

   $$\text{Answer} = \sqrt[5]{x^2}$$

2. Express the number $\sqrt[3]{a^5b^4}$ in exponential form

   Solution

   $$\sqrt[3]{a^5b^4} = (a^5b^4)^{\frac{1}{3}} = a^{\frac{5}{3}}b^{\frac{4}{3}}$$

"JUST THE MATHS"

SLIDES NUMBER

1.4

ALGEBRA 4
(Logarithms)

by

A.J.Hobson

UNIT 1.4 - ALGEBRA 4 - LOGARITHMS

1.4.1 COMMON LOGARITHMS

We normally count with a base of 10.

Each successive digit of a number corresponds to that digit multiplied by a certain power of 10.

ILLUSTRATION

$$73,520 = 7 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10^1.$$

Problem

Can a given number can be expressed as a single power of 10, not necessarily an integer power ?

The power will need to be **positive** since powers of 10 are not normally negative (or even zero).

ILLUSTRATION

By calculator,

$$1.99526 \simeq 10^{0.3}$$

and

$$2 \simeq 10^{0.30103}.$$

DEFINITION
In general, when

$$x = 10^y,$$

for some positive number x, we say that y is the **"logarithm to base 10"** of x (or **" Common Logarithm"** of x) and we write

$$y = \log_{10} x.$$

ILLUSTRATIONS

1. $\log_{10} 1.99526 = 0.3$ from the earlier illustrations.
2. $\log_{10} 2 = 0.30103$ from the earlier illustrations.
3. $\log_{10} 1 = 0$ simply because $10^0 = 1$.

1.4.2 LOGARITHMS IN GENERAL

DEFINITION
If B is a fixed positive number and x is another positive number such that

$$x = B^y,$$

we say that y is the "logarithm to base B of x and we write

$$y = \log_B x.$$

ILLUSTRATIONS

1. $\log_B 1 = 0$ simply because $B^0 = 1$.
2. $\log_B B = 1$ simply because $B^1 = B$.
3. $\log_B 0$ doesn't really exist because no power of B could ever be equal to zero.

   But, since a very large negative power of B will be a very small positive number, we usually write

$$\log_B 0 = -\infty$$

1.4.3 USEFUL RESULTS

(a) For any positive number x,

$$x = B^{\log_B x}.$$

Proof

In $x = B^y$, replace y by $\log_B x$.

(b) For any number y,

$$y = \log_B B^y.$$

In $y = \log_B x$, replace x by B^y.

1.4.4 PROPERTIES OF LOGARITHMS

(a) The Logarithm of Product.

$$\log_B p.q = \log_B p + \log_B q.$$

Proof:

From Result (a) of the previous section,

$$p.q = B^{\log_B p}.B^{\log_B q} = B^{\log_B p + \log_B q}.$$

(b) The Logarithm of a Quotient

$$\log_B \frac{p}{q} = \log_B p - \log_B q.$$

Proof:

From Result (a) of the previous section,

$$\frac{p}{q} = \frac{B^{\log_B p}}{B^{\log_B q}} = B^{\log_B p - \log_B q}.$$

(c) The Logarithm of an Exponential

$$\log_B p^n = n \log_B p,$$

where n need not be an integer.

Proof:

From Result (a) of the previous section,

$$p^n = \left(B^{\log_B p}\right)^n = B^{n \log_B p}.$$

(d) The Logarithm of a Reciprocal

$$\log_B \frac{1}{q} = -\log_B q.$$

Proof:

Method 1.
Left-hand side =
$\log_B 1 - \log_B q = 0 - \log_B q = -\log_B q$

Method 2.
Left-hand side = $\log_B q^{-1} = -\log_B q$.

(e) Change of Base

$$\log_B x = \frac{\log_A x}{\log_A B}.$$

Proof:

Suppose $y = \log_B x$.

Then, $x = B^y$.

Hence,

$$\log_A x = \log_A B^y = y \log_A B.$$

Thus

$$y = \frac{\log_A x}{\log_A B}.$$

Note:
Logarithms to any base are directly proportional to logarithms to another base.

1.4.5 NATURAL LOGARITHMS

In scientific work, only two bases of logarithms are used.

One is base 10 (for **"Common"** Logarithms).

The other is a base $e = 2.71828......$ (for **"Natural"** Logarithms) arising out of calculus.

The Natural Logarithm of x is denoted by $\log_e x$ or $\ln x$.

Note:
By change of base formula,

$$\log_{10} x = \frac{\log_e x}{\log_e 10} \qquad \text{and} \qquad \log_e x = \frac{\log_{10} x}{\log_{10} e}.$$

EXAMPLES

1. Solve for x the "indicial equation"

$$4^{3x-2} = 26^{x+1}.$$

Solution
Take logarithms of both sides,

$$(3x-2)\log_{10} 4 = (x+1)\log_{10} 26;$$
$$(3x-2)0.6021 = (x+1)1.4150;$$
$$1.8063x - 1.2042 = 1.4150x + 1.4150;$$
$$(1.8603 - 1.4150)x = 1.4150 + 1.2042;$$
$$0.3913x = 2.6192;$$
$$x = \frac{2.6192}{0.3913} \simeq 6.6936$$

2. Rewrite the expression

$$4x + \log_{10}(x+1) - \log_{10} x - \frac{1}{2}\log_{10}(x^3 + 2x^2 - x)$$

as the common logarithm of a single mathematical expression.

Solution

Convert every term to 1× a logarithm.

$$4x = \log_{10} 10^{4x}$$

$$\frac{1}{2}\log_{10}(x^3 + 2x^2 - x) = \log_{10}\left(x^3 + 2x^2 - x\right)^{\frac{1}{2}}.$$

Hence,

$$\log_{10} \frac{10^{4x}(x+1)}{x\sqrt{(x^3 + 2x^2 - x)}}.$$

3. Rewrite without logarithms the equation

$$2x + \ln x = \ln(x - 7).$$

Solution

Convert both sides to the natural logarithm of a single mathematical expression.

$$\text{L.H.S.} = 2x + \ln x = \ln e^{2x} + \ln x = \ln xe^{2x}.$$

Hence,

$$xe^{2x} = x - 7.$$

4. Solve for x the equation

$$6 \ln 4 + \ln 2 = 3 + \ln x.$$

Solution

Using $6 \ln 4 = \ln 4^6$ and $3 = \ln e^3$,

$$\ln 2(4^6) = \ln xe^3.$$

Hence,

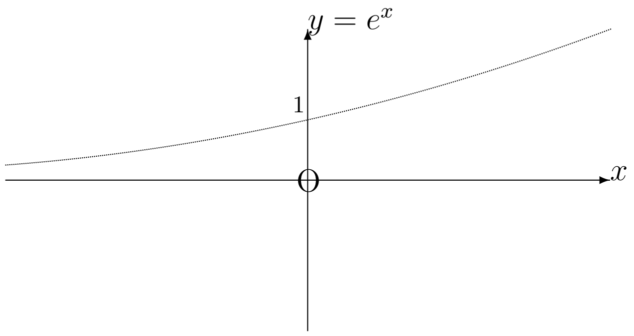
$$2(4^6) = xe^3,$$

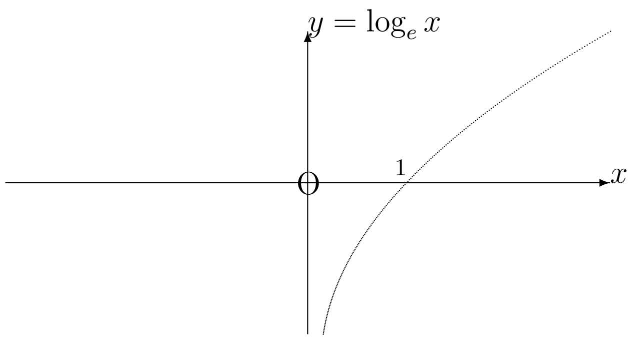
so that

$$x = \frac{2(4^6)}{e^3} \simeq 407.856$$

1.4.6 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

The graphs of $y = e^x$ and $y = \log_e x$ are as follows:





1.4.7 LOGARITHMIC SCALES

In a certain kind of graphical work, some use is made of a linear scale along which numbers can be allocated according to their logarithmic distances from a chosen origin of measurement.

For logarithms to base 10, the number 1 is placed at the zero of measurement (since $\log_{10} 1 = 0$).

The number 10 is placed at the first unit of measurement (since $\log_{10} 10 = 1$).

The number 100 is placed at the second unit of measurement (since $\log_{10} 100 = 2$) and so on.

Numbers such as $10^{-1} = 0.1$, $10^{-2} = 0.01$ etc. are placed at the points corresponding to -1 and -2 etc. respectively on an ordinary linear scale.

The logarithmic scale appears in **"cycles"**

Each cycle corresponds to a range of numbers between two consecutive powers of 10.

Intermediate numbers are placed at intervals which correspond to their logarithm values.

0.1 0.2 0.3 0.4 1 2 3 4 10

Notes:
(i) A given set of numbers will determine how many cycles are required.

For example .3, .6, 5, 9, 23, 42, 166 will require **four** cycles.

(ii) Commercially printed logarithmic scales do not specify the base of logarithms.

"JUST THE MATHS"

SLIDES NUMBER

1.5

ALGEBRA 5
(Manipulation of algebraic expressions)

by

A.J.Hobson

UNIT 1.5 - ALGEBRA 5
MANIPULATION OF ALGEBRAIC EXPRESSIONS

1.5.1 SIMPLIFICATION OF EXPRESSIONS

Remove brackets and collect together any terms which have the same format

Elementary Illustrations

1. $a + a + a + 3 + b + b + b + b + 8 \equiv 3a + 4b + 11.$
2. $11p^2 + 5q^7 - 8p^2 + q^7 \equiv 3p^2 + 6q^7.$
3. $a.(2a - b) + b.(a + 5b) - a^2 - 4b^2 \equiv 2a^2 - a.b + b.a + 5b^2 - a^2 - 4b^2 \equiv a^2 + b^2.$

Further illustrations

1. $x(2x + 5) + x^2(3 - x) \equiv 2x^2 + 5x + 3x^2 - x^3 \equiv 5x^2 + 5x - x^3.$
2. $x^{-1}(4x - x^2) - 6(1 - 3x) \equiv 4 - x - 6 + 18x \equiv 17x - 2.$

Two or more brackets multiplied together

$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd.$

EXAMPLES:

1. $(x+3)(x-5) \equiv x^2 + 3x - 5x - 15 \equiv x^2 - 2x - 15.$
2. $(x^3 - x)(x+5) \equiv x^4 - x^2 + 5x^3 - 5x.$
3. $(x+a)^2 \equiv (x+a)(x+a) \equiv x^2 + ax + ax + a^2 \equiv x^2 + 2ax + a^2$; a **"Perfect Square"**.
4. $(x+a)(x-a) \equiv x^2 + ax - ax - a^2 \equiv x^2 - a^2$; the **"Difference of two squares"**.

1.5.2 FACTORISATION

Introduction

"Factor" means "Multiplier".

Examples

1. $3x + 12 \equiv 3(x+4).$
2. $8x^2 - 12x \equiv x(8x - 12) \equiv 4x(2x - 3).$
3. $5x^2 + 15x^3 \equiv x^2(5 + 15x) \equiv 5x^2(1 + 3x).$
4. $6x + 3x^2 + 9xy \equiv x(6 + 3x + 9y) \equiv 3x(2 + x + 3y).$

Note:
When none of the factors can be broken down into simpler factors, the original expression is said to have been factorised into **"irreducible factors"**.

Factorisation of quadratic expressions

A "Quadratic Expression" is an expression of the form

$$ax^2 + bx + c.$$

The quadratic expression has "coefficients a, b and c".

EXAMPLES:

(a) When the coefficient of x^2 is 1

1. $x^2+5x+6 \equiv (x+m)(x+n) \equiv x^2+(m+n)x+mn$;
   $5 = m + n$ and $6 = mn$;
   By inspection, $m = 2$ and $n = 3$.
   Hence $x^2 + 5x + 6 \equiv (x + 2)(x + 3)$.
2. $x^2+4x-21 \equiv (x+m)(x+n) \equiv x^2+(m+n)x+mn$;
   $4 = m + n$ and $-21 = mn$;
   By inspection, $m = -3$ and $n = 7$.
   Hence $x^2 + 4x - 21 \equiv (x - 3)(x + 7)$.

Note:
For simple cases, carry out the factorisation entirely by inspection.

$$x^2 + 2x - 8 \equiv (x+?)(x+?) \equiv (x-2)(x+4).$$

$$x^2 + 10x + 25 \equiv (x+5)^2.$$

$$x^2 - 64 \equiv (x-8)(x+8).$$

$$x^2 - 13x + 2 \text{ won't factorise.}$$

(b) When the coefficient of x^2 is not 1

Determine the possible pairs of factors of the coefficient of x^2 and the possible pairs of factors of the constant term.

EXAMPLES

1. To factorise the expression $2x^2 + 11x + 12$,
   Try $(2x+1)(x+12)$, $(2x+12)(x+1)$, $(2x+6)(x+2)$, $(2x+2)(x+6)$, $(2x+4)(x+3)$ and $(2x+3)(x+4)$.

$$2x^2 + 11x + 12 \equiv (2x+3)(x+4).$$

2. To factorise the expression $6x^2 + 7x - 3$,

Try $(6x+3)(x-1)$, $(6x-3)(x+1)$, $(6x+1)(x-3)$, $(6x-1)(x+3)$, $(3x+3)(2x-1)$, $(3x-3)(2x+1)$, $(3x+1)(2x-3)$ and $(3x-1)(2x+3)$.

$$6x^2+7x-3 \equiv (3x-1)(2x+3).$$

1.5.3 COMPLETING THE SQUARE IN A QUADRATIC EXPRESSION

We use

$$(x+a)^2 \equiv x^2+2ax+a^2$$

and

$$(x-a)^2 \equiv x^2-2ax+a^2.$$

ILLUSTRATIONS

1. $x^2+6x+9 \equiv (x+3)^2$.
2. $x^2-8x+16 \equiv (x-4)^2$.
3. $4x^2-4x+1 \equiv 4\left[x^2-x+\frac{1}{4}\right] \equiv 4\left(x-\frac{1}{2}\right)^2$.
4. $x^2+6x+11 \equiv (x+3)^2+2$.
5. $x^2-8x+7 \equiv (x-4)^2-9$.
6. $4x^2-4x+5 \equiv 4\left[x^2-x+\frac{5}{4}\right]$

$$\equiv 4\left[\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + \tfrac{5}{4}\right]$$
$$\equiv 4\left[\left(x - \tfrac{1}{2}\right)^2 + 1\right]$$
$$\equiv 4\left(x - \frac{1}{2}\right)^2 + 4.$$

1.5.4 ALGEBRAIC FRACTIONS

Revision

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

EXAMPLES

1.
$$\frac{5}{25 + 15x} \equiv \frac{1}{5 + 3x},$$
assuming that $x \neq -\frac{5}{3}$.

2.
$$\frac{4x}{3x^2 + x} \equiv \frac{4}{3x + 1},$$
assuming that $x \neq 0 \;\text{ or }\; -\frac{1}{3}$.

3.
$$\frac{x + 2}{x^2 + 3x + 2} \equiv \frac{x + 2}{(x + 2)(x + 1)} \equiv \frac{1}{x + 1},$$
assuming that $x \neq -1 \;\text{ or }\; -2$.

4.

$$\frac{3x+6}{x^2+3x+2} \times \frac{x+1}{2x+8}$$

$$\equiv \frac{3(x+2)(x+1)}{2(x+4)(x+1)(x+2)}$$

$$\equiv \frac{3}{2(x+4)},$$

assuming that $x \neq -1, -2$ or -4.

5.

$$\frac{3}{x+2} \div \frac{x}{2x+4}$$

$$\equiv \frac{3}{x+2} \times \frac{2x+4}{x}$$

$$\equiv \frac{3}{x+2} \times \frac{2(x+2)}{x} \equiv \frac{6}{x},$$

assuming that $x \neq 0$ or -2.

6.

$$\frac{4}{x+y} - \frac{3}{y}$$

$$\equiv \frac{4y - 3(x+y)}{(x+y)y}$$

$$\equiv \frac{y-3x}{(x+y)y}.$$

7.

$$\frac{x}{x+1} + \frac{4-x^2}{x^2-x-2}$$

$$\equiv \frac{x(x-2)}{(x+1)(x-2)} + \frac{4-x^2}{(x+1)(x-2)}$$

$$\equiv \frac{x^2-2x+4-x^2}{(x+1)(x-2)}$$

$$\equiv \frac{2(2-x)}{(x+1)(x-2)} = -\frac{2}{x+1}$$

assuming that $x \neq 2 \;$ or $\; -1$.

"JUST THE MATHS"

SLIDES NUMBER

1.6

ALGEBRA 6
(Formulae and algebraic equations)

by

A.J.Hobson

UNIT 1.6 - ALGEBRA 6
FORMULAE AND ALGEBRAIC EQUATIONS

1.6.1 TRANSPOSITION OF FORMULAE

The following steps may be carried out on both sides of a given formula:

(a) Addition or subtraction of the same value;

(b) Multiplication or division by the same value;

(c) The raising of both sides to equal powers;

(d) Taking logarithms of both sides.

EXAMPLES

1. Make x the subject of the formula

$$y = 3(x + 7).$$

Solution

$$\frac{y}{3} = x + 7;$$

$$x = \frac{y}{3} - 7.$$

2. Make y the subject of the formula

$$a = b + c\sqrt{x^2 - y^2}.$$

Solution

$$a - b = c\sqrt{x^2 - y^2};$$

$$\frac{a-b}{c} = \sqrt{x^2 - y^2};$$

$$\left(\frac{a-b}{c}\right)^2 = x^2 - y^2;$$

$$\left(\frac{a-b}{c}\right)^2 - x^2 = -y^2;$$

$$x^2 - \left(\frac{a-b}{c}\right)^2 = y^2;$$

$$y = \pm\sqrt{x^2 - \left(\frac{a-b}{c}\right)^2}.$$

3. Make x the subject of the formula

$$e^{2x-1} = y^3.$$

Solution

Taking natural logarithms of both sides of the formula

$$2x - 1 = 3\ln y.$$

Hence

$$x = \frac{3\ln y + 1}{2}.$$

Note:

For scientific formulae, ignore the negative square roots.

1.6.2 SOLUTION OF LINEAR EQUATIONS

If $ax + b = c$, then $x = \frac{c-b}{a}$.

EXAMPLES

1. Solve the equation

$$5x + 11 = 20.$$

$$\text{Ans}: \quad x = \frac{20-11}{5} = \frac{9}{5} = 1.8$$

2. Solve the equation

$$3 - 7x = 12.$$

$$\text{Ans}: \ x = \frac{12 - 3}{-7} = \frac{9}{-7} \simeq -1.29$$

1.6.3 SOLUTION OF QUADRATIC EQUATIONS

Standard form is $ax^2 + bx + c = 0$.

(a) By Factorisation

EXAMPLES

1. Solve the quadratic equation

$$6x^2 + x - 2 = 0.$$

In factorised form,

$$(3x + 2)(2x - 1) = 0.$$

Hence, $x = -\frac{2}{3}$ or $x = \frac{1}{2}$.

2. Solve the quadratic equation

$$15x^2 - 17x - 4 = 0.$$

In factorised form

$$(5x + 1)(3x - 4) = 0.$$

Hence, $x = -\frac{1}{5}$ or $x = \frac{4}{3}$.

(b) By Completing the square

EXAMPLES

1. Solve the quadratic equation

$$x^2 - 4x - 1 = 0.$$

Equation can be written

$$(x-2)^2 - 5 = 0.$$

Thus,

$$x - 2 = \pm\sqrt{5}.$$

$$\text{Ans}: \quad x = 2 \pm \sqrt{5} \simeq 4.236 \quad \text{or} \quad -0.236$$

2. Solve the quadratic equation

$$4x^2 + 4x - 2 = 0.$$

Equation can be written

$$4\left[x^2 + x - \frac{1}{2}\right] = 0;$$

$$4\left[\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right] = 0.$$

Hence,

$$\left(x + \frac{1}{2}\right)^2 = \frac{3}{4};$$

$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}};$$

$$x = -\frac{1}{2} \pm \sqrt{\frac{3}{4}} \quad \text{or} \quad \frac{-1 \pm \sqrt{3}}{2}.$$

(c) By the Quadratic Formula

Given

$$ax^2 + bx + c = 0,$$

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0;$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = 0;$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}};$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}};$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:

$b^2 - 4ac$ is called the **"Discriminant"**.

The discriminant gives two solutions, one solution (coincident pair) or no (real) solutions according as its value is positive, zero or negative.

EXAMPLES

Use the quadratic formula to solve the following:

1.

$$2x^2 - 3x - 7 = 0.$$

Solution

$$x = \frac{3 \pm \sqrt{9+56}}{4} = \frac{3 \pm \sqrt{65}}{4}$$

$$= \frac{3 \pm 8.062}{4} \simeq 2.766 \quad \text{or} \quad -1.266$$

2.

$$9x^2 - 6x + 1 = 0.$$

$$x = \frac{6 \pm \sqrt{36-36}}{18} = \frac{6}{18} = \frac{1}{3} \quad \text{only.}$$

3.

$$5x^2 + x + 1 = 0.$$

Solution

$$x = \frac{-1 \pm \sqrt{1-20}}{10}.$$

No real solutions.

"JUST THE MATHS"

SLIDES NUMBER

1.7

ALGEBRA 7
(Simultaneous linear equations)

by

A.J.Hobson

UNIT 1.7 - ALGEBRA 7
SIMULTANEOUS LINEAR EQUATIONS

1.7.1 TWO SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

First eliminate one of the variables (eg. x) in order to calculate the other.

$$\begin{aligned} cax + cby &= cp, \\ acx + ady &= aq. \end{aligned}$$

$$y(cb - ad) = cp - aq;$$

$$y = \frac{cp - aq}{cb - ad} \quad \text{if} \quad cb - ad \neq 0.$$

To find x, substitute back or eliminate y.

Degenerate Case.

If $cb - ad = 0$, the left hand sides of the two equations are proportional to each other.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 6x - 2y &= 1, \qquad (1) \\ 4x + 7y &= 9. \qquad (2) \end{aligned}$$

$$\begin{aligned} 24x - 8y &= 4, \qquad (4) \\ 24x + 42y &= 54. \qquad (5) \end{aligned}$$

Hence, $-50y = -50$ and $y = 1$.

Substituting into (1), $6x - 2 = 1$ giving $6x = 3$.

Hence, $x = \frac{1}{2}$.

Alternative Method

$$\begin{aligned} 42x - 14y &= 7, \qquad (5) \\ -8x - 14y &= -18. \qquad (6) \end{aligned}$$

Hence, $50x = 25$, so $x = \frac{1}{2}$.

Substituting into (1) gives $3 - 2y = 1$, so $y = 1$.

1.7.2 THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

$$
\begin{aligned}
a_1x + b_1y + c_1z &= k_1, \\
a_2x + b_2y + c_2z &= k_2, \\
a_3x + b_3y + c_3z &= k_3.
\end{aligned}
$$

Eliminate one of the variables from two different pairs of the three equations.

EXAMPLE

Solve, for x, y and z, the simultaneous linear equations

$$
\begin{aligned}
x - y + 2z &= 9, && (1) \\
2x + y - z &= 1, && (2) \\
3x - 2y + z &= 8. && (3)
\end{aligned}
$$

Solution

Eliminating z from equations (2) and (3),

$$5x - y = 9. \qquad (4)$$

Eliminating z from equations (1) and (2),

$$5x + y = 11. \qquad (5)$$

Adding (4) to (5),

$$10x = 20 \quad \text{or} \quad x = 2.$$

Subtracting (4) from (5),

$$2y = 2 \quad \text{or} \quad y = 1.$$

Substituting x and y into (3),

$$z = 8 - 3x + 2y = 8 - 6 + 2 = 4$$

Thus,

$$x = 2, \quad y = 1 \quad \text{and} \quad z = 4.$$

1.7.3 ILL-CONDITIONED EQUATIONS

Rounding errors may swamp the values of the variables being solved for.

EXAMPLE

$$\begin{aligned} x + y &= 1, \\ 1.001x + y &= 2 \end{aligned}$$

have the common solution $x = 1000$, $y = -999$.

$$\begin{aligned} x + y &= 1, \\ x + y &= 2 \end{aligned}$$

have no solution at all.

$$\begin{aligned} x + y &= 1, \\ 0.999x + y &= 2 \end{aligned}$$

have solutions $x = -1000$, $y = 1001$.

"JUST THE MATHS"

SLIDES NUMBER

1.8

ALGEBRA 8
(Polynomials)

by

A.J.Hobson

UNIT 1.8 - ALGEBRA 8
POLYNOMIALS

Introduction

General form,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n,$$

a **"polynomial of degree n in x"**, having **"coefficients"** $a_0, a_1, a_2, a_3, ...a_n$, usually constant.

Note:
Polynomials of degree 1, 2 and 3 are called respectively "linear", "quadratic" and "cubic" polynomials.

1.8.1 THE FACTOR THEOREM

If $P(x)$ denotes an algebraic polynomial which has the value zero when $x = \alpha$, then $x - \alpha$ is a factor of the polynomial and
$P(x) \equiv (x - \alpha)\times$ another polynomial, $Q(x)$, of one degree lower.

$x = \alpha$ is called a **"root"** of the polynomial.

1.8.2 APPLICATION TO QUADRATIC AND CUBIC EXPRESSIONS

(a) Quadratic Expressions

To locate a root, try $x = 0, 1, -1, 2, -2, 3, -3, 4, -4, ...$

EXAMPLES

1. $x^2 + 2x - 3$ is zero when $x = 1$; hence $x - 1$ is a factor.
   The complete factorisation is $(x - 1)(x + 3)$.
2. $3x^2 + 20x - 7$ is zero when $x = -7$; hence $(x + 7)$ is a factor.
   The complete factorisation is
   $$(x + 7)(3x - 1).$$

(b) Cubic Expressions

Standard form is

$$ax^3 + bx^2 + cx + d.$$

EXAMPLES

1. $x^3 + 3x^2 - x - 3$ is zero when $x = 1$.
   Hence, $(x - 1)$ is a factor.

Thus,

$$x^3 + 3x^2 - x - 3 \equiv (x-1)(px^2 + qx + r)$$

for some constants p, q and r.

Comparing coefficients on both sides,

$$x^3 + 3x^2 - x - 3 \equiv (x-1)(x^2 + 4x + 3)$$

$$\equiv (x-1)(x+1)(x+3).$$

2. $x^3 + 4x^2 + 4x + 1$ is zero when $x = -1$ and so $x + 1$ must be a factor.

Hence

$$x^3 + 4x^2 + 4x + 1 \equiv (x+1)(px^2 + qx + r)$$

for some constants p, q and r.

Comparing coefficients on both sides,

$$x^3 + 4x^2 + 4x + 1 \equiv (x+1)(x^2 + 3x + 1).$$

1.8.3 CUBIC EQUATIONS

EXAMPLES

1. Solve the cubic equation

$$x^3 + 3x^2 - x - 3 = 0.$$

Solution

One solution is $x = 1$ and so $(x - 1)$ must be a factor.

$$(x - 1)(x^2 + 4x + 3) = 0;$$

$$(x - 1)(x + 1)(x + 3) = 0.$$

Solutions are $x = 1$, $x = -1$ and $x = -3$.

2. Solve the cubic equation

$$2x^3 - 7x^2 + 5x + 54 = 0.$$

Solution

One solution is $x = -2$ and so $(x+2)$ must be a factor.

$$(x + 2)(2x^2 - 11x + 27) = 0;$$

$$x = -2 \quad \text{or} \quad x = \frac{11 \pm \sqrt{121 - 216}}{4} \quad \text{not} \quad \text{real}.$$

1.8.4 LONG DIVISION OF POLYNOMIALS

(a) Exact Division

EXAMPLES

1. Divide the cubic expression $x^3 + 3x^2 - x - 3$ by $x - 1$.

   Solution

$$
\begin{array}{r}
x^2 + 4x + 3 \\
x - 1\overline{)x^3 + 3x^2 - x - 3} \\
\underline{x^3 - x^2 } \\
4x^2 - x - 3 \\
\underline{4x^2 - 4x } \\
3x - 3 \\
\underline{3x - 3} \\
0
\end{array}
$$

   Hence,

   $x^3+3x^2-x-3 \equiv (x-1)(x^2+4x+3) \equiv (x-1)(x+1)(x+3)$

2. Solve, completely, the cubic equation

$$x^3 + 4x^2 + 4x + 1 = 0$$

   Solution

   One solution is $x = -1$ so that $(x + 1)$ is a factor.

$$
\begin{array}{r}
x^2 + 3x + 1 \\
x+1\,\overline{)\,x^3 + 4x^2 + 4x + 1} \\
\underline{x^3 + x^2 } \\
3x^2 + 4x + 1 \\
\underline{3x^2 + 3x } \\
x + 1 \\
\underline{x + 1} \\
0
\end{array}
$$

Hence,

$$(x+1)(x^2+3x+1) = 0;$$

$x = -1$ and $x = \frac{-3\pm\sqrt{9-4}}{2} \simeq -0.382 \quad \text{or} \quad -2.618$

(b) Non-exact Division

Here, the remainder will not be zero.

EXAMPLES

1. Divide the polynomial $6x+5$ by the polynomial $3x-1$

   Solution

$$
\begin{array}{r}
2 \\
3x-1\,\overline{)\,6x + 5} \\
\underline{6x - 2} \\
7
\end{array}
$$

Hence,

$$\frac{6x+5}{3x-1} \equiv 2 + \frac{7}{3x-1}.$$

2. Divide $3x^2 + 2x$ by $x + 1$.

Solution

$$\begin{array}{r l}
 & \quad\quad\quad 3x - 1 \\
x+1 & \overline{)\,3x^2 + 2x} \\
 & \underline{3x^2 + 3x} \\
 & \quad\quad -x \\
 & \quad\quad \underline{-x - 1} \\
 & \quad\quad\quad 1
\end{array}$$

Hence,

$$\frac{3x^2+2x}{x+1} \equiv 3x - 1 + \frac{1}{x+1}.$$

3. Divide $x^4 + 2x^3 - 2x^2 + 4x - 1$ by $x^2 + 2x - 3$.

Solution

$$\begin{array}{r l}
 & \quad\quad\quad\quad x^2 \quad\quad + 1 \\
x^2+2x-3 & \overline{)\,x^4 + 2x^3 - 2x^2 + 4x - 1} \\
 & \underline{x^4 + 2x^3 - 3x^2} \\
 & \quad\quad\quad\quad x^2 + 4x - 1 \\
 & \quad\quad\quad\quad \underline{x^2 + 2x - 3} \\
 & \quad\quad\quad\quad\quad\quad 2x + 2
\end{array}$$

Hence

$$\frac{x^4+2x^3-2x^2+4x-1}{x^2+2x-3} \equiv x^2 + 1 + \frac{2x+2}{x^2+2x-3}.$$

"JUST THE MATHS"

SLIDES NUMBER

1.9

ALGEBRA 9
(The theory of partial fractions)

by

A.J.Hobson

UNIT 1.9 - ALGEBRA 9
THE THEORY OF PARTIAL FRACTIONS

1.9.1 INTRODUCTION

Applies chiefly to a **"Proper Rational Function"** (numerator powers lower than denominator powers)

"Improper Rational Function" = the sum of a polynomial and a proper rational function (by long division)

ILLUSTRATION

$$\frac{1}{2x+3} + \frac{3}{x-1} \equiv \frac{7x+8}{(2x+3)(x-1)}.$$

1.9.2 STANDARD TYPES OF PARTIAL FRACTION PROBLEM

(a) Denominator of the given rational function has all linear factors.

EXAMPLE

$$\frac{7x+8}{(2x+3)(x-1)} \equiv \frac{A}{2x+3} + \frac{B}{x-1}.$$

Solution

$$7x + 8 \equiv A(x-1) + B(2x+3).$$

Substituting $x = 1$ gives

$$7 + 8 = B(2 + 3).$$

Hence,

$$B = \frac{7 + 8}{2 + 3} = \frac{15}{5} = 3.$$

Substituting $x = -\frac{3}{2}$ gives

$$7 \times -\frac{3}{2} + 8 = A(-\frac{3}{2} - 1).$$

Hence,

$$A = \frac{7 \times -\frac{3}{2} + 8}{-\frac{3}{2} - 1} = \frac{-\frac{5}{2}}{-\frac{5}{2}} = 1.$$

$$\frac{7x + 8}{(2x + 3)(x - 1)} = \frac{1}{2x + 3} + \frac{3}{x - 1}.$$

Alternatively, use the "Cover-up" Rule

(b) Denominator of the given rational function contains one linear and one quadratic factor

EXAMPLE

$$\frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)} \equiv \frac{A}{x - 5} + \frac{Bx + C}{x^2 + 2x + 7}.$$

Solution

$$3x^2 + 9 \equiv A(x^2 + 2x + 7) + (Bx + C)(x - 5).$$

$x = 5$ gives

$$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7);$$

So, $84 = 42A$ or $A = 2$.

Equating coefficients of x^2, $3 = A + B$ and hence $B = 1$.

Equating constant terms (the coefficients of x^0), $9 = 7A - 5C = 14 - 5C$ and hence $C = 1$.

Therefore,

$$\frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)} \equiv \frac{2}{x - 5} + \frac{x + 1}{x^2 + 2x + 7}.$$

Note: A may be found by the cover-up rule, B and C by inspection.

(c) Denominator of the given rational function contains a repeated linear factor

EXAMPLE

$$\frac{9}{(x+1)^2(x-2)}.$$

Solution

First observe that

$$\frac{Ax+B}{(x+1)^2}.$$

may be written

$$\frac{A(x+1)+B-A}{(x+1)^2} \equiv \frac{A}{x+1}+\frac{B-A}{(x+1)^2} \equiv \frac{A}{x+1}+\frac{C}{(x+1)^2}.$$

Therefore, write

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-2}.$$

$$9 \equiv A(x+1)(x-2) + C(x-2) + D(x+1)^2.$$

$x = -1$ gives $9 = -3C$ so that $C = -3$.

$x = 2$ gives $9 = 9D$ so that $D = 1$.

Equating coefficients of x^2 gives $0 = A + D$ so that $A = -1$.

Hence,

$$\frac{9}{(x+1)^2(x-2)} \equiv -\frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}.$$

Note: D could have been obtained by the cover-up rule.

(d) Keily's Method (uses Cover-up Rule)

EXAMPLE

$$\frac{9}{(x+1)^2(x-2)}.$$

Solution

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right].$$

$$\equiv \frac{1}{x+1}\left[\frac{-3}{x+1} + \frac{3}{x-2}\right].$$

$$\equiv -\frac{3}{(x+1)^2} + \frac{3}{(x+1)(x-2)}.$$

$$\equiv -\frac{3}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-2}$$

as before.

Warning

$$\frac{9x^2}{(x+1)^2(x-2)}$$

leads to an improper rational function

$$\frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right].$$

"JUST THE MATHS"

SLIDES NUMBER

1.10

ALGEBRA 10
(Inequalities 1)

by

A.J.Hobson

UNIT 1.10 - ALGEBRA 10
INEQUALITIES 1.

1.10.1 INTRODUCTION

$a < b$ means "a is less than b".

$b > a$ means "b is greater than a".

These are **"strict inequalities"**.

eg. if a is the number of days in a particular month and b is the number of hours in that month, then $b > a$.

"Weak inequalities" are written in one of the forms

$$a \leq b \quad b \leq a \quad a \geq b \quad b \geq a.$$

eg. if a is the number of students who enrolled for a particular module in a university and b is the number of students who eventually passed that module, then $a \geq b$.

1.10.2 ALGEBRAIC RULES FOR INEQUALITIES

Suppose $a < b$; then,

1. $a + c < b + c$ for any number c.

2. $ac < bc$ when c is positive but $ac > bc$ when c is negative.

3. $\frac{1}{a} > \frac{1}{b}$ provided a and b are **both positive.**

Note:
If a is negative and b is positive, $\frac{1}{a} < \frac{1}{b}$

EXAMPLES

1. Simplify the inequality

$$2x + 3y > 5x - y + 7.$$

Solution

$$-3x + 4y > 7 \quad \text{or} \quad 3x - 4y + 7 < 0$$

2. Solve the inequality

$$\frac{1}{x-1} < 2$$

assuming that $x \neq 1$.

Solution

(a) If $x > 1$, i.e. $x - 1$ is positive,

$$1 < 2(x-1) \quad \text{or} \quad x - 1 > \frac{1}{2}.$$

Hence,

$$x > \frac{3}{2}.$$

(b) If $x < 1$, i.e. $x - 1$ is negative,

Any negative number is bound to be less than a positive number.

Hence inequality always true when $x < 1$.

Conclusion:

$x < 1$ and $x > \frac{3}{2}$.

3. Solve the inequality

$$2x - 7 \leq 3.$$

Solution

Adding 7, then dividing by 2 gives $x \leq 5$.

4. Solve the inequality

$$\frac{x-1}{x-6} \geq 0$$

Solution

Left hand side equals zero only when $x = 1$.

(a) If $x - 1 > 0$ and $x - 6 > 0$, then $x > 6$.

(b) If $x - 1 < 0$ and $x - 6 < 0$; then $x < 1$.

Note:

As x passes through 6 there is a sudden change from $-\infty$ to $+\infty$.

Conclusion:

The inequality is satisfied when either $x > 6$ or $x \leq 1$.

1.10.3 INTERVALS

(a) $a < x < b$ denotes an **"open interval"** of all the values of x between a and b but excluding a and b themselves.

(a, b) also means the same thing.

eg. if x is a purely decimal quantity, $-1 < x < 1$.

(b) $a \leq x \leq b$ denotes a **"closed interval"** of all the values of x from a to b inclusive.

$[a, b]$ also means the same thing.

eg. $\sqrt{1 - x^2}$ has real values only when $-1 \leq x \leq 1$.

Note: $a < x \leq b$ or $a \leq x < b$ i.e. $(a, b]$ and $[a, b)$ are **"half-open"** or **"half-closed"**.

(c) $x > a$ $x \geq a$ $x < a$ $x \leq a$ are called **"infinite intervals"**.

"JUST THE MATHS"

SLIDES NUMBER

1.11

ALGEBRA 11
Inequalities 2

by

A.J.Hobson

UNIT 1.11 - ALGEBRA 11
INEQUALITIES 2.

1.11.1 RECAP ON MODULUS, ABSOLUTE VALUE OR NUMERICAL VALUE

$$| x |= x \quad \text{if} \quad x \geq 0;$$

$$| x |= -x \quad \text{if} \quad x \leq 0.$$

Notes:

(i) Alternatively $| x |= +\sqrt{x^2}$;

(ii) It can be shown that, $| a + b | \leq | a | + | b |$; the **"Triangle Inequality"**.

1.11.2 INTERVAL INEQUALITIES

(a) Using the Modulus notation

We investigate the inequality

$$| x - a |< k,$$

where a is any number and k is a positive number.

Case 1. $x - a > 0$**.**

$$x - a < k, \quad \text{that is,} \quad x < a + k.$$

Case 2. $x - a < 0$**.**

$$-(x-a) < k, \quad \text{that is,} \quad a - x < k \quad \text{or} \quad x > a - k.$$

Conclusion

$$\mid x - a \mid < k \quad \text{means} \quad a - k < x < a + k.$$

Similarly for $\mid x - a \mid \leq k$.

EXAMPLE

Obtain the closed interval represented by the statement

$$\mid x + 3 \mid \leq 10$$

Solution

Using $a = -3$ and $k = 10$, we have

$$-3 - 10 \leq x \leq -3 + 10.$$

That is,

$$-13 \leq x \leq 7.$$

(b) Using Factorised Polynomials

EXAMPLE

Find the range of values of x for which

$$(x+3)(x-1)(x-2) > 0$$

Solution

Critical values are $x = -3, 1, 2$ dividing the line into
$x < -3, \quad -3 < x < 1, \quad 1 < x < 2, \quad x > 2;$

$x < -3$ gives (neg)(neg)(neg) and therefore < 0;

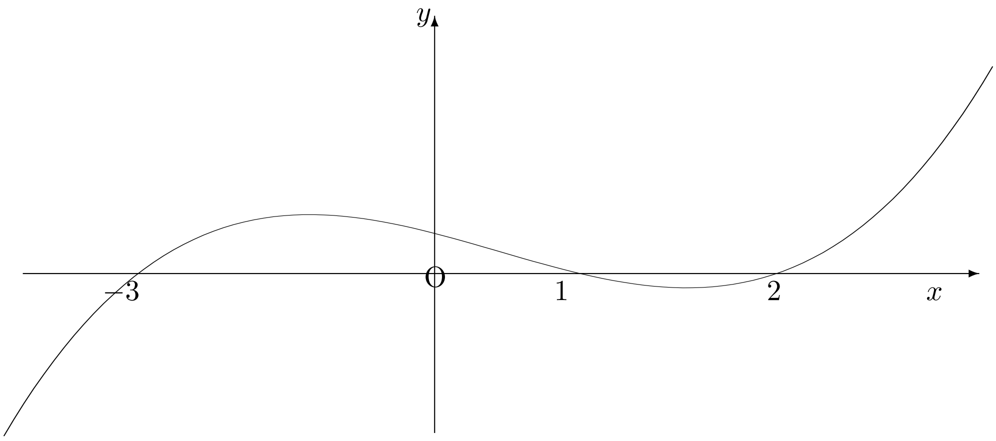
$-3 < x < 1$ gives (pos)(neg)(neg) and therefore > 0;

$1 < x < 2$ gives (pos)(pos)(neg) and therefore < 0;

$x > 2$ gives (pos)(pos)(pos) and therefore > 0.

$$\text{Ans}: \quad -3 < x < 1 \quad \text{and} \quad x > 2.$$

Note:
Alternatively, sketch the graph of the polynomial.



"JUST THE MATHS"

SLIDES NUMBER

2.1

SERIES 1
(Elementary progressions and series)

by

A.J.Hobson

UNIT 2.1 - SERIES 1
ELEMENTARY PROGRESSIONS AND SERIES

2.1.1 ARITHMETIC PROGRESSIONS

The **"sequence"** of numbers,

$$a, a + d, a + 2d, a + 3d, ...$$

is said to form an **"arithmetic progression"**.

The symbol a represents the **"first term"**.

The symbol d represents the **"common difference"**

The **"n-th term"** is given by the expression

$$a + (n - 1)d.$$

EXAMPLES

1. Determine the n-th term of the arithmetic progression

$$15, 12, 9, 6, ...$$

Solution

The n-th term is

$$15 + (n - 1)(-3) = 18 - 3n.$$

2. Determine the n-th term of the arithmetic progression

$$8, 8.125, 8.25, 8.375, 8.5, ...$$

Solution

The n-th term is

$$8 + (n-1)(0.125) = 7.875 + 0.125n.$$

3. The 13th term of an arithmetic progression is 10 and the 25th term is 20; calculate

   (a) the common difference;

   (b) the first term;

   (c) the 17th term.

Solution

$$a + 12d = 10$$

and

$$a + 24d = 20.$$

Hence,

(a) $12d = 10$, so $d = \frac{10}{12} = \frac{5}{6} \simeq 0.83$

(b) $a + 12 \times \frac{5}{6} = 10$, so $a + 10 = 10$ and $a = 0$.

(c) 17th term $= 0 + 16 \times \frac{5}{6} = \frac{80}{6} = \frac{40}{3} \simeq 13.3$

2.1.2 ARITHMETIC SERIES

If the terms of an arithmetic progression are added together, we obtain an **"arithmetic series"**

The total sum of the first n terms is denoted by S_n.

$S_n = a+[a+d]+[a+2d]+...+[a+(n-2)d]+[a+(n-1)d].$

TRICK

Write down the formula **forwards and backwards**.

$S_n = a+[a+d]+[a+2d]+...+[a+(n-2)d]+[a+(n-1)d].$

$S_n = [a+(n-1)d]+[a+(n-2)d]+...+[a+2d]+[a+d]+a.$

Adding gives $2S_n =$

$$[2a+(n-1)d]+...+[2a+(n-1)d]+[2a+(n-1)d].$$

R.H.S gives n repetitions of the same expression.

Hence,

$$2S_n = n[2a+(n-1)d]$$

or

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

Alternatively,

$$S_n = \frac{n}{2}[\text{FIRST} \quad + \quad \text{LAST}].$$

This is n times the average of the first and last terms.

EXAMPLES

1. Determine the sum of the natural numbers from 1 to 100.

   Solution

   The sum is given by

   $$\frac{100}{2} \times [1 + 100] = 5050.$$

2. How many terms of the arithmetic series

   $$10 + 12 + 14 + ...$$

   must be taken so that the sum of the series is 252 ?

   Solution

   The first term is clearly 10 and the common difference is 2.

   If n is the required number of terms,

$$252 = \frac{n}{2}[20 + (n-1) \times 2];$$

$$252 = \frac{n}{2}[2n + 18] = n(n+9);$$

$$n^2 + 9n - 252 = 0$$

or

$$(n-12)(n+21) = 0;$$

$n = 12$ ignoring $n = -21$.

3. A contractor agrees to sink a well 250 metres deep at a cost of £2.70 for the first metre, £2.85 for the second metre and an extra 15p for each additional metre. Find the cost of the last metre and the total cost.

   Solution

   We need an arithmetic series of 250 terms whose first term is 2.70 and whose common difference is 0.15

   The cost of the last metre is the 250-th term.

   Cost of last metre $= £[2.70 + 249 \times 0.15] = £40.05$

   The total cost $= £\frac{250}{2} \times [2.70 + 40.05] = £5343.75$

2.1.3 GEOMETRIC PROGRESSIONS

The sequence of numbers

$$a, ar, ar^2, ar^3, \ldots$$

is said to form a **"geometric progression"**.

The symbol a represents the **"first term"**.

The symbol r represents the **"common ratio"**.

The **"n-th term"** is given by the expression

$$ar^{n-1}.$$

EXAMPLES

1. Determine the n-th term of the geometric progression

$$3, -12, 48, -192, \ldots$$

Solution

The n-th term is

$$3(-4)^{n-1}.$$

This will be positive when n is an odd number and negative when n is an even number.

2. Determine the seventh term of the geometric progression

$$3, 6, 12, 24, \ldots$$

Solution

The seventh term is

$$3(2^6) = 192.$$

3. The third term of a geometric progression is 4.5 and the ninth is 16.2. Determine the common ratio.

   Solution

$$ar^2 = 4.5$$

and

$$ar^8 = 16.2$$

$$\frac{ar^8}{ar^2} = \frac{16.2}{4.5}$$

$$r^6 = 3.6$$

$$r = \sqrt[6]{3.6} \simeq 1.238$$

4. The expenses of a company are £200,000 a year. It is decided that each year they shall be reduced by 5% of those for the preceding year.
   What will be the expenses during the fourth year, the first reduction taking place at the end of the first year.

   Solution

   We use a geometric progression with first term 200,000 and common ratio 0.95

The expenses during the fourth year will be the fourth term of the progression.
Expenses in fourth year = £200,000 $\times(0.95)^3$ = £171475.

2.1.4 GEOMETRIC SERIES

If the terms of a geometric progression are added together, we obtain what is called a **"geometric series"**.

The total sum of a geometric series with n terms is denoted by the S_n.

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}.$$

TRICK

Write down both S_n and rS_n.

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n;$$

$$S_n - rS_n = a - ar^n;$$

$$S_n = \frac{a(1-r^n)}{1-r}.$$

Alternatively (eg. when $r > 1$)

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

EXAMPLES

1. Determine the sum of the geometric series
$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$
Solution
The sum is given by
$$S_6 = \frac{4(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{4(1 - 0.0156)}{0.5} \simeq 7.875$$

2. A sum of money £C is invested for n years at an interest of $100r\%$, compounded annually. What will be the total interest earned by the end of the n-th year ?
Solution
At the end of year 1, the interest earned will be Cr.
At the end of year 2, the interest earned will be $(C + Cr)r = Cr(1 + r)$.
At the end of year 3, the interest earned will be $C(1 + r)r + C(1 + r)r^2 = Cr(1 + r)^2$.
At the end of year n, the interest earned will be $Cr(1 + r)^{n-1}$.

The total interest earned by the end of year n will be

$$Cr + Cr(1+r) + Cr(1+r)^2 + ... + Cr(1+r)^{n-1}.$$

This is a geometric series of n terms with first term Cr and common ratio $1+r$.

The total interest earned by the end of year n will be

$$\frac{Cr((1+r)^n - 1)}{r} = C((1+r)^n - 1).$$

Note:

The same result can be obtained using only a geometric progression:

At the end of year 1, the total amount will be $C + Cr = C(1+r)$.

At the end of year 2, the total amount will be $C(1+r) + C(1+r)r = C(1+r)^2$.

At the end of year 3, the total amount will be $C(1+r)^2 + C(1+r)^2 r = C(1+r)^3$.

At the end of year n, the total amount will be $C(1+r)^n$.

Total interest earned will be $C(1+r)^n - C = C((1+r)^n - 1)$ as before.

The sum to infinity of a geometric series

In a geometric series with n terms, suppose $|r| < 1$.

As n approaches ∞, r^n approaches 0.

Hence,

$$S_\infty = \frac{a}{1 - r}.$$

EXAMPLES

1. Determine the sum to infinity of the geometric series

$$5 - 1 + \frac{1}{5} - \ldots\ldots$$

   Solution

   The sum to infinity is

$$\frac{5}{1 + \frac{1}{5}} = \frac{25}{6} \simeq 4.17$$

2. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If, in a certain year, its output was £25,000, what could be reckoned as its total future output ?

Solution

The total output, in pounds, for subsequent years will be given by

$$25000 \times 0.75 + 25000 \times (0.75)^2 + 25000 \times (0.75)^3 + ...$$

$$= \frac{25000 \times 0.75}{1 - 0.75} = 75000.$$

2.1.5 MORE GENERAL PROGRESSIONS AND SERIES

Introduction

Not all progressions and series encountered in mathematics are either arithmetic or geometric.

For example

$$1^2, 2^2, 3^2, 4^2, ..., n^2$$

is not arithmetic or geometric.

An **arbitrary** progression of n numbers which conform to some regular pattern is often denoted by

$$u_1, u_2, u_3, u_4, ..., u_n.$$

There may or may not be a simple formula for S_n.

The Sigma Notation (Σ).

1.

$$a+(a+d)+(a+2d)+......+(a+[n-1]d) = \sum_{r=1}^{n} (a+[r-1]d).$$

2.

$$a + ar + ar^2 + ...ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

3.

$$1^2 + 2^2 + 3^2 + ...n^2 = \sum_{r=1}^{n} r^2.$$

4.

$$-1^3 + 2^3 - 3^3 + 4^3 + ...(-1)^n n^3 = \sum_{r=1}^{n} (-1)^r r^3.$$

Notes:
(i) We sometimes count the terms of a series from zero rather than 1.

For example:

$$a + (a + d) + (a + 2d) + ...a + [n - 1]d = \sum_{r=0}^{n-1} (a + rd).$$

$$a + ar + ar^2 + ar^3 + ...ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

In general,

$$u_0 + u_1 + u_2 + u_3 + ... + u_{n-1} = \sum_{r=0}^{n-1} u_r.$$

(ii) We may also use the sigma notation for **"infinite series"**

For example:

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \quad \text{or} \quad \sum_{r=0}^{\infty} \frac{1}{3^r}.$$

STANDARD RESULTS

It may be shown that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

The first of these is the formula for the sum of an arithmetic series with first term 1 and last term n.

The second is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

The third is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1.$$

EXAMPLE

Determine the sum to n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + \ldots$$

Solution

The series is

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r$$

$$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r.$$

Using the three standard results, the summation becomes

$$\left[\frac{1}{2}n(n+1)\right]^2 + 3\left[\frac{1}{6}n(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{4}n(n+1)[n(n+1)+4n+2+4] = \frac{1}{4}n(n+1)[n^2+5n+6].$$

This simplifies to

$$\frac{1}{4}n(n+1)(n+2)(n+3).$$

"JUST THE MATHS"

SLIDES NUMBER

2.2

SERIES 2
(Binomial series)

by

A.J.Hobson

UNIT 2.2 - SERIES 2 - BINOMIAL SERIES

INTRODUCTION

In this section, we expand (multiplying out) an expression of the form

$$(A + B)^n.$$

A and B can be either mathematical expressions or numerical values.

n is a given number which need not be a positive integer.

2.2.1 PASCAL'S TRIANGLE

ILLUSTRATIONS

1. $(A + B)^1 \equiv$

$$A + B.$$

2. $(A + B)^2 \equiv$

$$A^2 + 2AB + B^2.$$

3. $(A + B)^3 \equiv$

$$A^3 + 3A^2B + 3AB^2 + B^3.$$

4. $(A + B)^4 \equiv$

$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4.$$

OBSERVATIONS

(i) The expansions begin with the maximum possible power of A and end with the maximum possible power of B.

(ii) The powers of A **decrease** in steps of 1 while the powers of B **increase** in steps of 1.

(iii) The coefficients follow the diagramatic pattern called

PASCAL'S TRIANGLE:

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

Each line begins and ends with the number 1.

Each of the other numbers is the sum of the two numbers above it in the previous line.

The next line would be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

5. $(A + B)^5 \equiv$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$

(iv) For

$$(A - B)^n$$

the terms are alternately positive and negative.

6. $(A-B)^6 \equiv$

$$A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6.$$

2.2.2 BINOMIAL FORMULAE

A more general method which can be applied to any value of n is the binomial formula.

DEFINITION

If n is a positive integer, the product

$$1.2.3.4.5..........n$$

is denoted by the symbol $n!$ and is called
"n factorial".

Note:

This definition could not be applied to the case when $n = 0$.

$0!$ is defined separately by the statement

$$0! = 1.$$

There is no meaning to $n!$ when n is a negative integer.

(a) Binomial formula for $(A+B)^n$ when n is a positive integer.

It can be shown that

$$(A+B)^n \equiv$$

$$A^n + nA^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2 +$$

$$\frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + ... + B^n.$$

Notes:

(i) This is the same result as given by Pascal's Triangle.

(ii) The last term is

$$\frac{n(n-1)(n-2)(n-3).......3.2.1}{n!}A^{n-n}B^n = A^0B^n = B^n.$$

(iii) The coefficient of $A^{n-r}B^r$ in the expansion is

$$\frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

and this is sometimes denoted by the symbol $\binom{n}{r}$.

(iv) A commonly used version is

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...+x^n.$$

EXAMPLES

1. Expand fully the expression $(1+2x)^3$.

   Solution

   $$(A+B)^3 \equiv A^3+3A^2B+3AB^2+B^3.$$

   Replace A by 1 and B by $2x$.

   $$(1+2x)^3 \equiv 1+3(2x)+3(2x)^2+(2x)^3$$

   $$\equiv 1+6x+12x^2+8x^3.$$

2. Expand fully the expression $(2-x)^5$.

   Solution

   $$(A+B)^5 \equiv A^5+5A^4B+10A^3B^2+10A^2B^3+5AB^4+B^5.$$

   Replace A by 2 and B by $-x$.

   $$(2-x)^5 \equiv 2^5+5(2)^4(-x)+10(2)^3(-x)^2+$$

   $$10(2)^2(-x)^3+5(2)(-x)^4+(-x)^5.$$

   That is,

   $$(2-x)^5 \equiv 32-80x+80x^2-40x^3+10x^4-x^5.$$

(b) Binomial formula for $(A+B)^n$ when n is negative or a fraction.

This time, the series will be an **infinite** series.

RESULT

If n is negative or a fraction and x lies strictly between $x=-1$ and $x=1$, it can be shown that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+......$$

EXAMPLES

1. Expand $(1+x)^{\frac{1}{2}}$ as far as the term in x^3.

   Solution

   $$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3+......$$

   $$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-......$$

   provided $-1 < x < 1$.

2. Expand $(2-x)^{-3}$ as far as the term in x^3 stating the values of x for which the series is valid.

Solution

First convert the expression $(2-x)^{-3}$ to one in which the leading term in the bracket is 1.

$$(2-x)^{-3} \equiv \left[2\left(1-\frac{x}{2}\right)\right]^{-3}$$

$$\equiv \frac{1}{8}\left(1+\left[-\frac{x}{2}\right]\right)^{-3}.$$

The required binomial expansion is

$$\frac{1}{8}[1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{x}{2}\right)^2+$$

$$\frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{x}{2}\right)^3+......].$$

That is,

$$\frac{1}{8}\left[1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+......\right].$$

The expansion is valid provided $-x/2$ lies strictly between -1 and 1.

Hence, $-2 < x < 2$.

(c) Approximate Values

The Binomial Series may be used to calculate simple approximations, as illustrated by the following example:

EXAMPLE

Evaluate $\sqrt{1.02}$ correct to five places of decimals.

Solution

Using $1.02 = 1 + 0.02$, we may say that

$$\sqrt{1.02} = (1 + 0.02)^{\frac{1}{2}}.$$

That is,

$$\sqrt{1.02} = 1 + \frac{1}{2}(0.02) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1.2}(0.02)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2.3}(0.02)^3 + \ldots$$

$$= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) - \ldots$$

$$= 1 + 0.01 - 0.00005 + 0.0000005 - \ \ldots$$

$$\simeq 1.010001 - 0.000050 = 1.009951$$

Hence, $\sqrt{1.02} \simeq 1.00995$

"JUST THE MATHS"

SLIDES NUMBER

2.3

SERIES 3
(Elementary convergence and divergence)

by

A.J.Hobson

UNIT 2.3 - ELEMENTARY CONVERGENCE AND DIVERGENCE

Introduction

An infinite series may be specified by either

$$u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

or

$$u_0 + u_1 + u_2 + \ldots = \sum_{r=0}^{\infty} u_r.$$

In the first of these, u_r is the r-th term while, in the second, u_r is the $(r+1)$-th term.

ILLUSTRATIONS

1.
$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} = \sum_{r=0}^{\infty} \frac{1}{r+1}.$$

2.
$$2 + 4 + 6 + 8 + \ldots = \sum_{r=1}^{\infty} 2r = \sum_{r=0}^{\infty} 2(r+1).$$

3.
$$1 + 3 + 5 + 7 + \ldots = \sum_{r=1}^{\infty} (2r-1) = \sum_{r=0}^{\infty} (2r+1).$$

2.3.1 THE DEFINITIONS OF CONVERGENCE AND DIVERGENCE

An infinite series may have a **"sum to infinity"** even though it is not possible to reach the end of the series.

For example, in

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \ = \sum_{r=1}^{\infty} \frac{1}{2^r},$$

$$S_n = \frac{\frac{1}{2}(1-\frac{1}{2}^n)}{1-\frac{1}{2}} = 1 - \frac{1}{2^n}.$$

As n becomes larger and larger, S_n approaches 1.

We say that the **"limiting value"** of S_n as n **"tends to infinity"** is 1; and we write

$$\lim_{n\to\infty} S_n = 1.$$

Since this limiting value is a **finite** number, we say that the series **"converges"** to 1.

DEFINITION (A)
For the infinite series

$$\sum_{r=1}^{\infty} u_r,$$

the expression

$$u_1 + u_2 + u_3 + \ldots + u_n$$

is called its **"n-th partial sum"**.

DEFINITION (B)
If the n-th Partial Sum of an infinite series tends to a finite limit as n tends to infinity, the series is said to **"converge"**. In **all** other cases, the series is said to **"diverge"**.

ILLUSTRATIONS

1.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{r=1}^{\infty} \frac{1}{2^r} \text{ converges.}$$

2.

$$1 + 2 + 3 + 4 + \ldots\ldots = \sum_{r=1}^{\infty} r \text{ diverges.}$$

3.

$$1 - 1 + 1 - 1 + \ldots\ldots = \sum_{r=1}^{\infty} (-1)^{n-1} \text{ diverges.}$$

Notes:

(i) Illustration 3 shows that a series which diverges does not necessarily diverge to infinity.

(ii) Whether a series converges or diverges depends less on the starting terms than it does on the later terms.

For example

$$7 - 15 + 2 + 39 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ . \ .$$

converges to $7 - 15 + 2 + 39 + 1 = 33 + 1 = 34$.

(iii) It is sometimes possible to test an infinite series for convergence or divergence without having to determine its sum to infinity.

2.3.2 TESTS FOR CONVERGENCE AND DIVERGENCE

First, we shall consider series of **positive** terms only.

TEST 1 - The r-th Term Test

An infinite series,

$$\sum_{r=1}^{\infty} u_r,$$

cannot converge unless

$$\lim_{r \to \infty} u_r = 0.$$

Outline Proof:

The series will converge only if the r-th partial sums, S_r, tend to a finite limit, L (say), as r tends to infinity.

Since $u_r = S_r - S_{r-1}$, then u_r must tend to $L - L = 0$ as r tends to infinity.

ILLUSTRATIONS

1. The convergent series

$$\sum_{r=1}^{\infty} \frac{1}{2^r}$$

is such that

$$\lim_{r \to \infty} \frac{1}{2^r} = 0.$$

2. The divergent series

$$\sum_{r=1}^{\infty} r$$

is such that

$$\lim_{r \to \infty} r \neq 0.$$

3. The series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is such that

$$\lim_{r \to \infty} \frac{1}{r} = 0,$$

but this series is **divergent** (see later).

N.B. The converse of the r-th Term Test is not true.

TEST 2 - The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\sum_{r=1}^{\infty} v_r$$

is a second series which is known to **converge**.

Then the first series converges provided that $u_r \leq v_r$.

Similarly, suppose

$$\sum_{r=1}^{\infty} w_r$$

is a series which is known to **diverge**

Then the first series diverges provided that $u_r \geq w_r$.

Note:
It may be necessary to ignore the first few values of r.

Outline Proof of Comparison Test:

Think of u_r and v_r as the heights of two sets of rectangles, all with a common base-length of one unit.

(i) If the series

$$\sum_{r=1}^{\infty} v_r$$

is **convergent** it represents a **finite** total area of an infinite number of rectangles.

The series

$$\sum_{r=1}^{\infty} u_r$$

represents a **smaller** area and, hence, is also finite.

(ii) A similar argument holds when

$$\sum_{r=1}^{\infty} w_r$$

is a **divergent** series and $u_r \geq w_r$.

A divergent series of **positive** terms can diverge only to $+\infty$ so that the set of rectangles determined by u_r generates an area that is greater than an area which is already infinite.

EXAMPLES

1. Show that the series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ ..$$

is divergent.

Solution

The given series may be written as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ \ldots$$

a series whose terms are all $\geq \frac{1}{2}$.

But the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ ..$$

is a divergent series and, hence, the series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent.

2. Given that

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is a convergent series, show that

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

is also a convergent series.

Solution

Firstly, for $r = 1, 2, 3, 4, \ldots,$

$$\frac{1}{r(r+1)} < \frac{1}{r.r} = \frac{1}{r^2}.$$

Hence the terms of the series

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

are smaller in value than those of a known convergent series. It therefore converges also.

Note:

It may be shown that the series

$$\sum_{r=1}^{\infty} \frac{1}{r^p}$$

is convergent whenever $p > 1$ and divergent whenever $p \leq 1$.

TEST 3 - D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = L;$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

There is no conclusion if $L = 1$.

Outline Proof:

(i) If $L > 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be greater than 1.

Thus, $u_{r+1} > u_r$ for a large enough value of r.

Hence, the terms cannot ultimately be decreasing; so Test 1 shows that the series diverges.

(ii) If $L < 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be less than 1.

Thus, $u_{r+1} < u_r$ for a large enough value or r.

Let this occur first when $r = s$.

From this value onwards, the terms steadily decrease in value.

We can certainly find a positive number, h, between L and 1 such that

$$\frac{u_{s+1}}{u_s} < h, \frac{u_{s+2}}{u_{s+1}} < h, \frac{u_{s+3}}{u_{s+2}} < h, \ \ldots$$

That is,

$$u_{s+1} < hu_s, \ u_{s+2} < hu_{s+1}, \ u_{s+3} < hu_{s+2}, \ \ldots,$$

which gives

$$u_{s+1} < hu_s, \ u_{s+2} < h^2 u_s, \ u_{s+3} < h^3 u_s, \ \ldots$$

But, since $L < h < 1$,

$$hu_s + h^2 u_s + h^3 u_s + \ \ldots$$

is a convergent geometric series.

Therefore, by the Comparison Test,

$$u_{s+1} + u_{s+2} + u_{s+3} + \ \ldots = \sum_{r=1}^{\infty} \ u_{s+r} \ \text{ converges.}$$

(iii) If $L = 1$, there will be no conclusion since we have already encountered examples of both a convergent series **and** a divergent series which give $L = 1$.

In particular,

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r}{r+1} = \lim_{r \to \infty} \frac{1}{1 + \frac{1}{r}} = 1.$$

Also,

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r^2}{(r+1)^2} = \lim_{r \to \infty} \left(\frac{r}{r+1}\right)^2$$

$$= \lim_{r \to \infty} \left(\frac{1}{1 + \frac{1}{r}}\right)^2 = 1$$

Note:

To calculate the limit as r tends to infinity of any ratio of two polynomials in r, divide the numerator and the denominator by the highest power of r.

For example,

$$\lim_{r\to\infty} \frac{3r^3+1}{2r^3+1} = \lim_{r\to\infty} \frac{3+\frac{1}{r^3}}{2+\frac{1}{r^3}} = \frac{3}{2}.$$

ILLUSTRATIONS

1. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} \frac{r}{2^r},$$

$$\frac{u_{r+1}}{u_r} = \frac{r+1}{2^{r+1}}.\frac{2^r}{r} = \frac{r+1}{2r}.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r+1}{2r} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{2} = \frac{1}{2}.$$

The limiting value is less than 1 so that the series converges.

2. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} 2^r,$$

$$\frac{u_{r+1}}{u_r} = \frac{2^{r+1}}{2^r} = 2.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} 2 = 2.$$

The limiting value is greater than 1 so that the series diverges.

"JUST THE MATHS"

SLIDES NUMBER

2.4

SERIES 4
(Further convergence and divergence)

by

A.J.Hobson

UNIT 2.4 - SERIES 4 - FURTHER CONVERGENCE AND DIVERGENCE

2.4.1 SERIES OF POSITIVE AND NEGATIVE TERMS

TEST 1 - The r-th Term Test (Revisited)

The r-th Term Test in Unit 2.3 may used for series whose terms are not necessarily all positive.

Outline Proof:

The formula

$$u_r = S_r - S_{r-1}$$

is valid for any series.

The series cannot converge unless the partial sums S_r and S_{r-1} both tend to the same finite limit as r tends to infinity.

Hence, u_r tends to zero as r tends to infinity.

Alternating Series

A simple kind of series with both positive and negative terms is one whose terms are alternately positive and negative.

Test 4 - The Alternating Series Test

If

$$u_1 - u_2 + u_3 - u_4 + \ . \ . \ , \quad \text{where} \ \ u_r > 0,$$

is such that

$$u_r > u_{r+1} \ \ \textbf{and} \ \ u_r \to 0 \ \ \text{as} \ \ r \to \infty,$$

then the series converges.

Outline Proof:

(a) Re-group the series as

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ . \ . \ .;$$

That is,

$$\sum_{r=1}^{\infty} \ v_r$$

where $v_1 = u_1 - u_2, v_2 = u_3 - u_4, v_3 = u_5 - u_6, \ . \ . \ . \ .$

v_r is positive, so that

$$S_r = v_1 + v_2 + v_3 + \ . \ . \ . \ + v_r$$

increases as r increases.

(b) Alternatively, re-group the series as

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ . \ . \ . \ ;$$

That is,

$$u_1 - \sum_{r=1}^{\infty} w_r$$

where $w_1 = u_2 - u_3, w_2 = u_4 - u_5, w_3 = u_6 - u_7, \ . \ . \ . \ .$

$S_r = u_1 - (w_1 + w_2 + w_3 + \ . \ . \ . + w_r)$ is less than u_1 since positive quantities are being subtracted from it.

(c) We conclude that the partial sums of the original series are steadily increasing but are never greater than u_1.

They must therefore tend to a finite limit as r tends to infinity; that is, the series converges.

ILLUSTRATION

The series

$$\sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots$$

is convergent since

$$\frac{1}{r} > \frac{1}{r+1} \ \ \textbf{and} \ \ \frac{1}{r} \rightarrow 0 \ \text{ as } \ r \rightarrow \infty.$$

2.4.2 ABSOLUTE AND CONDITIONAL CONVERGENCE

DEFINITION (A)

If

$$\sum_{r=1}^{\infty} u_r$$

is a series with both positive and negative terms, it is said to be **"absolutely convergent"** if

$$\sum_{r=1}^{\infty} |u_r|$$

is convergent.

DEFINITION (B)

If

$$\sum_{r=1}^{\infty} u_r$$

is a convergent series of positive and negative terms, but

$$\sum_{r=1}^{\infty} |u_r|$$

is a divergent series, then the first of these two series is said to be **"conditionally convergent"**.

ILLUSTRATIONS

1. The series

$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

converges absolutely since the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

converges.

2. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ \ldots$$

is conditionally convergent.

It converges (by the Alternating Series Test), but the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots \ \text{diverges.}$$

Notes:

(i) It may be shown that any series of positive and negative terms which is **absolutely** convergent will also be convergent.

(ii) Any test for the convergence of a series of positive terms may be used as a test for the absolute convergence of a series of both positive and negative terms.

2.4.3 TESTS FOR ABSOLUTE CONVERGENCE

The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that $|u_r| \leq v_r$ where

$$\sum_{r=1}^{\infty} v_r$$

is a convergent series of positive terms. Then, the given series is absolutely convergent.

D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = L.$$

Then the given series is absolutely convergent if $L < 1$.

Note:

If $L > 1$, then $|u_{r+1}| > |u_r|$ for large enough values of r showing that the **numerical** values of the terms steadily increase.

This implies that u_r does **not** tend to zero as r tends to infinity

Hence, (by the r-th Term Test) the series diverges.

If $L = 1$, there is no conclusion.

EXAMPLES

1. Show that the series

$$\frac{1}{1\times 2}-\frac{1}{2\times 3}-\frac{1}{3\times 4}+\frac{1}{4\times 5}-\frac{1}{5\times 6}-\frac{1}{6\times 7}+\ ..$$

is absolutely convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{1}{r(r+1)}$$

This is always less than $\frac{1}{r^2}$, the r-th term of a known convergent series.

2. Show that the series

$$\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\frac{4}{17}+\ ..$$

is conditionally convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{r}{r^2+1},$$

which tends to zero as r tends to infinity.

Also,

$$\frac{r}{r^2+1} > \frac{r+1}{(r+1)^2+1}$$

since this may be reduced to the true statement $r^2 + r > 1$.

Hence, by the alternating series test, the series converges.

However,

$$\frac{r}{r^2+1} > \frac{r}{r^2+r} = \frac{1}{r+1}$$

and, hence, by the Comparison Test, the series of absolute values is divergent since

$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$

is divergent.

2.4.4 POWER SERIES

A series of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ .\ .\ =\sum_{r=0}^{\infty}\ a_rx^r\ \text{ or }\ \sum_{r=1}^{\infty}\ a_{r-1}x^{r-1},$$

where x is usually a variable quantity, is called a **"power series in x with coefficients,**
$a_0,\ a_1,\ a_2,\ a_3,\ .\ .\ .$".

Notes:

(i) By summing the series from $r = 0$ to infinity, the constant term at the beginning (if there is one) can be considered as the term in x^0.

The various tests for convergence and divergence still apply in this alternative notation.

(ii) A power series will not necessarily be convergent (or divergent) for **all** values of x

It is usually required to determine the specific **range** of values of x for which the series converges

This can most frequently be done using D'Alembert's Ratio Test

ILLUSTRATION

For the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots = \sum_{r=1}^{\infty} (-1)^{r-1}\frac{x^r}{r},$$

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{(-1)^r x^{r+1}}{r+1}.\frac{r}{(-1)^{r-1}x^r}\right| = \left|\frac{r}{r+1}x\right|,$$

which tends to $|x|$ as r tends to infinity.

Thus, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

If $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots$$

which converges by the Alternating Series Test.

If $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ \ldots$$

which diverges.

The **precise** range of convergence for the given series is therefore $-1 < x \leq 1$.

"JUST THE MATHS"

SLIDES NUMBER

3.1

TRIGONOMETRY 1
(Angles & trigonometric functions)

by

A.J.Hobson

UNIT 3.1 - TRIGONOMETRY 1 - ANGLES AND TRIGONOMETRIC FUNCTIONS

3.1.1 INTRODUCTION

The following results will be assumed without proof:

(i) The Circumference, C, and Diameter, D, of a circle are directly proportional to each other through the formula

$$C = \pi D$$

or, if the radius is r,

$$C = 2\pi r.$$

(ii) The area, A, of a circle is related to the radius, r, by means of the formula

$$A = \pi r^2.$$

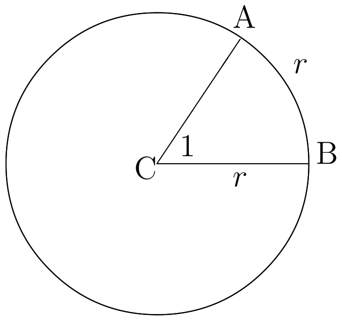
3.1.2 ANGULAR MEASURE

(a) Astronomical Units

The **"degree"** is a $\frac{1}{360}$th part of one complete revolution. It is based on the study of planetary motion where 360 is approximately the number of days in a year.

(b) Radian Measure

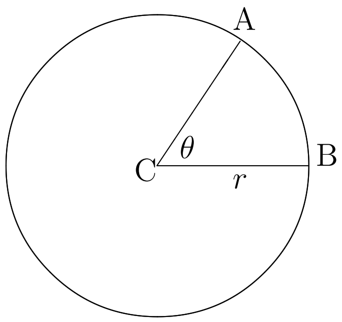
A **"radian"** is the angle subtended at the centre of a circle by an arc which is equal in length to the radius.



RESULTS

(i) There are 2π radians in one complete revolution;

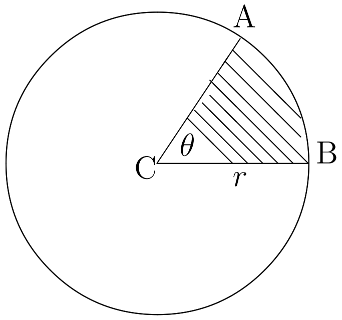
or π radians is equivalent to $180°$



(ii) In the above diagram, the arclength from A to B will be given by

$$\frac{\theta}{2\pi} \times 2\pi r = r\theta,$$

assuming that θ is measured in radians.



(iii) In the above diagram, the area of the sector ABC is given by

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

(c) Standard Angles

The scaling factor for converting degrees to radians is

$$\frac{\pi}{180}$$

and the scaling factor for converting from radians to degrees is

$$\frac{180}{\pi}.$$

ILLUSTRATIONS

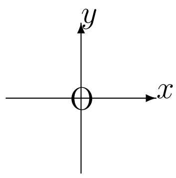
1. 15° is equivalent to $\frac{\pi}{180} \times 15 = \frac{\pi}{12}$.
2. 30° is equivalent to $\frac{\pi}{180} \times 30 = \frac{\pi}{6}$.
3. 45° is equivalent to $\frac{\pi}{180} \times 45 = \frac{\pi}{4}$.
4. 60° is equivalent to $\frac{\pi}{180} \times 60 = \frac{\pi}{3}$.

5. 75° is equivalent to $\frac{\pi}{180} \times 75 = \frac{5\pi}{12}$.

6. 90° is equivalent to $\frac{\pi}{180} \times 90 = \frac{\pi}{2}$.

(d) Positive and Negative Angles

Using cartesian axes Ox and Oy, the **"first quadrant"** is that for which x and y are both positive, and the other three quadrants are numbered from the first in an anti-clockwise sense.



From the positive x-direction, we measure angles positively in the anticlockwise sense and negatively in the clockwise sense. Special names are given the to type of angles obtained as follows:

1. Angles in the range between 0° and 90° are called **"positive acute"** angles

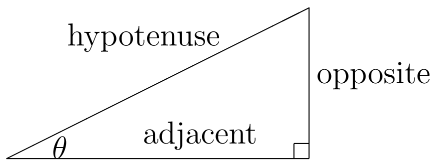
2. Angles in the range between 90° and 180° are called **"positive obtuse"** angles.

3. Angles in the range between 180° and 360° are called **"positive reflex"** angles.

4. Angles measured in the clockwise sense have similar names but preceded by the word **"negative"**.

3.1.3 TRIGONOMETRIC FUNCTIONS

For future reference, we shall assume, without proof, the result known as **"Pythagoras' Theorem"**. This states that the square of the length of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the lengths of the other two sides.



DEFINITIONS

(a) **"Sine"**

$$\sin\theta \equiv \frac{\text{opposite}}{\text{hypotenuse}};$$

(b) **"Cosine"**

$$\cos\theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}};$$

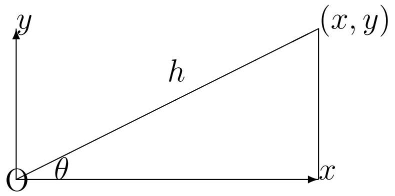
(c) **"Tangent"**

$$\tan\theta \equiv \frac{\text{opposite}}{\text{adjacent}}.$$

Notes:
(i) To remember the above, use

S.O.H.C.A.H.T.O.A.

(ii) The definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be extended to angles of any size:



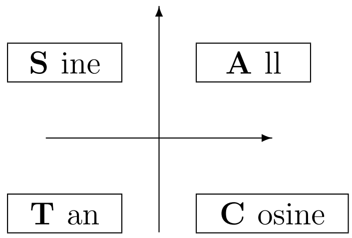
$$\sin\theta \equiv \frac{y}{h};$$

$$\cos\theta \equiv \frac{x}{h};$$

$$\tan\theta \equiv \frac{y}{x} \equiv \frac{\sin\theta}{\cos\theta}.$$

Trigonometric functions can also be called **"trigonometric ratios"**.

(iii) Basic trigonometric functions have positive values in the following quadrants.



(iv) Three other trigonometric functions are sometimes used and are defined as the reciprocals of the three basic functions as follows:

"Secant"

$$\sec\theta \equiv \frac{1}{\cos\theta};$$

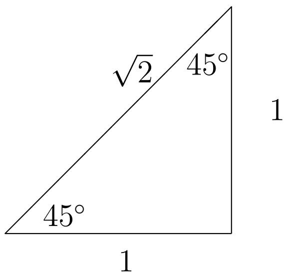
"Cosecant"

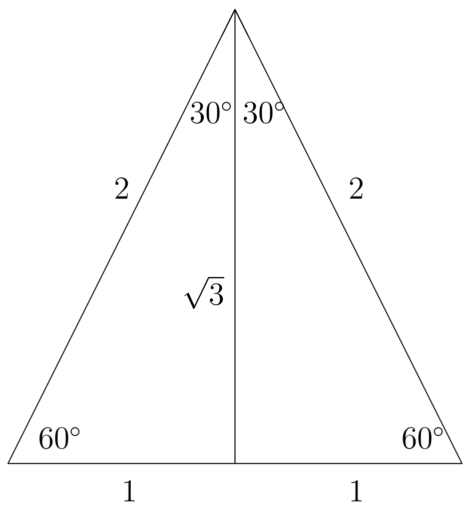
$$\mathrm{cosec}\theta \equiv \frac{1}{\sin\theta};$$

"Cotangent"

$$\cot\theta \equiv \frac{1}{\tan\theta}.$$

(v) The values of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the particular angles 30°, 45° and 60° are easily obtained without calculator from the following diagrams:





The diagrams show that

(a) $\sin 45° = \frac{1}{\sqrt{2}}$; (b) $\cos 45° = \frac{1}{\sqrt{2}}$; (c) $\tan 45° = 1$;

(d) $\sin 30° = \frac{1}{2}$; (e) $\cos 30° = \frac{\sqrt{3}}{2}$; (f) $\tan 30° = \frac{1}{\sqrt{3}}$;

(g) $\sin 60° = \frac{\sqrt{3}}{2}$; (h) $\cos 60° = \frac{1}{2}$; (i) $\tan 60° = \sqrt{3}$.

"JUST THE MATHS"

SLIDES NUMBER

3.2

TRIGONOMETRY 2
(Graphs of trigonometric functions)

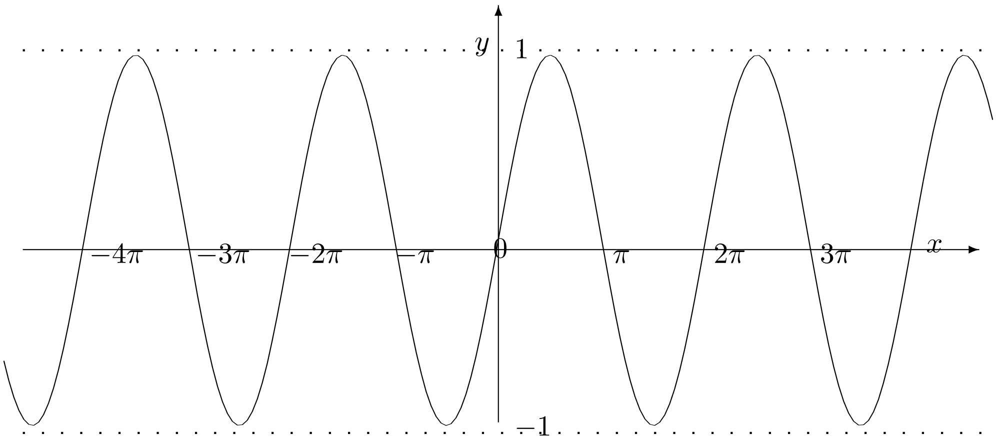
by

A.J.Hobson

UNIT 3.2 - TRIGONOMETRY 2
GRAPHS OF TRIGONOMETRIC FUNCTIONS

3.2.1 GRAPHS OF ELEMENTARY TRIGONOMETRIC FUNCTIONS

1. $y = \sin\theta$



Results and Definitions

(i)

$$\sin(\theta + 2\pi) \equiv \sin\theta.$$

$\sin\theta$ is a **"periodic function with period 2π"**.

(ii) Other periods are $\pm 2n\pi$ where n is any integer.

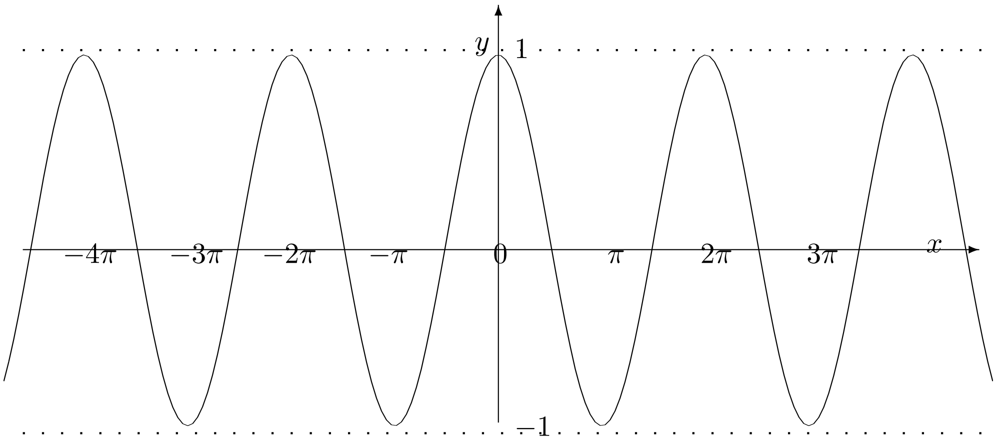
(iii) The smallest positive period is called the **"primitive period"** or **"wavelength"**.

(iv)

$$\sin(-\theta) \equiv -\sin\theta$$

and $\sin\theta$ is called an **"odd function"**.

2. $y = \cos\theta$



Results and Definitions

(i)

$$\cos(\theta + 2\pi) \equiv \cos\theta.$$

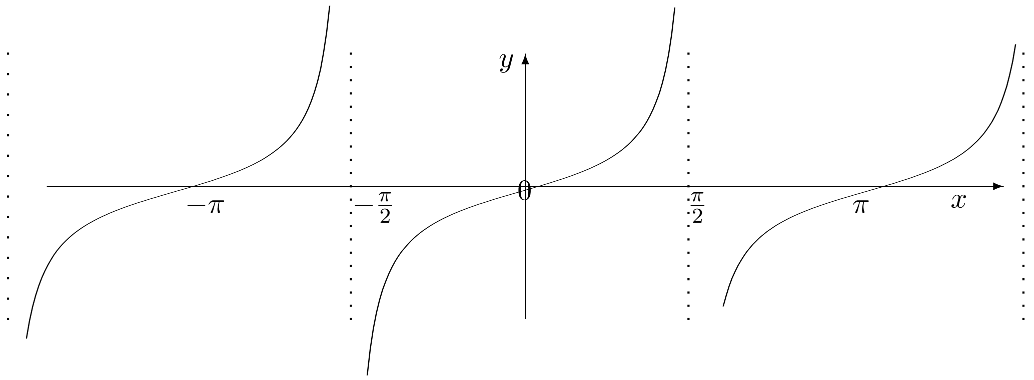
$\cos\theta$ is a periodic function with primitive period 2π.

(ii)

$$\cos(-\theta) \equiv \cos\theta$$

and $\cos\theta$ is called an **"even function"**.

3. $y = \tan\theta$



Results and Definitions

(i)

$$\tan(\theta + \pi) \equiv \tan\theta.$$

$\tan\theta$ is a periodic function with primitive period π.

(ii)

$$\tan(-\theta) \equiv -\tan\theta$$

and $\tan\theta$ is called an **"odd function"**.

3.2.2 GRAPHS OF MORE GENERAL TRIGONOMETRIC FUNCTIONS

In scientific work, it is possible to encounter functions of the form

$$\boxed{\text{Asin}(\omega\theta + \alpha)} \text{ and } \boxed{\text{Acos}(\omega\theta + \alpha)},$$

where ω and α are constants.

EXAMPLES

1. Sketch the graph of

$$y = 5\cos(\theta - \pi).$$

Solution

(i) the graph will have the same shape as the basic cosine wave;

(ii) the graph will lie between $y = -5$ and $y = 5$ so has an **"amplitude"** of 5;

(iii) the graph will cross the θ axis at the points for which

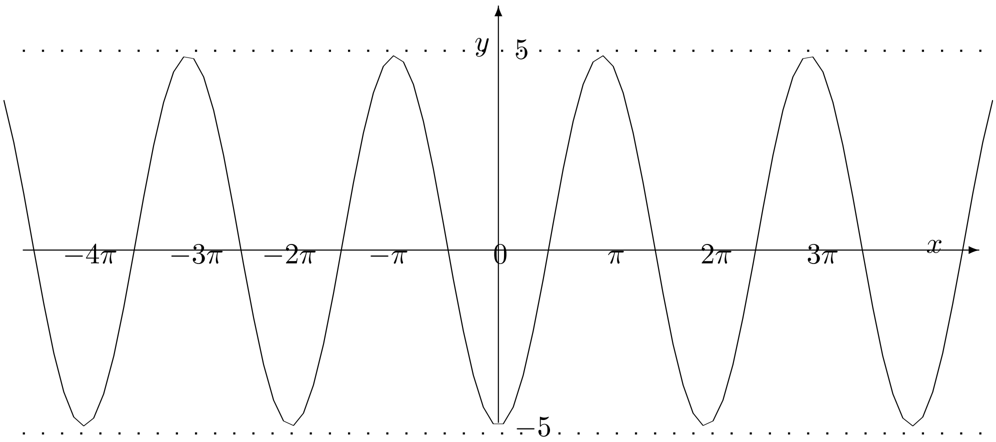
$$\theta - \pi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$$

that is

$$\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$$

(iv) The y-axis must be placed between the smallest **negative** intersection with the θ-axis and the smallest **positive** intersection with the θ-axis (in proportion to their values).

In this case, the y-axis must be placed half way between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$.



Note:

$$5\cos(\theta - \pi) \equiv -5\cos\theta$$

so that graph is an "upsidedown" cosine wave with an amplitude of 5.

Not all examples can be solved in this way.

2. Sketch the graph of

$$y = 3\sin(2\theta + 1).$$

Solution

(i) the graph will have the same shape as the basic sine wave;

(ii) the graph will have an amplitude of 3;

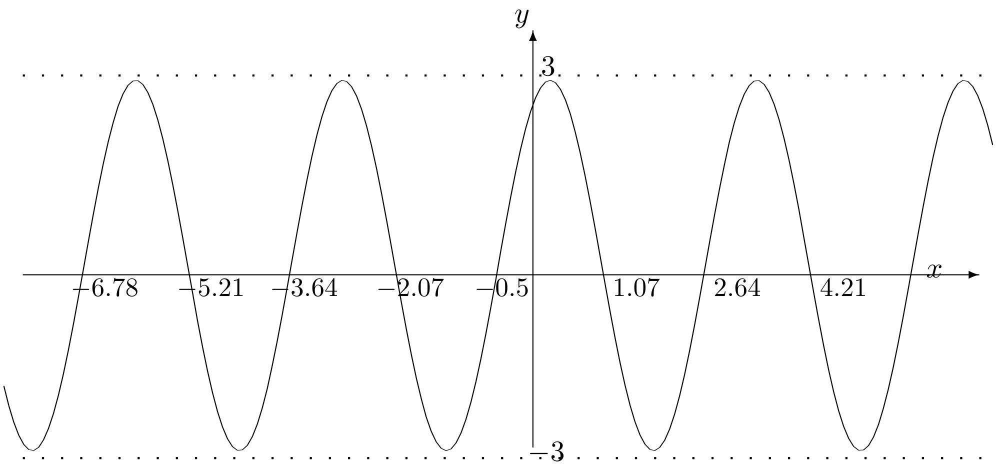
(iii) The graph will cross the θ-axis at the points for which

$$2\theta + 1 = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, ...$$

That is, $\theta =$

$$...-6.78, -5.21, -3.64, -2.07, -0.5, 1.07, 2.64, 4.21, 5.78...$$

(iv) The y-axis must be placed between $\theta = -0.5$ and $\theta = 1.07$ but at about one third of the way from $\theta = -0.5$



"JUST THE MATHS"

SLIDES NUMBER

3.3

TRIGONOMETRY 3
(Approximations & inverse functions)

by

A.J.Hobson

UNIT 3.3 - TRIGONOMETRY 3 APPROXIMATIONS AND INVERSE FUNCTIONS

3.3.1 APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}.....$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}......$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + ...$$

N.B. θ must be in radians.

If θ is small

$$\sin\theta \simeq \theta;$$

$$\cos\theta \simeq 1;$$

$$\tan\theta \simeq \theta.$$

Better approximations using more terms of the infinite series.

EXAMPLE

Assuming θ^n is negligible when $n > 4$,

$$5 + 2\cos\theta - 7\sin\theta \simeq 5 + 2 - \theta^2 + \frac{\theta^4}{12} - 7\theta + 7\frac{\theta^3}{6}$$

$$= \frac{1}{12}\left[\theta^4 + 14\theta^3 - 12\theta^2 - 84\theta + 84\right].$$

3.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

(a)

$$\text{Sin}^{-1}x$$

denotes any angle whose sine value is the number x.

It is necessary that $-1 \leq x \leq 1$.

(b)

$$\text{Cos}^{-1}x$$

denotes any angle whose cosine value is the number x.

It is necessary that $-1 \leq x \leq 1$.

(c)

$$\text{Tan}^{-1}x$$

denotes any angle whose tangent value is x.

x may be any value.

Note:
There will be two **basic** values of an inverse function from two different quadrants.

Either value may be increased or decreased by a whole multiple of 360° (2π).

EXAMPLES

1. $\text{Sin}^{-1}(\frac{1}{2}) = 30^\circ \pm n360^\circ$ or $150^\circ \pm n360^\circ$.
2. $\text{Tan}^{-1}(\sqrt{3}) = 60^\circ \pm n360^\circ$ or $240^\circ \pm n360^\circ$.
   Alternatively, $\text{Tan}^{-1}(\sqrt{3}) = 60^\circ \pm n180^\circ$.

   Another Type of Question
3. Obtain all of the solutions to the equation

   $$\cos 3x = -0.432$$

   which lie in the interval $-180^\circ \leq x \leq 180^\circ$.

   Solution

   $3x$ is any one of the angles (within an interval $-540^\circ \leq 3x \leq 540^\circ$) whose cosine is equal to -0.432.

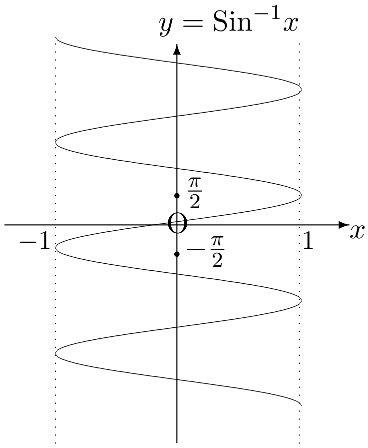
   By calculator, the simplest angle is 115.59°

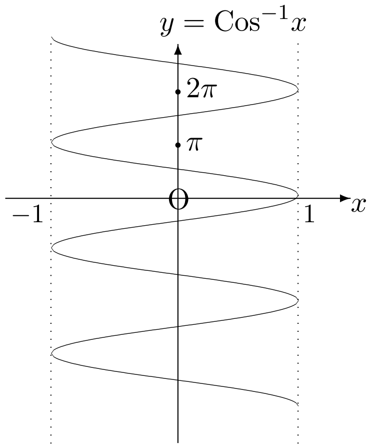
   The complete set is

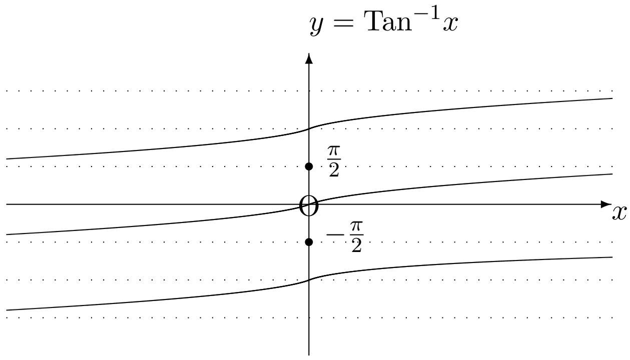
   $$\pm 115.59^\circ \quad \pm 244.41^\circ \quad \pm 475.59^\circ$$

   $$\text{giving} \quad x = \pm 38.5^\circ \quad \pm 81.5^\circ \quad \pm 158.5^\circ$$

Note: The graphs of inverse trigonometric functions are discussed fully in Unit 10.6, but we include them here for the sake of completeness.







PRINCIPAL VALUE.

This is the unique value which lies in a specified range.

Principal values use the lower-case initial letter of each inverse function.

(a) $\theta = \sin^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b) $\theta = \cos^{-1}x$ lies in the range $0 \leq \theta \leq \pi$.

(c) $\theta = \tan^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

EXAMPLES

1. $\sin^{-1}(\frac{1}{2}) = 30^\circ$ or $\frac{\pi}{6}$.
2. $\tan^{-1}(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$.
3. Obtain u in terms of v when $v = 5\cos(1 - 7u)$.

   Solution

$$\frac{v}{5} = \cos(1 - 7u);$$

$$\text{Cos}^{-1}\left(\frac{v}{5}\right) = 1 - 7u;$$

$$\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1 = -7u;$$

$$u = -\frac{1}{7}\left[\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1\right].$$

"JUST THE MATHS"

SLIDES NUMBER

3.4

TRIGONOMETRY 4
(Solution of triangles)

by

A.J.Hobson

UNIT 3.4 - TRIGONOMETRY 4

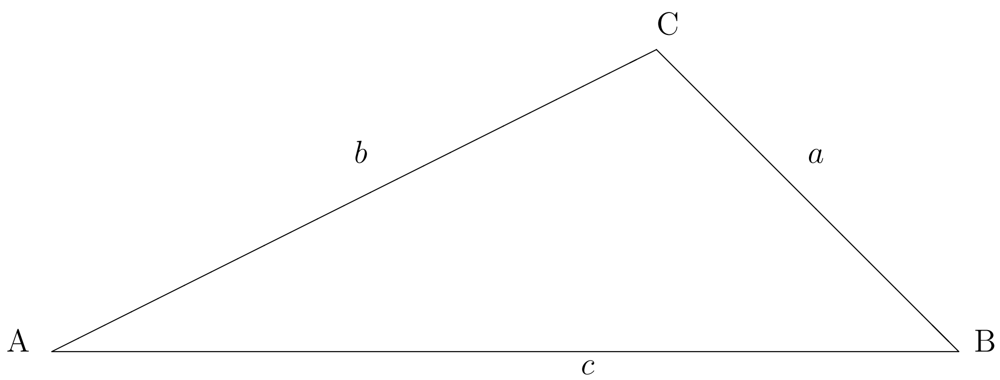
SOLUTION OF TRIANGLES

3.4.1 INTRODUCTION

The **"solution of a triangle"** is defined to mean the complete set of data relating to the lengths of its three sides and the values of its three interior angles.

It can be shown that interior angles add up to 180°.

For an arbitrary triangle with "vertices" A,B and C and sides of length a, b and c, we draw



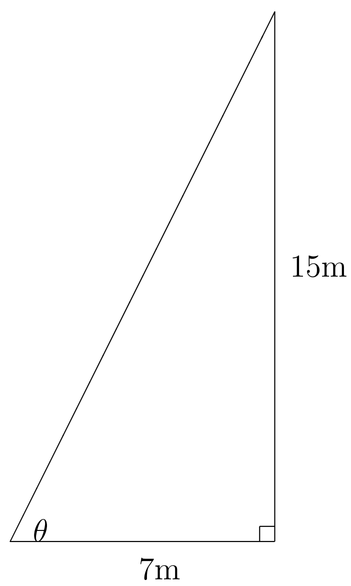
The angles at A,B and C will be denoted by $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$.

3.4.2 RIGHT-ANGLED TRIANGLES

EXAMPLE

From the top of a vertical pylon, 15 meters high, a guide cable is to be secured into the (horizontal) ground at a distance of 7 meters from the base of the pylon.
What will be the length of the cable and what will be its inclination (in degrees) to the horizontal ?

Solution



From Pythagoras' Theorem, the length of the cable will be

$$\sqrt{7^2 + 15^2} \simeq 16.55\text{m}.$$

The angle of inclination to the horizontal will be θ, where

$$\tan\theta = \frac{15}{7} \ \text{ and } \ \theta \simeq 65^\circ.$$

3.4.3 THE SINE AND COSINE RULES

(a) The Sine Rule

$$\frac{a}{\sin \widehat{A}} = \frac{b}{\sin \widehat{B}} = \frac{c}{\sin \widehat{C}}.$$

(b) The Cosine Rule

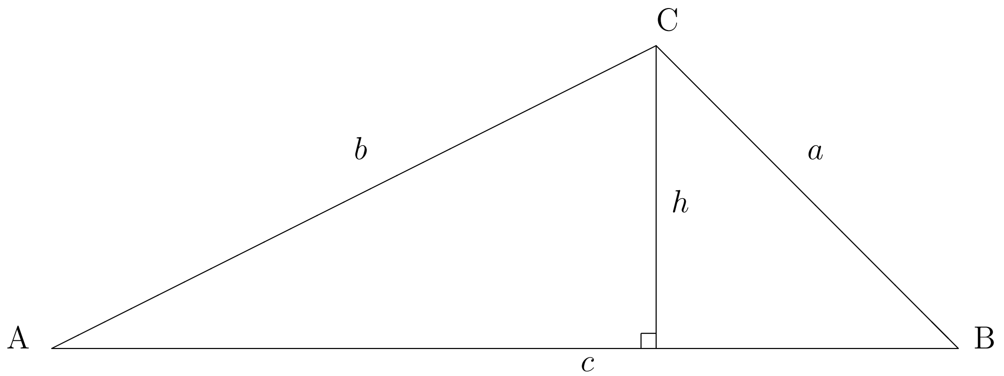
$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A};$$
$$b^2 = c^2 + a^2 - 2ca\cos\widehat{B};$$
$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C}.$$

Observation

Whenever the angle on the right-hand-side is a right-angle, the Cosine Rule reduces to Pythagoras' Theorem.

The Proof of the Sine Rule

First, draw the perpendicular (of length h) from the vertex C onto the side AB.



$$\frac{h}{b} = \sin \widehat{A} \quad \text{and} \quad \frac{h}{a} = \sin \widehat{B}.$$

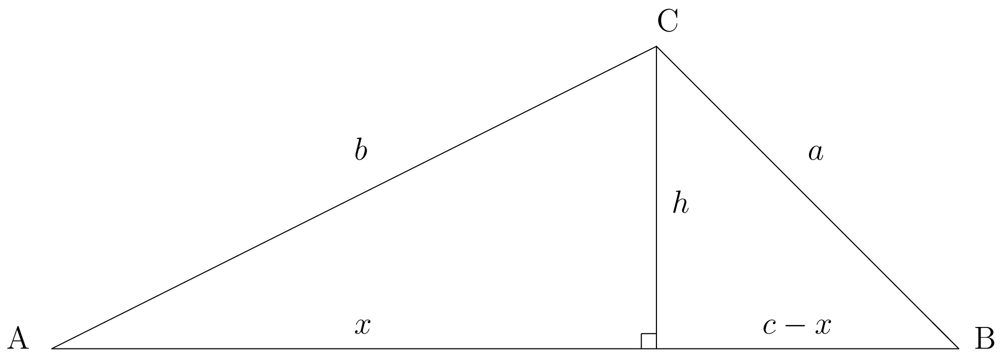
$$b \sin \widehat{A} = a \sin \widehat{B}.$$

$$\frac{b}{\sin \widehat{B}} = \frac{a}{\sin \widehat{A}}.$$

The rest of the Sine Rule can be obtained by considering the perpendicular drawn from a different vertex.

The Proof of the Cosine Rule

Let the side AB have lengths x and $c - x$ either side of the foot of the perpendicular drawn from C.



$$h^2 = b^2 - x^2$$

and

$$h^2 = a^2 - (c - x)^2.$$

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2.$$

$$a^2 = b^2 + c^2 - 2xc.$$

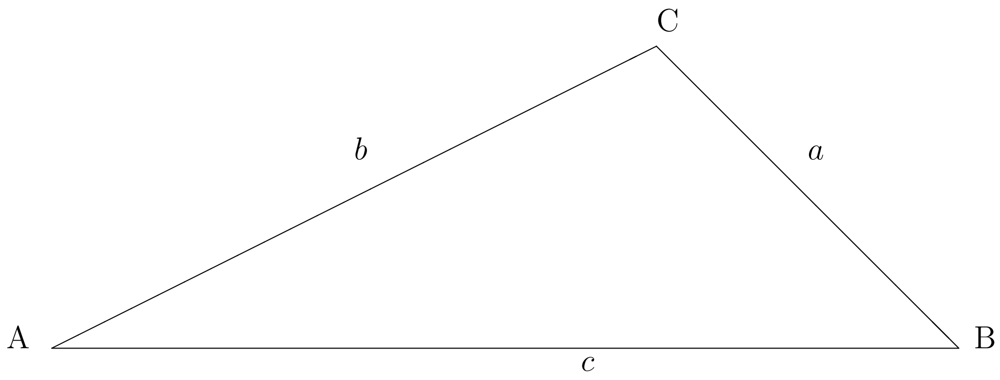
But $x = b \cos \widehat{A}$, and so

$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A}.$$

EXAMPLES

1. Solve the triangle ABC in the case when
   $\widehat{A} = 20°, \widehat{B} = 30°$ and $c = 10$cm

   Solution



Firstly, the angle $\widehat{C} = 130°$ since interior angles add up to 180°.

By the Sine Rule,

$$\frac{a}{\sin 20°} = \frac{b}{\sin 30°} = \frac{10}{\sin 130°};$$

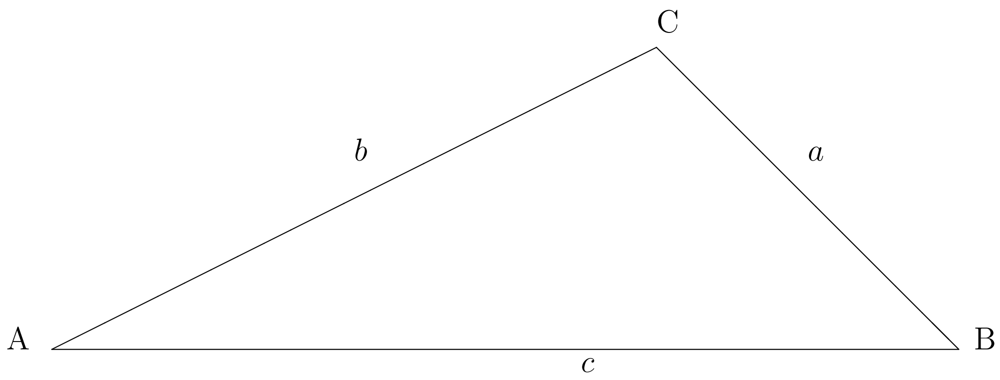
$$\frac{a}{0.342} = \frac{b}{0.5} = \frac{10}{0.766};$$

$$a = \frac{10 \times 0.342}{0.766} \cong 4.47\text{cm};$$

$$b = \frac{10 \times 0.5}{0.766} \cong 6.53\text{cm}.$$

2. Solve the triangle ABC in the case when $b = 9\text{cm}, \quad c = 5\text{cm}$ and $\widehat{A} = 70°$.

Solution



By the Cosine Rule,

$$a^2 = 25 + 81 - 90\cos 70°;$$

$$a^2 = 106 - 30.782 = 75.218;$$

$$a \simeq 8.673 \simeq 8.67\text{cm}.$$

By the Sine Rule,

$$\frac{8.673}{\sin 70°} = \frac{9}{\sin \widehat{B}} = \frac{5}{\sin \widehat{C}};$$

$$\sin \widehat{B} = \frac{9 \times \sin 70°}{8.673} = \frac{9 \times 0.940}{8.673} \simeq 0.975$$

This suggests $\widehat{B} \simeq 77.19^\circ$ and
$\widehat{C} \simeq 180^\circ - 70^\circ - 77.19^\circ \simeq 32.81^\circ$

Also allow the possibility that $\widehat{B} \simeq 102.81^\circ$ and $\widehat{C} \simeq 7.19^\circ$

However, alternative solution unacceptable since not consistent with all of the Sine Rule.

Thus,

$$a \simeq 8.67\text{cm}, \quad \widehat{B} \simeq 77.19^\circ, \quad \widehat{C} \simeq 32.81^\circ$$

Note: It is possible to encounter examples for which more than one solution **does** exist.

"JUST THE MATHS"

SLIDES NUMBER

3.5

TRIGONOMETRY 5
(Trigonometric identities & wave-forms)

by

A.J.Hobson

UNIT 3.5 - TRIGONOMETRY 5
TRIGONOMETRIC IDENTITIES
AND WAVE-FORMS

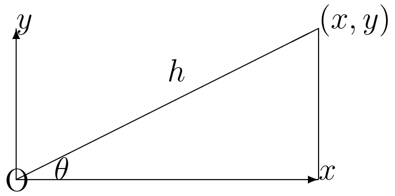
3.5.1 TRIGONOMETRIC IDENTITIES

ILLUSTRATION

Prove that

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Proof:



$$\cos\theta = \frac{x}{h} \text{ and } \sin\theta = \frac{y}{h};$$

$$x^2 + y^2 = h^2;$$

$$\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1;$$

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Other Variations

(a) $\cos^2\theta \equiv 1 - \sin^2\theta$; (rearrangement).

(b) $\sin^2\theta \equiv 1 - \cos^2\theta$; (rearrangement).

(c) $\sec^2\theta \equiv 1 + \tan^2\theta$; (divide by $\cos^2\theta$).

(d) $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$; (divide by $\sin^2\theta$).

Other Trigonometric Identities

$$\sec\theta \equiv \frac{1}{\cos\theta}$$

$$\text{cosec}\theta \equiv \frac{1}{\sin\theta}$$

$$\cot\theta \equiv \frac{1}{\tan\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \text{cosec}^2\theta$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$

$$\equiv 1 - 2\sin^2 A$$

$$\equiv 2\cos^2 A - 1$$

$$\sin A \equiv 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos A \equiv \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$$

$$\equiv 1 - 2\sin^2 \frac{1}{2} A$$

$$\equiv 2\cos^2 \frac{1}{2} A - 1$$

$$\tan A \equiv \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A}$$

$$\sin A + \sin B \equiv 2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B \equiv 2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B \equiv 2 \cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \equiv -2 \sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$$

$$\sin A \cos B \equiv \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$
$$\cos A \sin B \equiv \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$
$$\cos A \cos B \equiv \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$
$$\sin A \sin B \equiv \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$
$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

EXAMPLES

1. Show that

$$\sin^2 2x \equiv \frac{1}{2}(1 - \cos 4x).$$

Solution

$$\cos 4x \equiv 1 - 2\sin^2 2x.$$

2. Show that

$$\sin\left(\theta + \frac{\pi}{2}\right) \equiv \cos\theta.$$

Solution

The left hand side can be expanded as

$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2};$$

The result follows, because $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

3. Simplify the expression

$$\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha}.$$

Solution

Expression becomes

$$\frac{2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\cos\left(\frac{2\alpha-3\alpha}{2}\right)}{-2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\sin\left(\frac{2\alpha-3\alpha}{2}\right)}$$

$$\equiv \frac{2\sin\left(\frac{5\alpha}{2}\right).\cos\left(\frac{-\alpha}{2}\right)}{-2\sin\left(\frac{5\alpha}{2}\right).\sin\left(\frac{-\alpha}{2}\right)}$$

$$\equiv \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\equiv \cot\left(\frac{\alpha}{2}\right).$$

4. Express $2\sin 3x \cos 7x$ as the difference of two sines.

Solution

$$2\sin 3x \cos 7x \equiv \sin(3x + 7x) + \sin(3x - 7x).$$

Hence,

$$2\sin 3x \cos 7x \equiv \sin 10x - \sin 4x.$$

3.5.2 AMPLITUDE, WAVE-LENGTH, FREQUENCY AND PHASE ANGLE

Importance is attached to trigonometric functions of the form

$$A\sin(\omega t+\alpha) \quad \text{and} \quad A\cos(\omega t+\alpha),$$

where A, ω and α are constants and t is usually a time variable.

The expanded forms are

$$A\sin(\omega t+\alpha) \equiv A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$

and

$$A\cos(\omega t+\alpha) \equiv A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha.$$

(a) The Amplitude

A, represents the maximum value (numerically) which can be attained by each of the above trigonometric functions.

A is called the **"amplitude"** of each of the functions.

(b) The Wave Length (Or Period)

If t increases or decreases by a whole multiple of $\frac{2\pi}{\omega}$, then $(\omega t+\alpha)$ increases or decreases by a whole multiple of 2π; and hence the functions remain unchanged in value.

A graph, against t, of either $A\sin(\omega t+\alpha)$ or $A\cos(\omega t+\alpha)$ would be repeated in shape at regular intervals of length $\frac{2\pi}{\omega}$.

The repeated shape of the graph is called the **"wave profile"** and $\frac{2\pi}{\omega}$ is called the **"wave-length"**, or **"period"** of each of the functions.

(c) The Frequency

If t is a time variable, then the wave length (or period) represents the time taken to complete a single wave-profile.

Consequently, the number of wave-profiles completed in one unit of time is given by $\frac{\omega}{2\pi}$.

$\frac{\omega}{2\pi}$ is called the **"frequency"** of each of the functions.

Note:
ω is called the **"angular frequency"**;

ω represents the change in the quantity $(\omega t + \alpha)$ for every unit of change in the value of t.

(d) The Phase Angle

α affects the starting value, at $t = 0$, of the trigonometric functions $A\sin(\omega t + \alpha)$ and $A\cos(\omega t + \alpha)$.

Each of these is said to be **"out of phase"**, by an amount, α, with the trigonometric functions $A\sin\omega t$ and $A\cos\omega t$ respectively.

α is called the **"phase angle"** of each of the two original trigonometric functions;

it can take infinitely many values differing only by a whole multiple of 360° or 2π.

EXAMPLES

1. Express $\sin t + \sqrt{3}\cos t$ in the form $A\sin(t + \alpha)$, with α in degrees, and hence solve the equation,

$$\sin t + \sqrt{3}\cos t = 1,$$

for t in the range $0^\circ \leq t \leq 360^\circ$.

Solution

We require that

$$\sin t + \sqrt{3}\cos t \equiv A\sin t\cos\alpha + A\cos t\sin\alpha$$

Hence,

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = \sqrt{3},$$

which gives $A^2 = 4$ (using $\cos^2\alpha + \sin^2\alpha \equiv 1$) and also $\tan\alpha = \sqrt{3}$.
Thus,

$$A = 2 \quad \text{and} \quad \alpha = 60^\circ \text{ (principal value)}.$$

To solve the given equation, we may now use

$$2\sin(t + 60^\circ) = 1,$$

so that

$$t + 60^\circ = \text{Sin}^{-1}\frac{1}{2} = 30^\circ + k360^\circ \quad \text{or} \quad 150^\circ + k360^\circ,$$

where k may be any integer.
For the range $0^\circ \leq t \leq 360^\circ$, we conclude that

$$t = 330^\circ \quad \text{or} \quad 90^\circ.$$

2. Express $a \sin \omega t + b \cos \omega t$ in the form $A \sin(\omega t + \alpha)$. Apply the result to the expression $3 \sin 5t - 4 \cos 5t$ stating α in degrees, correct to one decimal place, and lying in the interval from $-180°$ to $180°$.

Solution

$$A \sin(\omega t + \alpha) \equiv a \sin \omega t + b \cos \omega t;$$

$$A \sin \alpha = b \quad \text{and} \quad A \cos \alpha = a;$$

$$A^2 = a^2 + b^2;$$

$$A = \sqrt{a^2 + b^2}.$$

Also

$$\frac{A \sin \alpha}{A \cos \alpha} = \frac{b}{a};$$

$$\alpha = \tan^{-1} \frac{b}{a}.$$

Note:

The particular angle chosen must ensure that $\sin\alpha = \frac{b}{A}$ and $\cos\alpha = \frac{a}{A}$ have the correct sign.

For $3\sin 5t - 4\cos 5t$, we have

$$A = \sqrt{3^2 + 4^2}$$

and

$$\alpha = \tan^{-1}\left(-\frac{4}{3}\right).$$

But $\sin\alpha \left(= -\frac{4}{5}\right)$ and $\cos\alpha \left(= \frac{3}{5}\right)$ so that $-90^\circ < \alpha < 0$;

that is $\alpha = -53.1^\circ$.

We conclude that

$$3\sin 5t - 4\cos 5t \equiv 5\sin(5t - 53.1^\circ)$$

3. Solve the equation

$$4\sin 2t + 3\cos 2t = 1$$

for t in the interval from $-180°$ to $180°$.

Solution

Expressing the left hand side of the equation in the form $A\sin(2t+\alpha)$, we require

$$A = \sqrt{4^2+3^2} = 5 \;\text{ and }\; \alpha = \tan^{-1}\frac{3}{4}.$$

Also $\sin\alpha\left(=\frac{3}{5}\right)$ and $\cos\alpha\left(=\frac{4}{5}\right)$ so that $0 < \alpha < 90°$.

Hence, $\alpha = 36.87°$ and

$$5\sin(2t+36.87°) = 1.$$

$$t = \frac{1}{2}\left[\text{Sin}^{-1}\frac{1}{5} - 36.87°\right].$$

$$\text{Sin}^{-1}\frac{1}{5} = 11.53° + k360° \;\text{ and }\; 168.46° + k360°,$$

where k may be any integer.

But, for t values which are numerically less than $180°$, we use $k = 0$ and $k = 1$ in the first and $k = 0$ and $k = -1$ in the second.

$$t = -12.67°,\; 65.80°,\; 167.33° \;\text{ and }\; -114.21°$$

"JUST THE MATHS"

SLIDES NUMBER

4.1

HYPERBOLIC FUNCTIONS 1 (Definitions, graphs and identities)

by

A.J.Hobson

UNIT 4.1 - HYPERBOLIC FUNCTIONS 1 DEFINTIONS, GRAPHS AND IDENTITIES

4.1.1 INTRODUCTION

We introduce a new group of mathematical functions, based on the functions

$$e^x \text{ and } e^{-x}.$$

Their properties resemble, very closely, those of the standard trigonometric functions.

Just as trigonometric functions can be related to the geometry of a circle, the new functions can be related to the geometry of a **hyperbola**.

4.1.2 DEFINITIONS

(a) Hyperbolic Cosine

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

The name of the function is pronounced **"cosh"**.

(b) Hyperbolic Sine

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

The name of the function is pronounced **"shine"**.

(c) Hyperbolic Tangent

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

The name of the function is pronounced **than"**.

In terms of exponentials, it is easily shown that

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(d) Other Hyperbolic Functions

(i) Hyperbolic secant , pronounced **"shek"**.

$$\text{sech}x \equiv \frac{1}{\cosh x}.$$

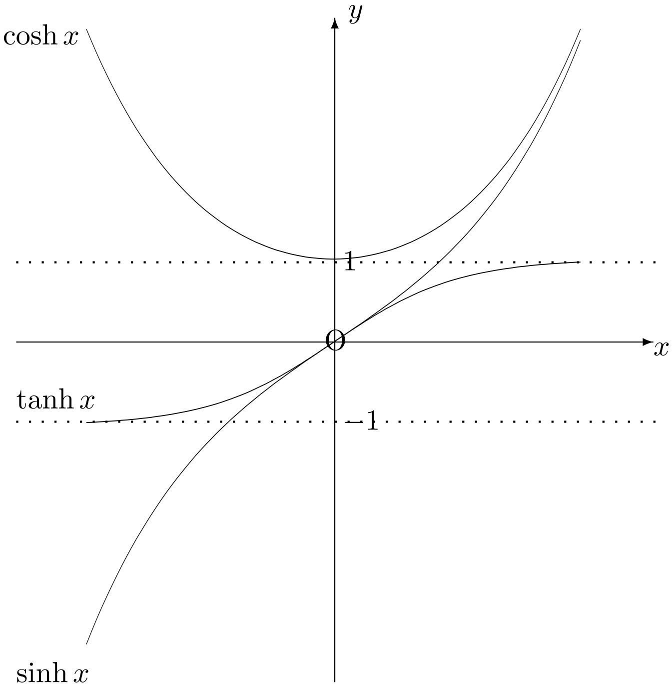
(ii) Hyperbolic cosecant , pronounced **'coshek"**.

$$\text{cosech}x \equiv \frac{1}{\sinh x}.$$

(iii) Hyperbolic cotangent , pronounced **"coth"**.

$$\text{coth}x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}.$$

4.1.3 GRAPHS OF HYPERBOLIC FUNCTIONS



The graph of $\cosh x$ exists only for y greater than or equal to 1.

The graph of $\tanh x$ exists only for y lying between -1 and $+1$.

The graph of $\sinh x$ covers the whole range of x and y values from $-\infty$ to $+\infty$.

4.1.4 HYPERBOLIC IDENTITIES

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \equiv e^x.$$

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \equiv e^{-x}.$$

3.

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Proof

Multiply together the results of the previous two illustrations;

$$e^x.e^{-x} = 1;$$

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x.$$

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity,

$$1 - \tanh^2 x \equiv \text{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity,

$$\coth^2 x - 1 \equiv \text{cosech}^2 x.$$

4.

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

Proof:

The right hand side is

$$\frac{e^x - e^{-x}}{2}.\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}.\frac{e^y - e^{-y}}{2}.$$

That is,

$$\frac{e^{(x+y)} + e^{(x-y)} - e^{(-x+y)} - e^{(-x-y)}}{4}$$

$$+\frac{e^{(x+y)} - e^{(x-y)} + e^{(-x+y)} - e^{(-x-y)}}{4}.$$

This simplifies to

$$\frac{2e^{(x+y)} - 2e^{(-x-y)}}{4}.$$

That is,

$$\frac{e^{(x+y)} - e^{-(x+y)}}{2} \equiv \sinh(x+y).$$

5.

$$\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y.$$

Proof

The proof is similar to Illustration 4.

6.

$$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}.$$

Proof

The proof is similar to Illustration 4.

4.1.5 OSBORN'S RULE

Starting with any trigonometric identity, change cos to cosh and sin to sinh.

Then, if the trigonometric identity contains (or implies) two sine functions multiplied together, change the sign in front of the relevant term from + to − or vice versa.

ILLUSTRATIONS

1.

$$\cos^2 x + \sin^2 x \equiv 1$$

leads to

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

2.

$$\sin(x - y) \equiv \sin x \cos y - \cos x \sin y$$

leads to

$$\sinh(x - y) \equiv \sinh x \cosh y - \cosh x \sinh y.$$

3.

$$\sec^2 x \equiv 1 + \tan^2 x$$

leads to

$$\text{sech}^2 x \equiv 1 - \tanh^2 x.$$

“JUST THE MATHS”

SLIDES NUMBER

4.2

HYPERBOLIC FUNCTIONS 2
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 4.2 - HYPERBOLIC FUNCTIONS 2
INVERSE HYPERBOLIC FUNCTIONS

4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are $\text{Cosh}^{-1}x$, $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$.

It may be shown that

(a)

$$\text{Cosh}^{-1}x = \pm\ln(x + \sqrt{x^2 - 1}).$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1}).$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

Notes:

(i) The positive value of $\text{Cosh}^{-1}x$ is called the **"principal value"** and is denoted by $\cosh^{-1}x$ (using a lower-case c).

(ii) $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$ have only **one** value but, for uniformity, we denote them by $\sinh^{-1}x$ and $\tanh^{-1}x$

4.2.2 THE PROOFS OF THE STANDARD FORMULAE

(a) Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0.$$

Hence,

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms

$$y = \ln(x \pm \sqrt{x^2 - 1}) = \pm \ln(x + \sqrt{x^2 - 1}),$$

since $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are reciprocals of each other, their product being the value 1.

(b) Inverse Hyperbolic Sine

If we let $y = \text{Sinh}^{-1}x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y}$$

or

$$(e^y)^2 - 2xe^y - 1 = 0.$$

Hence,

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

However, $x - \sqrt{x^2 + 1} < 0$ and cannot, therefore, be equated to a power of e.

Taking natural logarithms,

$$y = \ln(x + \sqrt{x^2 + 1}).$$

(c) Inverse Hyperbolic Tangent

If we let $y = \text{Tanh}^{-1}x$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

Hence,

$$x\left(e^{2y} + 1\right) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1 + x}{1 - x}.$$

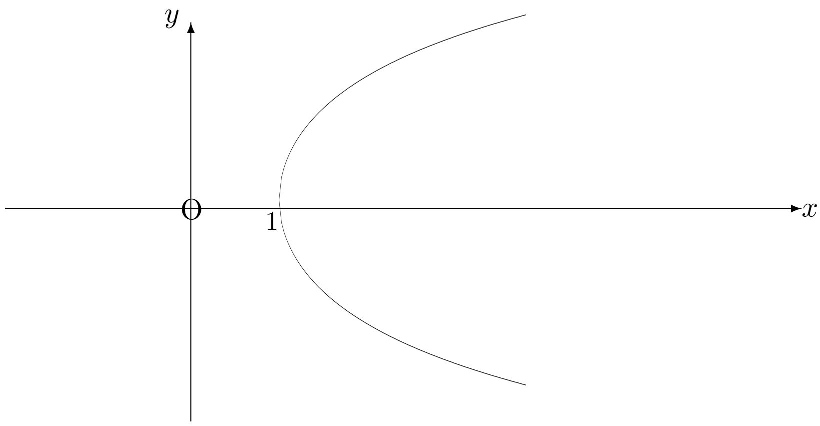
Taking natural logarithms,

$$y = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

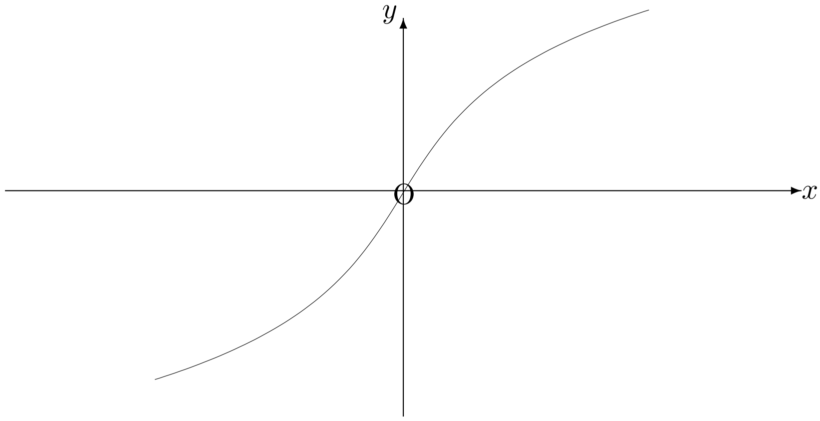
Note:
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

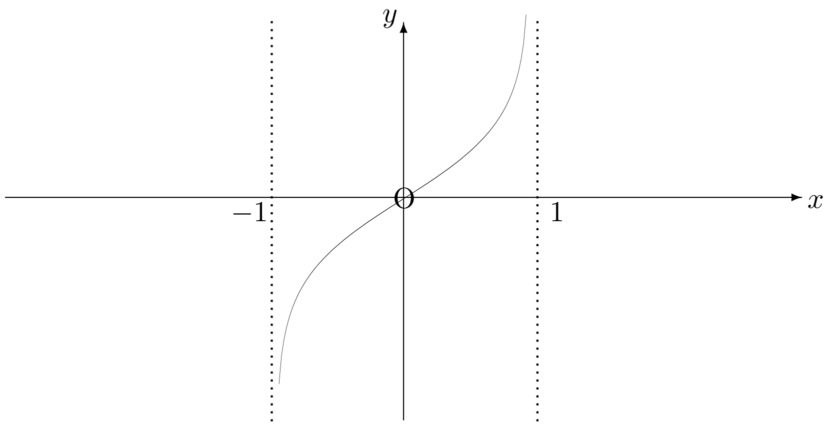
(a) $y = \text{Cosh}^{-1}x$



(b) $y = \mathrm{Sinh}^{-1} x$



(c) $y = \mathrm{Tanh}^{-1} x$



"JUST THE MATHS"

SLIDES NUMBER

5.1

GEOMETRY 1
(Co-ordinates, distance & gradient)

by

A.J.Hobson

UNIT 5.1 - GEOMETRY 1

CO-ORDINATES, DISTANCE AND GRADIENT

5.1.1 CO-ORDINATES

(a) Cartesian Co-ordinates

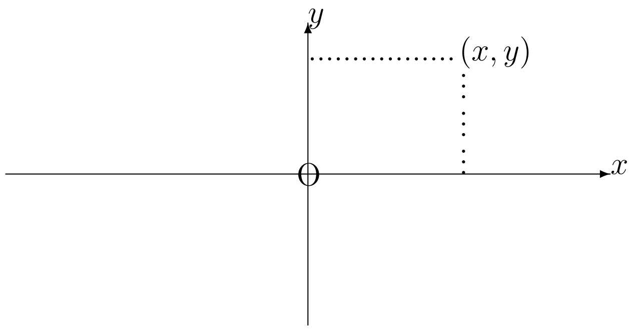
The position of a point P in a plane may be specified completely if we know its perpendicular distances from two chosen fixed straight lines

We distinguish between positive distances on one side of each line and negative distances on the other side of each line.

It is not essential that the two chosen fixed lines should be at right- angles to each other

We usually take them to be so for the sake of convenience.

Consider the following diagram:



The horizontal directed line, Ox, is called the **“x-axis”**.

Distances to the right of the origin (point O) are taken as positive.

The vertical directed line, Oy, is called the **“y-axis”**.

Distances above the origin (point O) are taken as positive.

The notation (x, y) denotes a point whose perpendicular distances from Oy and Ox are x and y respectively, the **“cartesian co-ordinates”** of the point.

(b) Polar Co-ordinates

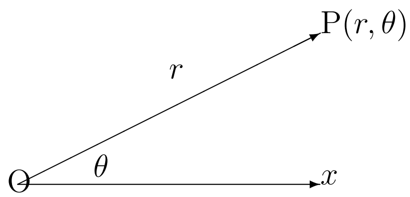
This is an alternative method of fixing the position of a point P in a plane.

First choose a point, O, called the **“pole”**.

Then choose a directed line , Ox, emanating from the pole in one direction only

Ox is called the **“initial line”**.

Consider the following diagram:



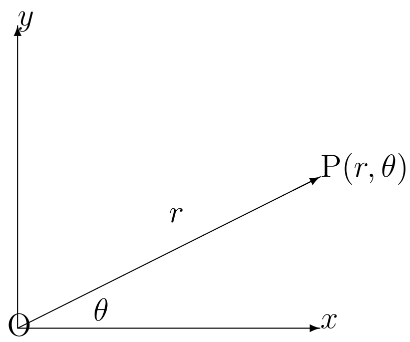
The position of P is determined by its distance r from the pole and the angle, θ which the line OP makes with the initial line.

θ is measured positively in a counter-clockwise sense or negatively in a clockwise sense from the initial line.

(r, θ) denotes the “polar co-ordinates” of the point.

5.1.2 THE RELATIONSHIP BETWEEN POLAR & CARTESIAN CO-ORDINATES

Superimpose one diagram upon the other.



(a) $x = r.\cos\theta$ and $y = r.\sin\theta$;

(b) $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

EXAMPLES

1. Express the equation

$$2x + 3y = 1$$

in polar co-ordinates.

Solution

Substituting for x and y separately,

$$2r\cos\theta + 3r\sin\theta = 1.$$

That is,

$$r = \frac{1}{2\cos\theta + 3\sin\theta}.$$

2. Express the equation

$$r = \sin\theta$$

in cartesian co-ordinates.

Solution

We could try substituting for r and θ separately; but it is easier to rewrite the equation as

$$r^2 = r\sin\theta,$$

which gives

$$x^2 + y^2 = y.$$

5.1.3 THE DISTANCE BETWEEN 2 POINTS

Given two points (x_1, y_1) and (x_2, y_2), the quantity $\mid x_2 - x_1 \mid$ is called the **"horizontal separation"** of the two points.

The quantity $\mid y_2 - y_1 \mid$ is called the **"vertical separation"** of the two points.

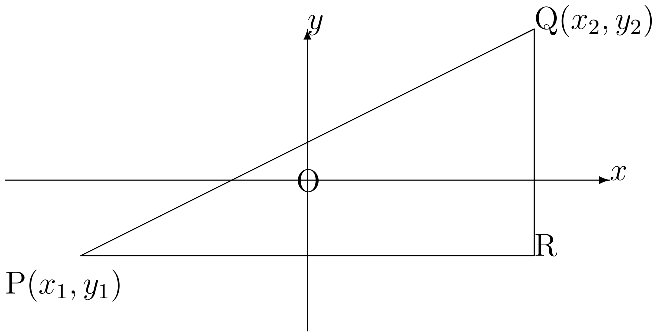
We are assuming that the x-axis is horizontal.

The expressions remain valid even when one or more of the co-ordinates is negative.

For example, the horizontal separation of the points $(5, 7)$ and $(-3, 2)$ is given by $\mid -3 - 5 \mid = 8$.

This agrees with the two points being on opposite sides of the y-axis.

The actual distance between (x_1, y_1) and (x_2, y_2) may be calculated from Pythagoras' Theorem, using their horizontal and vertical separations.



In the diagram,

$$\text{PQ}^2 = \text{PR}^2 + \text{RQ}^2.$$

That is,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

giving

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note: There is no need to include the modulus signs of the horizontal and vertical separations.

It does not matter which way round the points are labelled.

EXAMPLE
Calculate the distance, d, between the two points $(5, -3)$ and $(-11, -7)$.

Solution

$$d = \sqrt{(5+11)^2 + (-3+7)^2}.$$

That is,

$$d = \sqrt{256 + 16} = \sqrt{272} \cong 16.5$$

5.1.4 GRADIENT

The gradient of the straight-line segment PQ is defined to be the tangent of the angle which PQ makes with the positive x-direction.

When the co-ordinates of the two points are $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$, the gradient, m, is given by either

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

We distinguish between positive & negative gradient.

EXAMPLE
Obtain the gradient of the straight-line segment joining the two points $(8, -13)$ and $(-2, 5)$ and hence calculate the angle, θ, which the segment makes with the positive x-direction.

Solution

$$m = \frac{5 + 13}{-2 - 8} = \frac{-13 - 5}{8 + 2} = -1.8$$

Hence,

$$\tan\theta = -1.8$$

giving

$$\theta = \tan^{-1}(-1.8) \simeq 119°.$$

"JUST THE MATHS"

SLIDES NUMBER

5.2

GEOMETRY 2
(The straight line)

by

A.J.Hobson

UNIT 5.2 - GEOMETRY 2

THE STRAIGHT LINE

5.2.1 PREAMBLE

It is not possible to give a satisfactory diagramatic definition of a straight line

The attempt is likely to assume a knowledge of linear measurement which, itself, depends on the concept of a straight line

DEFINITION

A straight line is a set of points with cartesian co-ordinates (x, y) satisfying an equation of the form

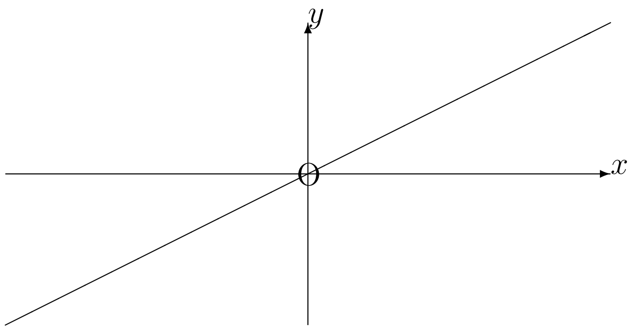
$$ax + by + c = 0,$$

where a, b and c are constants

This equation is called a **"linear equation"** and the symbol (x, y) itself, rather than a dot on the page, is an arbitrary point of the line.

5.2.2 STANDARD EQUATIONS OF A STRAIGHT LINE

(a) Having a given gradient and passing through the origin



Let the gradient be m.

All points (x, y) on the straight line
(but no others) satisfy the relationship,

$$\frac{y}{x} = m.$$

That is,

$$\boxed{y = mx.}$$

EXAMPLE

Determine, in degrees, the angle, θ, which the straight line,

$$\sqrt{3}y = x,$$

makes with the positive x-direction.

Solution

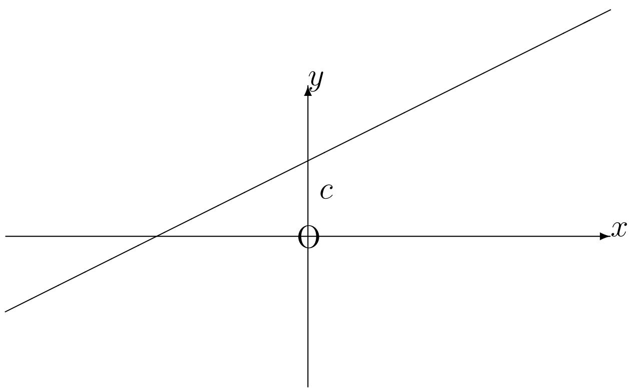
The gradient of the straight line is given by

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

Hence,

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ.$$

(b) Having a given gradient, and a given intercept on the vertical axis



Let the gradient be m and the intercept be c.

In the previous section, add c to all of the y co-ordinates.

Hence the equation of the straight line is

$$\boxed{y = mx + c.}$$

EXAMPLE
Determine the gradient, m, and intercept c on the y-axis of the straight line whose equation is

$$7x - 5y - 3 = 0.$$

Solution

On rearranging the equation, we have

$$y = \frac{7}{5}x - \frac{3}{5}.$$

Hence,

$$m = \frac{7}{5}$$

and

$$c = -\frac{3}{5}.$$

This straight line will intersect the y-axis **below** the origin because the intercept is negative.

(c) Having a given gradient and passing through a given point

Let the gradient be m and let the given point be (x_1, y_1).

Then

$$y = mx + c,$$

where

$$y_1 = mx_1 + c.$$

Subtracting the second from the first, we obtain

$$\boxed{y - y_1 = m(x - x_1).}$$

EXAMPLE

Determine the equation of the straight line having gradient $\frac{3}{8}$ and passing through the point $(-7, 2)$.

Solution

From the formula,

$$y - 2 = \frac{3}{8}(x + 7).$$

That is,

$$8y - 16 = 3x + 21 \text{ or } 8y = 3x + 37.$$

(d) Passing through two given points

Let the two given points be (x_1, y_1) and (x_2, y_2).

Then, the gradient is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

more usually written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Note:
The same result is obtained no matter which way round the given points are taken as (x_1, y_1) and (x_2, y_2).

EXAMPLE

Determine the equation of the straight line joining the two points $(-5, 3)$ and $(2, -7)$, stating the values of its gradient and its intercept on the y-axis.

Solution

Method 1.

$$\frac{y-3}{-7-3} = \frac{x+5}{2+5},$$

giving

$$7(y-3) = -10(x+5) \text{ or } 10x + 7y + 29 = 0.$$

Method 2.

$$\frac{y+7}{3+7} = \frac{x-2}{-5-2},$$

giving

$$-7(y+7) = 10(x-2) \text{ or } 10x + 7y + 29 = 0.$$

Rewriting the equation as

$$y = -\frac{10}{7}x - \frac{29}{7},$$

the gradient is $-\frac{10}{7}$ and the intercept is $-\frac{29}{7}$.

(e) The parametric equations of a straight line

In the previous section, the common value of the two fractions

$$\frac{y - y_1}{y_2 - y_1} \quad \text{and} \quad \frac{x - x_1}{x_2 - x_1}$$

is called the **"parameter"** of the point (x, y) and is usually denoted by t.

Equating each fraction separately to t

$$x = x_1 + (x_2 - x_1)t \quad \text{and} \quad y = y_1 + (y_2 - y_1)t.$$

These are called the **"parametric equations"** of the straight line

(x_1, y_1) and (x_2, y_2) are known as the **"base points"** of the parametric representation of the line.

Notes:

(i) In the above parametric representation, (x_1, y_1) has parameter $t = 0$ and (x_2, y_2) has parameter
$t = 1$.

(ii) Other parametric representations of the same line can be found by using the given base points in the opposite order, or by using a different pair of points on the line as base points.

EXAMPLES

1. Use parametric equations to find two other points on the line joining $(3, -6)$ and $(-1, 4)$.

   Solution

   One possible parametric representation of the line is

   $$x = 3 - 4t \quad y = -6 + 10t.$$

   To find another two points, substitute any two values of t other than 0 or 1.

   For example, with $t = 2$ and $t = 3$,

   $$x = -5, y = 14 \quad \text{and} \quad x = -9, y = 24.$$

   A pair of suitable points is therefore $(-5, 14)$ and $(-9, 24)$.

2. The co-ordinates, x and y of a moving particle are given, at time t, by the equations

$$x = 3 - 4t \quad \text{and} \quad y = 5 + 2t.$$

Determine the gradient of the straight line along which the particle moves.

Solution

Eliminating t, we have

$$\frac{x-3}{-4} = \frac{y-5}{2}.$$

That is,

$$2(x-3) = -4(y-5),$$

giving

$$y = -\frac{2}{4}x + \frac{26}{4}.$$

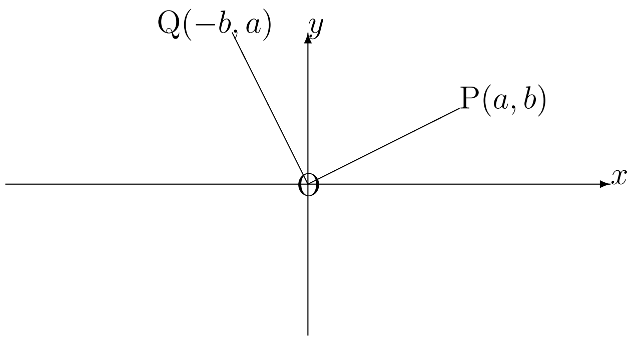
Hence, the gradient of the line is

$$-\frac{2}{4} = -\frac{1}{2}.$$

5.2.3 PERPENDICULAR STRAIGHT LINES

The perpendicularity of two straight lines is not dependent on either their length or their precise position in the plane.

We consider two straight line segments of equal length passing through the origin.



In the diagram, the gradient of OP $= \frac{b}{a}$ and the gradient of OQ $= \frac{a}{-b}$.

Hence,
the product of the gradients is equal to -1

or

Each gradient is minus the reciprocal of the other gradient.

EXAMPLE

Determine the equation of the straight line which passes through the point $(-2, 6)$ and is perpendicular to the straight line,

$$3x + 5y + 11 = 0.$$

Solution

The gradient of the given line is $-\frac{3}{5}$.

Hence, the gradient of a perpendicular line is $\frac{5}{3}$.

Thus, the required line has equation

$$y - 6 = \frac{5}{3}(x + 2),$$

giving

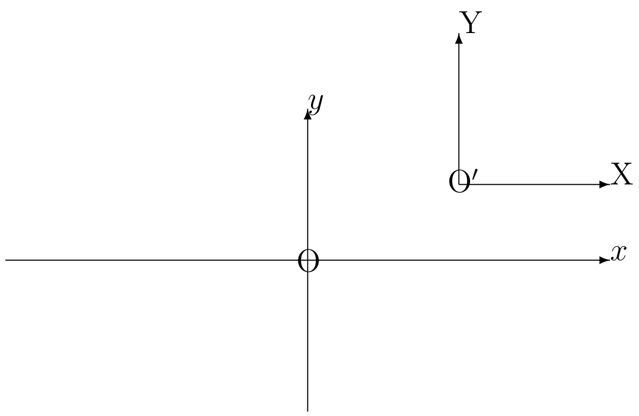
$$3y - 18 = 5x + 10.$$

That is,

$$3y = 5x + 28.$$

5.2.4 CHANGE OF ORIGIN

Given a cartesian system of reference with axes Ox and Oy, it may sometimes be convenient to consider a new set of axes O$'X$ parallel to Ox and O$'Y$ parallel to Oy with new origin at O$'$ whose co-ordinates are (h, k) referred to the original set of axes.



$$X = x - h \quad \text{and} \quad Y = y - k$$

or

$$x = X + h \quad \text{and} \quad y = Y + k.$$

EXAMPLE

Given the straight line,

$$y = 3x + 11,$$

determine its equation referred to new axes with new origin at the point $(-2, 5)$.

Solution

Using

$$x = X - 2 \quad \text{and} \quad y = Y + 5$$

we obtain

$$Y + 5 = 3(X - 2) + 11.$$

That is,

$$Y = 3X,$$

which is a straight line through the new origin with gradient 3.

Note:

The point $(-2, 5)$ is **on** the original line

Hence, the new line passes through the new origin.

Its gradient would not alter in the change of origin.

"JUST THE MATHS"

SLIDES NUMBER

5.3

GEOMETRY 3
(Straight line laws)

by

A.J.Hobson

UNIT 5.3 GEOMETRY 3

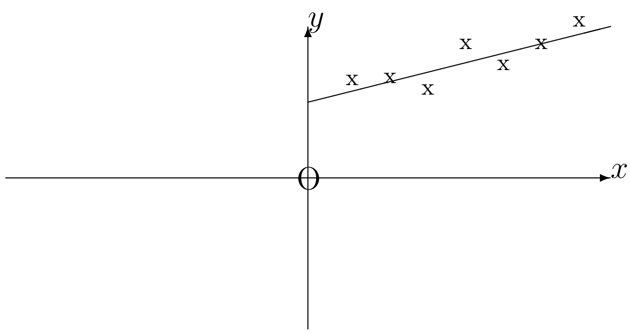
STRAIGHT LINE LAWS

5.3.1 INTRODUCTION

Suppose that two variables x and y are connected by a **"straight line law"**,

$$y = mx + c.$$

To estimate m and c, we could plot a graph of y against x and obtain the **"best straight line"** passing through (or near) the plotted points.



The "best straight line" averages out experimental errors.

Points which are out of character with the rest are usually ignored.

Measuring the gradient, m, and the intercept, c, on the y-axis is **not** always the wisest way of proceeding and should be avoided.

The reasons are as follows:

(i) The intercept may be "off the page".

(ii) Symbols other than x or y may leave doubts as to which is the equivalent of the y-axis and which is the equivalent of the x-axis.

Standard Method

Take two sets of readings, (x_1, y_1) and (x_2, y_2), from the best straight line drawn; then solve the simultaneous linear equations

$$\begin{aligned} y_1 &= mx_1 + c, \\ y_2 &= mx_2 + c. \end{aligned}$$

Choosing the two points as far apart as possible will reduce errors in calculation due to the use of small quantities.

5.3.2 LAWS REDUCIBLE TO LINEAR FORM

Experimental laws which are not linear can sometimes be reduced to linear form.

EXAMPLES

1.

$$y = ax^2 + b.$$

Method:

Let $X = x^2$ so that $y = aX + b$.

We may obtain a straight line by plotting y against X.

2.

$$y = ax^2 + bx.$$

Method:

Since

$$\frac{y}{x} = ax + b,$$

we may let $Y = \frac{y}{x}$, giving $Y = ax + b$.

A straight line will be obtained if we plot Y against x.

Note:

If one of the sets of readings is $(x, y) = (0, 0)$, we must ignore it in this example.

3.

$$xy = ax + b.$$

Methods:

(a) Letting $xy = Y$ gives $Y = ax + b$, and we could plot Y against x.

(b) Writing $y = a + \frac{b}{x}$, we could let $\frac{1}{x} = X$, giving $y = a + bX$.

In this case, a straight line is obtained if we plot y against X.

4.

$$y = ax^b.$$

Method:

Taking logarithms of both sides (base 10 will do here),

$$\log_{10} y = \log_{10} a + b \log_{10} x.$$

Letting $\log_{10} y = Y$ and $\log_{10} x = X$ gives

$$Y = \log_{10} a + bX.$$

A straight line will be obtained by plotting Y against X.

5.

$$y = ab^x.$$

Method:

Here again, logarithms may be used to give

$$\log_{10} y = \log_{10} a + x \log_{10} b.$$

Letting $\log_{10} y = Y$, we have

$$Y = \log_{10} a + x \log_{10} b.$$

A straight line is obtained if we plot Y against x.

6.

$$y = ae^{bx}.$$

Method:

Taking **natural** logarithms of both sides,

$$\log_e y = \log_e a + bx,$$

also written

$$\ln y = \ln a + bx.$$

Letting $\ln y = Y$, we obtain a straight line by plotting Y against x.

5.3.3 THE USE OF LOGARITHMIC GRAPH PAPER

In logarithmic examples, we may use a special kind of graph paper on which there is printed a logarithmic scale along one or both of the axis directions.

0.1 0.2 0.3 0.4 1 2 3 4 10

A logarithmic scale evaluates the logarithms of the numbers assigned to it provided these numbers are allocated to each **"cycle"** of the scale in successive powers of 10.

Data which includes numbers spread over several different successive powers of ten will need graph paper which has at least that number of cycles in the appropriate axis direction.

EXAMPLE

The numbers 0.03, 0.09, 0.17, 0.33, 1.82, 4.65, 12, 16, 20, 50 will need **four** cycles on a logarithmic scale.

These restrictions make logarithmic graph paper less economical to use than ordinary graph paper.

On a logarithmic scale, we plot the **actual** values of the variables whose logarithms we would otherwise have needed to look up.

This will give the straight line graph from which we take the usual two sets of readings.

The two sets of readings are substituted into the form of the experimental equation which occurs immediately after taking logarithms of both sides.

Any base of logarithms may be used since logarithms to two different bases are proportional to each other.

EXAMPLES

1.

$$y = ax^b.$$

Method

(i) Taking logarithms (to base 10) of both sides,

$$\log_{10} y = \log_{10} a + b \log_{10} x.$$

(ii) Plot a graph of y against x, both on logarithmic scales.

(iii) Estimate the "best straight line".

(iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations,

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + b \log_{10} x_1, \\ \log_{10} y_2 &= \log_{10} a + b \log_{10} x_2.\end{aligned}$$

If it is possible to choose readings which are powers of 10, so much the better, but this is not essential.

2.

$$y = ab^x.$$

Method

(i) Taking logarithms (to base 10) of both sides,

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the the "best straight line".

(iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations,

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + x_1 \log_{10} b, \\ \log_{10} y_2 &= \log_{10} a + x_2 \log_{10} b.\end{aligned}$$

If it is possible to choose zero for the x_1 value, so much the better, but this is not essential.

3.

$$y = ae^{bx}.$$

Method

(i) Taking natural logarithms of both sides,

$$\ln y = \ln a + bx.$$

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the "best straight line".

(iv) Read off two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations,

$$\begin{aligned} \ln y_1 &= \ln a + bx_1, \\ \ln y_2 &= \ln a + bx_2. \end{aligned}$$

If it possible to choose zero for the x_1 value, so much the better, but this is not essential.

"JUST THE MATHS"

SLIDES NUMBER

5.4

GEOMETRY 4
(Elementary linear programming)

by

A.J.Hobson

UNIT 5.4 - GEOMETRY 10

ELEMENTARY LINEAR PROGRAMMING

5.4.1 FEASIBLE REGIONS

(i) The equation, $y = mx + c$, of a straight line is satisfied only by points which lie on the line. But it is useful to investigate the conditions under which a point with co-ordinates (x, y) may lie on one side of the line or the other.

(ii) For example, the inequality $y < mx + c$ is satisfied by points which lie **below** the line and the inequality $y > mx + c$ is satisfied by points which lie **above** the line.

(iii) Linear inequalities of the form $Ax + By + C < 0$ or $Ax + By + C > 0$ may be interpretted in the same way by converting, if necessary, to one of the forms in (ii).

(iv) Weak inequalities of the form $Ax + By + C \leq 0$ or $Ax + By + C \geq 0$ include the points which lie on the line itself as well as those lying on one side of it.

(v) Several simultaneous linear inequalties may be used to determine a region of the xy-plane throughout which all of the inequalities are satisfied. The region is called the **"feasible region"**.

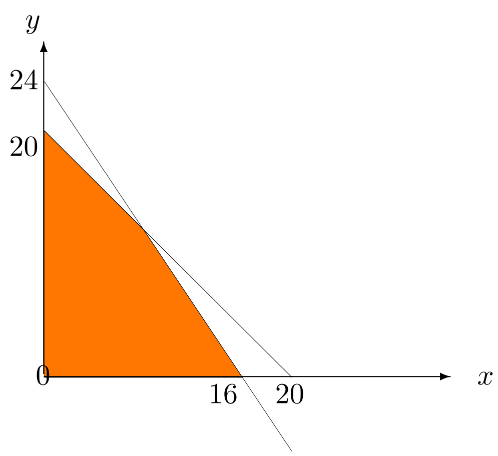
EXAMPLES

1. Determine the feasible region for the simultaneous inequalities

$$x \geq 0, \; y \geq 0, \; x + y \leq 20, \text{ and } 3x + 2y \leq 48.$$

Solution

We require the points of the first quadrant which lie on or below the straight line,
$y = 20 - x$ and on or below the straight line,
$y = -\frac{3}{2}x + 16.$
The feasible region is shown as the shaded area in the following diagram:



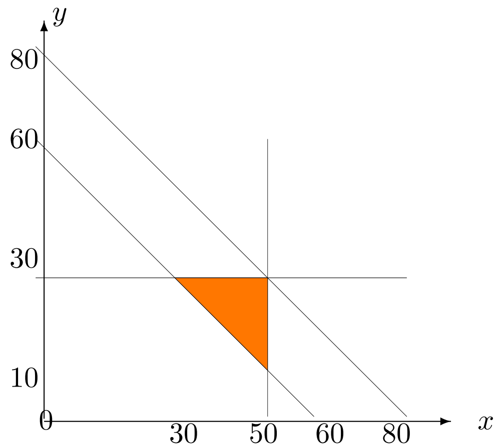
2. Determine the feasible region for the following simultaneous inequalities:

$$0 \leq x \leq 50, \ 0 \leq y \leq 30, \ x + y \leq 80, \ x + y \geq 60$$

Solution

We require the points which lie on or to the left of the straight line $x = 50$, on or below the straight line $y = 30$, on or below the straight line $y = 80 - x$ and on or above the straight line $y = 60 - x$.

The feasible region is shown as the shaded area in the following diagram:



5.4.2 OBJECTIVE FUNCTIONS

An important application of the feasible region discussed in the previous section is that of maximising (or minimising) a linear function of the form $px + qy$ subject to a set of simultaneous linear inequalities. Such a function is known as an **"objective function"**.

Essentially, it is required that a straight line with gradient $-\frac{p}{q}$ is moved across the appropriate feasible region until it reaches the highest possible point of that region for a maximum value or the lowest possible point for a minimum value. This will imply that the straight line $px + qy = r$ is such that r is the optimum value required.

However, for convenience, it may be shown that the optimum value of the objective function always occurs at one of the corners of the feasible region so that we simply evaluate it at each corner and choose the maximum (or minimum) value.

EXAMPLES

1. A farmer wishes to buy a number of cows and sheep. Cows cost £18 each and sheep cost £12 each.

   The farmer has accommodation for not more than 20 animals, and cannot afford to pay more than £288.

   If he can reasonbly expect to make a profit of £11 per cow and £9 per sheep, how many of each should he buy in order to make his total profit as large as possible ?

   Solution

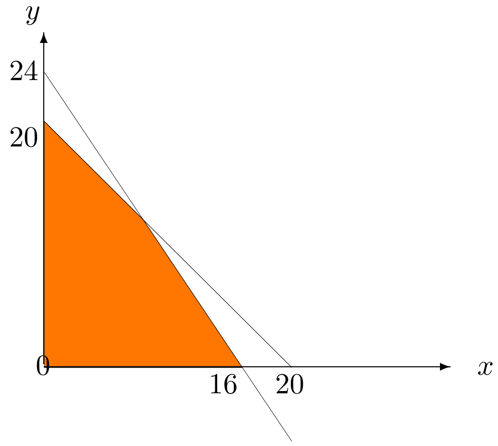
   Suppose he needs to buy x cows and y sheep; then, his profit is the objective function $P \equiv 11x + 9y$.

   Also,

$$x \geq 0, \; y \geq 0, \; x + y \leq 20,$$

$$\text{and } 18x + 12y \leq 288 \text{ or } 3x + 2y \leq 48.$$

   Thus, we require to maximize $P \equiv 11x + 9y$ in the feasible region for the first example of the previous section.



The corners of the region are the points $(0,0),\ (16,0), (0,20)$ and $(8,12)$ the last of these being the point of intersection of the two straight lines $x+y=20$ and $3x+2y=48$.

The maximum value occurs at the point $(8,12)$ and is equal to $88+108=196$. Hence, the farmer should buy 8 cows and 12 sheep

2. A cement manufacturer has two depots D_1 and D_2 which contain current stocks of 80 tons and 20 tons of cement respectively.

Two customers C_1 and C_2 place orders for 50 and 30 tons respectively

The transport cost is £1 per ton, per mile and the distances, in miles between D_1, D_2, C_1 and C_2 are given by the following table:

	C_1	C_2
D_1	40	30
D_2	10	20

From which depots should the orders be dispatched in order to minimise the transport costs ?

Solution

Suppose that D_1 distributes x tons to C_1 and y tons to C_2; then D_2 must distribute $50 - x$ tons to C_1 and $30 - y$ tons to C_2.

All quantities are positive and the following inequalties must be satisfied:

$$x \leq 50, \ y \leq 30, \ x + y \leq 80,$$

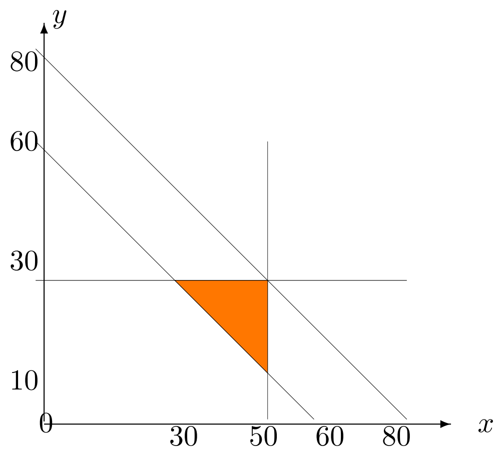
$$80 - (x + y) \leq 20 \text{ or } x + y \geq 60.$$

The total transport costs, T, are made up of $40x, \ 30y, \ 10(50 - x)$ and $20(30 - y)$.

That is, $T \equiv 30x + 10y + 1100$.

This is the objective function to be minimised.

From the diagram in the second example of the previous section, we need to evaluate the objective function at the points $(30, 30)$, $(50, 30)$ and $(50, 10)$.



The minimum occurs, in fact, at the point $(30, 30)$ so that D_1 should send 30 tons to C_1 and 30 tons to C_2 while D_2 should send 20 tons to C_1 but 0 tons to C_2.

"JUST THE MATHS"

SLIDES NUMBER

5.5

GEOMETRY 5
(Conic sections - the circle)

by

A.J.Hobson

UNIT 5.5 - GEOMETRY 4

CONIC SECTIONS - THE CIRCLE

5.5.1 INTRODUCTION

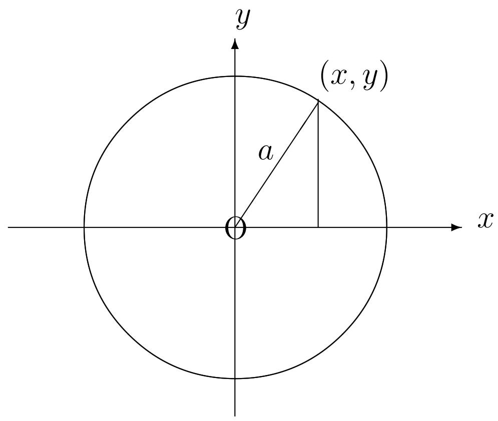
The Circle, the Parabola, the Ellipse and the Hyperbola could be generated, if desired, by considering plane sections through a cone; and, because of this, they are often called **"conic sections"** or even just **"conics"**.

DEFINITION

A circle is the path traced out by (or **"locus"** of) a point which moves at a fixed distance, called the **"radius"**, from a fixed point, called the **"centre"**.

5.5.2 STANDARD EQUATIONS FOR A CIRCLE

(a) Circle with centre at the origin and having radius a.



Using Pythagoras's Theorem in the diagram,

$$x^2 + y^2 = a^2.$$

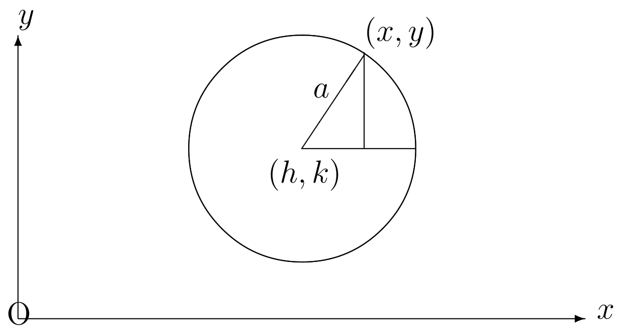
Note:
The angle θ in the diagram could be used as a parameter for the point (x, y) to give the parametric equations,

$$x = a\cos\theta, \; y = a\sin\theta.$$

Each point on the curve has infinitely many possible parameter values, all differing by a multiple of 2π.

We usually choose $-\pi < \theta \leq \pi$.

(b) Circle with centre (h, k) having radius a.



Using a temporary change of origin to the point (h, k) with X-axis and Y-axis, the circle would have equation

$$X^2 + Y^2 = a^2,$$

with reference to the new axes.

But, from previous work,

$$X = x - h, \text{ and } Y = y - k.$$

Hence, with reference to the original axes, the circle has equation

$$(x - h)^2 + (y - k)^2 = a^2;$$

or, in its expanded form,

$$x^2 + y^2 - 2hx - 2ky + c = 0,$$

where
$$c = h^2 + k^2 - a^2.$$

Notes:
(i) The parametric equations of this circle with reference to the temporary new axes would be
$$X = a\cos\theta, \;\; Y = a\sin\theta.$$

Hence, the parametric equations of the circle with reference to the original axes are
$$x = h + a\cos\theta, \;\; y = k + a\sin\theta.$$

(ii) If the equation of a circle is in the form
$$(x - h)^2 + (y - k)^2 = a^2,$$
it is easy to identify the centre, (h, k) and the radius, a.

If the equation is in its expanded form, we **complete the square in the x and y terms** in order to return to the first form.

EXAMPLES

1. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$x^2 + y^2 + 4x + 6y + 4 = 0.$$

Solution

$$x^2 + 4x \equiv (x + 2)^2 - 4.$$

$$y^2 + 6y \equiv (y + 3)^2 - 9.$$

$$(x + 2)^2 + (y + 3)^2 = 9.$$

Hence the centre is the point $(-2, -3)$ and the radius is 3.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$5x^2 + 5y^2 - 10x + 15y + 1 = 0.$$

Solution

Dividing throughout by the coefficient of the x^2 and y^2 terms,

$$x^2 + y^2 - 2x + 3y + \frac{1}{5} = 0.$$

$$x^2 - 2x \equiv (x-1)^2 - 1.$$

$$y^2 + 3y \equiv \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{61}{20}.$$

Hence, the centre is the point $\left(1, -\frac{3}{2}\right)$ and the radius is $\sqrt{\frac{61}{20}} \cong 1.75$

Note:

Not every equation of the form

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

represents a circle.

For some combinations of h, k and c, the radius would not be a real number.

In fact,

$$a = \sqrt{h^2 + k^2 - c}$$

which could easily turn out to be unreal.

"JUST THE MATHS"

SLIDES NUMBER

5.6

GEOMETRY 6
(Conic sections - the parabola)

by

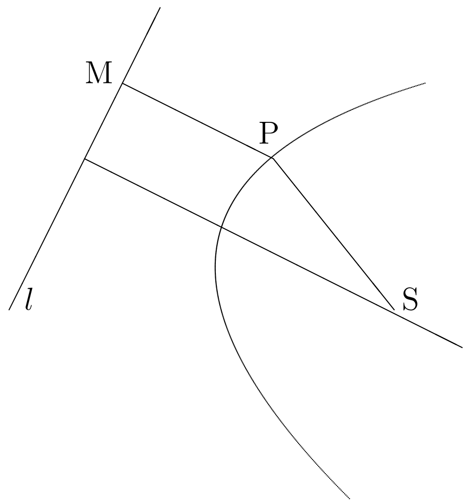
A.J.Hobson

UNIT 5.6 - GEOMETRY 6

CONIC SECTIONS - THE PARABOLA

5.6.1 INTRODUCTION

The Standard Form for the equation of a Parabola



DEFINITION

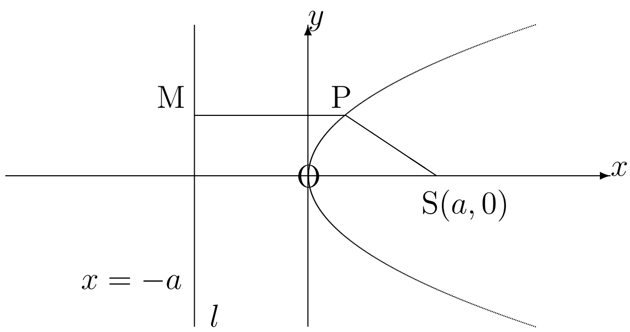
A parabola is the path traced out by (or **"locus"** of) a point, P, whose distance, SP, from a fixed point, S, called the **"focus"**, is equal to its perpendicular distance, PM, from a fixed line, l, called the **"directrix"**.

For convenience, we take the directrix l to be a vertical line, with the perpendicular onto it from the focus, S, being the x-axis.

For the y-axis, we take the line parallel to the directrix passing through the mid-point of the perpendicular from the focus onto the directrix.

The mid-point of the perpendicular from the focus onto the directrix is one of the points on the parabola.

Hence, the curve passes through the origin.



Let the focus be the point $(a, 0)$.

SP = PM gives

$$\sqrt{(x-a)^2+y^2} = x + a.$$

Squaring both sides

$$(x-a)^2+y^2 = x^2+2ax+a^2,$$

or

$$x^2-2ax+a^2+y^2 = x^2+2ax+a^2.$$

This reduces to

$$y^2 = 4ax.$$

All other versions of the equation of a parabola will be based on this version.

Notes:

(i) If a is negative, the bowl of the parabola faces in the opposite direction towards negative x values.

(ii) Any equation of the form $y^2 = kx$, where k is a constant, represents a parabola with vertex at the origin and axis of symmetry along the x-axis.

Its focus will lie at the point $\left(\frac{k}{4}, 0\right)$.

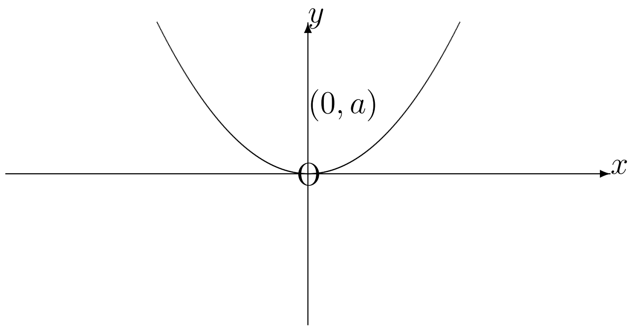
(iii) The parabola $y^2 = 4ax$ may be represented parametrically by the pair of equations

$$x = at^2, \;\; y = 2at.$$

The parameter, t, has no significance in the diagram.

5.6.2 OTHER FORMS OF THE EQUATION OF A PARABOLA

(a) Vertex at $(0, 0)$ with focus at $(0, a)$



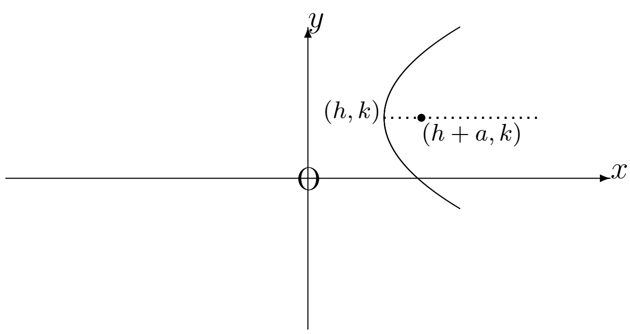
The equation is

$$x^2 = 4ay.$$

The parametric equations are

$$x = 2at, \ \ y = at^2.$$

(b) Vertex at (h, k) with focus at $(h + a, k)$



Using a temporary change of origin to the point (h, k), with X-axis and Y-axis, the parabola would have equation

$$Y^2 = 4aX.$$

With reference to the original axes, the parabola has equation

$$(y - k)^2 = 4a(x - h).$$

Notes:

(i) In the expanded form of this equation we may complete the square in the y terms to identify the vertex and focus.

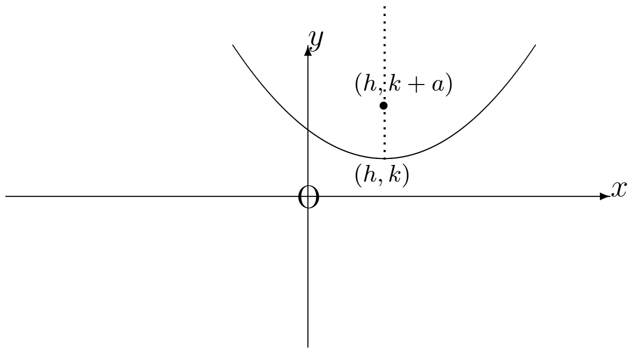
(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = at^2, \ \ Y = 2at.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + at^2, \ \ y = k + 2at.$$

(c) Vertex at (h, k) with focus at $(h, k + a)$



Using a temporary change of origin to the point (h, k) with X-axis and Y-axis, the parabola would have equation

$$X^2 = 4aY.$$

With reference to the original axes, the parabola has equation

$$(x - h)^2 = 4a(y - k).$$

Notes:

(i) In the expanded form of this equation, we may complete the square in the x terms to identify the vertex and focus.

(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = 2at, \ \ Y = at^2.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + 2at, \;\; y = k + at^2.$$

EXAMPLES

1. Give a sketch of the parabola whose cartesian equation is

$$y^2 - 6y + 3x = 10,$$

showing the co-ordinates of the vertex, focus and intesections with the x-axis and y-axis.

Solution

First, we complete the square in the y terms, obtaining

$$y^2 - 6y \equiv (y-3)^2 - 9.$$

Hence, the equation becomes

$$(y-3)^2 - 9 + 3x = 10.$$

That is,

$$(y-3)^2 = 19 - 3x,$$

or

$$(y-3)^2 = 4\left(-\frac{3}{4}\right)\left(x - \frac{19}{3}\right).$$

Thus, the vertex lies at the point $\left(\frac{19}{3}, 3\right)$ and the focus lies at the point $\left(\frac{19}{3} - \frac{3}{4}, 3\right)$; that is, $\left(\frac{67}{12}, 3\right)$.

The parabola $y^2 - 6y + 3x = 10$ intersects the x-axis where $y = 0$; that is,

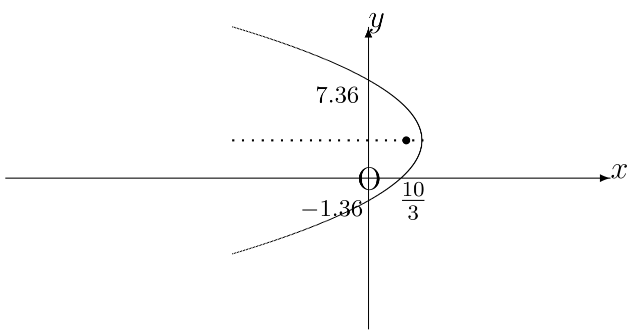
$$3x = 10,$$

giving $x = \frac{10}{3}$.
The parabola $y^2 - 6y + 3x = 10$ intersects the y-axis where $x = 0$; that is,

$$y^2 - 6y - 10 = 0.$$

This is a quadratic equation with solutions

$$y = \frac{6 \pm \sqrt{36 + 40}}{2} \cong 7.36 \text{ or } -1.36$$



2. Use the parametric equations of the parabola

$$x^2 = 8y$$

to determine its points of intersection with the straight line

$$y = x + 6.$$

Solution

The parametric equations are $x = 4t$, $y = 2t^2$.

Substituting the parametric equations into the equation of the straight line,

$$2t^2 = 4t + 6.$$

That is,

$$t^2 - 2t - 3 = 0,$$

or

$$(t - 3)(t + 1) = 0,$$

which is a quadratic equation in t having solutions $t = 3$ and $t = -1$.

The points of intersection are therefore $(12, 18)$ and $(-4, 2)$.

"JUST THE MATHS"

SLIDES NUMBER

5.7

GEOMETRY 7
(Conic sections - the ellipse

by

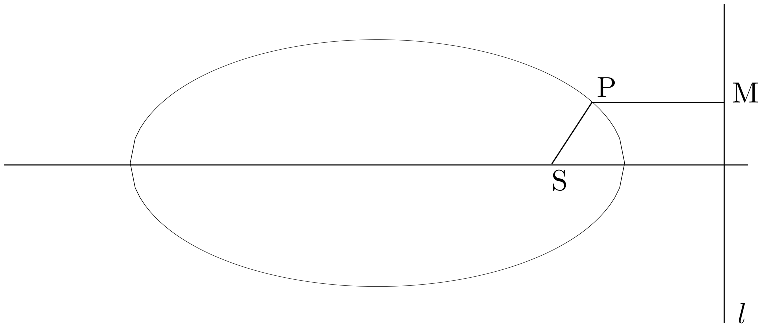
A.J.Hobson

UNIT 5.7 - GEOMETRY 7

CONIC SECTIONS - THE ELLIPSE

5.7.1 INTRODUCTION

The Standard Form for the equation of an Ellipse



DEFINITION

The Ellipse is the path traced out by (or **"locus"** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

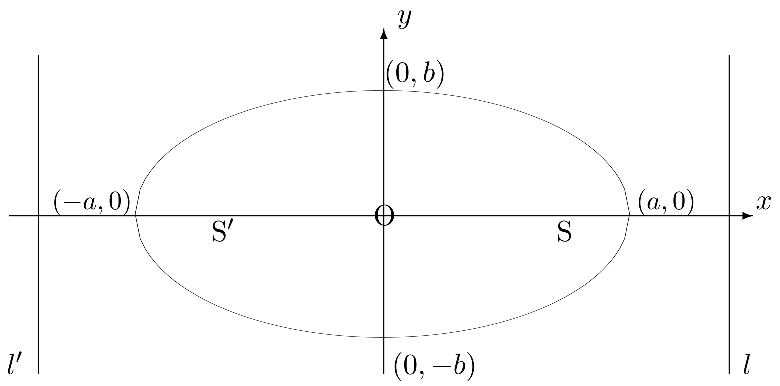
where $\epsilon < 1$ is a constant called the **"eccentricity"** of the ellipse.

The fixed line, l, is called a **"directrix"** of the ellipse and the fixed point, S, is called a **"focus"** of the ellipse.

The ellipse has two foci and two directrices because the curve is symmetrical about a line parallel to l **and** about the perpendicular line from S onto l.

The following diagram illustrates two foci, S and S$'$, together with two directrices, l and l'.

The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with associated parametric equations

$$x = a\cos\theta, \;\; y = b\sin\theta.$$

The curve intersects the axes at $(\pm a, 0)$ and $(0, \pm b)$.

The larger of a and b defines the length of the **"semi-major axis"**.

The smaller of a and b defines the length of the **"semi-minor axis"**.

The eccentricity, ϵ, is obtainable from the formula

$$b^2 = a^2\left(1 - \epsilon^2\right).$$

The foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$.

5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a\cos\theta, \;\; y = k + b\sin\theta.$$

Ellipses will usually be encountered in the **expanded** form of the standard cartesian equation.

It will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the ellipse.

EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

Solution

Completing the square in the x terms gives

$$3x^2 + 12x \equiv 3\left[x^2 + 4x\right]$$

$$\equiv 3\left[(x + 2)^2 - 4\right]$$

$$\equiv 3(x + 2)^2 - 12.$$

Completing the square in the y terms gives

$$y^2 - 2y \equiv (y - 1)^2 - 1.$$

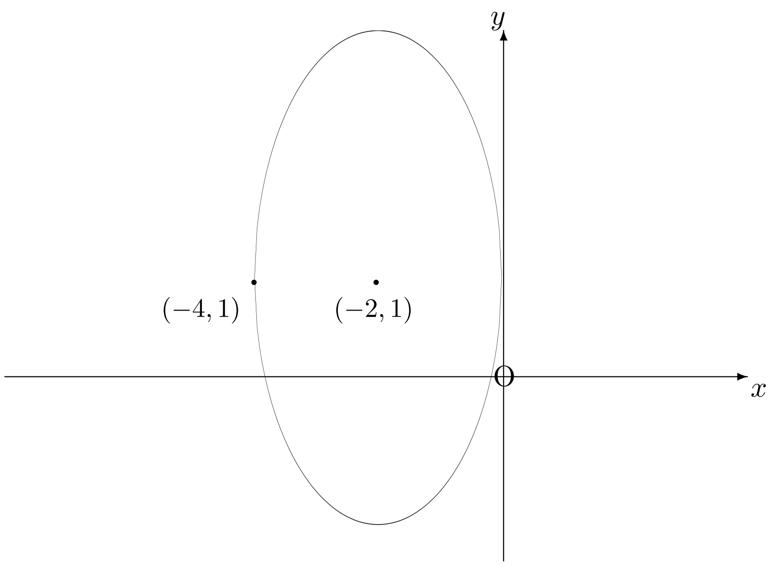
Hence, the equation of the ellipse becomes

$$3(x+2)^2 + (y-1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is at $(-2, 1)$ and the semi-axes have lengths $a = 2$ and $b = \sqrt{12}$.



"JUST THE MATHS"

SLIDES NUMBER

5.8

GEOMETRY 8
(Conic sections - the hyperbola)

by

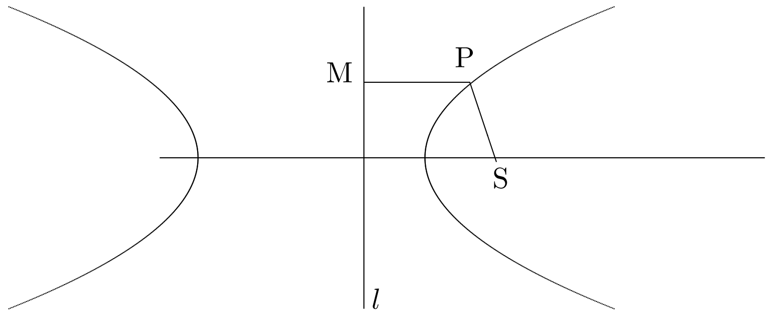
A.J.Hobson

UNIT 5.8 - GEOMETRY 8

CONIC SECTIONS - THE HYPERBOLA

5.8.1 INTRODUCTION

The Standard Form for the equation of a Hyperbola



DEFINITION

The hyperbola is the path traced out by (or "locus" of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

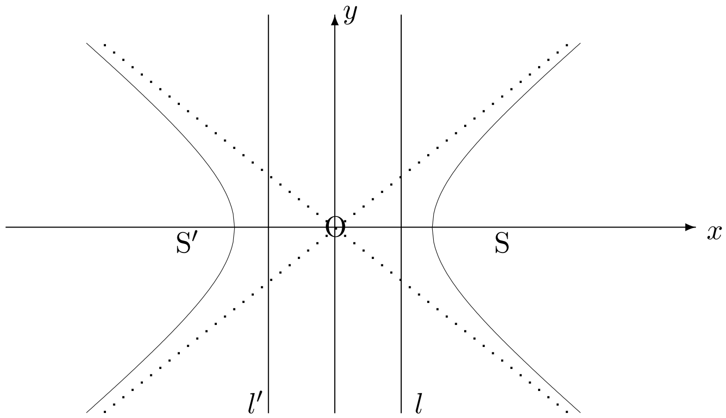
where $\epsilon > 1$ is a constant called the **"eccentricity"** of the hyperbola.

The fixed line, l, is called a **"directrix"** of the hyperbola and the fixed point, S, is called a **"focus"** of the hyperbola.

The hyperbola has two foci and two directrices because the curve is symmetrical about a line parallel to l **and** about the perpendicular line from S onto l

The following diagram illustrates two foci S and S$'$ together with two directrices l and l'.

The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the hyperbola has equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\text{sec}\theta, \;\; y = b\tan\theta.$$

For students who are familiar with “hyperbolic functions”, a set of parametric equations for the hyperbola is

$$x = a\text{cosh}t, \;\; y = b\text{sinh}t.$$

The curve intersects the x-axis at $(\pm a, 0)$ but does not intersect the y-axis at all.

The eccentricity, ϵ, is obtainable from the formula

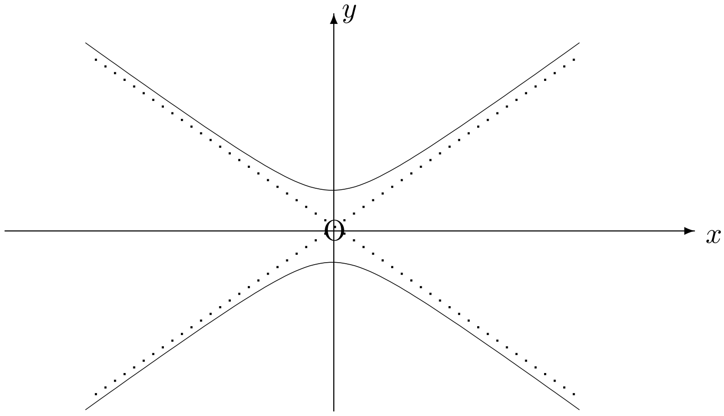
$$b^2 = a^2\left(\epsilon^2 - 1\right).$$

The foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$.

Note:
A hyperbola with centre $(0, 0)$, symmetrical about Ox and Oy, but intersecting the y-axis rather than the x-axis, has equation,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The roles of x and y are simply reversed.



5.8.2 ASYMPTOTES

At infinity, the hyperbola approaches two straight lines through the centre of the hyperbola called **"asymptotes"**.

It can be shown that both of the hyperbolae

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

have asymptotes whose equations are:

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

The equations of the asymptotes of a hyperbola are easily remembered by factorising the **left-hand side** of its equation, then equating each factor to zero.

5.8.3 MORE GENERAL FORMS FOR THE EQUATION OF A HYPERBOLA

The equation of a hyperbola, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

with associated parametric equations

$$x = h + a\text{sec}\theta, \;\; y = k + b\tan\theta$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

with associated parametric equations

$$x = h + a\tan\theta, \;\; y = k + b\text{sec}\theta.$$

Hyperbolae will usually be encountered in the expanded form of the standard cartesian equations.

It will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the hyperbola.

EXAMPLE

Determine the co-ordinates of the centre and the equations of the asymptotes of the hyperbola whose equation is

$$4x^2 - y^2 + 16x + 6y + 6 = 0.$$

Solution

Completing the square in the x terms gives

$$4x^2 + 16x \equiv 4\left[x^2 + 4x\right]$$

$$\equiv 4\left[(x+2)^2 - 4\right]$$

$$\equiv 4(x+2)^2 - 16.$$

Completing the square in the y terms gives

$$-y^2 + 6y \equiv -\left[y^2 - 6y\right]$$

$$\equiv -\left[(y-3)^2 - 9\right]$$

$$\equiv -(y-3)^2 + 9.$$

Hence the equation of the hyperbola becomes

$$4(x+2)^2 - (y-3)^2 = 1$$

or

$$\frac{(x+2)^2}{\left(\frac{1}{2}\right)^2} - \frac{(y-3)^2}{1^2} = 1.$$

The centre is located at the point $(-2, 3)$.

The asymptotes are

$$2(x+2) - (y-3) = 0 \text{ and } 2(x+2) + (y-3) = 0.$$

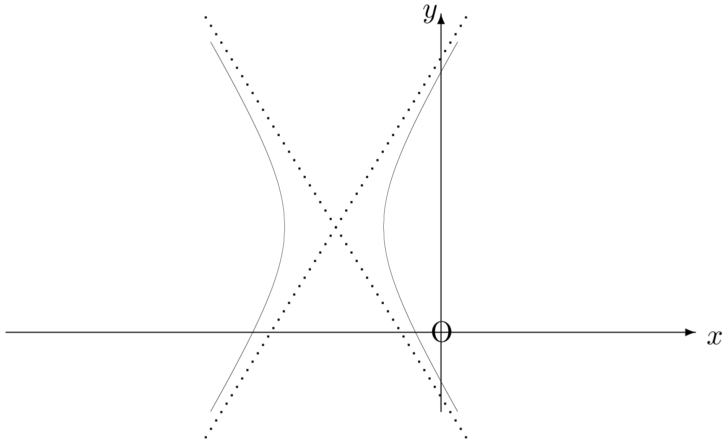
In other words,

$$2x - y + 7 = 0 \text{ and } 2x + y + 1 = 0.$$

To sketch the graph of a hyperbola, it is not always enough to have the position of the centre and the equations of the asymptotes.

It may also be necessary to investigate some of the intersections of the curve with the co-ordinate axes.

In the current example, it is possible to determine intersections at $(-0.84, 0)$, $(-7.16, 0)$, $(0, -0.87)$ and $(0, 6.87)$.



5.8.4 THE RECTANGULAR HYPERBOLA

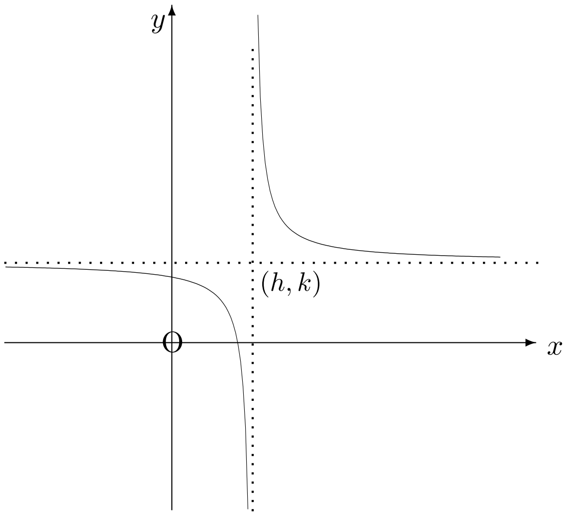
For some hyperbolae, the asymptotes are at
right-angles to each other.

In this case, the **asymptotes themselves** could be used as the x-axis and y-axis.

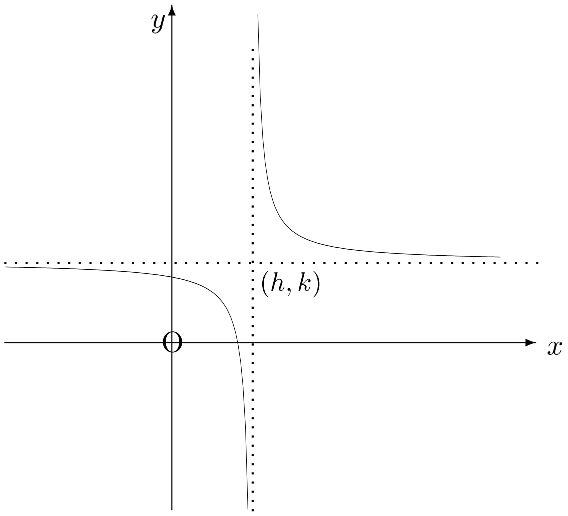
When this choice of reference system, the hyperbola, centre $(0, 0)$, has the equation

$$xy = C,$$

where C is a constant.



Similarly, a rectangular hyperbola with centre at the point (h, k) and asymptotes used as the axes of reference, has the equation, $(x - h)(y - k) = C$.



Note:
A suitable pair of parametric equations for the rectangular hyperbola, $(x - h)(y - k) = C$, are

$$x = t + h, \quad y = k + \frac{C}{t}.$$

EXAMPLES

1. Determine the centre of the rectangular hyperbola whose equation is

$$7x - 3y + xy - 31 = 0.$$

Solution

The equation factorises into the form

$$(x-3)(y+7) = 10.$$

Hence, the centre is located at the point $(3, -7)$.

2. A certain rectangular hyperbola has parametric equations,

$$x = 1 + t, \quad y = 3 - \frac{1}{t}.$$

Determine its points of intersection with the straight line $x + y = 4$.

Solution

Substituting for x and y into the equation of the straight line, we obtain

$$1 + t + 3 - \frac{1}{t} = 4 \quad \text{or} \quad t^2 - 1 = 0.$$

Hence, $t = \pm 1$ giving points of intersection at $(2, 2)$ and $(0, 4)$.

"JUST THE MATHS"

SLIDES NUMBER

5.9

GEOMETRY 9
(Curve sketching in general)

by

A.J.Hobson

UNIT 5.9 - GEOMETRY 8

CURVE SKETCHING IN GENERAL

Introduction

Here, we consider the approximate shape of a curve, whose equation is known, rather than an accurate "plot".

5.9.1 SYMMETRY

A curve is symmetrical about the x-axis if its equation contains only even powers of y.

A curve is symmetrical about the y-axis if its equation contains only even powers of x.

A curve is symmetrical with respect to the origin if its equation is unaltered when both x and y are changed in sign.

Symmetry with respect to the origin means that, if a point (x, y) lies on the curve, so does the point $(-x, -y)$.

ILLUSTRATIONS

1. The curve
$$x^2\left(y^2-2\right)=x^4+4$$
is symmetrical about both the x-axis and the y-axis.

   Once the shape of the curve is known in the first quadrant, the rest of the curve is obtained from this part by reflecting it in both axes.

   The curve is also symmetrical with respect to the origin.

2. The curve
$$xy=5$$
is symmetrical with respect to the origin but not about either of the co-ordinate axes.

5.9.2 INTERSECTIONS WITH THE CO-ORDINATE AXES

Any curve intersects the x-axis where $y=0$ and the y-axis where $x=0$.

Sometimes the curve has no intersection with one or more of the co-ordinate axes.

This will be borne out by an inability to solve for x when $y=0$ or y when $x=0$ (or both).

ILLUSTRATION

The circle,

$$x^2 + y^2 - 4x - 2y + 4 = 0,$$

meets the x-axis where

$$x^2 - 4x + 4 = 0.$$

That is,

$$(x - 2)^2 = 0,$$

giving a double intersection at the point $(2, 0)$.

This means that the circle **touches** the x-axis at $(2, 0)$.

The circle meets the y-axis where

$$y^2 - 2y + 4 = 0.$$

That is,

$$(y - 1)^2 = -3,$$

which is impossible, since the left hand side is bound to be positive when y is a real number.

Thus there are no intersections with the y-axis.

5.9.3 RESTRICTIONS ON THE RANGE OF EITHER VARIABLE

We illustrate as follows:

ILLUSTRATIONS

1. The curve whose equation is
$$y^2 = 4x$$
requires that x shall not be negative;
that is, $x \geq 0$.

2. The curve whose equation is
$$y^2 = x\left(x^2 - 1\right)$$
requires that the right hand side shall not be negative.
This will be so when either $x \geq 1$ or $-1 \leq x \leq 0$.

5.9.4 THE FORM OF THE CURVE NEAR THE ORIGIN

For small values of x (or y), the higher powers of the variable can be neglected to give a rough idea of the shape of the curve near to the origin.

ILLUSTRATION

The curve whose equation is

$$y = 3x^3 - 2x$$

approximates to the straight line,

$$y = -2x,$$

for very small values of x.

5.9.5 ASYMPTOTES

DEFINITION

An **"asymptote"** is a straight line which is approached by a curve at a very great distance from the origin.

Asymptotes Parallel to the Co-ordinate Axes

Consider the curve whose equation is

$$y^2 = \frac{x^3(3 - 2y)}{x - 1}.$$

(a) By inspection, we see that the straight line $x = 1$ "meets" this curve at an infinite value of y, making it an asymptote parallel to the y-axis.

(b) Now re-write the equation as

$$x^3 = \frac{y^2(x-1)}{3-2y}.$$

This suggests that the straight line $y = \frac{3}{2}$ "meets" the curve at an infinite value of x, making it an asymptote parallel to the x axis.

(c) Another method for (a) and (b) is to write the equation of the curve in a form without fractions.

In this case,

$$y^2(x-1) - x^3(3-2y) = 0.$$

We then equate to zero the coefficients of the highest powers of x and y.

That is,

the coefficient of y^2 gives $x - 1 = 0$.

the coefficient of x^3 gives $3 - 2y = 0$.

This method may be used with any curve to find asymptotes parallel to the co-ordinate axes.

If there aren't any such asymptotes, the method will not work.

(ii) Asymptotes in General for a Polynomial Curve

Suppose a given curve has an equation of the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y.

To find the intersections with this curve of a straight line

$$y = mx + c,$$

we substitute $mx + c$ in place of y.

We obtain a polynomial equation in x, say

$$a_0 + a_1 x + a_2 x^2 + \dots\dots + a_n x^n = 0.$$

For the line $y = mx + c$ to be an asymptote, this equation must have **coincident solutions at infinity**.

Replace x by $\frac{1}{u}$ and multiply throughout by u^n.

$$a_0 u^n + a_1 u^{n-1} + a_2 u^{n-2} + \dots\dots + a_{n-1} u + a_n = 0.$$

This equation must have coincident solutions at $u = 0$.

Hence

$$a_n = 0 \quad \text{and} \quad a_{n-1} = 0.$$

Conclusion
To find the asymptotes (if any) to a polynomial curve, we first substitute $y = mx + c$ into the equation of the curve.

Then, in the polynomial equation obtained, we **equate to zero the two leading coefficents** (that is, the coefficients of the highest two powers of x) and solve for m amd c.

EXAMPLE

Determine the equations of the asymptotes to the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution

Substituting $y = mx + c$ gives

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1.$$

That is,

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

Equating to zero the two leading coefficients; that is, the coefficients of x^2 and x, we obtain

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \text{ and } \frac{2mc}{b^2} = 0.$$

No solution is obtainable if $m = 0$ in the second statement since it implies $\frac{1}{a^2} = 0$ in the first statement.

Therefore, let $c = 0$ in the second statement, and $m = \pm\frac{b}{a}$ in the first statement.

The asymptotes are therefore

$$y = \pm\frac{b}{a}x \text{ that is } \frac{x}{a} \pm \frac{y}{b} = 0.$$

"JUST THE MATHS"

SLIDES NUMBER

5.10

GEOMETRY 10
(Graphical solutions)

by

A.J.Hobson

UNIT 5.10 - GEOMETRY 10

GRAPHICAL SOLUTIONS

5.10.1 INTRODUCTION

An algebraic equation in a variable quantity, x, may be written in the general form

$$f(x) = 0$$

where $f(x)$ is an algebraic expression involving x;
we call it a **"function of x"**; (see Unit 10.1).

In the following, $f(x)$ will be either

(a) a **linear** function of the form $ax + b$, where a and b are constants,

or

(b) a **quadratic** function of the form $ax^2 + bx + c$ where a, b and c are constants.

The solutions of the equation $f(x) = 0$ are those values of x which cause $f(x)$ to take the value zero.

The solutions are also the values of x for which the graph of the equation

$$y = f(x)$$

meets the x-axis.

5.10.2 THE GRAPHICAL SOLUTION OF LINEAR EQUATIONS

To solve the equation

$$ax + b = 0,$$

we may plot the graph of the equation $y = ax + b$ to find the point at which it meets the x-axis.

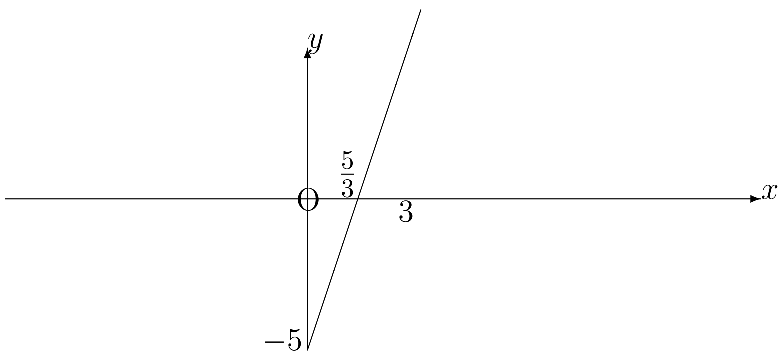
EXAMPLES

1. By plotting the graph of $y = 3x - 5$ from $x = 0$ to $x = 3$, solve the linear equation

$$3x - 5 = 0.$$

Solution

x	0	1	2	3
y	-5	-2	1	4



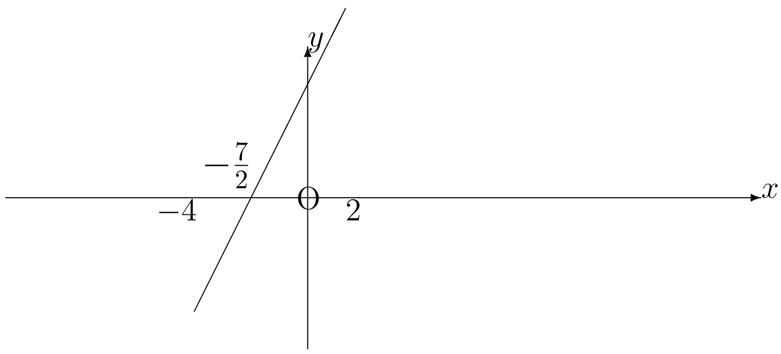
Hence $x \simeq 1.7$

2. By plotting the graph of $y = 2x + 7$ from $x = -4$ to $x = 2$, solve the linear equation

$$2x + 7 = 0.$$

Solution

x	-4	-3	-2	-1	0	1	2
y	-1	1	3	5	7	10	11



Hence $x = -3.5$

5.10.3 THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

To solve the quadratic equation

$$ax^2 + bx + c = 0,$$

we may plot the graph of the equation $y = ax^2 + bx + c$ and determine the points at which it crosses the x-axis.

Alternatively, we may plot the graphs of $y = ax^2 + bx$ and $y = -c$ to determine their points of intersection.

The alternative method is convenient since the first graph passes through the origin.

EXAMPLE

By plotting the graph of $y = x^2 - 4x$ from $x = -2$ to $x = 6$, solve the quadratic equations,

(a)

$$x^2 - 4x = 0;$$

(b)

$$x^2 - 4x + 2 = 0;$$

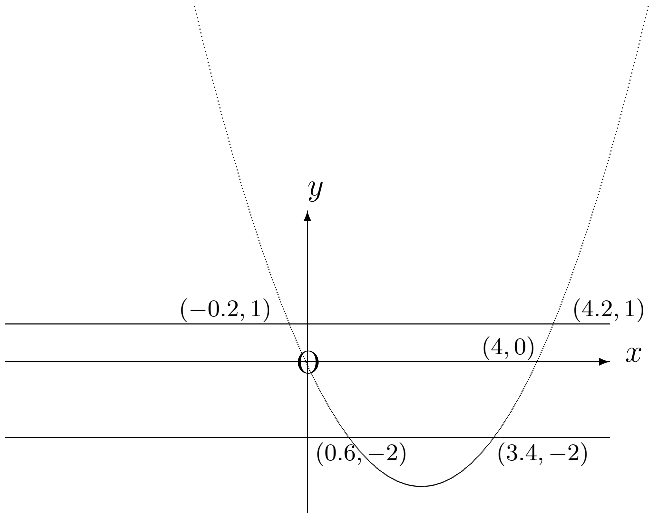
(c)

$$x^2 - 4x - 1 = 0.$$

Solution

x	-2	-1	0	1	2	3	4	5	6
y	12	5	0	-3	-2	-3	0	5	12

For parts (b) and (c), we shall also need the graphs of $y = -2$ and $y = 1$.



Hence the three sets of solutions are:

(a)

$$x = 0 \quad \text{and} \quad x = 4;$$

(b)

$$x \simeq 3.4 \quad \text{and} \quad x \simeq 0.6;$$

(c)

$$x \simeq 4.2 \quad \text{and} \quad x \simeq -0.2$$

5.10.4 THE GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

Problem

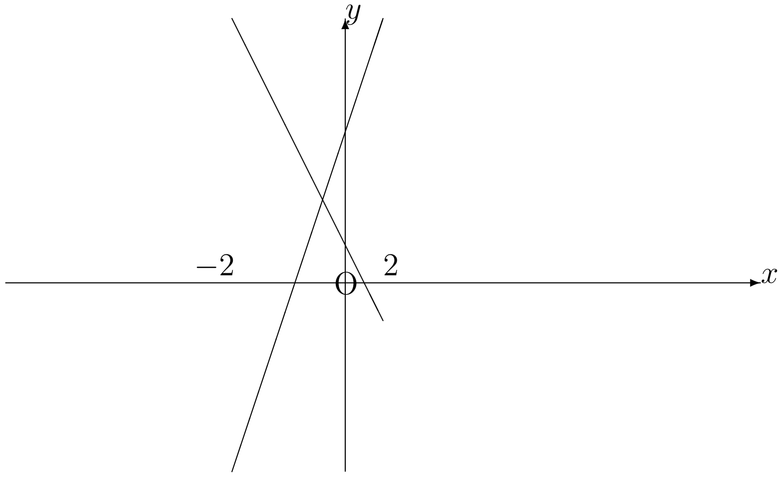
To solve either a pair of simultaneous linear equations or a pair of simultaneous equations consisting of one linear and one quadratic equation.

EXAMPLES

1. By plotting the graphs of $5x + y = 2$ and $-3x + y = 6$ from $x = -2$ to $x = 2$, determine the common solution of the two equations.

   Solution

x	-2	-1	0	1	2
$y_1 = 2 - 5x$	12	7	2	-3	-8
$y_2 = 6 + 3x$	0	3	6	9	12



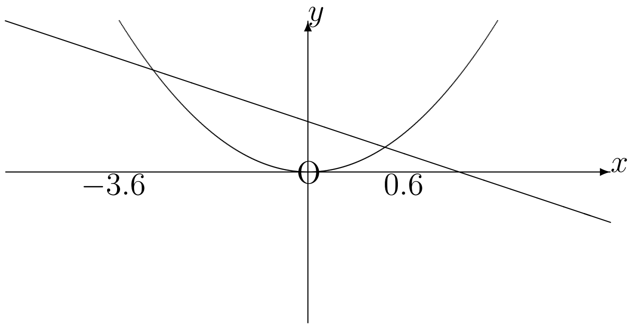
Hence $x = -0.5$ and $y = 4.5$

2. By plotting the graphs of the equations $y = x^2$ and $y = 2 - 3x$ from $x = -4$ to $x = 2$, determine their common solutions and hence solve the quadratic equation

$$x^2 + 3x - 2 = 0.$$

Solution

x	-4	-3	-2	-1	0	1	2
$y_1 = x^2$	16	9	4	1	0	1	4
$y_2 = 2 - 3x$	14	11	8	5	2	-1	-4



Hence $x \simeq 0.6$ and $x \simeq -3.6$

“JUST THE MATHS”

UNIT NUMBER

5.11

GEOMETRY 11
(Polar curves)

by

A.J.Hobson

UNIT 5.11 - GEOMETRY 11 - POLAR CURVES

5.11.1 INTRODUCTION

For conversion from cartesian co-ordinates, x and y, to polar co-ordinates, r and θ, we use the formulae,

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta,$$

For the reverse process, we may use the formulae,

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Sometimes the reverse process may be simplified by using a mixture of both sets of formulae.

We shall consider the graphs of certain relationships between r and θ without necessarily refering to the equivalent of those relationships in cartesian co-ordinates.

The graphs obtained will be called **"polar curves"**.

Note:

For the present context it will be necessary to assign a meaning to a point (r, θ), in polar co-ordinates, when r is negative.

We plot the point at a distance of $|r|$ along the $\theta - 180°$ line.

This implies that, when r is negative, the point (r, θ) is the same as the point $(|r|, \theta - 180°)$

5.11 2 THE USE OF POLAR GRAPH PAPER

For equations in which r is expressed in terms of θ, we plot r against θ using a graph paper divided into small cells by concentric circles and radial lines.

The radial lines are usually spaced at intervals of 15°.

The concentric circles allow a scale to be chosen to measure the distances, r, from the pole.

EXAMPLES

1. Sketch the graph of the equation $r = 2\sin\theta$.

   Solution

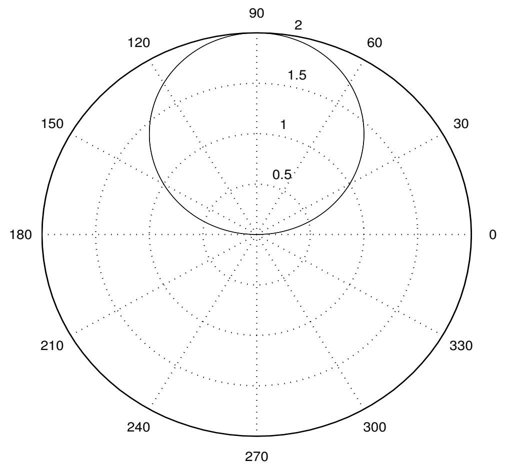
   First we construct a table of values of r and θ, in steps of 15°, from 0° to 360°.

θ	0°	15°	30°	45°	60°	75°	90°
r	0	0.52	1	1.41	1.73	1.93	2

θ	105°	120°	135°	150°	165°	180°	195°
r	1.93	1.73	1.41	1	0.52	0	−0.52

θ	210°	225°	240°	255°	270°	285°
r	−1	−1.41	−1.73	−1.93	−2	−1.93

θ	300°	315°	330°	345°	360°
r	−1.73	−1.41	−1	−0.52	0



Notes:

(i) The curve is a circle whose cartesian equation turns out to be

$$x^2 + y^2 - 2y = 0.$$

(ii) Since half of the values of r are negative, the circle is described twice over.

For example, the point $(-0.52, 195°)$ is the same as the point $(0.52, 15°)$.

2. Sketch the graph of the following equations:

(a)

$$r = 2(1 + \cos\theta);$$

(b)

$$r = 1 + 2\cos\theta;$$

(c)

$$r = 5 + 3\cos\theta.$$

Solution

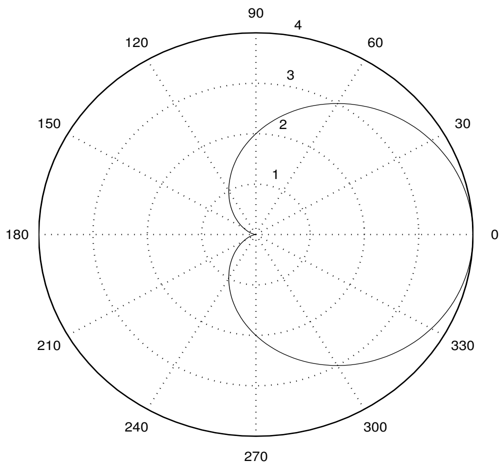
(a) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	4	3.93	3.73	3.42	3	2.52	2

θ	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	1.48	1	0.59	0.27	0.07	0

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$
r	0.07	0.27	0.59	1	1.48	2

θ	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	2.52	3	3.42	3.73	3.93	4



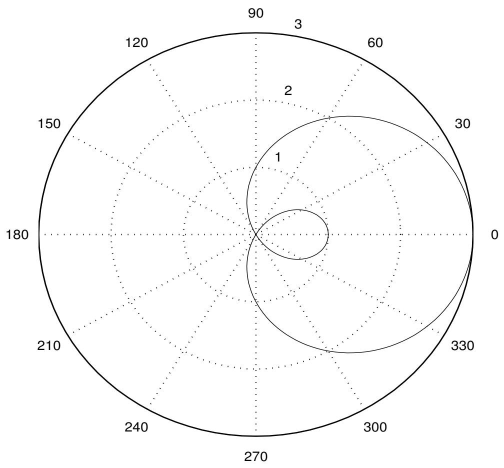
(b) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	3	2.93	2.73	2.41	2	1.52	1

θ	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	30.48	0	-0.41	-0.73	-0.93	-1

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$
r	-0.93	-0.73	-0.41	0	0.48	1

θ	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	1.52	2	2.41	2.73	2.93	3



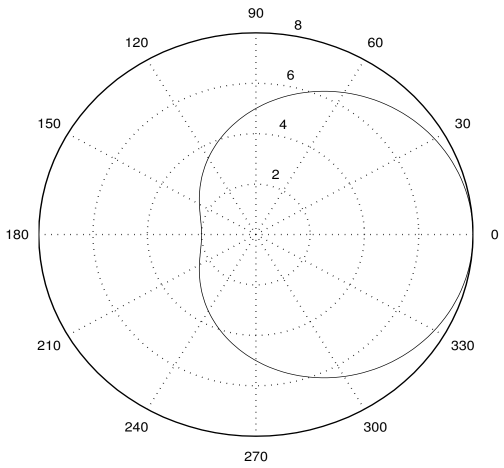
(c) The table of values is as follows:

θ	0°	15°	30°	45°	60°	75°	90°
r	8	7.90	7.60	7.12	6.5	5.78	5

θ	105°	120°	135°	150°	165°	180°
r	4.22	3.5	2.88	2.40	2.10	2

θ	195°	210°	225°	240°	255°	270°
r	2.10	2.40	2.88	3.5	4.22	5

θ	285°	300°	315°	330°	345°	360°
r	5.78	6.5	7.12	7.60	7.90	8



Note:

Each of the three curves in the above example is known as a **"limacon"**.

They illustrate special cases of the more general curve, $r = a + b\cos\theta$, as follows:

(i) If $a = b$, the limacon may also be called a **"cardioid"** (heart-shape). At the pole, the curve possesses a **"cusp"**.

(ii) If $a < b$, the limacon contains a **"re-entrant loop"**.

(iii) If $a > b$, the limacon contains neither a cusp nor a re-entrant loop.

For other well-known polar curves, together with any special titles associated with them, refer to the answers to the exercises associated with this unit.

"JUST THE MATHS"

SLIDES NUMBER

6.1

COMPLEX NUMBERS 1
(Definitions and algebra)

by

A.J.Hobson

UNIT 6.1 - COMPLEX NUMBERS 1
DEFINITIONS AND ALGEBRA

6.1.1 THE DEFINITION OF A COMPLEX NUMBER

INTRODUCTION

Equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

are called “Differential Equations”.

One method is to solve, first, the quadratic equation with coefficients a, b and c and, hence, solutions

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

ILLUSTRATION

Consider

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13 = 2\sin x.$$

We solve, first, the quadratic equation with coefficients 1, -6 and 13.

The solutions are

$$\frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4\sqrt{-1}}{2} = 3 \pm 2\sqrt{-1}.$$

Notes:

(i) The symbol $\sqrt{-1}$ will be regarded as an **"imaginary"** number.

(ii) $\sqrt{-1}$ will be denoted by j.

(iii) Solutions (involving j) of a quadratic equation, will always be of the form $a + bj$ or $a + jb$, where a and b are ordinary numbers of elementary arithmetic.

DEFINITIONS

1. The term **"complex number"** denotes any expression of the form $a + bj$ or $a + jb$ where a and b are ordinary numbers of elementary arithmetic (including zero) and $j^2 = -1$.
2. If $a = 0$, then bj or jb is **"purely imaginary"**.
3. If $b = 0$ then $a + j0 = a + 0j = a$ is **"real"**.

4. For $a + bj$ or $a + jb$, a is the **"real part"** and b is the **"imaginary part"**.
5. $a \pm bj$ or $a \pm jb$ form a pair of **"complex conjugates"**.

Note:
Other convenient notations include

$$z = x + jy \text{ and } \overline{z} = x - jy.$$

6.1.2 THE ALGEBRA OF COMPLEX NUMBERS

INTRODUCTION

(a) EQUALITY

Two complex numbers are defined to be equal if they have the same real part <u>and</u> the same imaginary part.

That is,

$$a + jb = c + jd \text{ if and only if } a = c \text{ and } b = d.$$

EXAMPLE

Determine x and y such that

$$(2x - 3y) + j(x + 5y) = 11 - j14.$$

Solution

EQUATING REAL AND IMAGINARY PARTS,

$$\begin{aligned} 2x - 3y &= 11, \\ x + 5y &= -14. \end{aligned}$$

These give $x = 1$ and $y = -3$.

(b) ADDITION AND SUBTRACTION

We add (or subtract) the real parts and the imaginary parts.

That is,

$$(a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d).$$

EXAMPLE

$(-7 + j2) + (10 - j5) = 3 - j3 = 3(1 - j)$ and
$(-7 + j2) - (10 - j5) = -17 + j7$.

(c) MULTIPLICATION

This is defined by

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad).$$

EXAMPLES

1.
$$(5+j9)(2+j6) = (10-54)+j(18+30) = -44+j48.$$
2.
$$(3 - j8)(1 + j4) = (3 + 32) + j(-8 + 12) = 35 + j4.$$
3.
$$(a + jb)(a - jb) = a^2 + b^2.$$

Note:
The product of a complex number and its complex conjugate is always a real number consisting of the sum of the squares of the real and imaginary parts.

(d) DIVISION

The method is to multiply both the numerator and the denominator of the complex ratio by the
conjugate of the denominator, giving

$$\frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} . \frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}.$$

EXAMPLES

1.

$$\frac{5+j3}{2+j7}=\frac{5+j3}{2+j7}.\frac{2-j7}{2-j7}$$

$$=\frac{(10+21)+j(6-35)}{2^2+7^2}=\frac{31-j29}{53}.$$

Hence, the real part is $\frac{31}{53}$ and the imaginary part is $-\frac{29}{53}$.

2.

$$\frac{6+j}{j2-4}=\frac{6+j}{j2-4}.\frac{-j2-4}{-j2-4}$$

$$=\frac{(-24+2)+j(-4-12)}{(-2)^2+(-4)^2}=\frac{-22-j16}{20}.$$

Hence the real part is $-\frac{22}{20}=-\frac{11}{10}$ and the imaginary part is $-\frac{16}{20}=-\frac{4}{5}$.

"JUST THE MATHS"

SLIDES NUMBER

6.2

COMPLEX NUMBERS 2
(The Argand Diagram)

by

A.J.Hobson

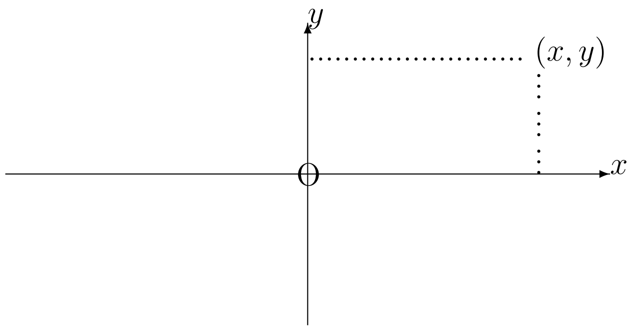
UNIT 6.2
COMPLEX NUMBERS 2

THE ARGAND DIAGRAM

6.2.1 INTRODUCTION

There is a **"one-to-one correspondence"** between the complex number $x + jy$ and the point with co-ordinates (x, y).

Hence it is possible to represent the complex number $x + jy$ by the point (x, y) in a geometrical diagram called the Argand Diagram



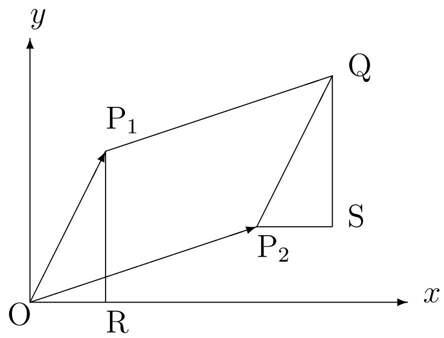
DEFINITIONS:

1. The x-axis is called the **"real axis"**; the points on it represent real numbers.
2. The y-axis is called the **"imaginary axis"**; the points on it represent purely imaginary numbers.

6.2.2 GRAPHICAL ADDITON AND SUBTRACTION

If two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, are represented in the Argand Diagram by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively, then the sum, $z_1 + z_2$, of the complex numbers will be represented by the point $Q(x_1 + x_2, y_1 + y_2)$.

If O is the origin, it is possible to show that Q is the fourth vertex of the parallelogram having OP_1 and OP_2 as adjacent sides.



In the diagram, the triangle ORP_1 has exactly the same shape as the triangle P_2SQ. Hence, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

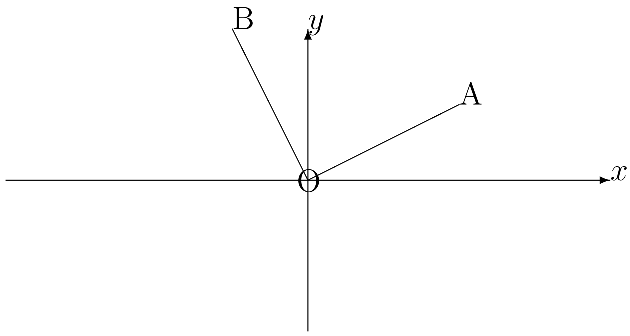
Note:
The difference $z_1 - z_2$ of the two complex numbers may similarly be found using $z_1 + (-z_2)$.

6.2.3 MULTIPLICATION BY j

Given any complex number $z = x + jy$, we observe that

$$jz = j(x + jy) = -y + jx.$$

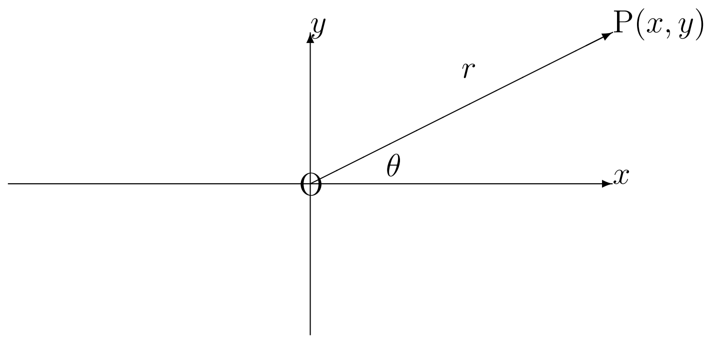
Thus, if z is represented in the Argand Diagram by the point with co-ordinates A(x, y), then jz is represented by the point with co-ordinates B$(-y, x)$.



But OB is in the position which would be occupied by OA if it were rotated through 90° in a counter-clockwise direction.

We conclude that, in the Argand Diagram, multiplication by j of a complex number rotates, through 90° in a counter-clockwise direction, the straight line segment joining the origin to the point representing the complex number.

6.2.4 MODULUS AND ARGUMENT



(a) Modulus

The distance, r, is called the **"modulus"** of z and is denoted by either $|z|$ or $|x + jy|$.

$$r = |z| = |x + jy| = \sqrt{x^2 + y^2}.$$

ILLUSTRATIONS

1.

$$|3 - j4| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

2.

$$|1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

3.

$$|j7| = |0 + j7| = \sqrt{0^2 + 7^2} = \sqrt{49} = 7.$$

(b) Argument

The **"argument"** (or **"amplitude"**) of a complex number, z, is defined to be the angle θ, measured positively counter-clockwise sense.

$$\tan\theta = \frac{y}{x}; \quad \text{that is,} \quad \theta = \tan^{-1}\frac{y}{x}.$$

Note:
For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360°.

The complete set of possible values is denoted by Arg$z$, using an upper-case A.

The particular value of the argument which lies in the interval $-180° < \theta \leq 180°$ is called the **"principal value"** of the argument and is denoted by $\arg z$ using a lower-case a.

The particular value 180°, in preference to −180°, represents the principal value of the argument of a negative real number.

ILLUSTRATIONS

1. $\text{Arg}(\sqrt{3}+j) =$

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ + k360^\circ,$$

where k may be any integer. But we note that

$$\arg(\sqrt{3}+j) = 30^\circ \ \text{ only.}$$

2. $\text{Arg}(-1+j) =$

$$\tan^{-1}(-1) = 135^\circ + k360^\circ$$

but **not** $-45^\circ + k360^\circ$, since the complex number $-1+j$ is represented by a point in the second quadrant of the Argand Diagram.

We note also that

$$\arg(-1+j) = 135^\circ \ \text{ only.}$$

3. $\text{Arg}(-1-j) =$

$$\tan^{-1}(1) = 225^\circ + k360^\circ \ \text{ or } \ -135^\circ + k360^\circ$$

but **not** $45^\circ + k360^\circ$, since the complex number $-1-j$ is represented by a point in the third quadrant of the Argand Diagram.

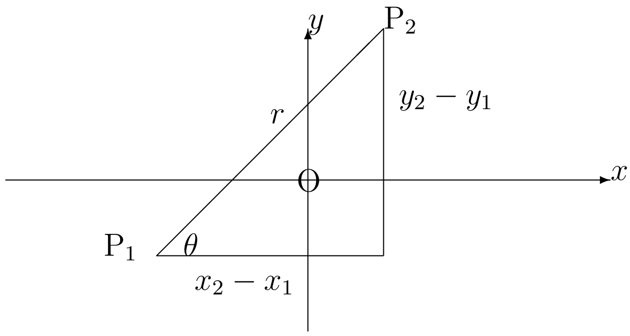
We note also that

$$\arg(-1-j) = -135^\circ \ \text{ only.}$$

Note:
The directed straight line segment described from the point P_1 (representing the complex number $z_1 = x_1 + jy_1$) to the point P_2 (representing the complex number $z_2 = x_2 + jy_2$) has length, r, equal to $|z_2 - z_1|$, and is inclined to the positive direction of the real axis at an angle, θ, equal to $\arg(z_2 - z_1)$.

This follows from the relationship

$$z_2 - z_1 = (x_2 - x_1) + j(y_2 - y_1).$$



“JUST THE MATHS”

SLIDES NUMBER

6.3

COMPLEX NUMBERS 3
(The polar & exponential forms)

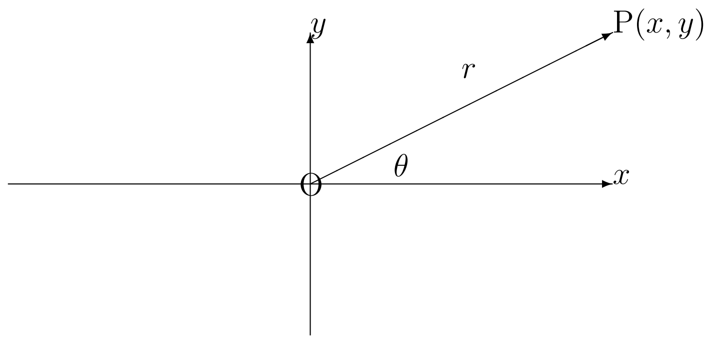
by

A.J.Hobson

UNIT 6.3 - COMPLEX NUMBERS 3

THE POLAR AND EXPONENTIAL FORMS

6.3.1 THE POLAR FORM



From the diagram,

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{y}{r} = \sin\theta;$$

$$x = r\cos\theta, \quad y = r\sin\theta;$$

$$x + jy = r(\cos\theta + j\sin\theta).$$

$x + jy$ is called the **"rectangular form"** or **"cartesian form"**.

$r(\cos\theta + j\sin\theta)$ ($r\angle\theta$ for short) is called the **"polar form"**.

θ may be positive, negative or zero and may be expressed in either degrees or radians.

EXAMPLES

1. Express the complex number $z = \sqrt{3} + j$ in polar form.

   Solution

   $$|z| = r = \sqrt{3+1} = 2$$

   and

   $$\text{Arg}z = \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ + k360^\circ,$$

   where k may be any integer.

   Alternatively, using radians,

   $$\text{Arg}z = \frac{\pi}{6} + k2\pi,$$

   where k may be any integer.

   Hence, in polar form, $z =$

   $2(\cos[30^\circ + k360^\circ] + j\sin[30^\circ + k360^\circ]) = 2\angle[30^\circ + k360^\circ]$

Alternatively, $z =$

$$2\left(\cos\left[\frac{\pi}{6} + k2\pi\right] + j\sin\left[\frac{\pi}{6} + k2\pi\right]\right) = 2\angle\left[\frac{\pi}{6} + k2\pi\right].$$

2. Express the complex number $z = -1 - j$ in polar form.

Solution

$$|z| = r = \sqrt{1+1} = \sqrt{2}$$

and

$$\text{Arg}z = \theta = \tan^{-1}(1) = -135^\circ + k360^\circ,$$

where k may be any integer.

Alternatively

$$\text{Arg}z = -\frac{3\pi}{4} + k2\pi,$$

where k may be any integer.

Hence, in polar form,

$$z = \sqrt{2}(\cos[-135^\circ + k360^\circ] + j\sin[-135^\circ + k360^\circ])$$

$$= \sqrt{2}\angle[-135^\circ + k360^\circ]$$

or

$$z = \sqrt{2}\left(\cos\left[-\frac{3\pi}{4} + k2\pi\right] + j\sin\left[-\frac{3\pi}{4} + k2\pi\right]\right)$$

$$= \sqrt{2}\angle\left[-\frac{3\pi}{4} + k2\pi\right].$$

Note:
If it is required that the polar form should contain only the **principal** value of the argument, θ, then, provided $-180° < \theta \leq 180°$ or $-\pi < \theta \leq \pi$, the component $k360°$ or $k2\pi$ of the result is simply omitted.

6.3.2 THE EXPONENTIAL FORM

It may be shown that

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

These are the **definitions** of $e^z, \sin z$ and $\cos z$.

In the equivalent series for $\sin x$ and $\cos x$, the value x (real), must be in **radians and not degrees**.

Deductions

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

But $j^2 = -1$, so

$$e^{j\theta} = 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

Hence,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

provided θ is expressed in radians and not degrees.

The complex number $x + jy$, having modulus r and argument $\theta + k2\pi$ may thus be expressed not only in polar form but also in

the exponential form, $re^{j\theta}$.

ILLUSTRATIONS

1.

$$\sqrt{3} + j = 2e^{j\left(\frac{\pi}{6} + k2\pi\right)}.$$

2.

$$-1 + j = \sqrt{2}e^{j\left(\frac{3\pi}{4} + k2\pi\right)}.$$

3.

$$-1 - j = \sqrt{2}e^{-j\left(\frac{3\pi}{4} + k2\pi\right)}.$$

Note:

If it is required that the exponential form should contain only the **principal** value of the argument, θ, then, provided $-\pi < \theta \leq \pi$, the component $k2\pi$ of the result is simply omitted.

6.3.3 PRODUCTS AND QUOTIENTS IN POLAR FORM

Let

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) = r_1\angle\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) = r_2\angle\theta_2.$$

(a) The Product

$$z_1.z_2 = r_1.r_2(\cos\theta_1 + j\sin\theta_1).(\cos\theta_2 + j\sin\theta_2)$$

That is,

$$z_1.z_2 = r_1.r_2([\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2]$$

$$+j[\sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2]).$$

$$z_1.z_2 = r_1.r_2(\cos[\theta_1+\theta_2]+j\sin[\theta_1+\theta_2]) = r_1.r_2\angle[\theta_1+\theta_2].$$

To determine the product of two complex numbers in polar form, we construct the product of their modulus values and the sum of their argument values.

(b) The Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}.$$

Multiplying the numerator and denominator by $\cos\theta_2 - j\sin\theta_2$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}([\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2]$$

$$+j[\sin\theta_1.\cos\theta_2 - \cos\theta_1.\sin\theta_2]).$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]) = \frac{r_1}{r_2}\angle[\theta_1 - \theta_2].$$

To determine the quotient of two complex numbers in polar form, we construct the quotient of their modulus values and the difference of their argument values.

ILLUSTRATIONS

1.

$$(\sqrt{3}+j).(-1-j) = 2\angle 30^\circ.\sqrt{2}\angle(-135^\circ)$$

$$= 2\sqrt{2}\angle(-105^\circ).$$

For all of the complex numbers in this example, including the result, the argument appears as the principal value.

2.

$$\frac{\sqrt{3}+j}{-1-j} = \frac{2\angle 30^\circ}{\sqrt{2}\angle(-135^\circ)} = \sqrt{2}\angle 165^\circ.$$

For all of the complex numbers in this example, including the result, the argument appears as the principal value.

3.

$$(-1-j).(-\sqrt{3}-j) = \sqrt{2}\angle(-135^\circ).2\angle(-150^\circ)$$

$$= 2\sqrt{2}\angle(-285^\circ).$$

This must be converted to $2\sqrt{2}\angle(75^\circ)$ if the principal value of the argument is required.

"JUST THE MATHS"

SLIDES NUMBER

6.4

COMPLEX NUMBERS 4
(Powers of complex numbers)

by

A.J.Hobson

UNIT 6.4 - COMPLEX NUMBERS 4

POWERS OF COMPLEX NUMBERS

6.4.1 POSITIVE WHOLE NUMBER POWERS

Let

$$z = r\angle\theta.$$

Then,

$$z^2 = r.r\angle(\theta + \theta) = r^2\angle 2\theta;$$

$$z^3 = z.z^2 = r.r^2\angle(\theta + 2\theta) = r^3\angle 3\theta;$$

$$z^n = r^n\angle n\theta.$$

This result is due to De Moivre.

EXAMPLE

$$\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)^{19} = (1\angle\left[\frac{\pi}{4}\right])^{19} = 1\angle\left[\frac{19\pi}{4}\right]$$

$$= 1\angle\left[\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}.$$

6.4.2 NEGATIVE WHOLE NUMBER POWERS

If n is a negative whole number, let

$$n = -m,$$

where m is a positive whole number.

$$z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{r^m\angle m\theta};$$

$$z^n = \frac{1}{r^m(\cos m\theta + j\sin m\theta)}$$

$$= \frac{1}{r^m}.\frac{(\cos m\theta - j\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos[-m\theta]+j\sin[-m\theta])$$

But $-m = n$, and so

$$z^n = r^n(\cos n\theta + j\sin n\theta) = r^n\angle n\theta.$$

The result of the previous section remains true for negative whole number powers.

EXAMPLE

$$(\sqrt{3}+j)^{-3} = (2\angle 30^\circ)^{-3} = \frac{1}{8}\angle(-90^\circ) = -\frac{j}{8}.$$

6.4.3 FRACTIONAL POWERS AND DE MOIVRE'S THEOREM

Consider the complex number

$$z^{\frac{1}{n}},$$

where n is a positive whole number and $z = r\angle\theta$.

$z^{\frac{1}{n}}$ means any complex number which gives z itself when raised to the power n.

Such a complex number is called
"an n-th root of z".

One such possibility is

$$r^{\frac{1}{n}}\angle\frac{\theta}{n}.$$

But, in general,

$$z = r\angle(\theta + k360^\circ),$$

where k may be any integer;

Hence,

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\angle\frac{\theta + k360^\circ}{n}.$$

The roots given by $k = 0, 1, 2, 3 \ldots\ldots\ldots\ldots n-1$ are also given by
$k = n, n+1, n+2, n+3, \ldots\ldots., 2n-1, 2n,$
$2n+1, 2n+2, 2n+3, \ldots..$

There are precisely n n-th roots given by
$k = 0, 1, 2, 3 \ldots\ldots\ldots, n-1.$

EXAMPLE

Determine the cube roots (i.e. 3rd roots) of the complex number $j8$.

Solution

We first write $j8 = 8\angle(90° + k360°)$.

Hence,

$$(j8)^{\frac{1}{3}} = 8^{\frac{1}{3}}\angle\frac{(90° + k360°)}{3}, \text{ where } k = 0, 1, 2.$$

The three distinct cube roots are

$$2\angle 30°, 2\angle 150° \text{ and } 2\angle 270° = 2\angle(-90°).$$

They all have the same modulus of 2 but their arguments are spaced around the Argand Diagram at regular intervals of $\frac{360°}{3} = 120°$.

Notes:

(i) The n-th roots of a complex number will all have the same modulus, but their arguments will be spaced at regular intervals of $\frac{360^\circ}{n}$.

(ii) If $-180^\circ < \theta \leq 180^\circ$, $k = 0$ gives the **"principal n-th root"**.

(ii) If $\frac{m}{n}$ is a fraction in its lowest terms,

$$z^{\frac{m}{n}} = \left(z^{\frac{1}{n}}\right)^m \text{ or } (z^m)^{\frac{1}{n}}.$$

DE MOIVRE'S THEOREM

If $z = r\angle\theta$, then, for any rational number n, **one value** of z^n is $r^n\angle n\theta$.

"JUST THE MATHS"

SLIDES NUMBER

6.5

COMPLEX NUMBERS 5
(Applications to trigonometric identities)

by

A.J.Hobson

UNIT 6.5 - COMPLEX NUMBERS 5

APPLICATIONS TO TRIGONOMETRIC IDENTITIES

6.5.1 INTRODUCTION

Reminder: **"Pascal's Triangle"**.

If n is a positive whole number, the diagram

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{array}$$

provides the coefficients in the expansion of $(A + B)^n$.

The sequence of terms is

$$A^n, \ A^{n-1}B, \ A^{n-2}B^2, \ A^{n-3}B^3, ..., B^n.$$

EXAMPLE

$$(A + B)^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3.$$

6.5.2 EXPRESSIONS FOR $\cos n\theta$ AND $\sin n\theta$ IN TERMS OF $\cos\theta$ AND $\sin\theta$.

From De Moivre's Theorem,

$$(\cos\theta + j\sin\theta)^n \equiv \cos n\theta + j\sin n\theta.$$

Real part of L.H.S. $\equiv \cos n\theta$.

Imaginary part of L.H.S $\equiv \sin n\theta$.

EXAMPLE

$$(\cos\theta + j\sin\theta)^3 \equiv$$

$$\cos^3\theta + 3\cos^2\theta.(j\sin\theta) + 3\cos\theta.(j\sin\theta)^2 + (j\sin\theta)^3.$$

Hence,

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta.\sin^2\theta$$

or $4\cos^3\theta - 3\cos\theta$, using $\sin^2\theta \equiv 1 - \cos^2\theta$;

$$\sin 3\theta \equiv 3\cos^2\theta.\sin\theta - \sin^3\theta$$

or $3\sin\theta - 4\sin^3\theta$, using $\cos^2\theta \equiv 1 - \sin^2\theta$.

6.5.3 EXPRESSIONS FOR $\cos^n\theta$ AND $\sin^n\theta$ IN TERMS OF SINES AND COSINES OF WHOLE MULTIPLES OF θ.

Particularly useful in calculus problems.

Suppose

$$z \equiv \cos\theta + j\sin\theta \quad - \quad (1)$$

Then, by De Moivre's Theorem, or by direct manipulation,

$$\frac{1}{z} \equiv \cos\theta - j\sin\theta \quad - \quad (2)$$

Adding (1) and (2) together, then subtracting (2) from (1), we obtain

$z + \frac{1}{z} \equiv 2\cos\theta$	$z - \frac{1}{z} \equiv j2\sin\theta$

Also, by De Moivre's Theorem,

$$z^n \equiv \cos n\theta + j\sin n\theta \quad - \quad (3)$$

and

$$\frac{1}{z^n} \equiv \cos n\theta - j\sin n\theta \quad - \quad (4)$$

Adding (3) and (4) together, then subtracting (4) from (3), we obtain

$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$	$z^n - \frac{1}{z^n} \equiv j2\sin n\theta$

Note:

This last result <u>includes</u> the previous one for $n = 1$.

EXAMPLES

1. Determine an identity for $\sin^3\theta$.

   Solution

$$j^3 2^3 \sin^3\theta \equiv \left(z - \frac{1}{z}\right)^3,$$

where $z \equiv \cos\theta + j\sin\theta$.

That is,

$$-j8\sin^3\theta \equiv z^3 - 3z^2.\frac{1}{z} + 3z.\left(\frac{1}{z}\right)^2 - \frac{1}{z^3}$$

or, after cancelling common factors,

$$-j8\sin^3\theta \equiv z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \equiv \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right),$$

which gives

$$-j8\sin^3\theta \equiv j2\sin 3\theta - j6\sin\theta.$$

Hence,

$$\sin^3\theta \equiv \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right).$$

2. Determine an identity for $\cos^4\theta$.

Solution

$$2^4\cos^4\theta \equiv \left(z + \frac{1}{z}\right)^4,$$

where $z \equiv \cos\theta + j\sin\theta$.

That is,

$$16\cos^4\theta \equiv z^4 + 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 + 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

or, after cancelling common factors,

$$16\cos^4\theta \equiv z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4} \equiv z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6,$$

which gives

$$16\cos^4\theta \equiv 2\cos 4\theta + 8\cos 2\theta + 6.$$

Hence,

$$\cos^4\theta \equiv \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).$$

"JUST THE MATHS"

SLIDES NUMBER

6.6

COMPLEX NUMBERS 6
(Complex loci)

by

A.J.Hobson

UNIT 6.6 - COMPLEX NUMBERS 6

COMPLEX LOCI

6.6.1 INTRODUCTION

The directed line segment joining the point representing a complex number z_1 to the point representing a complex number z_2 is of length equal to $|z_2 - z_1|$ and is inclined to the positive direction of the real axis at an angle equal to $\arg(z_2 - z_1)$. (See Unit 6.2).

Variable complex numbers may be constrained to move along a certain path (or **"locus"**) in the Argand Diagram.

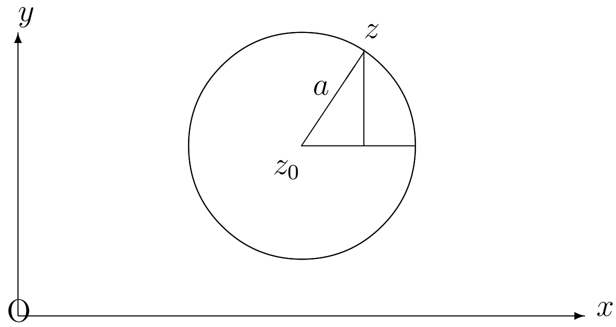
For many practical applications, such paths (or **"loci"**) will normally be either straight lines or circles.

Let $z = x + jy$ denote a **variable** complex number (represented by the point (x, y) in the Argand Diagram).

Let $z_0 = x_0 + jy_0$ denote a **fixed** complex number (represented by the point (x_0, y_0) in the Argand Diagram).

6.6.2 THE CIRCLE

Let the moving point representing z move on a circle, with radius a, whose centre is at the fixed point representing z_0.



Then,

$$|z - z_0| = a.$$

Note:
Substituting $z = x + jy$ and $z_0 = x_0 + jy_0$,

$$|(x - x_0) + j(y - y_0)| = a,$$

$$\text{or} \quad (x - x_0)^2 + (y - y_0)^2 = a^2.$$

ILLUSTRATION

$$|z - 3 + j4| = 7$$

represents a circle, with radius 7, whose centre is the point representing the complex number $3 - j4$.

In cartesian co-ordinates,

$$(x - 3)^2 + (y + 4)^2 = 49.$$

6.6.3 THE HALF-STRAIGHT-LINE

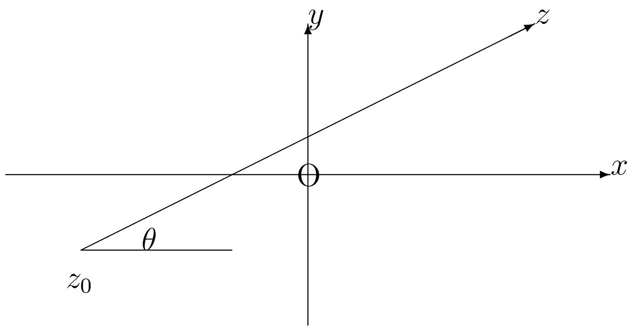
Let the **"directed"** straight line segment described **from** the fixed point representing z_0 **to** the moving point representing z be inclined at an angle θ to the positive direction of the real axis.

Then,

$$\arg(z - z_0) = \theta.$$

This equation is satisfied by **all** of the values of z for which the inclination of the directed line segment is genuinely θ and **not** $180^\circ - \theta$

$180^\circ - \theta$ corresponds to points on the other half of the straight line joining the two points.



Note:
Substituting $z = x + jy$ and $z_0 = x_0 + jy_0$,

$$\arg([x - x_0] + j[y - y_0]) = \theta.$$

That is,

$$\tan^{-1}\frac{y - y_0}{x - x_0} = \theta,$$

or

$$y - y_0 = \tan\theta(x - x_0).$$

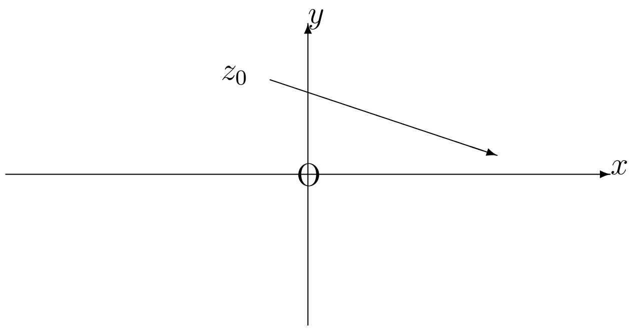
This is the equation of a straight line with gradient $\tan\theta$ passing through the point (x_0, y_0).

It represents only that half of the straight line for which $x - x_0$ and $y - y_0$ correspond, in sign as well as value, to the real and imaginary parts of a complex number whose argument is genuinely θ and not $180° - \theta$.

ILLUSTRATION

$$\arg(z + 1 - j5) = -\frac{\pi}{6}$$

represents the half-straight-line described from the point representing $z_0 = -1 + j5$ to the point representing $z = x + jy$ and inclined to the positive direction of the real axis at an angle of $-\frac{\pi}{6}$.



In terms of cartesian co-ordinates,

$$\arg([x + 1] + j[y - 5]) = -\frac{\pi}{6}.$$

We need $x + 1 > 0$ and $y - 5 < 0$.

We thus have the half-straight-line with equation

$$y - 5 = \tan\left(-\frac{\pi}{6}\right)(x + 1) = -\frac{1}{\sqrt{3}}(x + 1),$$

which lies to the right of, and below the point $(-1, 5)$.

6.6.4 MORE GENERAL LOCI

In general, we substitute $z = x + jy$ to obtain the cartesian equation of the locus.

ILLUSTRATIONS

1. The equation

$$\left|\frac{z-1}{z+2}\right| = 3$$

may be written

$$|z-1| = 3|z+2|.$$

That is,

$$(x-1)^2 + y^2 = 3[(x+2)^2 + y^2],$$

which simplifies to

$$2x^2 + 2y^2 + 14x + 13 = 0,$$

or

$$\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{23}{4},$$

representing a circle with centre $\left(-\frac{7}{2}, 0\right)$ and radius $\sqrt{\frac{23}{4}}$.

2. The equation

$$\arg\left(\frac{z-3}{z}\right) = \frac{\pi}{4}$$

may be written

$$\arg(z-3) - \arg z = \frac{\pi}{4}.$$

That is,

$$\arg([x-3]+jy) - \arg(x+jy) = \frac{\pi}{4}.$$

Taking tangents of both sides and using the trigonometric identity for $\tan(A-B)$,

$$\frac{\frac{y}{x-3} - \frac{y}{x}}{1 + \frac{y}{x-3}\frac{y}{x}} = 1.$$

On simplification,

$$x^2 + y^2 - 3x - 3y = 0,$$

or

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{2},$$

the equation of a circle with centre $\left(\frac{3}{2}, \frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$.
But $\frac{z-3}{z}$ cannot have an argument of $\frac{\pi}{4}$ unless its real and imaginary parts are **both** positive.

In fact,

$$\frac{z-3}{z} = \frac{(x-3)+jy}{x+jy}.\frac{x-jy}{x-jy} = \frac{x(x-3)+y^2+j3}{x^2+y^2},$$

which requires

$$x(x-3)+y^2 > 0.$$

That is,

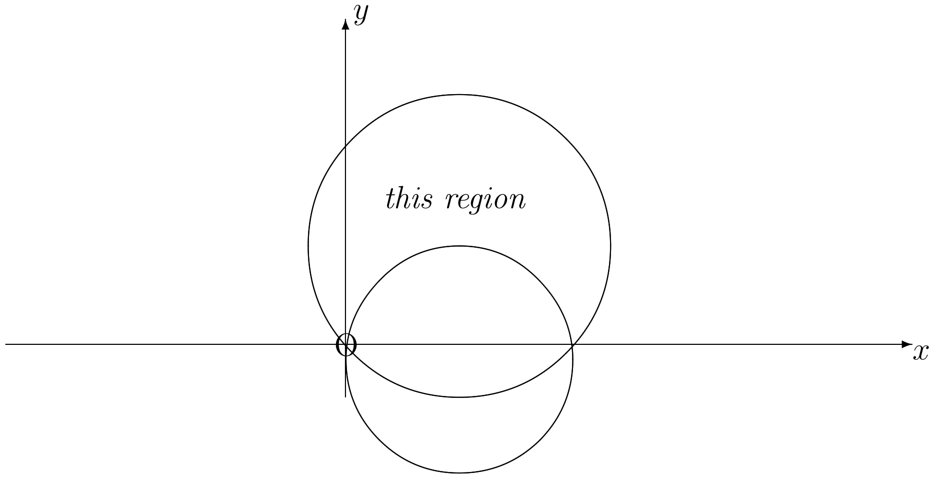
$$x^2+y^2-3x > 0,$$

or

$$\left(x-\frac{3}{2}\right)^2 + y^2 > \frac{9}{4}.$$

Conclusion

The locus is that part of the circle with centre $\left(\frac{3}{2},\frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$ which lies **outside** the circle with centre $\left(\frac{3}{2},0\right)$ and radius $\frac{3}{2}$.



"JUST THE MATHS"

SLIDES NUMBER

7.1

DETERMINANTS 1
(Second order determinants)

by

A.J.Hobson

UNIT 7.1 - DETERMINANTS 1

SECOND ORDER DETERMINANTS

7.1.1 PAIRS OF SIMULTANEOUS LINEAR EQUATIONS

Determinants may be introduced by considering

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - - -(2) \end{aligned}$$

Subtracting equation (2) $\times$ b_1 from equation (1) $\times$ b_2,

$$a_1b_2x - a_2b_1x + c_1b_2 - c_2b_1 = 0.$$

$$\text{Hence, } \ x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \ \text{ provided } \ a_1b_2 - a_2b_1 \neq 0.$$

Subtracting equation (2) $\times$ a_1 from equation (1) $\times$ a_2,

$$a_2b_1y - a_1b_2y + a_2c_1 - a_1c_2 = 0.$$

$$\text{Hemce, } \ y = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \ \text{ provided } \ a_1b_2 - a_2b_1 \neq 0.$$

The Symmetrical Form

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1},$$

provided $a_1b_2 - a_2b_1 \neq 0$.

7.1.2 THE DEFINITION OF A SECOND ORDER DETERMINANT

Let

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

The symbol on the left-hand-side may be called either a **"second order determinant"** or a **"2×2 determinant"**;

it has two **"rows"** (horizontally), two **"columns"** (vertically) and four **"elements"** (the numbers inside the determinant).

7.1.3 CRAMER'S RULE FOR TWO SIMULTANEOUS LINEAR EQUATIONS

The symmetrical solution to the two simultaneous linear equations may now be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0;$

or, in an abbreviated form,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

provided $\Delta_0 \neq 0$.

This determinant rule for solving two simultaneous linear equations is called **"Cramer's Rule"** and has equivalent forms for a larger number of equations.

Note:
The interpretation of Cramer's Rule in the case when $a_1b_2 - a_2b_1 = 0$ is a special case.

Observations

In Cramer's Rule,

1. To remember the determinant underneath x, cover up the x terms in the original simultaneous equations.
2. To remember the determinant underneath y, cover up the y terms in the original simultaneous equations.
3. To remember the determinant underneath 1 cover up the constant terms in the original simultaneous equations.
4. The final determinant is labelled Δ_0 as a reminder to evaluate it **first**.

   If $\Delta_0 = 0$, there is no point in evaluating Δ_1 and Δ_2.

EXAMPLES

1. Evaluate the determinant

$$\Delta = \begin{vmatrix} 7 & -2 \\ 4 & 5 \end{vmatrix}.$$

Solution

$$\Delta = 7 \times 5 - 4 \times (-2) = 35 + 8 = 43.$$

2. Express the value of the determinant

$$\Delta = \begin{vmatrix} -p & -q \\ p & -q \end{vmatrix}$$

in terms of p and q.

Solution

$$\Delta = (-p) \times (-q) - p \times (-q) = p.q + p.q = 2pq$$

3. Use Cramer's Rule to solve for x and y the simultaneous linear equations

$$\begin{aligned} 5x - 3y &= -3, \\ 2x - y &= -2. \end{aligned}$$

Solution

Rearrange the equations in the form

$$\begin{aligned} 5x - 3y + 3 &= 0, \\ 2x - y + 2 &= 0. \end{aligned}$$

Hence, by Cramer's Rule,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

where

$$\Delta_0 = \begin{vmatrix} 5 & -3 \\ 2 & -1 \end{vmatrix} = -5 + 6 = 1;$$

$$\Delta_1 = \begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -6 + 3 = -3;$$

$$\Delta_2 = \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} = 10 - 6 = 4.$$

Thus,

$$x = \frac{\Delta_1}{\Delta_0} = -3 \ \text{ and } \ y = -\frac{\Delta_2}{\Delta_0} = -4.$$

Special Cases

If $\Delta_0 = 0$, then the equations

$$
\begin{aligned}
a_1x + b_1y + c_1 &= 0, - - - - - - - - - - (1) \\
a_2x + b_2y + c_2 &= 0. - - - - - - - - - -(2)
\end{aligned}
$$

are such that

$$a_1b_2 - a_2b_1 = 0.$$

In other words,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

The x and y terms in one of the equations are proportional to the x and y terms in the other equation.

Two situations arise:

EXAMPLES

1. For the set of equations

$$
\begin{aligned}
3x - 2y &= 5, \\
6x - 4y &= 10,
\end{aligned}
$$

$\Delta_0 = 0$ but the second equation is simply a multiple of the first.

One of the equations is redundant and so there exists an **infinite number of solutions**.

Either of the variables may be chosen at random with the remaining variable being expressible in terms of it.

2. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 7, \end{aligned}$$

$\Delta_0 = 0$ but, from the second equation,

$$3x - 2y = 3.5,$$

which is inconsistent with

$$3x - 2y = 5.$$

In this case **there are no solutions at all**.

Summary of the Special Cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

"JUST THE MATHS"

SLIDES NUMBER

7.2

DETERMINANTS 2
(Consistency and third order determinants)

by

A.J.Hobson

UNIT 7.2 - DETERMINANTS 2

CONSISTENCY AND THIRD ORDER DETERMINANTS

7.2.1 CONSISTENCY FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the set of equations

$$
\begin{aligned}
a_1x + b_1y + c_1 &= 0, - - - - - - - - - - (1)\\
a_2x + b_2y + c_2 &= 0, - - - - - - - - - - (2)\\
a_3x + b_3y + c_3 &= 0, - - - - - - - - - - -(3)
\end{aligned}
$$

(Assume that any pair has a unique common solution by Cramer's Rule).

To be consistent, the common solution of any pair must also satisfy the remaining equation.

In particular, the common solution of equations (2) and (3) must also satisfy equation (1).

By Cramer's Rule in equations (2) and (3),

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

This solution will satisfy equation (1) provided that

$$a_1 \frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} - b_1 \frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + c_1 = 0.$$

In other words,

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is the determinant condition for the consistency of three simultaneous linear equations in two unknowns.

7.2.2 THE DEFINITION OF A THIRD ORDER DETERMINANT

In the consistency condition of the previous section, the expression on the left-hand-side is called a **"determinant of the third order"** and is denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

It has three **"rows"** (horizontally), three **"columns"** (vertically) and nine **"elements"** (the numbers inside the determinant).

The definition of a third order determinant may be stated in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notes:

(i) Other forms of the definition are also possible

(ii) Take each element of the first row in turn and multiply by its **"minor"**.

(iii) The Minor is the 2×2 order determinant obtained by covering up the row and column in which the element appears

(iv) the results are then combined according to a $+, -, +$ pattern.

(v) Rows are counted from the top to the bottom and columns are counted from left to the right.

(vi) Each row is read from the left to the right and each column is read from the top to the bottom.

ILLUSTRATION

The third element of the second column is b_3.

EXAMPLES

1. Evaluate the determinant

$$\Delta = \begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}.$$

Solution

$$\Delta = -3\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 5 & 3 \end{vmatrix} + 7\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}.$$

That is,

$$\Delta = -3(12 - 2) - 2(0 + 10) + 7(0 - 20) = -190.$$

2. Show that the simultaneous linear equations

$$\begin{aligned} 3x - y + 2 &= 0, \\ 2x + 5y - 1 &= 0, \\ 5x + 4y + 1 &= 0 \end{aligned}$$

are consistent (assuming that any two of the three have a common solution), and obtain the common solution.

Solution

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & -1 \\ 5 & 4 & 1 \end{vmatrix} =$$

$$3\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix}$$

$$= 3(5+4) + (2+5) + 2(8-25) = 27 + 7 - 34 = 0.$$

Thus, the equations are consistent.
Solving the first two:

$$\frac{x}{\begin{vmatrix} -1 & 2 \\ 5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}}.$$

That is,

$$\frac{x}{-9} = \frac{-y}{-7} = \frac{1}{17},$$

which gives

$$x = -\frac{9}{17} \text{ and } y = \frac{7}{17}.$$

Notes:

(i) The given set of equations are
"linearly dependent" (the third equation is the sum of the other two).

(ii) The rows of the determinant of coefficients and constants are linearly dependent
(Row 3 = Row 1 plus Row 2).

(iii) It may shown that the value of a determinant is zero if and only if its rows are linearly dependent.

(iv) An alternative way of proving consistency is to show linear dependence.

7.2.3 THE RULE OF SARRUS

So far,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} =$$

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

But the same terms can be obtained from following diagram:

$$\left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}.$$

Take the sum of the possible products of the trios of numbers in the direction $\searrow$

and subtract the sum of the possible products of the trios of numbers in the $\nearrow$ direction:

$$(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_2c_3a_1 + c_3a_2b_1).$$

These are exactly the same terms as those obtained by the original formula.

The **"Rule of Sarrus"** is ideal for electronic calculators.

EXAMPLE

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} \begin{matrix} -3 & 2 \\ 0 & 4 \\ 5 & -1 \end{matrix}$$

$$= ([-3].4.3 + 2.[-2].5 + 7.0.[-1])$$

$$-(5.4.7 + [-1].[-2].[-3] + 3.0.2)$$

$$= (-36 - 20 + 0) - (140 - 6 + 0)$$

$$= -56 - 134 = -190.$$

7.2.4 CRAMER'S RULE FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \end{aligned}$$

have a common solution, given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

which is called the "Key" to the solution and requires that $\Delta_0 \neq 0$.

Again the rule is known as "Cramer's Rule".

EXAMPLE

Using the Rule of Sarrus, obtain the common solution of the simultaneous linear equations

$$\begin{aligned} x+4y-z+2 &= 0, \\ -x-y+2z-9 &= 0, \\ 2x+y-3z+15 &= 0. \end{aligned}$$

Solution

The "Key" is

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

where

(i)

$$\Delta_0 = \left| \begin{array}{ccc} 1 & 4 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & -3 \end{array} \right| \begin{array}{cc} 1 & 4 \\ -1 & -1 \\ 2 & 1 \end{array}.$$

Hence,

$$\Delta_0 = (3+16+1) - (2+2+12) = 20 - 16 = 4$$

which is non-zero, and so we may continue:

(ii)

$$\Delta_1 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 2 & -9 \\ 1 & -3 & 15 \end{vmatrix} \begin{matrix} 4 & -1 \\ -1 & 2 \\ 1 & -3 \end{matrix}.$$

Hence,

$$\Delta_1 = (120 + 9 + 6) - (4 + 108 + 15) = 135 - 127 = 8.$$

(iii)

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 2 \\ -1 & 2 & -9 \\ 2 & -3 & 15 \end{vmatrix} \begin{matrix} 1 & -1 \\ -1 & 2 \\ 2 & -3 \end{matrix}.$$

Hence,

$$\Delta_2 = (30 + 18 + 6) - (8 + 27 + 15) = 54 - 50 = 4.$$

(iv)

$$\Delta_3 = \begin{vmatrix} 1 & 4 & 2 \\ -1 & -1 & -9 \\ 2 & 1 & 15 \end{vmatrix} \begin{matrix} 1 & 4 \\ -1 & -1 \\ 2 & 1 \end{matrix}.$$

Hence,

$$\Delta_3 = (-15 - 72 - 2) - (-4 - 9 - 60) = -89 + 73 = -16.$$

(v) The solutions are therefore

$$x = -\frac{\Delta_1}{\Delta_0} = -\frac{8}{4} = -2;$$

$$y = \frac{\Delta_2}{\Delta_0} = \frac{4}{4} = 1;$$

$$z = -\frac{\Delta_3}{\Delta_0} = -\frac{-16}{4} = 4.$$

Special Cases

If it should happen that $\Delta_0 = 0$, the rows of Δ_0 must be linearly dependent.

That is, the three groups of x, y and z terms must be linearly dependent.

Different situations arise according to the constant terms:

EXAMPLES

1. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 4 &= 0, \end{aligned}$$

the third equation is the difference between the first two and hence it is redundant.

There will be an infinite number of solutions. For example, we may choose z at random, solving for x and y;

$$x = \frac{11 - 5z}{5} \quad \text{and} \quad y = \frac{5z - 3}{5}.$$

2. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 7 &= 0, \end{aligned}$$

the third equation is inconsistent with the difference between the first two equations. That is,

$x - 3y + 4z - 7 = 0$ inconsistent with $x - 3y + 4z - 4 = 0$.

In this case, there are no common solutions.

3. For the simultaneous linear equations

$$\begin{aligned} x - 2y + 3z - 1 &= 0, \\ 2x - 4y + 6z - 2 &= 0, \\ 3x - 6y + 9z - 3 &= 0 \end{aligned}$$

we have only one independent equation.

There will be an infinite number of solutions.

Choose two of the variables at random, then determine the remaining variable.

Summary of the special cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

"JUST THE MATHS"

SLIDES NUMBER

7.3

DETERMINANTS 3
(Further evaluation of 3 x 3 determinants)

by

A.J.Hobson

UNIT 7.3 - DETERMINANTS 3

FURTHER EVALUATION OF THIRD ORDER DETERMINANTS

7.3.1 EXPANSION BY ANY ROW OR COLUMN

Reminders:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2).$$

This is the **"expansion by the first row"**.

ILLUSTRATION 1 - Expansion by the second row.

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

gives exactly the same result as in the original formula.

ILLUSTRATION 2 - Expansion by the third column

$$c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)$$

gives exactly the same result as

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

Note:

Similar patterns of symbols give the expansions by the remaining rows and columns.

Summary

A third order determinant may be expanded (that is, evaluated) if we first multiply each of the three elements in any row or (any column) by its minor;

then we combine the results according the following pattern of so-called **"place-signs"**.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

COFACTORS

Every **"signed-minor"** is called a **"cofactor"**.

When the place-sign is $+$, the minor and the cofactor are the same.

When the place-sign is $-$, the cofactor is <u>numerically</u> equal to the minor but opposite in sign.

For instance, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

(i) the minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

but the cofactor of b_1 is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

(ii) the minor and cofactor of b_2 are both equal to

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}.$$

7.3.2 ROW AND COLUMN OPERATIONS ON DETERMINANTS

INTRODUCTION

The following is especially useful for determinants where some or all of the elements are **variable** quantities.

STANDARD PROPERTIES OF DETERMINANTS

1. If all of the elements in a row or a column have the value zero, then the value of the determinant is equal to zero.

   Proof:

   Expand the determinant by the row or column of zeros.

2. If all but one of the elements in a row or column are equal to zero, then the value of the determinant is the product of the non-zero element in that row or column with its cofactor.

   Proof:

   Expand the determinant by the row or column containing the single non-zero element.

   The determinant is effectively equivalent to a determinant of one order lower.

   For example,

$$\begin{vmatrix} 5 & 1 & 0 \\ -2 & 4 & 3 \\ 6 & 8 & 0 \end{vmatrix} = -3\begin{vmatrix} 5 & 1 \\ 6 & 8 \end{vmatrix} = -3(40-6) = -102.$$

3. If a determinant contains two identical rows or two identical columns, then the value of the determinant is zero.

   Proof:

   Expand the determinant by a row or column other than the two identical ones.

   All of the cofactors have value zero.

   For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

4. If two rows, or two columns, are interchanged the value of the determinant is unchanged numerically but it is reversed in sign.

   Proof:

   Expand the determinant by a row or column other than the two which have been interchanged

   All of the cofactors will be changed in sign.

For example

$$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

5. If all of the elements in a row or column have a common factor, then this common factor may be removed from the determinant and placed outside.

   Proof:

   Expanding the determinant by the row or column which contains the common factor is equivalent to removing the common factor first, then expanding by the new row or column so created.

   For example,

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

   Note:

   If all elements in any row or column are multiplied by the same factor, then the value of the determinant is also multiplied by that factor.

6. If the elements of any row in a determinant are altered by adding to them (or subtracting from them) a common multiple of the corresponding elements in another row, then the value of the determinant is unaltered.

   A similar result applies to columns.

   ILLUSTRATION

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} =$$

$$[(a_1 + kb_1)b_2 - (a_2 + kb_2)b_1] = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

EXAMPLES

Let R_1, R_2 and R_3 denote Row 1, Row 2 and Row 3.

Let C_1, C_2 and C_3 denote Column 1, Column 2 and Column 3.

Let $\longrightarrow$ stand for "becomes".

The following examples use **"row operations"** and **"column operations"**.

1. Evaluate the determinant,

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix} \quad C_1 \quad \longrightarrow \quad C_1 \quad \div \quad 5;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \\ 3 & 2 & 3 \end{vmatrix} \quad R_2 \quad \longrightarrow \quad R_2 - 2R_1;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 3 & 2 & 3 \end{vmatrix} \quad R_3 \quad \longrightarrow \quad R_3 - 3R_1;$$

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 0 & -7 & -18 \end{vmatrix}$$

$$= 5(18 - 35) = 5 \times -17 = -85.$$

2. Solve, for x, the equation

$$\begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0.$$

Solution

Direct expansion gives a cubic equation in x.

Therefore, try to obtain factors of the equation **before** expanding the determinant.

Here, the three expressions in each column add up to the same quantity, namely $x + 2$.

Thus, add Row 2 to Row 1, then add Row 3 to the new Row 1.

This gives $x + 2$ as a factor of the first row.

$$0 = \begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} x+2 & x+2 & x+2 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x+2)$$

$$= (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix}$$

$$C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x+2)\begin{vmatrix} 1 & 0 & 0 \\ 5 & x-4 & -4 \\ -3 & -1 & x+1 \end{vmatrix}$$

$$= (x+2)[(x-4)(x+1) - 4] = (x-2)(x^2 - 3x - 8).$$

Hence,

$$x = -2 \text{ or } x = \frac{3 \pm \sqrt{9+32}}{2} = \frac{3 \pm \sqrt{41}}{2}.$$

3. Solve, for x, the equation

$$\begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} = 0.$$

Solution

The sum of the corresponding pairs of elements in the first two rows is the same, namely $x - 4$.

$$0 = \begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2$$

$$= \begin{vmatrix} x-4 & x-4 & x-4 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x-4)$$

$$= (x-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix}$$

$$C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x-4)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x & -1 \\ 7 & 1 & x \end{vmatrix}$$

$$= (x-4)(x^2+1)$$

$x = 4$ and $x = \pm j$.

4. Solve, for x, the equation

$$\begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix}.$$

Solution

The 2 in Row 1 may be used to reduce to zero the 4 underneath it in Row 2.

$$0 = \begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1$$

$$= \begin{vmatrix} x & 3 & 2 \\ 4-2x & x-2 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 \div (x-2)$$

$$= (x-2)\begin{vmatrix} x & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad C_1 \longrightarrow C_1 + 2C_2$$

$$= (x-2)\begin{vmatrix} x+6 & 3 & 2 \\ 0 & 1 & 0 \\ 4 & 1 & x-1 \end{vmatrix}$$

$$= (x-2)[(x+6)(x-1)-8] = (x-2)[x^2+5x-14]$$

$$= (x-2)(x+7)(x-2).$$

Thus,

$$x = 2 \ \text{(repeated)} \ \text{and} \ x = -7.$$

"JUST THE MATHS"

SLIDES NUMBER

7.4

DETERMINANTS 4
(Homogeneous linear equations)

by

A.J.Hobson

7.4.1 Trivial and non-trivial solutions

UNIT 7.4 - DETERMINANTS 4

HOMOGENEOUS LINEAR EQUATIONS

7.4.1 TRIVIAL AND NON-TRIVIAL SOLUTIONS

Consider <u>three</u> **"homogeneous"** linear equations of the form

$$\begin{aligned}
a_1x + b_1y + c_1z &= 0,\\
a_2x + b_2y + c_2z &= 0,\\
a_3x + b_3y + c_3z &= 0.
\end{aligned}$$

Observations

1. In Cramer's Rule, if $\Delta_0 \neq 0$, there will exist a unique solution, namely $x = 0$, $y = 0$, $z = 0$.

   Each of Δ_1, Δ_2 and Δ_3 will contain a column of zeros.

   But this solution is obvious and we call it the **"trivial solution"**.

2. Question: are there any **"non-trivial"** solutions.

3. Non-trivial solutions occur when the equations are not linearly independent.

   (a) If one of the equations is redundant, we could solve the remaining two in an infinite number of ways by choosing one of the variables at random.

   (b) If two of the equations are redundant, we could solve the remaining equation in an infinite number of ways by choosing two of the variables at random.

4. There will be non-trivial solutions if

$$\Delta_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

5. If non-trivial solutions exist, any solution

$$x = \alpha, \quad y = \beta, \quad z = \gamma$$

   will imply other solutions of the form

$$x = \lambda\alpha, \quad y = \lambda\beta, \quad z = \lambda\gamma,$$

   where λ is any non-zero number.

   This is because the right-hand-sides are all zero.

TYPE 1 - One of the three equations is redundant

The solution will be of the form

$$x : y : z = \alpha : \beta : \gamma.$$

That is,

$$\frac{x}{y} = \frac{\alpha}{\beta}, \quad \frac{y}{z} = \frac{\beta}{\gamma} \text{ and } \frac{x}{z} = \frac{\alpha}{\gamma}.$$

Method

(a) First eliminate z between two equations in order to find the ratio $x : y$

Then eliminate y between two equations in order to find the ratio $x : z$

Note:

For a slightly simpler method, see Example 1 later.

TYPE 2 - Two of the three equations are redundant

This case arises when the three homogeneous linear equations are multiples of one another.

If the only equation remaining is

$$ax + by + cz = 0,$$

we could choose any two of the variables at random and solve for the remaining variable.

For example, putting $y = 0$ we obtain
$x : y : z = -\frac{c}{a} : 0 : 1;$

Similarly, putting $z = 0$ we obtain
$x : y : z = -\frac{b}{a} : 1 : 0$

Now we can generate solutions with **any** values,
$y = \beta$, $z = \gamma$ and a suitable x.

In fact

$$x = -(\beta.\frac{b}{a} + \gamma.\frac{c}{a}), \quad y = \beta, \quad z = \gamma.$$

Note:
It may be shown that, for homogeneous linear simultaneous equations, no other types of solution exist.

EXAMPLES

1. Show that the homogeneous linear equations

$$\begin{aligned} 2x + y - z &= 0, \\ x - 3y + 2z &= 0, \\ x + 4y - 3z &= 0 \end{aligned}$$

have solutions other than $x = 0$, $y = 0$, $z = 0$ and determine the ratios $x : y : z$ for these non-trivial solutions.

Solution

(a) Using the Rule of Sarrus, $\Delta_0 =$.

$$\left|\begin{array}{ccc} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{array}\right| \begin{array}{cc} 2 & 1 \\ 1 & -3 \\ 1 & 4 \end{array} = (18+2-4)-(3+16-3) = 0.$$

Thus, the equations are linearly dependent and, hence, have non-trivial solutions.

Note:

We could, alternatively, have noticed that the first equation is the sum of the second and third equations.

(b) **Slightly simpler method than before**

In a set of ratios $\alpha : \beta : \gamma$, any of the three quantities ($\neq 0$) may be replaced by 1.

For example,

$$\alpha : \beta : \gamma = \frac{\alpha}{\gamma} : \frac{\beta}{\gamma} : 1$$

as long as $\gamma \neq 0$.

Let us now suppose that $z = 1$, giving

$$\begin{aligned} 2x + y - 1 &= 0, \\ x - 3y + 2 &= 0, \\ x + 4y - 3 &= 0. \end{aligned}$$

These give

$$x = \frac{1}{7} \text{ and } y = \frac{5}{7},$$

which means that

$$x : y : z = \frac{1}{7} : \frac{5}{7} : 1$$

That is,

$$x : y : z = 1 : 5 : 7$$

Any three numbers in these ratios form a solution.

2. Obtain the values of λ for which the homogeneous linear equations

$$\begin{aligned} (1-\lambda)x + y - 2z &= 0, \\ -x + (2-\lambda)y + z &= 0, \\ y - (1-\lambda)z &= 0 \end{aligned}$$

have non-trivial solutions.

Solution

First we solve the equation

$$0 = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda) \end{vmatrix} \quad R_1 \longrightarrow R_1 - R_3$$

$$= \begin{vmatrix} 1-\lambda & 0 & -1+\lambda \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (1-\lambda)$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & -1 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \quad C_3 \longrightarrow C_3 + C_1$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -(1-\lambda)(2-\lambda)(1+\lambda)$$

Hence, $\lambda = 1, \quad -1$ or 2.

3. Determine the general solution of the homogeneous linear equation

$$3x - 7y + z = 0.$$

Solution

Substituting $y = 0$, we obtain $3x + z = 0$ and hence $x : y : z = -\frac{1}{3} : 0 : 1$.

Substituting $z = 0$, we obtain $3x - 7y = 0$ and hence $x : y : z = \frac{7}{3} : 1 : 0$

The general solution may thus be given by

$$x = \frac{7\beta}{3} - \frac{\gamma}{3}, \quad y = \beta, \quad z = \gamma,$$

where β and γ are arbitrary numbers.

Note:

Other equivalent versions are possible according to which of the three variables are chosen to have arbitrary values.

"JUST THE MATHS"

SLIDES NUMBER

8.1

VECTORS 1
(Introduction to vector algebra)

by

A.J.Hobson

UNIT 8.1 - VECTORS 1 - INTRODUCTION TO VECTOR ALGEBRA

8.1.1 DEFINITIONS

1. A **"scalar"** quantity is one which has magnitude, but is not related to any direction in space.

   Examples: Mass, Speed, Area, Work.

2. A **"vector"** quantity is one which is specified by both a magnitude and a direction in space.

   Examples: Velocity, Weight, Acceleration.

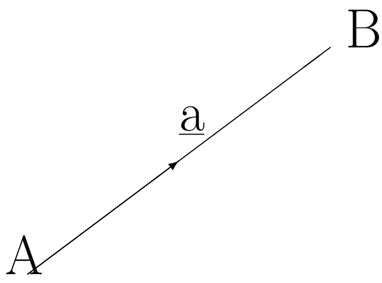
3. A vector quantity with a fixed point of application is called a **"position vector"**.

4. A vector quantity which is restricted to a fixed line of action is called a **"line vector"**.

5. A vector quantity which is defined only by its magnitude and direction is called a **"free vector"**.

   Note:

   Unless otherwise stated, all vectors in the remainder of these units will be free vectors.



6. A vector quantity can be represented diagramatically by a directed straight line segment in space (with an arrow head) whose direction is that of the vector and whose length represents is magnitude according to a suitable scale.

7. The symbols $\underline{a}$, $\underline{b}$, $\underline{c}$, ... will be used to denote vectors with magnitudes a,b,c....

   Sometimes we use $\underline{AB}$ for the vector drawn from the point A to the point B.

   Notes:

   (i) The magnitude of the vector $\underline{AB}$ is the length of the line AB.

   It can also be denoted by the symbol $|\underline{AB}|$.

   (ii) The magnitude of the vector $\underline{a}$ is the number a.

   It can also be denoted by the symbol $|\underline{a}|$.

8. A vector whose magnitude is 1 is called a **“unit vector”**.

   The symbol $\underline{\hat{a}}$ denotes a unit vector in the same direction as $\underline{a}$.

   A vector whose magnitude is zero is called a **“zero vector”** and is denoted by **O** or $\underline{O}$. It has indeterminate direction.

9. Two (free) vectors $\underline{a}$ and $\underline{b}$ are said to be **“equal”** if they have the same magnitude and direction.

   Note:

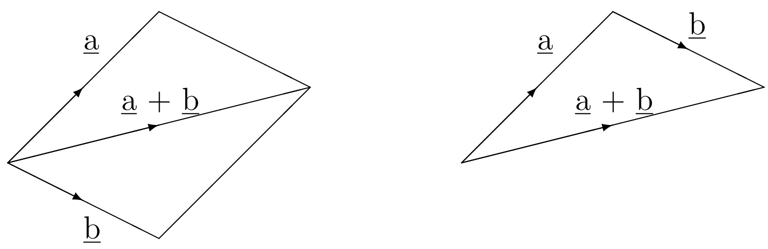
   Two directed straight line segments which are parallel and equal in length represent exactly the same vector.

10. A vector whose magnitude is that of $\underline{a}$ but with opposite direction is denoted by $-\underline{a}$.

8.1.2 ADDITION AND SUBTRACTION OF VECTORS

We define the sum of two arbitrary vectors diagramatically using either a parallelogram or a triangle.

This will then lead also to a definition of subtraction for two vectors.



Parallelogram Law   Triangle Law

Notes:
(i) The Triangle Law is more widely used than the Parallelogram Law.

$\underline{a}$ and $\underline{b}$ describe the triangle in the same sense.

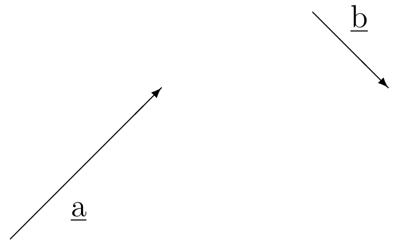
$\underline{a} + \underline{b}$ describes the triangle in the opposite sense.

(ii) To define subtraction, we use

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}).$$

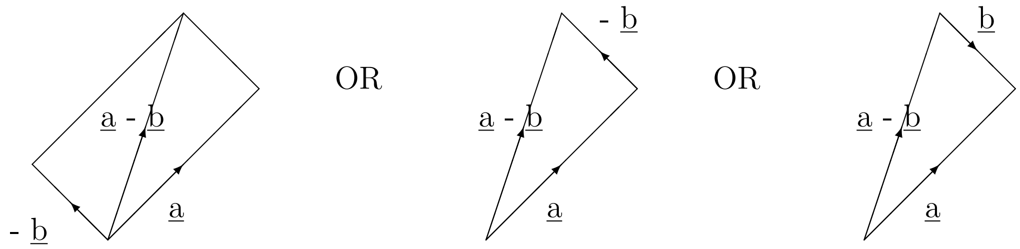
EXAMPLE

Determine $\underline{a} - \underline{b}$ for the following vectors:



Solution

We may construct the following diagrams:



Observations

(i) To determine $\underline{a} - \underline{b}$, we require that $\underline{a}$ and $\underline{b}$ describe the triangle in opposite senses while $\underline{a} - \underline{b}$ describes the triangle in the same sense as $\underline{b}$.

(ii) The sum of the three vectors describing the sides of a triangle in the same sense is the zero vector.

8.1.3 MULTIPLICATION OF A VECTOR BY A SCALAR

If m is any positive real number, $m\underline{\mathrm{a}}$ is defined to be a vector in the same direction as $\underline{\mathrm{a}}$, but of m times its magnitude.

$-m\underline{\mathrm{a}}$ is a vector in the opposite direction to $\underline{\mathrm{a}}$, but of m times its magnitude.

Note:
$\underline{\mathrm{a}} = \mathrm{a}\hat{\underline{\mathrm{a}}}$ and hence

$$\frac{1}{\mathrm{a}}.\underline{\mathrm{a}} = \hat{\underline{\mathrm{a}}}.$$

If any vector is multiplied by the reciprocal of its magnitude, we obtain a unit vector in the same direction.

This process is called **"normalising the vector"**.

8.1.4 LAWS OF ALGEBRA OBEYED BY VECTORS

(i) The Commutative Law of Addition

$$\underline{\mathrm{a}} + \underline{\mathrm{b}} = \underline{\mathrm{b}} + \underline{\mathrm{a}}.$$

(ii) The Associative Law of Addition

$$\underline{\mathrm{a}} + (\underline{\mathrm{b}} + \underline{\mathrm{c}}) = (\underline{\mathrm{a}} + \underline{\mathrm{b}}) + \underline{\mathrm{c}} = \underline{\mathrm{a}} + \underline{\mathrm{b}} + \underline{\mathrm{c}}.$$

(iii) The Associative Law of Multiplication by a Scalar

$$m(n\underline{\mathrm{a}}) = (mn)\underline{\mathrm{a}} = mn\underline{\mathrm{a}}.$$

(iv) The Distributive Laws for Multiplication by a Scalar

$$(m + n)\underline{\mathrm{a}} = m\underline{\mathrm{a}} + n\underline{\mathrm{a}}$$

and

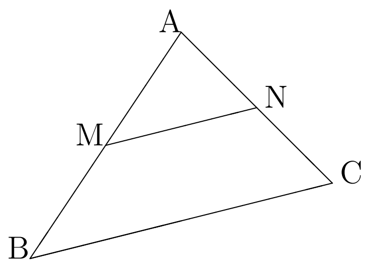
$$m(\underline{\mathrm{a}} + \underline{\mathrm{b}}) = m\underline{\mathrm{a}} + m\underline{\mathrm{b}}.$$

8.1.5 VECTOR PROOFS OF GEOMETRICAL RESULTS

EXAMPLES

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of its length.

   Solution



By the Triangle Law,

$$\underline{\text{BC}} = \underline{\text{BA}} + \underline{\text{AC}}$$

and

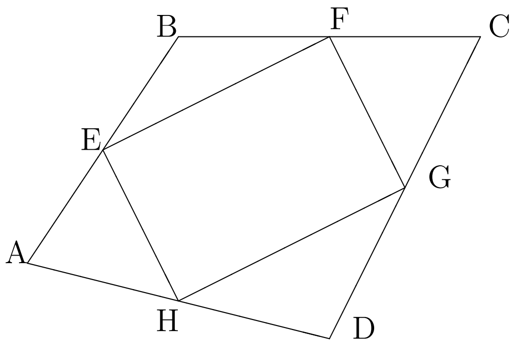
$$\underline{\text{MN}} = \underline{\text{MA}} + \underline{\text{AN}} = \frac{1}{2}\underline{\text{BA}} + \frac{1}{2}\underline{\text{AC}}.$$

Hence,

$$\underline{\text{MN}} = \frac{1}{2}(\underline{\text{BA}} + \underline{\text{AC}}) = \frac{1}{2}\underline{\text{BC}}.$$

2. ABCD is a quadrilateral (four-sided figure) and E,F,G,H are the midpoints of AB, BC, CD and DA respectively. Show that EFGH is a parallelogram.

Solution



By the Triangle Law,

$$\underline{\mathrm{EF}} = \underline{\mathrm{EB}} + \underline{\mathrm{BF}} = \frac{1}{2}\underline{\mathrm{AB}} + \frac{1}{2}\underline{\mathrm{BC}} = \frac{1}{2}(\underline{\mathrm{AB}} + \underline{\mathrm{BC}}) = \frac{1}{2}\underline{\mathrm{AC}}$$

and also

$$\underline{\mathrm{HG}} = \underline{\mathrm{HD}} + \underline{\mathrm{DG}} = \frac{1}{2}\underline{\mathrm{AD}} + \frac{1}{2}\underline{\mathrm{DC}} = \frac{1}{2}(\underline{\mathrm{AD}} + \underline{\mathrm{DC}}) = \frac{1}{2}\underline{\mathrm{AC}}.$$

Hence,

$$\underline{\mathrm{EF}} = \underline{\mathrm{HG}}.$$

"JUST THE MATHS"

SLIDES NUMBER

8.2

VECTORS 2
(Vectors in component form)

by

A.J.Hobson

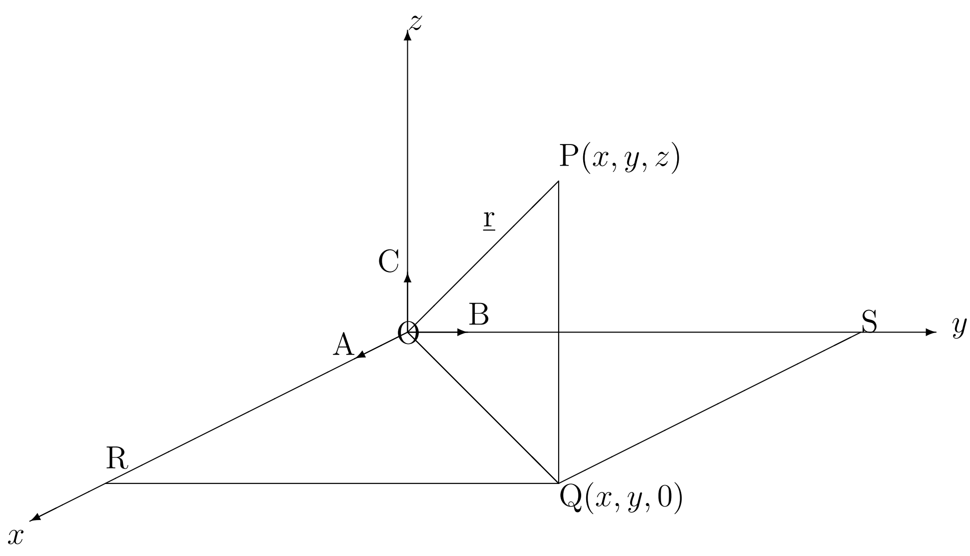
UNIT 8.2 - VECTORS 2
VECTORS IN COMPONENT FORM

8.2.1 THE COMPONENTS OF A VECTOR

Vectors in space are defined in terms of **unit vectors** placed along the axes Ox, Oy and Oz of a three-dimensional right-handed cartesian reference system.

These unit vectors will be denoted respectively by **i**, **j**, and **k**; ("bars" and "hats" may be omitted).

Consider the following diagram:



In the diagram, $\underline{\text{OA}} = \mathbf{i}$, $\underline{\text{OB}} = \mathbf{j}$ and $\underline{\text{OC}} = \mathbf{k}$.

P is the point with co-ordinates (x, y, z).

By the Triangle Law,

$$\underline{\mathrm{r}} = \underline{\mathrm{OP}} = \underline{\mathrm{OQ}} + \underline{\mathrm{QP}} = \underline{\mathrm{OR}} + \underline{\mathrm{RQ}} + \underline{\mathrm{QP}}.$$

That is,

$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

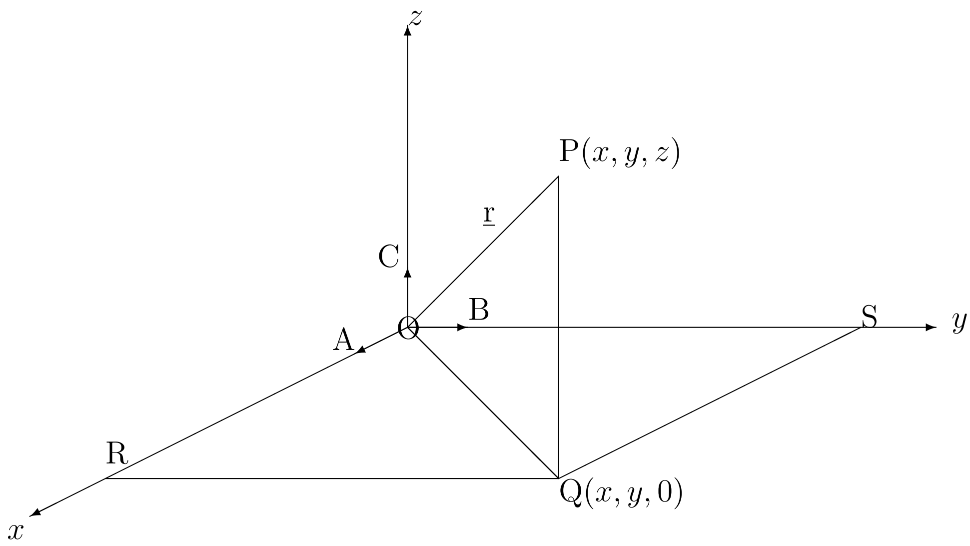
Note:

Vectors which emanate from the origin are not a special case since we are dealing with free vectors.

Nevertheless, $\underline{\mathrm{OP}}$ is called the position vector of the point P.

The numbers x, y and z are called the **"components"** of $\underline{\mathrm{OP}}$ (or of any other vector in space with the same magnitude and direction as $\underline{\mathrm{OP}}$).

8.2.2 THE MAGNITUDE OF A VECTOR IN COMPONENT FORM



By Pythagoras' Theorem,

$$(\mathrm{OP})^2 = (\mathrm{OQ})^2 + (\mathrm{QP})^2 = (\mathrm{OR})^2 + (\mathrm{RQ})^2 + (\mathrm{QP})^2$$

That is,

$$\mathrm{r} = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}.$$

EXAMPLE

Determine the magnitude of the vector

$$\underline{\mathrm{a}} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

and hence obtain a unit vector in the same direction.

Solution

$$|\underline{\mathrm{a}}| = \mathrm{a} = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}.$$

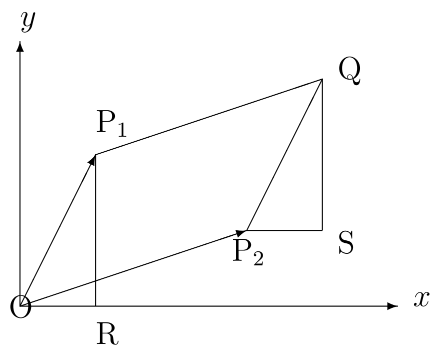
A unit vector in the same direction as $\underline{\mathrm{a}}$ is obtained by normalising $\underline{\mathrm{a}}$, that is, dividing it by its own magnitude.

The required unit vector is

$$\underline{\hat{\mathrm{a}}} = \frac{1}{\mathrm{a}}.\underline{\mathrm{a}} = \frac{5\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{30}}.$$

8.2.3 THE SUM AND DIFFERENCE OF VECTORS IN COMPONENT FORM

Consider, first, a situation in **two** dimensions:



In the diagram, suppose P_1 has co-ordinates (x_1, y_1) and suppose P_2 has co-ordinates (x_2, y_2).

Δ^{gl} ORP_1 has exactly the same shape as Δ^{gl} $\mathrm{P}_2\mathrm{SQ}$.

Hence, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

By the Parallelogram Law, $\underline{\text{OQ}}$ is the sum of $\underline{\text{OP}_1}$ and $\underline{\text{OP}_2}$.

That is,

$$(x_1\mathbf{i} + y_1\mathbf{j}) + (x_2\mathbf{i} + y_2\mathbf{j}) = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}.$$

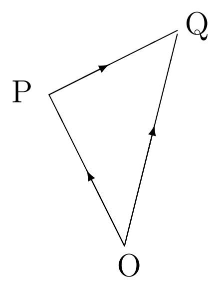
It can be shown that this result applies in three dimensions also.

To find the **difference** of two vectors, we calculate the difference of their separate components.

EXAMPLE

Two points P and Q in space have cartesian co-ordinates $(-3, 1, 4)$ and $(2, -2, 5)$ respectively.
Determine the vector $\underline{\text{PQ}}$.

Solution



$$\underline{\text{OP}} = -3\mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

$$\underline{\text{OQ}} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}.$$

By the triangle Law,

$$\underline{\text{PQ}} = \underline{\text{OQ}} - \underline{\text{OP}} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Note:
The vector drawn from the origin to the point $(5, -3, 1)$ is the **same** as the vector $\underline{\text{PQ}}$.

8.2.4 THE DIRECTION COSINES OF A VECTOR

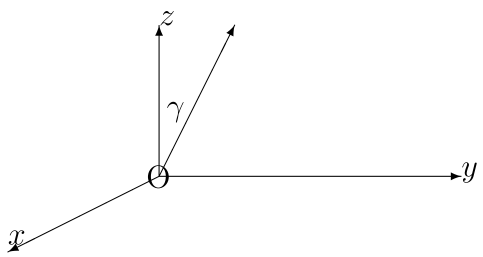
Suppose that

$$\underline{\mathrm{OP}} = \underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and suppose that $\underline{\mathrm{OP}}$ makes angles α, β and γ with Ox, Oy and Oz respectively.

Then

$$\cos\alpha = \frac{x}{\mathrm{r}}, \quad \cos\beta = \frac{y}{\mathrm{r}} \ \text{ and } \ \cos\gamma = \frac{z}{\mathrm{r}}.$$



$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **"direction cosines"** of $\underline{\mathrm{r}}$.

Any three numbers in the same ratio as the direction cosines are said to form a set of **"direction ratios"** for the vector $\underline{\mathrm{r}}$.

$x : y : z$ is one possible set of direction ratios.

EXAMPLE

The direction cosines of the vector

$$6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

are

$$\frac{6}{\sqrt{41}}, \ \frac{2}{\sqrt{41}} \ \text{ and } \ \frac{-1}{\sqrt{41}}$$

since the vector has magnitude $\sqrt{36 + 4 + 1} = \sqrt{41}$.

A set of direction ratios for this vector are $6 : 2 : -1$.

"JUST THE MATHS"

SLIDES NUMBER

8.3

VECTORS 3
(Multiplication of one vector by another)

by

A.J.Hobson

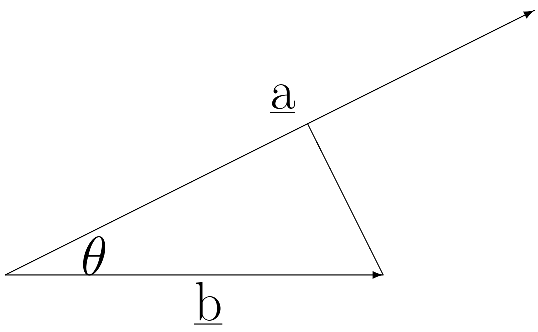
UNIT 8.3 - VECTORS 3

MULTIPLICATION OF ONE VECTOR BY ANOTHER

8.3.1 THE SCALAR PRODUCT (or "Dot" Product)

DEFINITION

The **"Scalar Product"**, $\underline{a} \bullet \underline{b}$, of two vectors, $\underline{a}$ and $\underline{b}$, is defined as $ab\cos\theta$ where θ is the angle between the directions of $\underline{a}$ and $\underline{b}$, drawn so that they have a common end-point and are directed away from that point.



$$0 \leq \theta \leq \pi$$

Scientific Application

If $\underline{b}$ were a force of magnitude b, then $b\cos\theta$ would be its resolution (or component) along the vector $\underline{a}$.

Hence $\underline{a} \bullet \underline{b}$ would represent the work done by $\underline{b}$ in moving an object along the vector $\underline{a}$.

8.3.2 DEDUCTIONS FROM THE DEFINITION OF DOT PRODUCT

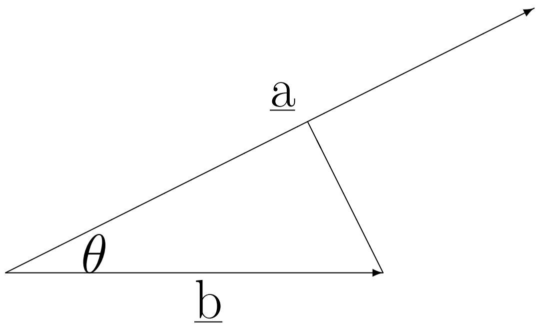
(i) $\underline{a} \bullet \underline{a} = a^2$.

Proof:

$$\underline{a} \bullet \underline{a} = a.a \cos 0 = a^2.$$

(ii) $\underline{a} \bullet \underline{b}$ can be interpreted as the magnitude of one vector times the perpendicular projection of the other vector onto it.

Proof:



$$0 \le \theta \le \pi$$

$b\cos\theta$ is the perpendicular projection of $\underline{b}$ onto $\underline{a}$ and $a\cos\theta$ is the perpendicular projection of $\underline{a}$ onto $\underline{b}$.

(iii) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$.

Proof:

This follows since $ab\cos\theta = ba\cos\theta$.

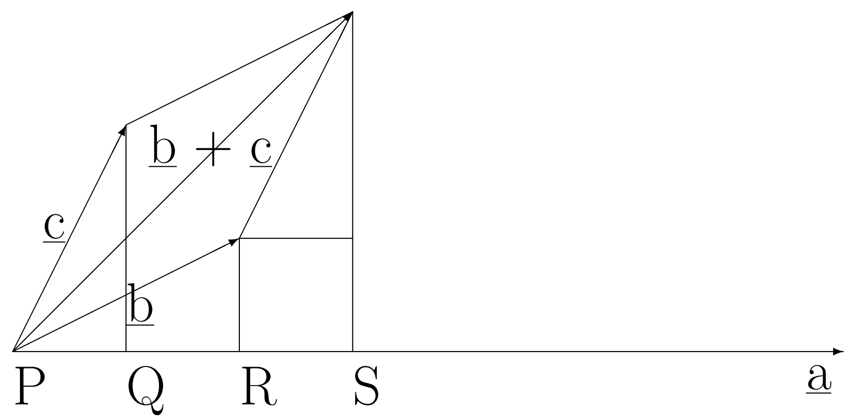
(iv) Two non-zero vectors are perpendicular if and only if their Scalar Product is zero.

Proof:

$\underline{a}$ is perpendicular to $\underline{b}$ if and only if the angle $\theta = \frac{\pi}{2}$.

That is, if and only if $\cos\theta = 0$ and hence, $ab\cos\theta = 0$.

(v) $\underline{a} \bullet (\underline{b} + \underline{c}) = \underline{a} \bullet \underline{b} + \underline{a} \bullet \underline{c}$.



The result follows from (ii) since the projections PR and PQ of $\underline{b}$ and $\underline{c}$ respectively onto $\underline{a}$ add up to the projection PS of $\underline{b} + \underline{c}$ onto $\underline{a}$.

Note: RS is equal in length to PQ.

(vi) The Scalar Product of any two of the standard unit vectors **i**, **j** and **k** is given by the following multiplication table:

$\bullet$	**i**	**j**	**k**
i	1	0	0
j	0	1	0
k	0	0	1

That is $\mathbf{i} \bullet \mathbf{i} = 1$, $\mathbf{j} \bullet \mathbf{j} = 1$ and $\mathbf{k} \bullet \mathbf{k} = 1$;

but,

$\mathbf{i} \bullet \mathbf{j} = 0$, $\mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \bullet \mathbf{k} = 0$.

8.3.3 THE STANDARD FORMULA FOR DOT PRODUCT

If

$$\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then,

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{b}} = a_1b_1 + a_2b_2 + a_3b_3.$$

Proof:

This result follows easily from the multiplication table for Dot Products.

Note: The angle between two vectors

If θ is the angle between the two vectors $\underline{\text{a}}$ and $\underline{\text{b}}$, then

$$\cos\theta = \frac{\underline{\text{a}} \bullet \underline{\text{b}}}{\text{ab}}.$$

Proof:

This result is just a restatement of the original definition of a Scalar Product.

EXAMPLE

If

$$\underline{\text{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{\text{b}} = 3\mathbf{j} - 4\mathbf{k},$$

then,

$$\cos\theta = \frac{2\times 0 + 2\times 3 + (-1)\times(-4)}{\sqrt{2^2+2^2+1^2}\sqrt{3^2+4^2}} = \frac{10}{15} = \frac{2}{3}.$$

Hence,

$$\theta = 48.19^\circ \text{ or } 0.84 \text{ radians}.$$

8.3.4 THE VECTOR PRODUCT (or "Cross Product")

DEFINITION

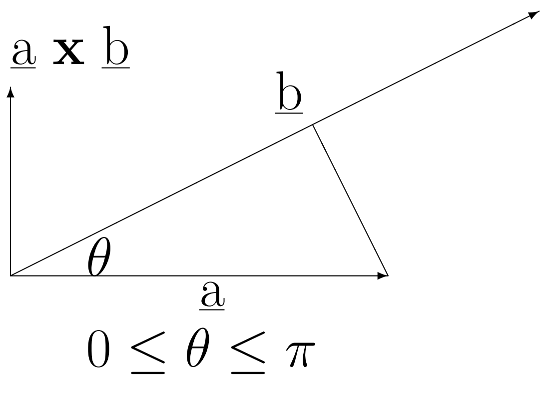
If θ is the angle between two vectors $\underline{a}$ and $\underline{b}$, drawn so that they have a common end point and are directed away from that point, then the **"Vector Product"** of $\underline{a}$ and $\underline{b}$ is defined to be a vector of magnitude

$$\text{ab} \sin\theta.$$

The direction of the Vector Product is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ and in a sense which obeyes the **"right-hand-thread screw rule"** in turning from $\underline{a}$ to $\underline{b}$.

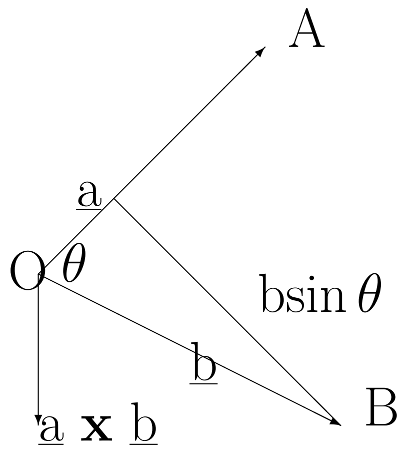
The Vector Product is denoted by

$$\underline{a} \ \mathbf{x} \ \underline{b}.$$



Scientific Application

Consider the following diagram:



Suppose that the vector $\underline{OA} = \underline{a}$ represents a force acting at the point O and that the vector $\underline{OB} = \underline{b}$ is the position vector of the point B.

Let the angle between the two vectors be θ.

Then the **"moment"** of the force $\underline{OA}$ about the point B is a vector whose magnitude is

$$ab\sin\theta$$

and whose direction is perpendicular to the plane of O, A and B in a sense which obeys the right-hand-thread screw rule in turning from $\underline{OA}$ to $\underline{OB}$.

That is,

$$\text{Moment} = \underline{a} \ \mathbf{x} \ \underline{b}.$$

8.3.5 DEDUCTIONS FROM THE DEFINITION OF CROSS PRODUCT

(i)

$$\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{b}} = -(\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{a}}) = (-\underline{\mathrm{b}}) \mathbf{\ x\ } \underline{\mathrm{a}} = \underline{\mathrm{b}} \mathbf{\ x\ } (-\underline{\mathrm{a}}).$$

Proof:
This follows easily by considering the implications of the right-hand-thread screw rule.

(ii) Two vectors are parallel if and only if their Cross Product is a zero vector.

Proof:
Two vectors are parallel if and only if the angle, θ, between them is zero or π.

In either case, $\sin\theta = 0$, which means that $\mathrm{ab}\sin\theta = 0$, i.e. $|\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{b}}| = 0$.

(iii) The Cross Product of a vector with itself is a zero vector.

Proof:

$$|\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{a}}| = \mathrm{a.a.}\sin 0 = 0.$$

(iv)

$$\underline{a} \mathbf{x} (\underline{b} + \underline{c}) = \underline{a} \mathbf{x} \underline{b} + \underline{a} \mathbf{x} \underline{c}.$$

Proof:
This is best proved using the standard formula for a Cross Product in terms of components
(see 8.3.6 below).

(v) The multiplication table for the Cross Products of the standard unit vectors **i**, **j** and **k** is as follows:

x	**i**	**j**	**k**
i	**O**	**k**	−**j**
j	− **k**	**O**	**i**
k	**j**	− **i**	**O**

That is
$\mathbf{i} \times \mathbf{i} = \mathbf{O}$, $\mathbf{j} \times \mathbf{j} = \mathbf{O}$, $\mathbf{k} \times \mathbf{k} = \mathbf{O}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$,
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\,\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\,\mathbf{i}$
and $\mathbf{i} \times \mathbf{k} = -\,\mathbf{j}$.

8.3.6 THE STANDARD FORMULA FOR CROSS PRODUCT

If

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

then

$$\underline{a} \mathbf{x} \underline{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This is usually abbreviated to

$$\underline{a} \ \mathbf{x} \ \underline{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

the symbol on the right hand side being called a **"determinant"** (see Unit 7.2).

EXAMPLES

1. If $\underline{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{j} - 4\mathbf{k}$, find $\underline{a} \ \mathbf{x} \ \underline{b}$.

   Solution

$$\underline{a} \ \mathbf{x} \ \underline{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{vmatrix} = (-8+3)\mathbf{i} - (-8-0)\mathbf{j} + (6-0)\mathbf{k}$$

$$= -5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}.$$

2. Show that, for any two vectors $\underline{a}$ and $\underline{b}$,

$$(\underline{a} + \underline{b}) \ \mathbf{x} \ (\underline{a} - \underline{b}) = 2(\underline{b} \ \mathbf{x} \ \underline{a}).$$

   Solution

   The left hand side =

$$\underline{a} \ \mathbf{x} \ \underline{a} - \underline{a} \ \mathbf{x} \ \underline{b} + \underline{b} \ \mathbf{x} \ \underline{a} - \underline{b} \ \mathbf{x} \ \underline{b}.$$

   That is,

$$\mathbf{O} + \underline{b} \ \mathbf{x} \ \underline{a} + \underline{b} \ \mathbf{x} \ \underline{a} = 2(\underline{b} \ \mathbf{x} \ \underline{a}).$$

3. Determine the area of the triangle defined by the vectors

$$\underline{\mathrm{a}} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Solution

If θ is the angle between the two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, then the area of the triangle is $\frac{1}{2}\mathrm{ab}\sin\theta$ from elementary trigonometry.

The area is therefore given by

$$\frac{1}{2}|\underline{\mathrm{a}} \;\mathbf{x}\; \underline{\mathrm{b}}|.$$

That is,

$$\text{Area} = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{matrix}\right\| = \frac{1}{2}|4\mathbf{i} + \mathbf{j} - 5\mathbf{k}|.$$

This gives

$$\text{Area} = \frac{1}{2}\sqrt{16 + 1 + 25} = \frac{1}{2}\sqrt{42} \simeq 3.24$$

"JUST THE MATHS"

SLIDES NUMBER

8.4

VECTORS 4
(Triple products)

by

A.J.Hobson

UNIT 8.4 - VECTORS 4

TRIPLE PRODUCTS

8.4.1 THE TRIPLE SCALAR PRODUCT

DEFINITION 1

Given three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, expressions such as

$$\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c}), \quad \underline{b} \bullet (\underline{c} \textbf{ x } \underline{a}), \quad \underline{c} \bullet (\underline{a} \textbf{ x } \underline{b})$$

or

$$(\underline{a} \textbf{ x } \underline{b}) \bullet \underline{c}, \quad (\underline{b} \textbf{ x } \underline{c}) \bullet \underline{a}, \quad (\underline{c} \textbf{ x } \underline{a}) \bullet \underline{b}$$

are called **“triple scalar products”** because their results are all scalar quantities.

The brackets are optional because there is no ambiguity without them.

We shall take $\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c})$ as the typical triple scalar product.

The formula for a triple scalar product

Suppose that

$$\underline{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \ \underline{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}, \ \underline{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

Then,

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the basic formula for scalar product, this becomes

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Notes:

(i) By properties of determinants (interchanging rows),

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}) = -\underline{a} \bullet (\underline{c} \mathbf{x} \underline{b}) = \underline{c} \bullet (\underline{a} \mathbf{x} \underline{b})$$

$$= -\underline{c} \bullet (\underline{b} \mathbf{x} \underline{a}) = \underline{b} \bullet (\underline{c} \mathbf{x} \underline{a}) = -\underline{b} \bullet (\underline{a} \mathbf{x} \underline{c}).$$

The **"cyclic permutations"** of $\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c})$ are all equal in numerical value and in sign; the remaining permutations are equal to $\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c})$ in numerical value, but opposite in sign.

(ii) $\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c})$, is often denoted by $[\underline{a}, \underline{b}, \underline{c}]$.

EXAMPLE

Evaluate the triple scalar product, $\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c})$, in the case when

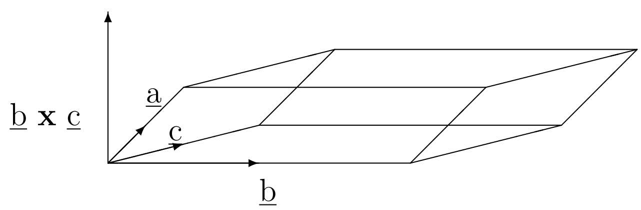
$$\underline{a} = 2\mathbf{i} + \mathbf{k}, \quad \underline{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \underline{c} = -\mathbf{i} + \mathbf{j}.$$

Solution

$$\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c}) = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 2.(-2) - 0.(2) + 1.(2) = -2.$$

A geometrical application of the triple scalar product

Suppose that $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie along three adjacent edges of a parallelepiped.



The area of the base of the parallelepiped is the **magnitude** of the vector $\underline{b}\ \mathbf{x}\ \underline{c}$ which is perpendicular to the base.

The perpendicular height of the parallelepiped is the projection of the vector $\underline{a}$ onto the vector $\underline{b}\ \mathbf{x}\ \underline{c}$.

The perpendicular height is

$$\frac{\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c})}{|\underline{b} \textbf{ x } \underline{c}|}.$$

The volume, V, of the parallelepiped is equal to the area of the base times the perpendicular height.

Hence,

$$V = \underline{a} \bullet (\underline{b} \textbf{ x } \underline{c}).$$

This is the result **numerically**, since the triple scalar product could turn out to be negative.

Note:
The above geometrical application also provides a condition that three given vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie in the same plane.

The condition that they are **"coplanar"** is that

$$\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c}) = 0.$$

That is, the three vectors would determine a parallelepiped whose volume is zero.

8.4.2 THE TRIPLE VECTOR PRODUCT

DEFINITION 2

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors, then the expression

$$\underline{a} \mathbf{x} (\underline{b} \mathbf{x} \underline{c})$$

is called the **"triple vector product"** of $\underline{a}$ with $\underline{b}$ and $\underline{c}$.

Notes:

(i) The triple vector product is clearly a vector quantity.

(ii) The brackets are important since it can be shown (in general) that

$$\underline{a} \mathbf{x} (\underline{b} \mathbf{x} \underline{c}) \neq (\underline{a} \mathbf{x} \underline{b}) \mathbf{x} \underline{c}.$$

ILLUSTRATION

Let the three vectors be position vectors, with the origin as a common end-point.

Then,

$\underline{a} \mathbf{x} (\underline{b} \mathbf{x} \underline{c})$ is perpendicular to both $\underline{a}$ and $\underline{b} \mathbf{x} \underline{c}$.

But $\underline{b} \mathbf{x} \underline{c}$ is already perpendicular to both $\underline{b}$ and $\underline{c}$.

That is, $\underline{a}$ **x** ($\underline{b}$ **x** $\underline{c}$) lies in the plane of $\underline{b}$ and $\underline{c}$.

Consequently, ($\underline{a}$ **x** $\underline{b}$) **x** $\underline{c}$, which is the same as $-\underline{c}$ **x** ($\underline{a}$ **x** $\underline{b}$), will lie in the plane of $\underline{a}$ and $\underline{b}$.

Hence, ($\underline{a}$ **x** $\underline{b}$) **x** $\underline{c}$ will, in general, be different from $\underline{a}$ **x** ($\underline{b}$ **x** $\underline{c}$).

The formula for a triple vector product

Suppose that

$$\underline{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \ \underline{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ \ \underline{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

Then,

$$\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c}) = \underline{a}\ \mathbf{x} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}.$$

The **i** component of this vector is equal to

$$a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) =$$
$$b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3).$$

By adding and subtracting $a_1b_1c_1$, the expression $b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)$ can be rearranged in the form

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1.$$

This is the **i** component of the vector

$$(\underline{a} \bullet \underline{c})\underline{b} - (\underline{a} \bullet \underline{b})\underline{c}.$$

Similar expressions can be obtained for the **j** and **k** components.

We conclude that

$$\underline{a} \mathbf{x} (\underline{b} \mathbf{x} \underline{c}) = (\underline{a} \bullet \underline{c})\underline{b} - (\underline{a} \bullet \underline{b})\underline{c}.$$

EXAMPLE

Determine the triple vector product of $\underline{a}$ with $\underline{b}$ and $\underline{c}$, where

$$\underline{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \underline{b} = -2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \underline{c} = 3\mathbf{k}.$$

Solution

$$\underline{a} \bullet \underline{c} = -3 \quad \text{and} \quad \underline{a} \bullet \underline{b} = 4.$$

$$\text{Hence,} \quad \underline{a} \mathbf{x} (\underline{b} \mathbf{x} \underline{c}) = -3\underline{b} - 4\underline{c} = 6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k}.$$

"JUST THE MATHS"

SLIDES NUMBER

8.5

VECTORS 5
(Vector equations of straight lines)

by

A.J.Hobson

UNIT 8.5 - VECTORS 5

VECTOR EQUATIONS OF STRAIGHT LINES

8.5.1 INTRODUCTION

We shall assume that

(a) the position vector of a variable point, $\mathrm{P}(x, y, z)$, is given by

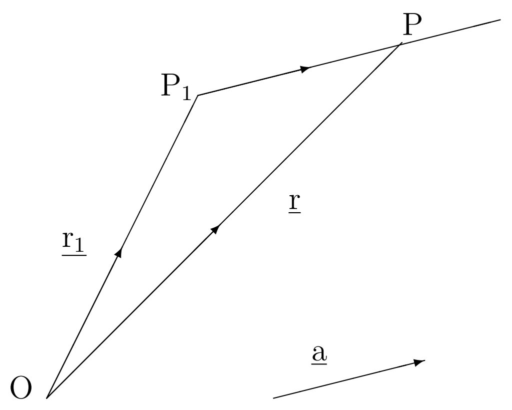
$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

(b) the position vectors of fixed points, such as $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$, are given by

$$\underline{\mathrm{r}_1} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \quad \underline{\mathrm{r}_2} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}.$$

8.5.2 THE STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

We consider a straight line passing through the point, P_1, with position vector, $\underline{\mathrm{r}_1}$, and parallel to the vector, $\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.



From the diagram,

$$\underline{\mathrm{OP}} = \underline{\mathrm{OP}_1} + \underline{\mathrm{P}_1\mathrm{P}}.$$

But,

$$\underline{\mathrm{P}_1\mathrm{P}} = t\underline{\mathrm{a}},$$

for some number t

Hence,

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}},$$

which is the vector equation of the straight line.

The components of $\underline{\mathrm{a}}$ form a set of direction ratios for the straight line.

Notes:

(i) The vector equation of a straight line passing through the **origin** and parallel to a given vector $\underline{\mathrm{a}}$ will be of the form

$$\underline{r} = t\underline{\mathrm{a}}.$$

(ii) By equating **i**, **j** and **k** components on both sides of the vector equation,

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t.$$

If these are solved for the parameter, t, we obtain

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}.$$

EXAMPLES

1. Determine the vector equation of the straight line passing through the point with position vector, $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, and parallel to the vector, $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

   Express the vector equation of the straight line in standard cartesian form.

   Solution

   The vector equation of the straight line is

   $$\underline{\mathrm{r}} = \mathbf{i} - 3\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

   or

   $$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (1 + 2t)\mathbf{i} + (-3 + 3t)\mathbf{j} + (1 - 4t)\mathbf{k}.$$

   Eliminating t from each component, we obtain the cartesian form of the straight line,

   $$\frac{x-1}{2} = \frac{y+3}{3} = \frac{z-1}{-4}.$$

2. The equations

$$\frac{3x+1}{2} = \frac{y-1}{2} = \frac{-z+5}{3}$$

determine a straight line. Express them in vector form and find a set of direction ratios for the straight line.

Solution

Rewriting the equations so that the coefficients of x, y and z are unity,

$$\frac{x+\frac{1}{3}}{\frac{2}{3}} = \frac{y-1}{2} = \frac{z-5}{-3}.$$

Hence, in vector form, the equation of the line is

$$\underline{\mathrm{r}} = -\frac{1}{3}\mathbf{i} + \mathbf{j} + 5\mathbf{k} + t\left(\frac{2}{3}\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\right).$$

Thus, a set of direction ratios for the straight line are $\frac{2}{3} : 2 : -3$ or $2 : 6 : -9$

3. Show that the two straight lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = \mathbf{j}, \quad \underline{\mathrm{a}_1} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

$$\underline{\mathrm{r}_2} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \underline{\mathrm{a}_2} = -2\mathbf{i} - 2\mathbf{j},$$

have a common point and determine its co-ordinates.

Solution

Any point on the first line is such that

$$x = t, \quad y = 1 + 2t, \quad z = -t,$$

for some parameter value, t; and any point on the second line is such that

$$x = 1 - 2s, \quad y = 1 - 2s, \quad z = 1,$$

for some parameter value, s.

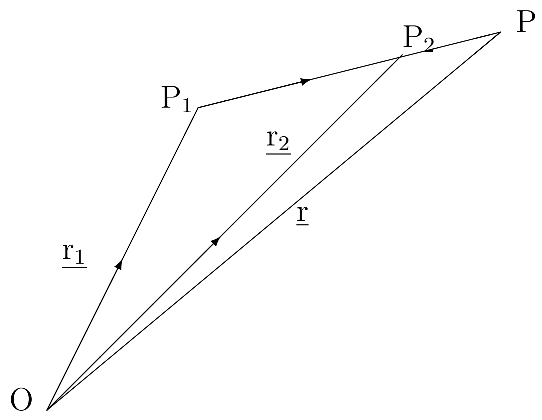
The lines have a common point if t and s exist such that these are the same point.

In fact, $t = -1$ and $s = 1$ are suitable values and give the common point $(-1, -1, 1)$.

8.5.3 THE STRAIGHT LINE PASSING THROUGH TWO GIVEN POINTS

If a straight line passes through the two given points, P_1 and P_2, it is certainly parallel to the vector

$$\underline{a} = \underline{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$



Thus, the vector equation of the straight line is

$$\underline{r} = \underline{r_1} + t\underline{a},$$

as before.

Notes:

(i) The parametric equations of the straight line passing through P_1 and P_2 are

$$x = x_1 + (x_2 - x_1)t,\ \ y = y_1 + (y_2 - y_1)t,\ \ z = z_1 + (z_2 - z_1)t.$$

The "base-points" of the parametric representation (that is, P_1 and P_2), have parameter values $t = 0$ and $t = 1$ respectively.

(ii) The standard cartesian form of the straight line passing through P_1 and P_2 is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

EXAMPLE

Determine the vector equation of the straight line passing through the two points, $P_1(3, -1, 5)$ and $P_2(-1, -4, 2)$.

Solution

$$\underline{OP_1} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

and

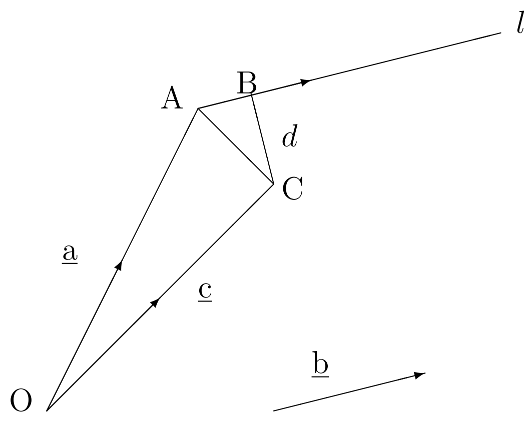
$$\underline{P_1P_2} = (-1-3)\mathbf{i} + (-4+1)\mathbf{j} + (2-5)\mathbf{k} = -4\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}.$$

Hence, the vector equation of the straight line is

$$\underline{r} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} - t(4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}).$$

8.5.4 THE PERPENDICULAR DISTANCE OF A POINT FROM A STRAIGHT LINE

For a straight line, l, passing through a given point, A, with position vector, $\underline{\text{a}}$ and parallel to a given vector, $\underline{\text{b}}$, we may determine the perpendicular distance, d, from this line, of a point, C, with position vector $\underline{\text{c}}$.



From the diagram, with Pythagoras' Theorem,

$$d^2 = (\text{AC})^2 - (\text{AB})^2.$$

But $\underline{\text{AC}} = \underline{\text{c}} - \underline{\text{a}}$, and so

$$(\text{AC})^2 = (\underline{\text{c}} - \underline{\text{a}}) \bullet (\underline{\text{c}} - \underline{\text{a}}).$$

Also, the length, AB, is the projection of $\underline{\text{AC}}$ onto the line, l, which is parallel to $\underline{\text{b}}$.

Hence,

$$\text{AB} = \frac{(\underline{\text{c}} - \underline{\text{a}}) \bullet \underline{\text{b}}}{\text{b}},$$

which gives the result

$$d^2 = (\underline{\text{c}} - \underline{\text{a}}) \bullet (\underline{\text{c}} - \underline{\text{a}}) - \left[\frac{(\underline{\text{c}} - \underline{\text{a}}) \bullet \underline{\text{b}}}{\text{b}}\right]^2.$$

From this result, d may be deduced.

EXAMPLE

Determine the perpendicular distance of the point, $(3, -1, 7)$, from the straight line passing through the two points, $(2, 2, -1)$ and $(0, 3, 5)$.

Solution

In the standard formula, we have

$$\underline{\text{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{j},$$

$$\underline{b} = (0-2)\mathbf{i} + (3-2)\mathbf{j} + (5-[-1])\mathbf{k} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$\text{b} = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41}$$

$$\underline{\text{c}} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k},$$

and

$$\underline{\mathrm{c}} - \underline{\mathrm{a}} = (3-2)\mathbf{i} + (-1-2)\mathbf{j} + (7-[-1])\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

Hence, the perpendicular distance, d, is given by

$d^2 =$

$$1^2 + (-3)^2 + 8^2 - \frac{(1)(-2) + (-3)(1) + (8)(6)}{\sqrt{41}} = 74 - \frac{43}{\sqrt{41}}$$

which gives $d \simeq 8.20$

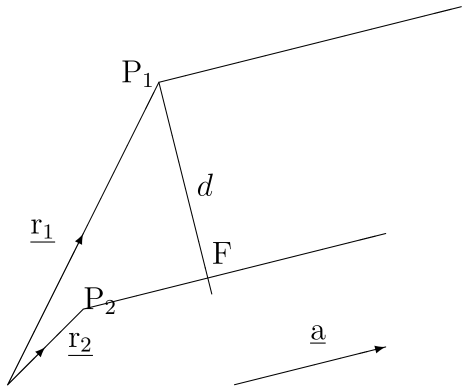
8.5.5 THE SHORTEST DISTANCE BETWEEN TWO PARALLEL STRAIGHT LINES

This will be the perpendicular distance from one of the lines of any point on the other line.

We may consider the perpendicular distance between the straight lines passing through the fixed points, with position vector $\underline{\mathrm{r}_1}$ and $\underline{\mathrm{r}_2}$, respectively and both parallel to the fixed vector, $\underline{\mathrm{a}}$.

These two lines will have vector equations

$$\underline{r} = \underline{r_1} + t\underline{a} \text{ and } \underline{r} = \underline{r_2} + t\underline{a}.$$



In the diagram, F is the foot of the perpendicular onto the second line from the point P_1 on the first line.

The length of this perpendicular is d.

Hence,

$$d^2 = (\underline{r_2} - \underline{r_1}) \bullet (\underline{r_2} - \underline{r_1}) - \left[\frac{(\underline{r_2} - \underline{r_1}) \bullet \underline{a}}{a}\right]^2.$$

EXAMPLE

Determine the shortest distance between the straight line passing through the point with position vector $\underline{r_1} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, parallel to the vector $\underline{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the straight line passing through the point with position vector $\underline{r_2} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, parallel to $\underline{b}$.

Solution

From the formula,

$$d^2 = (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$-\left[\frac{(-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}}\right]^2.$$

That is,

$$d^2 = (36 + 16 + 4) - \left[\frac{-6 + 4 - 2}{\sqrt{3}}\right]^2 = 56 - \frac{16}{3} = \frac{152}{3},$$

which gives

$$d \simeq 7.12$$

8.5.6 THE SHORTEST DISTANCE BETWEEN TWO SKEW STRAIGHT LINES

Two straight lines are said to be **"skew"** if they are not parallel and do not intersect each other.

It may be shown that such a pair of lines will always have a common perpendicular (that is, a straight line segment which meets both and is perpendicular to both).

Its length will be the shortest distance between the two skew lines.

Consider the straight lines, whose vector equations are

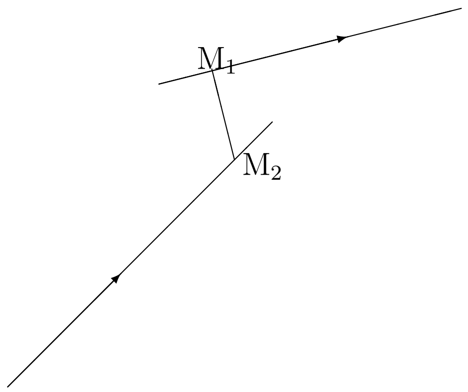
$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \ \text{ and } \ \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2}.$$

Let the point, M_1, on the first line and the point, M_2, on the second line be the ends of the common perpendicular.

Let M_1 amd M_2 have position vectors $\underline{\mathrm{m}_1}$ and $\underline{\mathrm{m}_2}$, respectively.

Then, for some values, t_1 and t_2, of the parameter,

$$\underline{\mathrm{m}_1} = \underline{\mathrm{r}_1} + t_1\underline{\mathrm{a}_1} \ \text{ and } \ \underline{\mathrm{m}_2} = \underline{\mathrm{r}_2} + t_2\underline{\mathrm{a}_2}.$$



Firstly, we have

$$\underline{\mathrm{M_1M_2}} = \underline{\mathrm{m}_2} - \underline{\mathrm{m}_1} = (\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1}.$$

Secondly, a vector which is perpendicular to both of the skew lines is $\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}$.

A unit vector perpendicular to both of the skew lines is

$$\frac{\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|}.$$

This implies that

$$(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1} = \pm d\frac{\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|},$$

where d is the shortest distance between the skew lines.

Finally, taking the scalar (dot) product of both sides of this result with the vector, $\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}$, we obtain

$$(\underline{r_2} - \underline{r_1}) \bullet (\underline{a_1} \mathbf{x} \underline{a_2}) = \pm d \frac{|\underline{a_1} \mathbf{x} \underline{a_2}|^2}{|\underline{a_1} \mathbf{x} \underline{a_2}|},$$

giving

$$d = \left| \frac{(\underline{r_2} - \underline{r_1}) \bullet (\underline{a_1} \mathbf{x} \underline{a_2})}{|\underline{a_1} \mathbf{x} \underline{a_2}|} \right|.$$

EXAMPLE

Determine the perpendicular distance between the two skew lines

$$\underline{r} = \underline{r_1} + t\underline{a_1} \text{ and } \underline{r} = \underline{r_2} + t\underline{a_2},$$

where

$$\underline{r_1} = 9\mathbf{j} + 2\mathbf{k}, \quad \underline{a_1} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},$$

$$\underline{r_2} = -6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}, \quad \underline{a_2} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

Solution

$$\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1} = -6\mathbf{i} - 14\mathbf{j} + 8\mathbf{k}$$

and

$$\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k},$$

so that

$$d = \frac{(-6)(-6) + (-14)(-15) + (8)(3)}{\sqrt{36 + 225 + 9}}$$

$$= \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}.$$

"JUST THE MATHS"

SLIDES NUMBER

8.6

VECTORS 6
(Vector equations of planes)

by

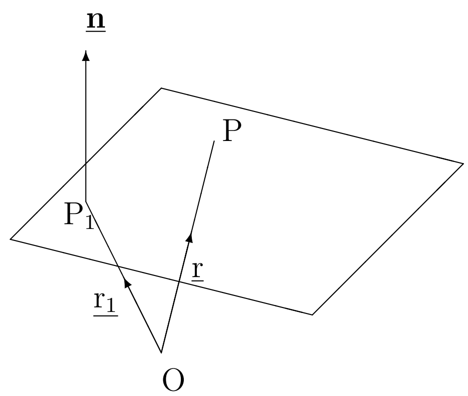
A.J.Hobson

UNIT 8.6 - VECTORS 6

VECTOR EQUATIONS OF PLANES

8.6.1 THE PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

A plane in space is completely specified if we know one point in it, together with a vector which is perpendicular to the plane.



In the diagram, the given point is P_1, with position vector r_1, and the given vector is $\underline{n}$.

The vector, $\underline{P_1P}$, is perpendicular to $\underline{n}$.

Hence,

$$(\underline{r} - \underline{r_1}) \bullet \underline{n} = 0$$

or

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = \underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}} = d \text{ say.}$$

Notes:

(i) When $\underline{\mathrm{n}}$ is a unit vector, d is the perpendicular projection of $\underline{\mathrm{r}_1}$ onto $\underline{\mathrm{n}}$.

That is, d is the perpendicular distance of the origin from the plane.

(ii) If

$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } \underline{\mathrm{n}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

the cartesian form for the equation of the above plane will be

$$ax + by + cz = d.$$

EXAMPLE

Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector, $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, and perpendicular to the vector $\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

Solution

The vector equation is

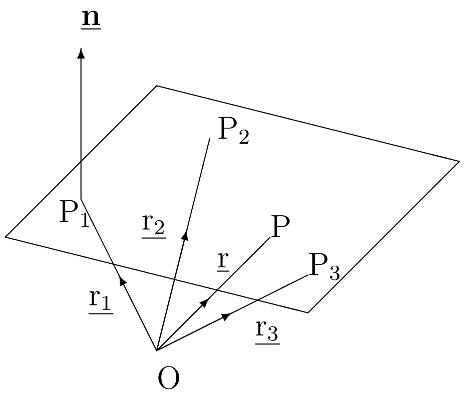
$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = (3)(1) + (-2)(-4) + (1)(-1) = 10$$

and, hence, the cartesian equation is

$$x - 4y - z = 10.$$

8.6.2 THE PLANE PASSING THROUGH THREE GIVEN POINTS

We consider a plane passing through the points, $\mathrm{P}_1(x_1, y_1, z_1)$, $\mathrm{P}_2(x_2, y_2, z_2)$ and $\mathrm{P}_3(x_3, y_3, z_3)$.



In the diagram, a suitable vector for $\underline{\mathrm{n}}$ is

$$\underline{\mathrm{P_1P_2}}\ \mathbf{x}\ \underline{\mathrm{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}.$$

The equation of the plane is

$$(\underline{\mathrm{r}} - \underline{\mathrm{r}_1}) \bullet \underline{\mathrm{n}} = 0.$$

That is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

From the properties of determinants, this is equivalent to

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

EXAMPLE

Determine the cartesian equation of the plane passing through the three points, $(0, 2, -1)$, $(3, 0, 1)$ and $(-3, -2, 0)$.

Solution

The equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{vmatrix} = 0,$$

which simplifies to

$$2x - 3y - 6z = 0.$$

This plane also passes through the origin.

8.6.3 THE POINT OF INTERSECTION OF A STRAIGHT LINE AND A PLANE

The vector equation of a straight line passing through the fixed point with position vector, $\underline{\mathrm{r}}_1$, and parallel to the fixed vector, $\underline{\mathrm{a}}$, is

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}}.$$

We require the point at which this line meets the plane

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d.$$

We require t to be such that

$$(\underline{r_1} + t\underline{a}) \bullet \underline{n} = d.$$

From this equation, the value of t and, hence, the point of intersection, may be found.

EXAMPLE

Determine the point of intersection of the plane

$$\underline{r} \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7$$

and the straight line passing through the point, $(4, -1, 3)$, which is parallel to the vector, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

Solution

We need to obtain t such that

$$(4\mathbf{i} - \mathbf{j} + 3\mathbf{k} + t[2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}]) \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7.$$

That is,

$$(4+2t)(1)+(-1-2t)(-3)+(3+5t)(-1) = 7 \text{ or } 4+3t = 7.$$

Thus, $t = 1$ and, hence, the point of intersection is $(4 + 2, -1 - 2, 3 + 5) = (6, -3, 8)$.

8.6.4 THE LINE OF INTERSECTION OF TWO PLANES

Let two non-parallel planes have vector equations

$$\underline{r} \bullet \underline{n_1} = d_1 \text{ and } \underline{r} \bullet \underline{n_2} = d_2.$$

Their line of intersection will be perpendicular to both $\underline{n_1}$ and $\underline{n_2}$.

The line of intersection will thus be parallel to $\underline{n_1} \mathbf{x} \underline{n_2}$.

To obtain the vector equation of this line, we must determine a point on it.

For convenience, we take the point (common to both planes) for which one of x, y or z is zero.

EXAMPLE

Determine the vector equation and, hence, the cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\underline{r} \bullet \underline{n_1} = 2 \text{ and } \underline{r} \bullet \underline{n_2} = 17,$$

where

$$\underline{n_1} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \underline{n_2} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Solution

Firstly,

$$\underline{\mathrm{n}_1} \ \mathbf{x} \ \underline{\mathrm{n}_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}.$$

Secondly, the cartesian equations of the two planes are

$$x + y + z = 2 \ \text{ and } \ 4x + y + 2z = 17.$$

When $z = 0$, these become

$$x + y = 2 \ \text{ and } \ 4x + y = 17.$$

These have common solution $x = 5$, $y = -3$.

Thus, a point on the line of intersection is $(5, -3, 0)$, which has position vector $5\mathbf{i} - 3\mathbf{j}$.

Hence, the vector equation of the line of intersection is

$$\underline{\mathrm{r}} = 5\mathbf{i} - 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}).$$

Finally, since $x = 5 + t$, $y = -3 + 2t$ and $z = -3t$, the line of intersection is represented, in standard cartesian form, by

$$\frac{x-5}{1} = \frac{y+3}{2} = \frac{z}{-3}.$$

8.6.5 THE PERPENDICULAR DISTANCE OF A POINT FROM A PLANE

We consider the plane whose equation is

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d$$

and the point, P_1, whose position vector is $\underline{\mathrm{r}_1}$.

The straight line through the point P_1 which is perpendicular to the plane has equation

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{n}}.$$

This line meets the plane at the point, P_0, with position vector $\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}$, where

$$(\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) \bullet \underline{\mathrm{n}} = d.$$

That is,

$$(\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}}) + t_0 \mathrm{n}^2 = d.$$

Hence,

$$t_0 = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}^2}.$$

Finally,

$$\underline{\mathrm{P}_0\mathrm{P}_1} = (\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) - \underline{\mathrm{r}_1} = t_0\underline{\mathrm{n}},$$

and its magnitude, $t_0\mathrm{n}$, will be the perpendicular distance, p, of the point P_1 from the plane.

In other words,

$$p = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}}.$$

Note:
In terms of cartesian co-ordinates, this formula is equivalent to

$$p = \frac{d - (ax_1 + by_1 + cz_1)}{\sqrt{a^2 + b^2 + c^2}}.$$

EXAMPLE

Determine the perpendicular distance, p, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

Solution

From the cartesian formula,

$$p = \frac{13 - [(1)(2) + (2)(-3) + (2)(4)]}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

"JUST THE MATHS"

SLIDES NUMBER

9.1

MATRICES 1
(Definitions & elementary matrix algebra)

by

A.J.Hobson

UNIT 9.1 - MATRICES 1 - DEFINITIONS AND ELEMENTARY MATRIX ALGEBRA

9.1.1 INTRODUCTION

(a) Presentation of Data

Sets of numerical information can often be presented as a rectangular **"array"** of numbers.

For example, a football results table:

TEAM	PLAYED	WON	DRAWN	LOST
Blackburn	22	11	6	5
Burnley	22	9	6	7
Chelsea	21	7	8	6
Leicester	21	6	8	7
Stoke	21	6	6	9

If the headings are taken for granted, we write simply

$$\begin{bmatrix} 22 & 11 & 6 & 5 \\ 22 & 9 & 6 & 7 \\ 21 & 7 & 8 & 6 \\ 21 & 6 & 8 & 7 \\ 21 & 6 & 6 & 9 \end{bmatrix}$$

This symbol is called a **"matrix"**.

(b) Presentation of Algebraic Results

In two-dimensional geometry, the **"vector"** $\underline{\text{OP}_0}$ can be moved to the position $\underline{\text{OP}_1}$ by means of a **"reflection"**, a **"rotation"**, a **"magnification"** or a combination of such operations.

The relationship between the co-ordinates (x_0, y_0) of P_0 and (x_1, y_1) of P_1 is given by

$$\begin{aligned} x_1 &= ax_0 + by_0, \\ y_1 &= cx_0 + dy_0. \end{aligned}$$

ILLUSTRATIONS

1. The equations

$$\begin{aligned} x_1 &= -x_0, \\ y_1 &= \quad\; y_0 \end{aligned}$$

represent a reflection in the y-axis.

2. The equations

$$\begin{aligned} x_1 &= kx_0, \\ y_1 &= \quad ky_0 \end{aligned}$$

represent a magnification when $\mid k \mid > 1$ and a contraction when $\mid k \mid < 1$.

3. The equations

$$\begin{aligned} x_1 &= x_0 \cos\theta - y_0 \sin\theta, \\ y_1 &= x_0 \sin\theta + y_0 \cos\theta \end{aligned}$$

represent a rotation of $\underline{\mathrm{OP}}_0$ through an angle θ in a counter-clockwise direction.

Such **"linear transformations"** are completely specified by a, b, c, d in the correct positions.

When referring to a linear transformation, we write the matrix symbol

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To show that a transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

operates on the vector $\underline{\mathrm{OP}}_0$, transforming it into the vector $\underline{\mathrm{OP}}_1$, we write

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

9.1.2 DEFINITIONS

(i) A matrix is a **"rectangular array of numbers"** arranged in rows (horizontally), columns (vertically) and enclosed in brackets.

ILLUSTRATIONS:

$$\begin{bmatrix} 1 & 3 & 7 \\ 9 & 9 & 10 \end{bmatrix}, \begin{bmatrix} 10 & 16 & 17 \\ 3 & 5 & 11 \\ 4 & 0 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}.$$

Note:

Rows are counted from top to bottom and columns are counted from left to right.

(ii) Any number within the array is called
an **"element"** of the matrix.

The $\underline{ij-th \text{ element}}$ is the element lying in the i-th row and the j-th column of the matrix.

(iii) If a matrix has m rows and n columns, it is called a **"matrix of order** $m \times n$**"** or simply an "$m \times n$ **matrix"**. It clearly has mn elements.

(iv) A matrix of order $m \times m$ is called a **"square"** matrix.

Note:
A matrix of order 1×1 is considered to be the same as a single number.

(v) A matrix of order $m \times 1$ is called
a **"column vector"** and a matrix of order $1 \times n$ is called a **"row vector"**.

(vi) An arbitrary matrix whose elements and order do not have to be specified may be denoted by a single capital letter such as A,B,C, etc.

An arbitrary matrix of order $m \times n$ may be denoted fully by the symbol

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In each double-subscript, the first number is the row number and the second is the column number.

If the matrix is square, the elements $a_{11}, a_{22}, a_{33}, ...a_{mm}$ form the **"leading diagonal"**.

They are called the **"diagonal elements"** and their sum is called the **"trace"** of the matrix.

An abbreviated form is $[a_{ij}]_{m \times n}$.

EXAMPLE

A matrix $\mathrm{A} = [a_{ij}]_{2\times 3}$ is such that $a_{ij} = i^2 + 2j$. Write out A in full.

Solution

$$\mathrm{A} = \begin{bmatrix} 3 & 5 & 7 \\ 6 & 8 & 10 \end{bmatrix}$$

.

(vii) Given a matrix, A, of order $m \times n$, the matrix of order $n \times m$ obtained from A by writing the rows as columns is called the **“transpose”** of A.

The transpose is denoted by A^{T} - some books use A' or $\tilde{\mathrm{A}}$.

ILLUSTRATION

If

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

then,

$$\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

(viii) A matrix, A, is said to be **"symmetric"** if $\mathrm{A} = \mathrm{A}^{\mathrm{T}}$.

ILLUSTRATION

$$\mathrm{A} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

is symmetric.

(ix) A matrix, A, is said to be **"skew-symmetric"** if the elements of A^{T} are minus the corresponding elements of A itself. This will mean that the leading diagonal elements must be zero.

ILLUSTRATION

$$\mathrm{A} = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

is skew-symmetric.

(x) A matrix is said to be **"diagonal"** if the elements which are not on the leading diagonal have value zero while the elements on the leading diagonal are not all equal to zero.

9.1.3 THE ALGEBRA OF MATRICES (Part One)

1. Equality

Two matrices are said to be equal if they have the same order and also pairs of elements in corresponding positions are equal in value.

In symbols,

$$[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}$$

provided

$$a_{ij} = b_{ij}.$$

ILLUSTRATION

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

implies that $x = 1, y = 2$ and $z = 3$.

2. Addition and Subtraction

The sum and difference of two matrices are defined only when they have the same order.

The sum of two matrices of order $m \times n$ is formed by adding together the pairs of elements in corresponding positions.

The difference of two matrices of order $m \times n$ is formed by subtracting the pairs of elements in corresponding positions.

In symbols,

$$[a_{ij}]_{m\times n} \pm [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{ij} \pm b_{ij}.$$

Note:
A matrix minus itself is called a **"null matrix"** and may be denoted, for short, by $[0]_{m\times n}$ or just $[0]$ when the order is understood.

ILLUSTRATION
A grocer has two shops, each selling apples, oranges and bananas.

The sales, in kilogrammes, of each fruit for the two shops on two separate days are represented by the matrices

$$\begin{bmatrix} 36 & 25 & 10 \\ 20 & 30 & 15 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 40 & 30 & 12 \\ 22 & 35 & 20 \end{bmatrix}$$

where the rows refer to the two shops and the columns refer to apples, oranges and bananas respectively.

The total sales, in kilogrammes, of each fruit for the two shops on both days together are represented by the matrix

$$\begin{bmatrix} 76 & 55 & 22 \\ 42 & 65 & 35 \end{bmatrix}.$$

The differences in sales of each fruit for the two shops between the second day and the first day are represented by the matrix

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & 5 & 5 \end{bmatrix}.$$

3. Additive Identities and Additive Inverses

(a) A null matrix, added to another matrix of the same order, leaves it unchanged.

We say that a null matrix behaves as an **"additive identity"**.

(b) Since $A - A = [0]$, we say that $-A$ is the **"additive inverse"** of A.

"JUST THE MATHS"

SLIDES NUMBER

9.2

MATRICES 2
(Further matrix algebra)

by

A.J.Hobson

9.2.1 Multiplication by a single number
9.2.2 The product of two matrices
9.2.3 The non-commutativity of matrix products
9.2.4 Multiplicative identity matrices

UNIT 9.2 - MATRICES 2
THE ALGEBRA OF MATRICES (Part Two)

9.2.1 MULTIPLICATION BY A SINGLE NUMBER

Multiplying a matrix of any order by a **positive whole number**, n, is equivalent to adding together n copies of the given matrix.

Every element would be multiplied by n.

To extend this idea to multiplication by **any** number, λ, we multiply every element by λ.

In symbols,

$$\lambda \left[a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n}$$

where

$$b_{ij} = \lambda a_{ij}$$

Note:

The rule for multiplying a matrix by a single number can also be used to remove common factors from the elements of a matrix.

ILLUSTRATION

$$\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

9.2.2 THE PRODUCT OF TWO MATRICES

We introduce the definition with a semi-practical illustration.

ILLUSTRATION

A motor manufacturer, with three separate factories, makes two types of car - one called "standard" and the other called "luxury".
In order to manufacture each type of car, he needs a certain number of units of material and a certain number of units of labour each unit representing £300.

A table of data to represent this information could be

Type	Materials	Labour
Standard	12	15
Luxury	16	20

The manufacturer receives an order from another country to supply 400 standard cars and 900 luxury cars.

He distributes the export order as follows:

Location	Standard	Luxury
Factory A	100	400
Factory B	200	200
Factory C	100	300

The number of units of material and labour needed to complete the order may be given by the following table:

Location	Materials	Labour
Factory A	$100 \times 12 + 400 \times 16$	$100 \times 15 + 400 \times 20$
Factory B	$200 \times 12 + 200 \times 16$	$200 \times 15 + 200 \times 20$
Factory C	$100 \times 12 + 300 \times 16$	$100 \times 15 + 300 \times 20$

In matrix notation,

$$\begin{bmatrix} 100 & 400 \\ 200 & 200 \\ 100 & 300 \end{bmatrix} \cdot \begin{bmatrix} 12 & 15 \\ 16 & 20 \end{bmatrix} =$$

$$\begin{bmatrix} 100 \times 12 + 400 \times 16 & 100 \times 15 + 400 \times 20 \\ 200 \times 12 + 200 \times 16 & 200 \times 15 + 200 \times 20 \\ 100 \times 12 + 300 \times 16 & 100 \times 15 + 300 \times 20 \end{bmatrix} =$$

$$\begin{bmatrix} 7600 & 9500 \\ 5600 & 7000 \\ 6000 & 7500 \end{bmatrix}.$$

OBSERVATIONS

(i) The product matrix has 3 rows because the first matrix on the left has 3 rows.

(ii) The product matrix has 2 columns because the second matrix on the left has 2 columns.

(iii) The product cannot be worked out unless the number of columns in the first matrix matches the number of rows in the second matrix.

(iv) The elements of the product matrix are systematically obtained by multiplying (in pairs) the corresponding elements of each row in the first matrix with each column in the second matrix.

We read each row of the first matrix from left to right and each column of the second matrix from top to bottom.

The Formal Definition of a Matrix Product

If A and B are matrices, then the product AB is defined (that is, it has a meaning) only when the number of columns in A is equal to the number of rows in B.

If A is of order $m \times n$ and B is of order $n \times p$, then AB is of order $m \times p$.

To obtain the element in the i-th row and j-th column of AB, we multiply corresponding elements of the i-th row of A and the j-th column of B then add up the results.

ILLUSTRATION

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 1 & -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -1 & 12 \\ 1 & -10 & 1 & 15 \end{bmatrix}$$

Note:
A matrix of any order can be multiplied by a matrix of order 1×1 since this is taken as a single number.

9.2.3 THE NON-COMMUTATIVITY OF MATRIX PRODUCTS

In elementary arithmetic, if a and b are two numbers, then $ab = ba$ (that is, the product **"commutes"**).

This is not so for matrices A and B.

(a) If A is of order $m \times n$, then B must be of order $n \times m$ if both AB **and** BA are to be defined.

(b) AB and BA will have different orders unless $m = n$, in which case the two products will be square matrices of order $m \times m$.

(c) Even if A and B are **both** square matrices of order $m \times m$, it will not normally be the case that AB is the same as BA.

EXAMPLE

$$\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} . \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 11 & 35 \end{bmatrix};$$

but,

$$\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} . \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 19 & 37 \end{bmatrix}.$$

Notes:
(i) To show only that $AB \neq BA$, we simply demonstrate that one pair of corresponding elements are unequal in value.

(ii) It turns out that the non-commutativity of matrix products is the only algebraic rule which causes problems.

Others are O.K. such as

$A + B \equiv B + A$;
the **"Commutative Law of Addition"**.

$A + (B + C) \equiv (A + B) + C$;
the **"Associative Law of Addition"**.

$A(BC) \equiv (AB)C$;
the **"Associative Law of Multiplication"**.

$A(B + C) \equiv AB + BC$ or $(A + B)C \equiv AC + BC$;
the **"Distributive Laws"**.

(iii) In the matrix product, AB, we say either that B is **"pre-multiplied"** by A or that A is **"post-multiplied"** by B.

9.2.4 MULTIPLICATIVE IDENTITY MATRICES

A square matrix with 1's on the leading diagonal and zeros elsewhere is called a **"multiplicative identity matrix"**.

An $n \times n$ multiplicative identity matrix is denoted by I_n.

Sometimes, the notation I, without a subscript, is sufficient.

EXAMPLES

$$\mathrm{I}_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathrm{I}_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{I}_4 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

I_n multiplies another matrix (with an appropriate number of rows or columns) to leave it unchanged.

We use just "identity matrix" (unless it is necessary to distinguish it from the **additive** identity matrix referred to earlier).

Another common name for an identity matrix is a **"unit matrix"**.

ILLUSTRATION

Suppose that

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Then, post-multiplying by I_2, it is easily checked that

$$AI_2 = A.$$

Similarly, pre-multiplying by I_3, it is easily checked that

$$I_3A = A.$$

In general, if A is of order $m \times n$, then

$$AI_n = I_mA = A.$$

"JUST THE MATHS"

SLIDES NUMBER

9.3

MATRICES 3
(Matrix inversion & simultaneous equations)

by

A.J.Hobson

UNIT 9.3 - MATRICES 3

MATRIX INVERSION AND SIMULTANEOUS EQUATIONS

9.3.1 INTRODUCTION

In Matrix Algebra, there is no such thing as **division** in the usual sense.

An equivalent operation called **"inversion"** is similar to the process where division by a value, a, is the same as multiplication by $\frac{1}{a}$.

For example, consider the equation

$$mx = k.$$

The solution is obviously

$$x = \frac{k}{m}.$$

Alternatively,

(a) Pre-multiply both sides of the given equation by m^{-1}

$$m^{-1}.(mx) = m^{-1}k.$$

(b) Rearrange this as

$$(m^{-1}.m)x = m^{-1}k.$$

(c) Use $m^{-1}.m = 1$ to give

$$1.x = m^{-1}k.$$

(d) Use $1.x = x$ to give

$$x = m^{-1}k.$$

Later, we see an almost identical sequence of steps, with matrices

Matrix inversion is developed from the rules for matrix multiplication.

9.3.2 MATRIX REPRESENTATION OF SIMULTANEOUS LINEAR EQUATIONS

In this section, we consider three simultaneous linear equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned}$$

These can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

or

$$\mathrm{MX} = \mathrm{K}.$$

Note:
Suppose N is such that NM = I.

Pre-multiply MX = K by N to give

$$\mathrm{N(MX)} = \mathrm{NK}.$$

That is,

$$\mathrm{(NM)X} = \mathrm{NK}.$$

In other words,

$$\mathrm{IX} = \mathrm{NK}.$$

Hence,

$$\mathrm{X} = \mathrm{NK}.$$

N exhibits a similar behaviour to the number m^{-1} encountered earlier; we replace N with M^{-1}.

9.3.3 THE DEFINITION OF A MULTIPLICATIVE INVERSE

The **"multiplicative inverse"** of a square matrix M is another matrix, denoted by M^{-1} which has the property

$$M^{-1}.M = I.$$

Notes:

(i) It is certainly **possible** for the product of two matrices to be an identity matrix
(see Unit 9.2, Exercises)

(ii) We may usually call M^{-1} the "inverse" of M rather than the "multiplicative inverse".

(iii) It can be shown that, when $M^{-1}.M = I$, it is also true that

$$M.M^{-1} = I.$$

(iv) A square matrix cannot have more than one inverse.

Assume that A had two inverses, B and C.

Then,

$$C = CI = C(AB) = (CA)B = IB = B.$$

9.3.4 THE FORMULA FOR A MULTIPLICATIVE INVERSE

(a) The inverse of a 2 x 2 matrix

Taking

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

and

$$\mathrm{M}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

we require that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} . \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} a_1P + b_1R &= 1, \\ a_2P + b_2R &= 0, \\ a_1Q + b_1S &= 0, \\ a_2Q + b_2S &= 1. \end{aligned}$$

These equations are satisfied by

$$P = \frac{b_2}{|\mathrm{M}|}, \quad Q = -\ \frac{b_1}{|\mathrm{M}|}, \quad R = -\ \frac{a_2}{|\mathrm{M}|} \quad S = \frac{a_1}{|\mathrm{M}|},$$

where

$$|\mathrm{M}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

$|\mathrm{M}|$ is called **“the determinant of the matrix** M”.

Summary

$$\mathrm{M}^{-1} = \frac{1}{|\mathrm{M}|} \begin{bmatrix} b_2 & -\ b_1 \\ -\ a_2 & a_1 \end{bmatrix}.$$

EXAMPLES

1. Write down the inverse of the matrix

$$M = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}.$$

Solution

$$\mid M \mid = 41.$$

Hence,

$$M^{-1} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.$$

Check

$$M^{-1}.M = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.\begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y &= 1, \\ x - 2y &= 5. \end{aligned}$$

Solution

The equations can be written MX = K, where

$$\mathrm{M} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathrm{X} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathrm{K} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

First, check that $|\mathrm{M}| \neq 0$.

$$| \mathrm{M} |= \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -6 - 1 = -7.$$

Thus,

$$\mathrm{M}^{-1} = -\frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}.$$

The solution of the simultaneous equations is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

That is,

$$x = 1 \quad y = -2.$$

(b) The inverse of a 3 x 3 Matrix

We use another version of Cramer's rule.

The simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

have the solution

$$\frac{x}{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}$$

$$= \frac{z}{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

METHOD

(i) The last determinant above is called **"the determinant of the matrix** M**"**, denoted by | M |.

In |M|, we let A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 denote the **"cofactors"** (or **"signed minors"**) of a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3 respectively.

(ii) For each of k_1, k_2 and k_3 the cofactor is the same as the corresponding cofactor in | M |.

(iii) The solutions for x, y and z can be written as follows:

$$x = \frac{1}{\mid \mathrm{M} \mid}\left(k_1A_1 + k_2A_2 + k_3A_3\right);$$

$$y = \frac{1}{\mid \mathrm{M} \mid}\left(k_1B_1 + k_2B_2 + k_3B_3\right);$$

$$z = \frac{1}{\mid \mathrm{M} \mid}\left(k_1C_1 + k_2C_2 + k_3C_3\right);$$

or, in matrix format,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\mid \mathrm{M} \mid}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

Compare this with

$$\mathrm{X} = \mathrm{M}^{-1}K.$$

We conclude that

$$M^{-1} = \frac{1}{\mid M \mid}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

Summary
Similar working would occur for larger or smaller systems of equations.

In general, the inverse of a square matrix is **the transpose of the matrix of cofactors times the reciprocal of the determinant of the matrix**.

Notes:
(i) If $\mid M \mid = 0$, then the matrix M does not have an inverse and is said to be **"singular"**.

If $|M| \neq 0$, M is said to be **"non-singular"**.

(ii) The transpose of the matrix of cofactors is called the **"adjoint"** of M, denoted by AdjM.

There is always an adjoint though not always an inverse.

When the inverse exists,

$$M^{-1} = \frac{1}{\mid M \mid}\text{AdjM}.$$

(iii) The inverse of a matrix of order 2×2 fits the above scheme also.

The cofactor of each element will be a single number associated with a **"place-sign"** according to the following pattern:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}.$$

Hence, if

$$\text{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

then,

$$\text{M}^{-1} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

The matrix part of the result can be obtained by interchanging the diagonal elements of M and reversing the signs of the other two elements.

EXAMPLE

Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y - z &= 1, \\ x - 2y + z &= 0, \\ 2x + 2y + z &= 13. \end{aligned}$$

Solution

The equations can be written MX = K, where

$$\mathrm{M} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathrm{K} = \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix}.$$

$$\mid \mathrm{M} \mid = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 3(-2-2) - 1(1-2) + (-1)(2+4).$$

$$|\mathrm{M}| = -17.$$

If C denotes the matrix of cofactors, then

$$C = \begin{bmatrix} \boxed{-4} & 1 & \boxed{6} \\ -3 & \boxed{5} & -4 \\ \boxed{-1} & -4 & \boxed{-7} \end{bmatrix}.$$

Notes:

(i) The framed elements indicate those for which the place sign is positive.

(ii) The remaining four elements are those for which the place sign is negative.

(iii) In finding the elements of C, **do not multiply the cofactors of the elements in** M **by the elements themselves**.

The Inverse is given by

$$M^{-1} = \frac{1}{\mid M \mid}\text{Adj}M = \frac{1}{-17}C^{T} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}.$$

The solution of the equations is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -17 \\ -51 \\ -85 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

"JUST THE MATHS"

SLIDES NUMBER

9.4

MATRICES 4
(Row operations)

by

A.J.Hobson

UNIT 9.4 - MATRICES 4
ROW OPERATIONS

9.4.1 MATRIX INVERSES
BY ROW OPERATIONS

DEFINITION

An **"elementary row operation"** on a matrix is any one of the following three possibilities:

(a) The interchange of two rows;

(b) The multiplication of the elements in any row by a non-zero number;

(c) The addition of the elements in any row to the corresponding elements in another row.

Note:
From types (b) and (c), the elements in any row may be **subtracted** from the corresponding elements in another row.

More generally, **multiples** of the elements in any row may be added to or subtracted from the corresponding elements in another row.

RESULT 1.
To perform an elementary row operation on a matrix **algebraically**, we may pre-multiply the matrix by an identity matrix on which the same elementary row operation has been already performed.

For example, in the matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

subtract twice the third row from the second row.

This could be regarded as the succession of two elementary row operations as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} . \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ 2a_3 & 2b_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 - 2a_3 & b_2 - 2b_3 \\ a_3 & b_3 \end{bmatrix}.$$

DEFINITION
An **"elementary matrix"** is a matrix obtained from an identity matrix by performing upon it one elementary row operation.

RESULT 2.
If a certain sequence of elementary row operations converts a given square matrix, M, into the corresponding identity matrix, then the same sequence of elementary row operations in the same order will convert the identity matrix into M^{-1}.

Proof:

Suppose that

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.M = \mathrm{I},$$

where $E_1, E_2, E_3, E_4, ..., E_{n-1}, En$ are elementary matrices.

Post-multiplying both sides with M^{-1},

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\mathrm{M}.\mathrm{M}^{-1} = \mathrm{I}.\mathrm{M}^{-1}.$$

In other words,

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\mathrm{I} = \mathrm{M}^{-1},$$

which proves the result.

EXAMPLES

1. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

Solution

First we write down the given matrix side-by-side with the corresponding identity matrix.

$$\begin{bmatrix} 3 & 7 & \vdots & 1 & 0 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Secondly, we try to arrange that the first element in the first column of this arrangement is 1.

This can be carried out by subtracting the second row from the first row.
(Instruction: $R_1 \rightarrow R_1 - R_2$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Thirdly, we try to convert the first column of the display into the first column of the identity matrix

This can be carried out by subtracting twice the first row from the second row.
(Instruction: $R_2 \rightarrow R_2 - 2R_1$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

Lastly, we try to convert the second column of the display into the second column of the identity matrix. This can be carried out by subtracting twice the second row from the first row
(Instruction: $R_1 \to R_1 - 2R_2$).

$$\begin{bmatrix} 1 & 0 & \vdots & 5 & -7 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

The inverse matrix is therefore

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}.$$

Notes:

(i) The technique used for a 2×2 matrix applies to square matrices of all orders with appropriate modifications.

(ii) The idea is to obtain, in chronological order, the columns of the identity matrix from the columns of the given matrix.

We do this by converting each diagonal element in the given matrix to 1 and then using multiples of 1 to reduce the remaining elements in the same column to zero.

(iii) Once any row has been used to reduce elements to zero, that row must not be used again as the operator;

2. Use elementary row operations to show that the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

has no inverse.

Solution

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 2 & 6 & \vdots & 0 & 1 \end{bmatrix};$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & -1 & 1 \end{bmatrix}.$$

There is no way now of continuing to convert the given matrix into the identity matrix; hence, there is no inverse.

3. Use elementary row operations to find the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{ccc|ccc} 4 & 1 & 6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right];$$

$R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right];$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & -2 & 1 & 2 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array}\right];$$

$R_2 \to 2R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array}\right];$$

$R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 2 & \vdots & 2 & -10 & 4 \end{bmatrix};$$

$R_3 \to R_3 \times \frac{1}{2}$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -5 & 2 \end{bmatrix};$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -1 & 7 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 5 & 2 \end{bmatrix}.$$

The required inverse matrix is therefore

$$\begin{bmatrix} -1 & 7 & -3 \\ -1 & 2 & 0 \\ 1 & -5 & 2 \end{bmatrix}.$$

9.4.2 GAUSSIAN ELIMINATION THE ELEMENTARY VERSION

The method will be introduced through the case of three equations in three unkowns, but may be applied in other cases as well.

First, consider the <u>special</u> equations

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ b_2y + c_2z &= k_2, \\ c_3z &= k_3. \end{aligned}$$

We may obtain z from the third equation.

Substituting for z into the second equation, we may find y.

Substituting for both y and z in the first equation we may find x.

The method of Gaussian Elimination reduces any set of linear equations to this triangular form by adding or subtracting suitable multiples of pairs of the equations.

The method is usually laid out in a tabular form using only the coefficients of the variables and the constant terms.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We try to arrange that the first coefficient in the first equation is 1.

Here, we could interchange the first two equations.

$$\begin{array}{ccc|c} \boxed{1} & -2 & -1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -1 & 1 & 8 \end{array}$$

This format is known as an **"augmented matrix"**. The matrix of coefficients has been augmented by the matrix of constant terms.

Using $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ gives a new table:

1	-2	-1	2
0	$\boxed{5}$	3	-1
0	5	4	2

$R_3 \to R_3 - R_2$ gives another new table:

1	-2	-1	2
0	5	3	-1
0	0	1	3

The numbers enclosed in the boxes are called the **"pivot elements"** and are used to reduce to zero the elements below them in the same column.

The final table provides a new set of equations, equivalent to the original:

$$\begin{aligned} x - 2y - z &= 2 \\ 5y + 3z &= -1 \\ z &= 3 \end{aligned}$$

Hence, $\boxed{z = 3 \quad , y = -2 \quad , x = 1}$.

INSERTING A CHECK COLUMN

With large numbers of equations, often involving awkward decimal quantities, the margin for error is greatly increased.

As a check on the arithmetic at each stage, we introduce some **additional** arithmetic which has to remain consistent with the calculations already being carried out.

Add together the numbers in each row in order to produce an extra column.

Each row operation is then performed on the extended rows.

In the new table, the final column should still be the sum of the numbers to the left of it.

The working for the previous example would be as follows:

$\boxed{1}$	-2	-1	2	0		1	-2	-1	2	0		1	-2	-1	2	0
2	1	1	3	7		0	$\boxed{5}$	3	-1	7		0	5	3	-1	7
3	-1	1	8	11,		0	5	4	2	11,		0	0	1	3	4

Using $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$
and $R_3 \rightarrow R_3 - R_2$.

"JUST THE MATHS"

SLIDES NUMBER

9.5

MATRICES 5
(Consistency and rank)

by

A.J.Hobson

UNIT 9.5 - MATRICES 5
CONSISTENCY AND RANK

9.5.1 THE CONSISTENCY OF SIMULTANEOUS LINEAR EQUATIONS

Introduction

Some sets of equations cannot be solved to give a unique solution.

The Gaussian Elimination method is able to detect such sets.

ILLUSTRATION 1.

$$
\begin{aligned}
3x - y + z &= 1, \\
2x + 2y - 5z &= 0, \\
5x + y - 4z &= 7.
\end{aligned}
$$

Gaussian Elimination gives

$$
\begin{array}{ccc|c|c}
3 & -1 & 1 & 1 & 4 \\
2 & 2 & -5 & 0 & -1 \\
5 & 1 & -4 & 7 & 9
\end{array}
$$

$R_1 \rightarrow R_1 - R_2$

1	-3	6	1	5
2	2	-5	0	-1
5	1	-4	7	9

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 5R_1$

1	-3	6	1	5
0	8	-17	-2	-11
0	16	-34	2	-16

$R_3 \to R_3 - 2R_2$

1	-3	6	1	5
0	8	-17	-2	-11
0	0	0	6	6

The third line seems to imply that $0.z = 6$; that is, $0 = 6$ which is impossible.

Hence the equations have no solution and are said to be **"inconsistent"**.

ILLUSTRATION 2.

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 1. \end{aligned}$$

Gaussian Elimination gives

3	-1	1	1	4
2	2	-5	0	-1
5	1	-4	1	3

$R_1 \longrightarrow R_1 - R_2$

1	-3	6	1	5
2	2	-5	0	-1
5	1	-4	1	3

$R_2 \longrightarrow R_2 - 2R_1$ and $R_3 \longrightarrow R_3 - 5R_1$

1	-3	6	1	5
0	8	-17	-2	-11
0	16	-34	-4	-22

$R_3 \longrightarrow R_3 - 2R_2$

1	-3	6	1	5
0	8	-17	-2	-11
0	0	0	0	0

Hence, the original set of equations is equivalent to

$$\begin{aligned} x - 3y + 6z &= 1, \\ 8y - 17z &= -2. \end{aligned}$$

First form of solution

For the two equations

$$\begin{aligned} x - 3y + 6z &= 1, \\ 8y - 17z &= -2. \end{aligned}$$

Any one of the three variables may be chosen at random, the other two being expressed in terms of it.

For instance, if we choose z at random, then

$$y = \frac{17z - 2}{8} \text{ giving } x = \frac{3z + 2}{8}.$$

Neater form of solution

Let $x = x_0$, $y = y_0$, $z = z_0$ be any **known** solution. For example, if $z = 0$, $y = -\frac{1}{4}$ and hence $x = \frac{1}{4}$.

Let us now substitute

$$\begin{aligned} x &= x_1 + x_0, \\ y &= y_1 + y_0, \\ z &= z_1 + z_0. \end{aligned}$$

We obtain

$$\begin{aligned} (x_1 + x_0) - 3(y_1 + y_0) + 6(z_1 + z_0) &= 1, \\ 8(y_1 + y_0) - 17(z_1 + z_0) &= -2. \end{aligned}$$

Because (x_0, y_0, z_0) is a known solution, this reduces to

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0. \end{aligned}$$

This is a set of **"homogeneous linear equations"**.

Although clearly satisfied by $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, we regard this as a **"trivial solution"** and ignore it.

An infinite number of non-trivial solutions can be found for each of which the variables x_1, y_1 and z_1 are in a certain set of ratios.

From

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0, \end{aligned}$$

the second equation gives

$y_1 = \frac{17}{8}z_1$ which means that $y_1 : z_1 = 17 : 8$.

Substituting into the first equation,

$x_1 - \frac{51}{8}z_1 + 6z_1 = 0$ which means that $x_1 = \frac{3}{8}z_1$ or $x_1 : z_1 = 3 : 8$.

Combining these conclusions

$$x_1 : y_1 : z_1 = 3 : 17 : 8.$$

Any three numbers in these ratios will serve as a set of values for x_1, y_1 and z_1.

In general, the neater form of solution can be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$

where α may be any non-zero number.

In our present example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 17 \\ 8 \end{bmatrix}.$$

Notes:
(i) Since at least one of the variables is going to be non-zero, we may begin by letting it equal 1.

In the equations

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0, \end{aligned}$$

suppose we let $z_1 = 1$.

Then

$$y_1 = \frac{17}{8} \text{ and } x_1 = \frac{3}{8}.$$

Hence,

$$x_1 : y_1 : z_1 = \frac{3}{8} : \frac{17}{8} : 1,$$

which can be rewritten

$$x_1 : y_1 : z_1 = 3 : 17 : 8.$$

(ii) It may happen that a set of simultaneous linear equations reduces to **only one** equation.

That is, each equation is a multiple of the first

ILLUSTRATION 3.

Suppose a set of equations reduces to

$$3x - 2y + 5z = 6.$$

A particular solution is $x_0 = 1, y_0 = 1, z_0 = 1.$

The general solution is given by

$$x = x_0 + x_1,\ y = y_0 + y_1,\ z = z_0 + z_1,$$

where

$$3x_1 - 2y_1 + 5z_1 = 0.$$

Let $x_1 = \alpha$ and $y_1 = \beta$ (random numbers).

Hence,

$$z_1 = \frac{2\beta - 3\alpha}{5}.$$

Using $x = \alpha$, $y = \beta$ and $z = \frac{2\beta - 3\alpha}{5}$, we may write

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

9.5.2 THE ROW-ECHELON FORM OF A MATRIX

Using Gaussian Elimination to solve

$$\mathrm{MX} = \mathrm{K},$$

we begin with the augmented matrix M|K.

Then we use elementary row operations to obtain more zeros at the beginning of each row than at the beginning of the previous row.

If desired, the first non-zero element in each row could be reduced to 1 on dividing that row throughout by the value of the first non-zero element.

DEFINITION
The "row echelon form" of a matrix is that for which the first non-zero element in each row is 1 and occurs to the right of the first non-zero element in the previous row.

Note:
In practice, for "row echelon form", the first non-zero element in each row need not necessarily be reduced to 1.

9.5.3 THE RANK OF A MATRIX

DEFINITION
The **"rank "** of a matrix is the number or rows which do not reduce to a complete row of zeros when the matrix has been converted to row echelon form.

ILLUSTRATIONS

1. In the previous Illustration 1, M had rank 2 but M|K had rank 3. The equations were inconsistent.
2. In the previous Illustration 2, M had rank 2 and M|K also had rank 2. The equations had an infinite number of solutions.

3. Whenever both M and M|K have rank 3, there is a unique solution to the simultaneous equations.

SUMMARY

1. The equations MX = K are inconsistent if

$$\text{rank M} < \text{rank M|K}.$$

2. The equations MX = K have an infinite number of solutions if

$$\text{rank M} = \text{rank M|K} < n,$$

where n is the number of equations.

3. The equations MX = K have a unique solution if

$$\text{rank M} = \text{rank M|K} = n,$$

where n is the number of equations.

"JUST THE MATHS"

SLIDES NUMBER

9.6

MATRICES 6
(Eigenvalues and eigenvectors)

by

A.J.Hobson

UNIT 9.6 - MATRICES 6

EIGENVALUES AND EIGENVECTORS

9.6.1 THE STATEMENT OF THE PROBLEM

Let A be any square matrix, and let X be a column vector with the same number of rows as there are columns in A.

For example, if A is of order $m \times m$, then X must be of order $m \times 1$ and AX will also be of order $m \times 1$.

We ask the question:

"Is it ever possible that AX can be just a scalar multiple of X ?"

We exclude the case when the elements of X are all zero.

ILLUSTRATIONS

1.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The formal statement of the problem

For a given square matrix, A, we investigate the existence of column vectors, X, such that

$$AX = \lambda X,$$

for some scalar quantity λ.

Each such column vector is called an **"eigenvector"** of the matrix, A.

Each corresponding value of λ is called an **"eigenvalue"** of the matrix, A.

Notes:

(i) The German word "eigen" means"hidden".

(ii) Other alternative names are "latent values and latent vectors" or "characteristic values and characteristic vectors".

(iii) In the discussion which follows, A will be, mostly, a matrix of order 3×3.

9.6.2 THE SOLUTION OF THE PROBLEM

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then the matrix equation, $AX = \lambda X$, means

$$\begin{aligned} a_1x + b_1y + c_1z &= \lambda x, \\ a_2x + b_2y + c_2z &= \lambda y, \\ a_3x + b_3y + c_3z &= \lambda z; \end{aligned}$$

or, on rearrangement,

$$\begin{aligned} (a_1 - \lambda)x + b_1y + c_1z &= 0, \\ a_2x + (b_2 - \lambda)y + c_2z &= 0, \\ a_3x + b_3y + (c_3 - \lambda)z &= 0. \end{aligned}$$

This is a set of homogeneous linear equations in x, y and z and may be written

$$(A - \lambda I)X = [0],$$

where I denotes the identity matrix of order 3×3.

From Unit 7.4, the three homogeneous linear equations have a solution other than $x = 0$, $y = 0$, $z = 0$ if

$$|\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0.$$

On expansion, this gives a cubic equation in λ called the **"characteristic equation"** of A.

The left-hand side of the characteristic equation is called the **"characteristic polynomial"** of A.

The characteristic equation of a 3×3 matrix, being a cubic equation, will (in general) have three solutions.

EXAMPLES

1. Determine the eigenvalues and eigenvectors of the matrix

$$\mathrm{A} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}.$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = \begin{vmatrix} 2 - \lambda & 4 \\ 5 & 3 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda + 2)(\lambda - 7).$$

The eigenvalues are therefore $\lambda = -2$ and $\lambda = 7$.

(b) The eigenvectors

Case 1. $\lambda = -2$

We require to solve the equation $x + y = 0$,

giving $x : y = -1 : 1$ and a corresponding eigenvector

$$X = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = 7$

We require to solve the equation $5x - 4y = 0$,

giving $x : y = 4 : 5$ and a corresponding eigenvector

$$X = \beta \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

where β is any **non-zero** scalar.

2. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}.$$

Direct expansion of the determinant gives the equation

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0,$$

which will factorise into

$$(1+\lambda)^2(8-\lambda) = 0.$$

Note:

Students who have studied row and column operations for determinants (see Unit 7.3) may obtain this by simplifying the determinant first.

One way is to subtract the third column from the first column and then add the third row to the first row.

The eigenvalues are therefore $\lambda = -1$(repeated) and $\lambda = 8$.

(b) The eigenvectors

Case 1. $\lambda = 8$

We solve the homogeneous equations

$$\begin{aligned} -5x + 2y + 4z &= 0, \\ 2x - 8y + 2z &= 0, \\ 4x + 2y - 5z &= 0. \end{aligned}$$

Eliminating x from the second and third equations gives $18y - 9z = 0$.

Eliminating y from the second and third equations gives $18x - 18z = 0$.

Since z appears twice, we may let $z = 1$ to give $y = \frac{1}{2}$ and $x = 1$ and, hence,

$$x : y : z = 1 : \frac{1}{2} : 1 = 2 : 1 : 2$$

The eigenvectors corresponding to $\lambda = 8$ are

$$\mathrm{X} = \alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = -1$

We solve the homogeneous equations

$$\begin{aligned} 4x + 2y + 4z &= 0, \\ 2x + y + 2z &= 0, \\ 4x + 2y + 4z &= 0. \end{aligned}$$

These are all the same equation, $2x + y + 2z = 0$.

First form of solution

Two of the variables may be chosen at random (say $y = \beta$ and $z = \gamma$).

Then the third variable may be expressed in terms of them; (in this case $x = -\frac{1}{2}\beta - \gamma$).

Neater form of solution

First obtain a pair of independent particular solutions by setting two of the variables, in turn, at 1 and another at 0.

For example $y = 1$ and $z = 0$ gives $x = -\frac{1}{2}$ while $y = 0$ and $z = 1$ gives $x = -1$.

Using the particular solutions $x = -\frac{1}{2}$, $y = 1$, $z = 0$ and $x = -1$, $y = 0$, $z = 1$ the general solution is given by

$$\mathrm{X} = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha \text{ and } \beta \text{ not both zero.}$$

Notes:

(i) Similar results in Case 2 could be obtained by choosing a **different** pair of the three variables at random.

(ii) Other special cases arise if the three homogeneous equations reduce to a single equation in which one or even two of the variables is absent.

ILLUSTRATIONS

1. If the homogeneous equations reduced to $y = 0$, the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   This allows $x = \alpha$ and $z = \gamma$ to be chosen at random, assuming that α and γ are not both zero simultaneously.

2. If the homogeneous equations reduced to

$$3x + 5z = 0,$$

then the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This allows $x = \alpha$ and $y = \beta$ to be chosen at random, assuming that α and β are not both zero simultaneously.

"JUST THE MATHS"

SLIDES NUMBER

9.7

MATRICES 7
(Linearly independent eigenvectors)
&
(Normalised eigenvectors)

by

A.J.Hobson

UNIT 9.7 - MATRICES 7

LINEARLY INDEPENDENT AND NORMALISED EIGENVECTORS

9.7.1 LINEARLY INDEPENDENT EIGENVECTORS

It is often useful to know if an $n \times n$ matrix, A, possesses a full set of n eigenvectors $X_1,\ X_2,\ X_3, \ldots, X_n$, which are **"linearly independent"**.

That is, they are **not** connected by any relationship of the form

$$a_1 X_1 + a_2 X_2 + a_3 X_3 + \ldots \equiv 0,$$

where $a_1,\ a_2,\ a_3, \ldots$ are constants.

If the eigenvalues of A are distinct, the eigenvectors are linearly independent; but, if any of the eigenvalues are repeated, further investigation may be necessary.

ILLUSTRATIONS

1. In Unit 9.6, it was shown that the matrix,

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$

has eigenvalues $\lambda = 8$ and $\lambda = -1$(repeated), with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar and

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where β and γ are not both equal to zero at the same time.

The matrix A possesses a set of three linearly independent eigenvectors which may be chosen as

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is reasonably obvious that these are linearly independent.

A formal check would be to show that the matrix

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

has rank 3.

2. It may be shown that the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

has eigenvalues $\lambda = 2$ (repeated) and $\lambda = 1$ with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \beta \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

where α and β are any non-zero numbers

In this case, it is not possible to obtain a full set of three linearly independent eigenvectors.

9.7.2 NORMALISED EIGENVECTORS

It is sometimes convenient to use a set of **"normalised"** eigenvectors, which means that, for each eigenvector, the sum of the squares of its elements is equal to 1.

An eigenvector may be normalised if we multiply it by (plus or minus) the reciprocal of the square root of the sum of the squares of its elements.

ILLUSTRATIONS

1. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$

is

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

2. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

"JUST THE MATHS"

SLIDES NUMBER

9.8

MATRICES 8
(Characteristic properties)
&
(Similarity transformations)

by

A.J.Hobson

UNIT 9.8 - MATRICES 8

CHARACTERISTIC PROPERTIES AND SIMILARITY TRANSFORMATIONS

9.8.1 PROPERTIES OF EIGENVALUES AND EIGENVECTORS

(i) The eigenvalues of a matrix are the same as those of its transpose.

Proof:

Given a square matrix, A, the eigenvalues of A^T are the solutions of the equation

$$|A^T - \lambda I| = 0.$$

But, since I is a symmetric matrix, this is equivalent to

$$|(A - \lambda I)^T| = 0.$$

The result follows, since a determinant is unchanged in value when it is transposed.

(ii) The Eigenvalues of the multiplicative inverse of a matrix are the reciprocals of the eigenvalues of the matrix itself.

Proof:

If λ is any eigenvalue of a square matrix, A, then

$$\mathrm{AX} = \lambda \mathrm{X},$$

for some column vector, X.

Premultiplying this relationship by A^{-1}, we obtain

$$\mathrm{A}^{-1}\mathrm{AX} = \mathrm{A}^{-1}(\lambda \mathrm{X}) = \lambda(\mathrm{A}^{-1}\mathrm{X}).$$

Thus,

$$\mathrm{A}^{-1}\mathrm{X} = \frac{1}{\lambda}\mathrm{X}.$$

(iii) The eigenvectors of a matrix and its multiplicative inverse are the same.

Proof:

This follows from the proof of **(ii)**, since

$$A^{-1}X = \frac{1}{\lambda}X$$

implies that X is an eigenvector of A^{-1}.

(iv) If a matrix is multiplied by a single number, the eigenvalues are multiplied by that number, but the eigenvectors remain the same.

Proof:

If A is multiplied by α, we may write the equation $AX = \lambda X$ in the form $\alpha AX = \alpha\lambda X$.

Thus, αA has eigenvalues, $\alpha\lambda$, and eigenvectors, X.

(v) If $\lambda_1, \quad \lambda_2, \quad \lambda_3, \ . \ .$ are the eigenvalues of the matrix A and n is a positive integer, then $\lambda_1^n, \quad \lambda_2^n, \quad \lambda_3^n, \ . \ .$ are the eigenvalues of A^n.

Proof:

If λ denotes any one of the eigenvalues of the matrix, A, then $AX = \lambda X$.

Premultiplying both sides by A, we obtain
$A^2X = A\lambda X = \lambda AX = \lambda^2X$.

Hence, λ^2 is an eigenvlaue of A^2.

Similarly $A^3X = \lambda^3X$, and so on.

(vi) If $\lambda_1, \ \lambda_2, \ \lambda_3, \ \ . \ . \ .$ are the eigenvalues of the $n \times n$ matrix A, I is the $n \times n$ multiplicative identity matrix and k is a single number, then the eigenvalues of the matrix $A + kI$ are $\lambda_1 + k, \ \lambda_2 + k, \ \lambda_3 + k, \ . \ . \ .$.

Proof:

If λ is any eigenvalue of A, then $AX = \lambda X$.

Hence,

$$(A + kI)X = AX + kX = \lambda X + kX = (\lambda + k)X.$$

(vii) A matrix is singular ($|A| = 0$) if and only if at least one eigenvalue is equal to zero.

Proof:

(a) If X is an eigenvector corresponding to an eigenvalue, $\lambda = 0$, then $AX = \lambda X = [0]$.

From the theory of homogeneous linear equations, it follows that $|A| = 0$.

(b) Conversely, if $|A| = 0$, the homogeneous system $AX = [0]$ has a solution for X other than $X = [0]$.

Hence, at least one eigenvalue must be zero.

(viii) If A is an orthogonal matrix ($AA^T = I$), then every eigenvalue is either $+1$ or -1.

Proof:

The statement $AA^T = I$ can be written $A^{-1} = A^T$ so that, by **(i)** and **(ii)**, the eigenvalues of A are equal to their own reciprocals

That is, they must have values $+1$ or -1.

(ix) If the elements of a matrix below the leading diagonal or the elements above the leading diagonal are all equal zero, then the eigenvalues are equal to the diagonal elements.

ILLUSTRATION

An **"upper-triangular matrix"**, A, of order 3×3, has the form

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}.$$

The characteristic equation is given by

$$0 = |\mathrm{A} - \lambda \mathrm{I}|$$

$$= \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ 0 & b_2 - \lambda & c_2 \\ 0 & 0 & c_3 - \lambda \end{vmatrix} = (a_1 - \lambda)(b_2 - \lambda)(c_3 - \lambda).$$

Hence, $\lambda = a_1,\ b_2$ or c_3.

A similar proof holds for a **"lower-triangular matrix"**.

Note:
A special case of both a lower-triangular matrix and an upper-triangular matrix is a diagonal matrix.

(x) The sum of the eigenvalues of a matrix is equal to the trace of the matrix (the sum of the diagonal elements) and the product of the eigenvalues is equal to the determinant of the matrix.

ILLUSTRATION

We consider the case of a 2×2 matrix, A, given by

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

The characteristic equation is

$$0 = \begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = \lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1).$$

But, for any quadratic equation, $a\lambda^2 + b\lambda + c = 0$, the sum of the solutions is equal to $-b/a$ and the product of the solutions is equal to c/a.

In this case, therefore, the sum of the solutions is $a_1 + b_2$ while the product of the solutions is $a_1 b_2 - a_2 b_1$.

9.8.2 SIMILAR MATRICES

DEFINITION

Two matrices, A and B, are said to be **"similar"** if

$$\text{B} = \text{P}^{-1}\text{AP},$$

for some non-singular matrix, P.

Notes:

(i) P is certainly square, so that A and B must also be square and of the same order as P.

(ii) The relationship $\text{B} = \text{P}^{-1}\text{AP}$ is regarded as a **"transformation"** of the matrix, A, into the matrix, B.

(iii) A relationship of the form $B = QAQ^{-1}$ may also be regarded as a similarity transformation on A, since Q is the multiplicative inverse of Q^{-1}.

THEOREM

Two similar matrices, A and B, have the same eigenvalues. Furthermore, if the similarity transformation from A to B is $B = P^{-1}AP$, then the eigenvectors, X and Y, of A and B respectively are related by the equation

$$Y = P^{-1}X.$$

Proof:

The eigenvalues, λ, and the eigenvectors, X, of A satisfy the relationship $AX = \lambda X$.

Hence,

$$P^{-1}AX = \lambda P^{-1}X.$$

Secondly, using the fact that $PP^{-1} = I$, we have

$$P^{-1}APP^{-1}X = \lambda P^{-1}X.$$

This may be written

$$(\mathrm{P}^{-1}\mathrm{AP})(\mathrm{P}^{-1}\mathrm{X}) = \lambda(\mathrm{P}^{-1}\mathrm{X})$$

or

$$\mathrm{BY} = \lambda\mathrm{Y},$$

where $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$ and $\mathrm{Y} = \mathrm{P}^{-1}\mathrm{X}$.

This shows that the eigenvalues of A are also the eigenvalues of B, and that the eigenvectors of B are of the form $\mathrm{P}^{-1}\mathrm{X}$.

Reminders

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and, in general, for a square matrix M,

$\mathrm{M}^{-1} = \frac{1}{|\mathrm{M}|}\times$ the transpose of the cofactor matrix.

"JUST THE MATHS"

SLIDES NUMBER

9.9

MATRICES 9
(Modal & spectral matrices)

by

A.J.Hobson

UNIT 9.9 - MATRICES 9

MODAL AND SPECTRAL MATRICES

9.9.1 ASSUMPTIONS AND DEFINITIONS

For convenience, we shall make, here, the following assumptions:

(a) The n eigenvalues, $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, of an $n \times n$ matrix, A, are arranged in order of decreasing value.

(b) Corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$ respectively, A possesses a full set of eigenvectors $\mathrm{X}_1,\ \mathrm{X}_2,\ \mathrm{X}_3, \ldots, \mathrm{X}_n$, which are **"linearly independent"**.

If two eigenvalues coincide, the order of writing down the corresponding pair of eigenvectors will be immaterial.

DEFINITION 1

The square matrix obtained by using, as its columns, any set of linearly independent eigenvectors of a matrix A is called a **"modal matrix"** of A, and may be denoted by M.

Notes:

(i) There are infinitely many modal matrices for a given matrix, A, since any multiple of an eigenvector is also an eigenvector.

(ii) It is sometimes convenient to use a set of normalised eigenvectors.

When using normalised eigenvectors, the modal matrix may be denoted by N and, for an $n \times n$ matrix, A, there are 2^n possibilities for N, since each of the n columns has two possibilities.

DEFINITION 2

If $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\lambda_n$ are the eigenvalues of an $n \times n$ matrix, A, then the diagonal matrix,

$$\left[\begin{array}{cccccc} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \lambda_n \end{array}\right],$$

is called the **"spectral matrix"** of A, and may be denoted by S.

EXAMPLE

For the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

determine a modal matrix, a modal matrix of normalised eigenvectors and the spectral matrix.

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0,$$

which may be shown to give

$$-(1+\lambda)(1-\lambda)(2-\lambda) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 1$ and $\lambda_3 = -1$ in order of decreasing value.

Case 1. $\lambda = 2$

We solve the simultaneous equations

$$\begin{aligned} -x + y - 2z &= 0, \\ -x + 0y + z &= 0, \\ 0x + y - 3z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 3 : 1$

Case 2. $\lambda = 1$

We solve the simultaneous equations

$$\begin{aligned} 0x + y - 2z &= 0, \\ -x + y + z &= 0, \\ 0x + y - 2z &= 0, \end{aligned}$$

which give $x : y : z = 3 : 2 : 1$

Case 3. $\lambda = -1$

We solve the simultaneous equations

$$\begin{aligned} 2x + y - 2z &= 0, \\ -x + 3y + z &= 0, \\ 0x + y + 0z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 0 : 1$

A modal matrix for A may therefore be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

A modal matrix of normalised eigenvectors may be given by

$$\mathrm{N} = \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{2}{\sqrt{14}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9.9.2 DIAGONALISATION OF A MATRIX

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, it is clear that a matrix, A, and its spectral matrix, S, have the same eigenvalues.

THEOREM

The matrix, A, is similar to its spectral matrix, S, the similarity transformation being

$$M^{-1}AM = S,$$

where M is a modal matrix for A.

ILLUSTRATION:

Suppose that X_1, X_2 and X_3 are linearly independent eigenvectors of a 3×3 matrix, A, corresponding to eigenvalues λ_1, λ_2 and λ_3, respectively.

Then,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad \text{and} \quad AX_3 = \lambda_3 X_3.$$

Also,

$$\mathrm{M} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

If M is premultiplied by A, we obtain a 3×3 matrix whose columns are $\mathrm{AX}_1,\ \mathrm{AX}_2,$ and AX_3.

That is,

$$\mathrm{AM} = [\,\mathrm{AX}_1 \quad \mathrm{AX}_2 \quad \mathrm{AX}_3\,] = [\,\lambda_1\mathrm{X}_1 \quad \lambda_2\mathrm{X}_2 \quad \lambda_3\mathrm{X}_3\,]$$

or

$$\mathrm{AM} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathrm{MS}.$$

We conclude that

$$\mathrm{M}^{-1}\mathrm{AM} = \mathrm{S}.$$

Notes:

(i) M^{-1} exists only because $\mathrm{X}_1,\ \mathrm{X}_2$ and X_3 are linearly independent.

(ii) The similarity transformation in the above theorem reduces the matrix, A, to **"diagonal form"** or **"canonical form"** and the process is often referred to as the **"diagonalisation"** of the matrix, A.

EXAMPLE

Verify the above Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution

From an earlier example, a modal matrix for A may be given by

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It may be shown that

$$M^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}$$

and, hence,

$$M^{-1}AM = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = S.$$

“JUST THE MATHS”

SLIDES NUMBER

9.10

MATRICES 10
(Symmetric matrices & quadratic forms)

by

A.J.Hobson

UNIT 9.10 - MATRICES 10

SYMMETRIC MATRICES AND QUADRATIC FORMS

9.10.1 SYMMETRIC MATRICES

We state the following without proof:

(i) All of the eigenvalues of a symmetric matrix are real and, hence, so are the eigenvectors.

(ii) A symmetric matrix of order $n \times n$ always has n linearly independent eigenvectors.

(iii) For a symmetric matrix, suppose that X_i and X_j are linearly independent eigenvectors associated with <u>different</u> eigenvectors; then

$$\mathrm{X}_i \mathrm{X}_j^{\mathrm{T}} \equiv x_i x_j + y_i y_j + z_i z_j = 0.$$

We say that X_i and X_j are
"mutually orthogonal".

If a symmetric matrix has any repeated eigenvalues, it is still possible to determine a full set of mutually orthogonal eigenvectors, but not <u>every</u> full set of eignvectors will have the orthogonality property.

(iv) A symmetric matrix always has a modal matrix whose columns are mutually orthogonal. When the eigenvalues are distinct, this is true for <u>every</u> modal matrix.

(v) A modal matrix, N, of normalised eigenvectors is an orthogonal matrix.

ILLUSTRATIONS

1. If N is of order 3×3, we have

$$\mathrm{N}^{\mathrm{T}}.N = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} . \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. It was shown, in Unit 9.6, that the matrix

$$\mathrm{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$, and $\lambda = -1$ (repeated), with associated eigenvectors

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} -\frac{1}{2}\beta - \gamma \\ \beta \\ \gamma \end{bmatrix}.$$

A set of **linearly independent** eigenvectors may therefore be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, X_1 is orthogonal to X_2 and X_3, but X_2 and X_3 are not orthogonal to each other.

However, we may find β and γ such that

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{is orthogonal to} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We simply require that

$$\frac{1}{2}\beta + 2\gamma = 0$$

or

$$\beta + 4\gamma = 0.$$

This will be so, for example, when $\beta = 4$ and $\gamma = -1$.

A new set of linearly independent mutually orthogonal eigenvectors can thus be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

9.10.2 QUADRATIC FORMS

An algebraic expression of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2yzx + 2hxy$$

is called a **"quadratic form"**.

In matrix notation, it may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv X^T A X,$$

and we note that the matrix, A, is symmetric.

In the scientific applications of quadratic forms, it is desirable to know whether such a form is

(a) always positive; (b) always negative;

(c) both positive and negative.

It may be shown that, if we change to new variables, (u, v, w), using a linear transformation,

$$\mathrm{X} = \mathrm{PU},$$

where P is some non-singular matrix, then the new quadratic form has the same properties aas the original, concerning its sign.

We now show that a good choice for P is a modal matrix, N, of normalised, linearly independent, mutually orthogonal eigenvectors for A.

Putting $\mathrm{X} = \mathrm{NU}$, the expression $\mathrm{X^TAX}$ becomes $\mathrm{U^TN^TANX}$.

But, since N is orthogonal when A is symmetric, $\mathrm{N^T} = \mathrm{N^{-1}}$ and hence $\mathrm{N^TAN}$ is the spectral matrix, S, for A.

The new quadratic form is therefore

$$\mathrm{U^TSU} \equiv [\,u \quad v \quad w\,].\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\equiv \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2.$$

Clearly, if all of the eigenvalues are positive, then the new quadratic form is always positive; and, if all of the eigenvalues are negative, then the new quadratic form is always negative.

The new quadratic form is called the **"canonical form under similarity"** of the original quadratic form.

"JUST THE MATHS"

SLIDES NUMBER

10.1

DIFFERENTIATION 1
(Functions and limits)

by

A.J.Hobson

UNIT 10.1 - DIFFERENTIATION 1
FUNCTIONS AND LIMITS

10.1.1 FUNCTIONAL NOTATION

Introduction

If a variable quantity, y, depends for its values on another variable quantity, x, we say that **"y is a function of x"**.

In general, we write $y = f(x)$.

This is pronounced "y equals f of x".

Notes:
(i) y is called the **"dependent variable"** and x is called the **"independent variable"**.

(ii) We do not always use the letter f.

ILLUSTRATIONS

1. $P = P(T)$ could mean that a pressure, P, is a function of absolute temperature, T;

2. $i = i(t)$ could mean that an electric current, i, is a function of time t;

3. the original statement, $y = f(x)$ could have been written $y = y(x)$.

The general format:

DEPENDENT VARIABLE = DEPENDENT VARIABLE(INDEPENDENT VARIABLE)

10.1.2 NUMERICAL EVALUATION OF FUNCTIONS

If α is a number, then $f(\alpha)$ denotes the value of the function $f(x)$ when $x = \alpha$ is substituted into it.

ILLUSTRATION

If

$$f(x) \equiv 4\sin 3x,$$

then,

$$f\left(\frac{\pi}{4}\right) = 4\sin\frac{3\pi}{4} = 4 \times \frac{1}{\sqrt{2}} \cong 2.828$$

10.1.3 FUNCTIONS OF A LINEAR FUNCTION

The notation

$$f(ax+b),$$

where a and b are constants, implies a **known** function, $f(x)$, in which x has been replaced by the linear function $ax+b$.

ILLUSTRATION

If

$$f(x) \equiv 3x^2 - 7x + 4,$$

then,

$$f(5x-1) \equiv 3(5x-1)^2 - 7(5x-1) + 4.$$

It usually best to leave the expression in the bracketed form.

10.1.4 COMPOSITE FUNCTIONS (or Functions of a Function) IN GENERAL

The symbol

$$f[g(x)]$$

implies a **known** function, $f(x)$, in which x has been replaced by **another known** function, $g(x)$.

ILLUSTRATION

If

$$f(x) \equiv x^2 + 2x - 5$$

and

$$g(x) \equiv \sin x,$$

then,

$$f[g(x)] \equiv \sin^2 x + 2\sin x - 5;$$

but also,

$$g[f(x)] \equiv \sin(x^2 + 2x - 5),$$

which is not identical to the first result.

Hence, in general,

$$f[g(x)] \not\equiv g[f(x)].$$

Exceptions

If

$$f(x) \equiv e^x \quad \text{and} \quad g(x) \equiv \log_e x$$

we obtain

$$f[g(x)] \equiv e^{\log_e x} \equiv x \quad \text{and} \quad g[f(x)] \equiv \log_e(e^x) \equiv x.$$

The functions $\log_e x$ and e^x are said to be **"inverses"** of each other.

10.1.5 INDETERMINATE FORMS

Certain fractional expressions involving functions can reduce to

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

These forms are meaningless or **“indeterminate”**.

Indeterminate forms need to be dealt with using **“limiting values”**.

(a) The Indeterminate Form $\frac{0}{0}$

In the fractional expression

$$\frac{f(x)}{g(x)},$$

suppose that $f(\alpha) = 0$ and $g(\alpha) = 0$.

It is impossible to evaluate the fraction when $x = \alpha$.

We may consider its values as x becomes increasingly close to α with out actually reaching it

We say that **“x tends to α”**.

Note:

By the **Factor Theorem** (Unit 1.8), $(x - \alpha)$ must be a factor of both numerator and denominator.

The result as x approaches α is denoted by

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)}.$$

EXAMPLE

Calculate

$$\lim_{x \to 1} \frac{x-1}{x^2+2x-3}.$$

Solution

First we factorise the denominator.

One of its factors must be $x-1$ because it takes the value zero when $x = 1$.

The result is therefore

$$\lim_{x \to 1} \frac{x-1}{(x-1)(x+3)}$$

$$= \lim_{x \to 1} \frac{1}{x+3} = \frac{1}{4}.$$

(b) The Indeterminate Form $\frac{\infty}{\infty}$

Problem

To evaluate either

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

or

$$\lim_{x \to -\infty} \frac{f(x)}{g(x)}.$$

EXAMPLE

Calculate

$$\lim_{x \to \infty} \frac{2x^2 + 3x - 1}{7x^2 - 2x + 5}.$$

Solution

We divide numerator and denominator by the highest power of x appearing.

$$\lim_{x \to \infty} \frac{2 + \frac{3}{x} - \frac{1}{x^2}}{7 - \frac{2}{x} + \frac{5}{x^2}} = \frac{2}{7}.$$

Notes:

(i) For the ratio of two polynomials of equal degree, the limiting value as $x \to \pm\infty$ is the ratio of the leading coefficients of x.

(ii) For two polynomials of unequal degree, we insert zero coefficients in appropriate places to consider them as being of equal degree.

The results then obtained will be either zero or infinity.

ILLUSTRATION

$$\lim_{x\to\infty} \frac{5x + 11}{3x^2 - 4x + 1} = \lim_{x\to\infty} \frac{0x^2 + 5x + 11}{3x^2 - 4x + 1} = \frac{0}{3} = 0.$$

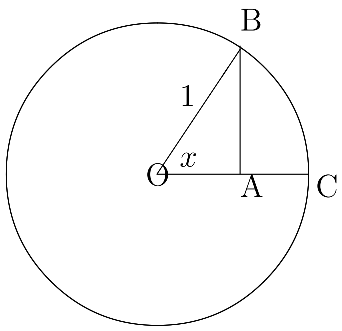
A Useful Standard Limit

In Unit 3.3, it is shown that, for very small values of x in radians, $\sin x \simeq x$.

This suggests that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

For a non-rigorous proof, consider the following diagram in which the angle x is situated at the centre of a circle with radius 1.



Length of line AB $= \sin x$.

Length of arc BC $= x$.

As x decreases almost to zero, these lengths become closer.

That is,

$$\sin x \to x \quad \text{as} \quad x \to 0$$

or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

10.1.6 EVEN AND ODD FUNCTIONS

Introduction

Any **even** power of x will be unchanged in value if x is replaced by $-x$.

Any **odd** power of x will be unchanged in numerical value, though altered in sign, if x is replaced by $-x$.

DEFINITION

A function $f(x)$ is said to be **"even"** if it satisfies the identity

$$f(-x) \equiv f(x).$$

ILLUSTRATIONS: x^2, $2x^6 - 4x^2 + 5$, $\cos x$.

DEFINITION

A function $f(x)$ is aid to be **"odd"** if it satisfies the identity

$$f(-x) \equiv -f(x).$$

ILLUSTRATIONS: x^3, $x^5 - 3x^3 + 2x$, $\sin x$.

Note:

It is not necessary for every function to be either even or odd. For example, the function $x + 3$ is neither even nor odd.

EXAMPLE

Express an arbitrary function, $f(x)$ as the sum of an even function and an odd function.

Solution

We may write

$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

The first term on the R.H.S. is unchanged if x is replaced by $-x$.

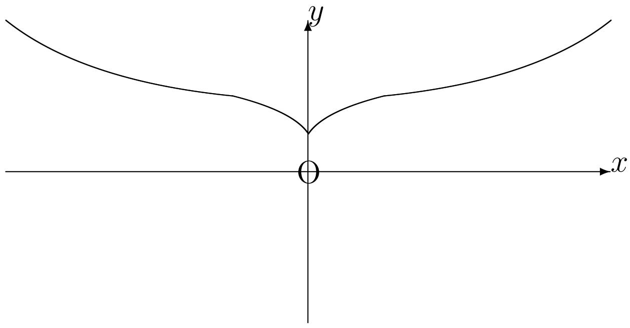
The second term on the R.H.S. is reversed in sign if x is replaced by $-x$.

We have thus expressed $f(x)$ as the sum of an even function and an odd function.

GRAPHS OF EVEN AND ODD FUNCTIONS

(i) The graph of the relationship $y = f(x)$, where $f(x)$ is **even**, will be symmetrical about the y-axis.

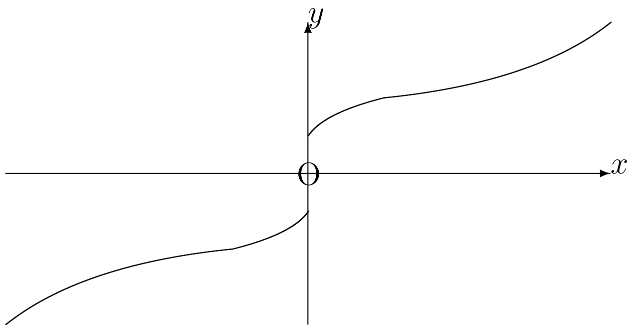
For every point (x, y) on the graph, there is also the point $(-x, y)$.



(ii) The graph of the relationship $y = f(x)$, where $f(x)$ is **odd**, will be symmetrical with respect to the origin.

For every point (x, y) on the graph, there is also the point $(-x, -y)$.

The part of the graph for $x < 0$ can be obtained from the part for $x > 0$ by reflecting it first in the x-axis and then in the y-axis.

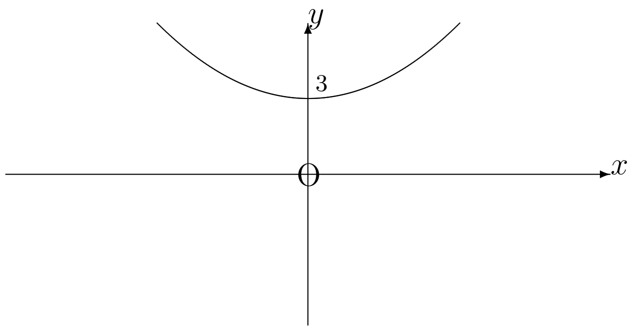


EXAMPLE

Sketch the graph, from $x = -3$ to $x = 3$ of the even function, $f(x)$, defined in the interval $0 < x < 3$ by the formula

$$f(x) \equiv 3 + x^3.$$

Solution



ALGEBRAICAL PROPERTIES OF ODD AND EVEN FUNCTIONS

1. The product of an even function and an odd function is an odd function.

Proof:

If $f(x)$ is even and $g(x)$ is odd, then

$$f(-x).g(-x) \equiv f(x).[-g(x)] \equiv -f(x).g(x).$$

2. The product of an even function and an even function is an even function.

Proof:

If $f(x)$ and $g(x)$ are both even functions, then

$$f(-x).g(-x) \equiv f(x).g(x).$$

3. The product of an odd function and an odd function is and even function.

Proof:

If $f(x)$ and $g(x)$ are both odd functions, then

$$f(-x).g(-x) \equiv [-f(x)].[-g(x)] \equiv f(x).g(x).$$

EXAMPLE

Show that the function

$$f(x) \equiv \sin^4 x. \tan x$$

is an **odd** function.

Solution

$$f(-x) \equiv \sin^4(-x). \tan(-x) \equiv -\sin^4 x. \tan x.$$

"JUST THE MATHS"

SLIDES NUMBER

10.2

DIFFERENTIATION 2
(Rates of change)

by

A.J.Hobson

UNIT 10.2 - DIFFERENTIATION 2

RATES OF CHANGE

10.2.1 INTRODUCTION

For the functional relationship

$$y = f(x),$$

we may plot y against x to obtain a curve (or straight line).

If y is the distance travelled, at time x, of a moving object, the rate of increase of y with respect to x becomes **speed**.

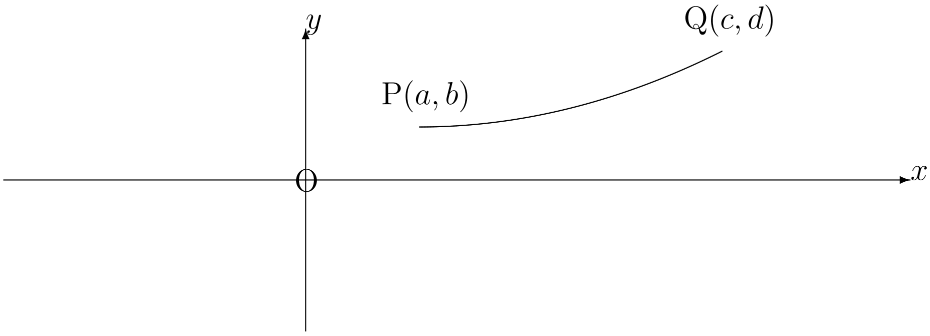
10.2.2 AVERAGE RATES OF CHANGE

For a vehicle travelling 280 miles in 7 hours,

$$\frac{280}{7} = 40$$

represents an **“average speed”** of 40 miles per hour over the whole journey.

Consider the relationship $y = f(x)$ between any two variables x and y.



Between P(a, b) and Q(c, d), an increase of $c - a$ in x gives rise to an increase of $d - b$ in y.

The average rate of increase of y with respect to x from P to Q is

$$\frac{d-b}{c-a}.$$

If y **decreases** as x increases (between P and Q), the average rate of increase will be negative

All rates of increase which are POSITIVE correspond to an INCREASING function.

All rates of increase which are NEGATIVE correspond to a DECREASING function.

Note:
For later work, P(x, y) and Q$(x+\delta x, y+\delta y)$ will denote points very close together.

The symbols δx and δy represent **"a small fraction of x"** and **"a small fraction of y"**, respectively.

δx is normally positive, but δy may turn out to be negative.

The average rate of increase may now be given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Average rate of increase =

$$\frac{(\text{new value of } y) \quad \text{minus} \quad (\text{old value of } y)}{(\text{new value of } x) \quad \text{minus} \quad (\text{old value of } x)}$$

EXAMPLE
Determine the average rate of increase of the function

$$y = x^2$$

between the following pairs of points on its graph:

(a) $(3, 9)$ and $(3.3, 10.89)$;

(b) $(3, 9)$ and $(3.2, 10.24)$;

(c) $(3, 9)$ and $(3.1, 9.61)$.

Solution

The results are

(a) $\frac{\delta y}{\delta x} = \frac{1.89}{0.3} = 6.3;$

(b) $\frac{\delta y}{\delta x} = \frac{1.24}{0.2} = 6.2;$

(c) $\frac{\delta y}{\delta x} = \frac{0.61}{0.1} = 6.1$

10.2.3 INSTANTANEOUS RATES OF CHANGE

Allowing Q to approach P along the curve, we may determine the **actual** rate of increase of y with respect to x at P.

The above solution suggests that the rate of increase of $y = x^2$ with respect to x at the point $(3, 9)$ is equal to 6.

This is called the **"instantaneous rate of increase of y with respect to x"** at the chosen point.

In general, we consider a limiting process in which an **infinite** number of points approach the chosen one along the curve.

The limiting process is represented by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

10.2.4 DERIVATIVES

(a) The Definition of a Derivative

If

$$y = f(x),$$

the **"derivative of y with respect to x"** at any point (x, y) on the graph of the function is defined to be the instantaneous rate of increase of y with respect to x at that point.

If a small increase of δx in x gives rise to a corresponding increase (positive or negative) of δy in y, the derivative will be given by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This limiting value is usually denoted by one of the three symbols

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad f'(x) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)].$$

Notes:

1. $\frac{\mathrm{d}}{\mathrm{d}x}$ is called a **"differential operator"**;
2. $f'(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]$ are normally used when the second variable, y, is not involved.
3. The derivative of a constant function must be zero.
4. The derivative represents the **gradient of the tangent at the point** (x, y) to the curve whose equation is
$$y = f(x).$$

(b) Differentiation from First Principles

EXAMPLES

1. Differentiate the function x^4 from first principles.

   Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4\right] = \lim_{\delta x \to 0} \frac{(x+\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^4 + 4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[4x^3 + 6x^2\delta x + 4x(\delta x)^2 + (\delta x)^3\right] = 4x^3.$$

Note:
The general formula is

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$

for any constant value n, not necessarily an integer.

2. Differentiate the function $\sin x$ from first principles.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

But,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

3. Differentiate from first principles the function

$$\log_b x,$$

where b is any base of logarithms.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_b x\right] = \lim_{\delta x \to 0} \frac{\log_b(x+\delta x) - \log_b x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log_b\left(1+\frac{\delta x}{x}\right)}{\delta x}.$$

But writing

$$\frac{\delta x}{x} = r \quad \text{that is} \quad \delta x = rx,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_b x\right] = \frac{1}{x} \lim_{r \to 0} \frac{\log_b(1+r)}{r}$$

$$= \frac{1}{x} \lim_{r \to 0} \log_b(1+r)^{\frac{1}{r}}.$$

For convenience, we may choose b so that the above limiting value is equal to 1.

This will occur when

$$b = \lim_{r \to 0} (1+r)^{\frac{1}{r}}.$$

The appropriate value of b turns out to be approximately 2.71828

This is the standard base of natural logarithms denoted by e.

Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_e x\right] = \frac{1}{x}.$$

Note:

In scientific work, the natural logarithm of x is usually denoted by $\ln x$.

"JUST THE MATHS"

SLIDES NUMBER

10.3

DIFFERENTIATION 3
(Elementary techniques of differentiation)

by

A.J.Hobson

UNIT 10.3 - DIFFERENTIATION 3

ELEMENTARY TECHNIQUES OF DIFFERENTIATION

10.3.1 STANDARD DERIVATIVES

$f(x)$	$f'(x)$
a const.	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

10.3.2 RULES OF DIFFERENTIATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Af(x) + Bg(x)\right] = A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

Proof:

The left-hand-side is equivalent to

$$\lim_{\delta x \to 0} \frac{[Af(x+\delta x) + Bg(x+\delta x)] - [Af(x) + Bg(x)]}{\delta x}$$

$$= A\left[\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right]+B\left[\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}\right].$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[Af(x)+Bg(x)] = A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

This is easily extended to **"linear combinations"** of three or more functions of x.

EXAMPLES

1. Write down the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

Solution

Using the linearity property, the standard derivative of x^n, and the derivative of a constant, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{\mathrm{d}}{\mathrm{d}x}[x^2] + 2\frac{\mathrm{d}}{\mathrm{d}x}[x^6] + 13\frac{\mathrm{d}}{\mathrm{d}x}[x^1] - \frac{\mathrm{d}}{\mathrm{d}x}[7]$$

$$= 12x + 12x^5 + 13.$$

2. Write down the derivative with respect to x of the function
$$\frac{5}{x^2} - 4\sin x + 2\ln x.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5}{x^2} - 4\sin x + 2\ln x\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[5x^{-2} - 4\sin x + 2\ln x\right]$$
$$= -10x^{-3} - 4\cos x + \frac{2}{x}$$
$$= \frac{-10}{x^3} - 4\cos x + \frac{2}{x}.$$

(b) Composite Functions (or Functions of a Function)

(i) Functions of a Linear Function

Expressions like $(5x+2)^{16}$, $\sin(2x+3)$, $\ln(7-4x)$ may be called **"functions of a linear function"**.

The general form is
$$f(ax+b),$$
where a and b are constants.

In the above illustrations, $f(x)$ would be x^{16}, $\sin x$ and $\ln x$ respectively.

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax + b.$$

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δy in y and δu in u.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

Assuming that δy and δu tend to zero as δx tends to zero,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} . \frac{\mathrm{d}u}{\mathrm{d}x}.$$

This rule is called the **"Function of a Function Rule"**, **"Composite Function Rule"** or **"Chain Rule"**.

EXAMPLES

1. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = (5x+2)^{16}$.
   Solution
   First write $y = u^{16}$ where $u = 5x + 2$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = 16u^{15}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = 16u^{15}.5 = 80(5x+2)^{15}$.
2. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin(2x+3)$.
   Solution
   First write $y = \sin u$ where $u = 2x + 3$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos u.2 = 2\cos(2x+3)$.
3. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln(7-4x)$.
   Solution
   First write $y = \ln u$ where $u = 7 - 4x$.
   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = -4$.
   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u}.(-4) = \frac{-4}{7-4x}$.

Note:
For quickness, treat $ax + b$ as if it were a single x, then multiply the final result by the constant value, a.

(ii) Functions of a Function in general

The formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}$$

may be used for the composite function

$$f[g(x)].$$

We write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (x^2 + 7x - 3)^4.$$

Solution

Let $y = u^4$ where $u = x^2 + 7x - 3$.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3.(2x + 7)$$

$$= 4(x^2 + 7x - 3)^3(2x + 7).$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \ln(x^2 - 3x + 1).$$

Solution

Let $y = \ln u$ where $u = x^2 - 3x + 1$.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}.(2x-3) = \frac{2x-3}{x^2-3x+1}.$$

3. Determine the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ in the case when

$$y = 2\sin(5x^2 - 1) + 19x.$$

Solution

Suppose $z = 2\sin(5x^2 - 1)$.

Let $z = 2\sin u$, where $u = 5x^2 - 1$.

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u.10x = 20x\cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 20x\cos(5x^2 - 1) + 19.$$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 20\cos 4 + 19 \simeq 5.927$

Calculator must be in **radian mode**.

Note:
For quickness, treat $g(x)$ as if it were a single x, then multiply by $g'(x)$

EXAMPLE

Determine the derivative of $\sin^3 x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^3 x\right] =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(\sin x)^3\right] =$$

$$3(\sin x)^2 . \cos x =$$

$$3\sin^2 x . \cos x .$$

"JUST THE MATHS"

SLIDES NUMBER

10.4

DIFFERENTIATION 4
(Products and quotients)
&
(Logarithmic differentiation)

by

A.J.Hobson

UNIT 10.4 - DIFFERENTIATION 4

PRODUCTS, QUOTIENTS AND LOGARITHMIC DIFFERENTIATION

10.4.1 PRODUCTS

Suppose

$$y = u(x)v(x),$$

where $u(x)$ and $v(x)$ are two functions of x.

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δu in u, δv in v and δy in y.

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(u + \delta u)(v + \delta v) - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{uv + u\delta v + v\delta u + \delta u\delta v - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} \right].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[u.v] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Hint: Think of this as
(FIRST times DERIVATIVE OF SECOND) plus (SECOND times DERIVATIVE OF FIRST)

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when
$$y = x^7 \cos 3x.$$
Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^7.-3\sin 3x+\cos 3x.7x^6 = x^6[7\cos 3x-3x\sin 3x].$$

2. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$ in the case when
$$y = (x^2 - 8)\ln(2x + 3).$$
Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - 8).\frac{1}{2x+3}.2 + \ln(2x + 3).2x$$
$$= 2\left[\frac{x^2 - 8}{2x + 3} + x\ln(2x + 3)\right].$$
When $x = -1$, this has value -14 since $\ln 1 = 0$.

10.4.2 QUOTIENTS

Suppose that

$$y = \frac{u(x)}{v(x)}.$$

We may write

$$y = u(x).[v(x)]^{-1}.$$

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u.(-1)[v]^{-2}.\frac{\mathrm{d}v}{\mathrm{d}x} + v^{-1}.\frac{\mathrm{d}u}{\mathrm{d}x},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

EXAMPLES

1. Show that the derivative with respect to x of $\tan x$ is $\sec^2 x$.

   Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x.\cos x - \sin x. - \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{2x+1}{(5x-3)^3}.$$

Solution

Using $u(x) \equiv 2x+1$ and $v(x) \equiv (5x-3)^3$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3)^3.2 - (2x+1).3(5x-3)^2.5}{(5x-3)^6}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 15(2x+1)}{(5x-3)^4} = -\frac{20x+21}{(5x-3)^4}.$$

Note:

A modified version of the Quotient Rule is for quotients in the form

$$\frac{u}{v^n}.$$

If

$$y = \frac{u}{v^n},$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - nu\frac{\mathrm{d}v}{\mathrm{d}x}}{v^{n+1}}.$$

In Example 2 above, we could write

$$u \equiv 2x+1 \quad v \equiv 5x-3 \quad \text{and} \quad n = 3.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 3(2x+1).5}{(5x-3)^4},$$

as before.

10.4.3 LOGARITHMIC DIFFERENTIATION

(a) Functions containing a variable index

First consider the **"exponential function"**, e^x.

Letting

$$y = e^x,$$

we may write

$$\ln y = x.$$

Differentiating both sides **with respect to** x, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y = e^x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x.$$

Notes:

(i) Differentiating $x = \ln y$ with respect to y,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

But it can be shown that, for most functions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}},$$

so that the same result is obtained as before.

(ii) The derivative of e^x may easily be used to establish the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2\, x.$$

We use the definitions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}, \quad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

and

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

FURTHER EXAMPLES

1. Write down the derivative with respect to x of the function

$$e^{\sin x}.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] = e^{\sin x}.\cos x.$$

2. Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (3x+2)^x.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x\ln(3x+2).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{3}{3x+2} + \ln(3x+2).1.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)^x\left[\frac{3x}{3x+2} + \ln(3x+2)\right].$$

(b) Products or Quotients with more than two elements

We illustrate with examples:

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in the case when

$$y = e^{x^2}.\cos x.(x+1)^5.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x^2 + \ln\cos x + 5\ln(x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 2x - \frac{\sin x}{\cos x} + \frac{5}{x+1}.$$

Hence,

$$\frac{dy}{dx} = e^{x^2}.\cos x.(x+1)^5\left[2x - \tan x + \frac{5}{x+1}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{e^x.\sin x}{(7x+1)^4}.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x + \ln\sin x - 4\ln(7x+1).$$

Differentiating both sides with repect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\cos x}{\sin x} - 4.\frac{7}{7x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x.\sin x}{(7x+1)^4}\left[1 + \cot x - \frac{28}{7x+1}\right].$$

Note:

In all examples on logarithmic differentiation, the original function will appear as a factor at the beginning of its derivative.

"JUST THE MATHS"

SLIDES NUMBER

10.5

DIFFERENTIATION 5
(Implicit and parametric functions)

by

A.J.Hobson

UNIT 10.5 - DIFFERENTIATION 5
IMPLICIT & PARAMETRIC FUNCTIONS

10.5.1 IMPLICIT FUNCTIONS

Some relationships between two variables x and y do not give y explicitly in terms of x (or x explicitly in terms of y).

ILLUSTRATIONS

1.

$$x^2 + y^2 = 16$$

is not explicit for either x or y.
We could, however, write

$$y = \pm\sqrt{16 - x^2} \quad \text{or} \quad x = \pm\sqrt{16 - y^2}.$$

2. By contrast, consider

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

There is no way of stating either variable explicitly in terms of the other.

Such relationships between x and y are said to be **'implicit relationships"** and **"implicit differentiation"** means differentiate each term in the relationship with respect to the same variable without attempting to rearrange the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + y^2 = 16.$$

Solution

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2y} = -\frac{x}{y}.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + 2xy^3 + y^5 = 4.$$

Solution

$$2x + 2\left[x.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.1\right] + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

On rearrangement,

$$\left[6xy^2 + 5y^4\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + 2y^3).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 2y^3}{6xy^2 + 5y^4}.$$

3. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

Solution

$$x^2.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.2x + 9\cos(5x - 7y).\left[5 - 7\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0.$$

On rearrangment,

$$\left[3x^2y^2 - 63\cos(5x - 7y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[2xy^3 + 45\cos(5x - 7y)\right].$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^3 + 45\cos(5x - 7y)}{3x^2y^2 - 63\cos(5x - 7y)}.$$

10.5.2 PARAMETRIC FUNCTIONS

Sometimes, the variables x and y can be expressed in terms of a third variable, usually t or θ, called a **"parameter"**.

In general, we write

$$x = x(t) \quad \text{and} \quad y = y(t).$$

From the Function of a Function Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x},$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in terms of t in the case when

$$x = 3t^2 \quad \text{and} \quad y = 6t.$$

Solution

$$\frac{dy}{dt} = 6 \quad \text{and} \quad \frac{dx}{dt} = 6t.$$

Hence,

$$\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of θ in the case when

$$x = \sin^3\theta \quad \text{and} \quad y = \cos^3\theta.$$

Solution

$$\frac{dx}{d\theta} = 3\sin^2\theta.\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -3\cos^2\theta.\sin\theta.$$

Hence,

$$\frac{dy}{dx} = \frac{-3\cos^2\theta.\sin\theta}{3\sin^2\theta.\cos\theta}.$$

That is,

$$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

"JUST THE MATHS"

SLIDES NUMBER

10.6

DIFFERENTIATION 6
(Inverse trigonometric functions)

by

A.J.Hobson

UNIT 10.6 - DIFFERENTIATION 6

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

10.6.1 SUMMARY OF RESULTS

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}},$$

where $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}},$$

where $0 \leq \cos^{-1}x \leq \pi$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2},$$

where $-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$.

10.6.2 THE DERIVATIVE OF AN INVERSE SINE

We shall consider the formula

$$y = \text{Sin}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case S in the formula (see later).

The formula is equivalent to

$$x = \sin y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \equiv \pm\sqrt{1-\sin^2 y} \equiv \pm\sqrt{1-x^2}.$$

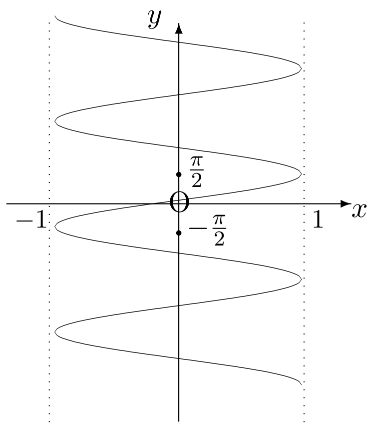
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sin}^{-1}x.$$

This may be obtained from the graph of $y = \sin x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

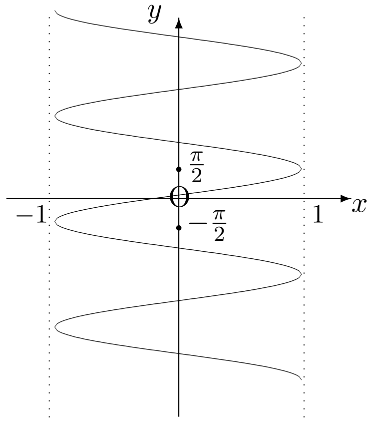


Observations

(i) x must lie in the interval $-1 \leq x \leq 1$.

(ii) For each x in $-1 \leq x \leq 1$, y has infinitely many values - spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{dy}{dx}$, one positive and the other negative.



(iv) On the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y and one (positive) value of $\frac{dy}{dx}$ for each x in $-1 \leq x \leq 1$.

The restricted part of the graph defines the **"principal value"** of the inverse sine function and is denoted by $\sin^{-1}x$ using a lower-case s.

Hence,

$$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

10.6.3 THE DERIVATIVE OF AN INVERSE COSINE

We shall consider the formula

$$y = \mathrm{Cos}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula (see later).

The formula is equivalent to

$$x = \cos y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \equiv \pm\sqrt{1-\cos^2 y} \equiv \pm\sqrt{1-x^2}.$$

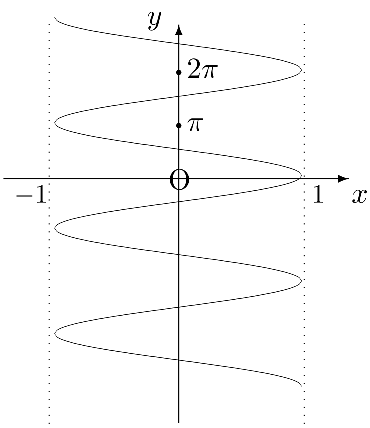
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cos}^{-1} x,$$

which may be obtained from the graph of $y = \cos x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

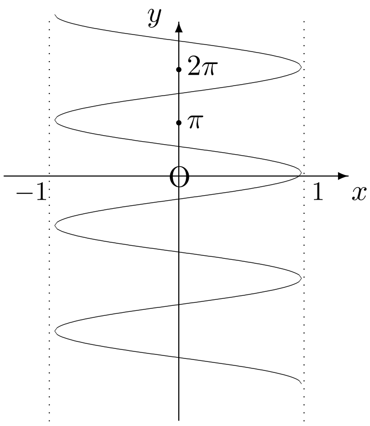


Observations

(i) x must lie in the interval $-1 \leq x \leq 1$.

(ii) For each x in $-1 \leq x \leq 1$, y has infinitely many values - spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one positive and the other negative.



(iv) On the part of the graph from $y = 0$ to $y = \pi$, we may distinguish the results from those of the inverse sine function.

There will be only one value of y with one (negative) value of $\frac{dy}{dx}$ for each x in $-1 \leq x \leq 1$.

The restricted part of the graph defines the **"principal value"** of the inverse cosine function and is denoted by $\cos^{-1}x$ using a lower-case c.

Hence,

$$\frac{d}{dx}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}.$$

10.6.4 THE DERIVATIVE OF AN INVERSE TANGENT

We shall consider the formula

$$y = \mathrm{Tan}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case T in the formula (see later).

The formula is equivalent to

$$x = \tan y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y \equiv 1 + \tan^2 y \equiv 1 + x^2.$$

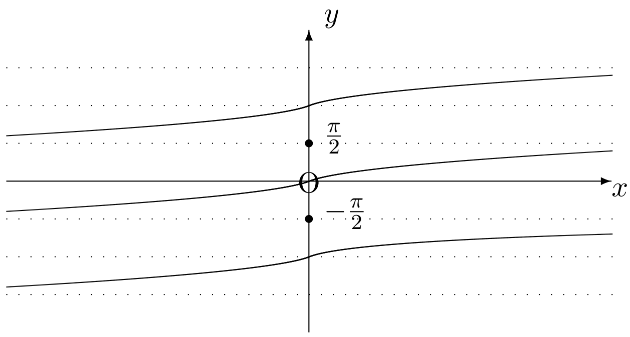
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tan}^{-1} x,$$

which may be obtained from the graph of $y = \tan x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

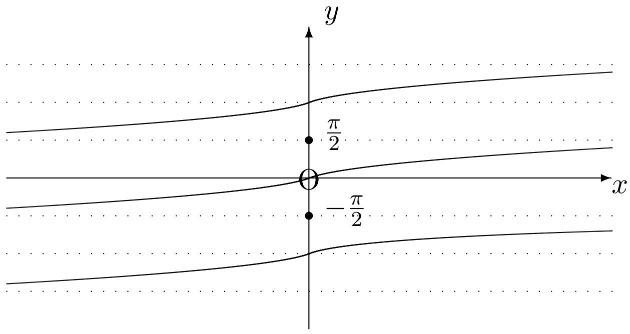


Observations

(i) x may lie anywhere in the interval $-\infty < x < \infty$.

(ii) For each x, y has infinitely many values - spaced at regular intervals of π.

(iii) For each x, there is only possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.



(iv) On the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y for each value of x.

The restricted part of the graph defines the **"principal value"** of the inverse tangent function and is denoted by $\tan^{-1}x$ using a lower-case t.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}.$$

ILLUSTRATIONS

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}2x] = \frac{2}{\sqrt{1-4x^2}}.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}(x+3)] = -\frac{1}{\sqrt{1-(x+3)^2}}.$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}(\sin x)] = \frac{\cos x}{1+\sin^2 x}.$$

4.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}\left(x^5\right)] = \frac{5x^4}{\sqrt{1-x^{10}}}.$$

"JUST THE MATHS"

SLIDES NUMBER

10.7

DIFFERENTIATION 7
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 10.7 - DIFFERENTIATION 7

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

10.7.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric and inverse hyperbolic functions should be considered as standard results, as follows:

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}},$$
where $-\infty < \sinh^{-1}x < \infty$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}},$$
where $\cosh^{-1}x \geq 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2}$$
where $-\infty < \tanh^{-1}x < \infty$.

10.7.2 THE DERIVATIVE OF AN INVERSE HYPERBOLIC SINE

We shall consider the formula

$$y = \text{Sinh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case S in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to

$$x = \sinh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y \equiv \sqrt{1 + \sin^2 y} \equiv \sqrt{1 + x^2},$$

noting that $\cosh y$ is never negative.

Thus,

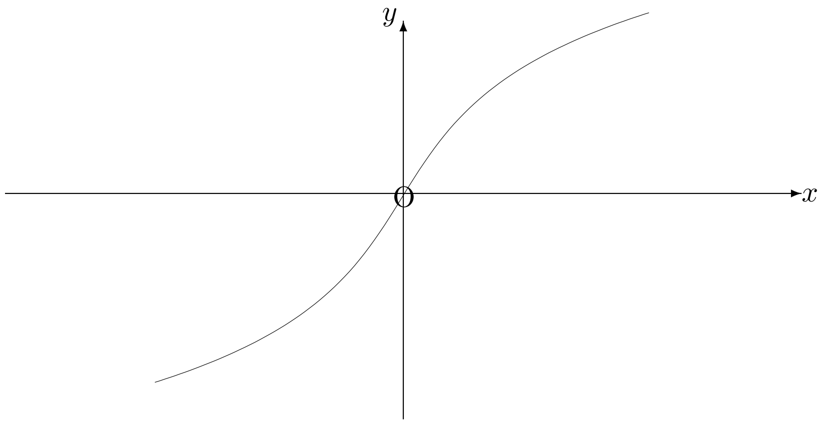
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 + x^2}}.$$

Consider now the graph of the formula

$$y = \text{Sinh}^{-1}x,$$

which may be obtained from the graph of $y = \sinh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x may lie anywhere in the interval $-\infty < x < \infty$.
2. For each value of x, the variable y has only one value.
3. For each value of x, there is only one possible value of $\frac{dy}{dx}$, which is positive.

4. There is no need to distinguish between a general value and a principal value of the inverse hyperbolic sine function since there is only one value of both the function and its derivative.

   However, it is customary to denote the inverse function by $\sinh^{-1}x$ using a lower-case s.

   Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}}.$$

10.7.3 THE DERIVATIVE OF AN INVERSE HYPERBOLIC COSINE

We shall consider the formula

$$y = \mathrm{Cosh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:

There is a special significance in using the upper-case C in the formula; see later.

The formula is equivalent to

$$x = \cosh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y \equiv \pm\sqrt{\cosh^2 y - 1} \equiv \pm\sqrt{x^2-1}.$$

Thus,

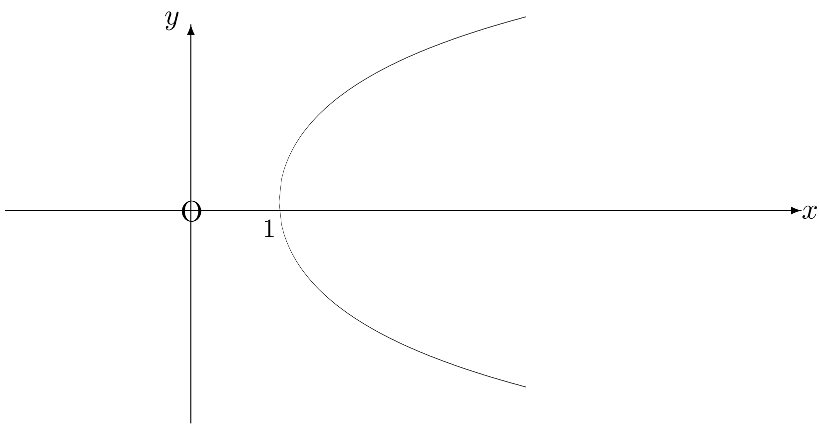
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}.$$

Consider now the graph of the formula

$$y = \text{Cosh}^{-1}x,$$

which may be obtained from the graph of $y = \cosh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x must lie in the interval $x \geq 1$.
2. For each value of x in the interval $x > 1$, the variable y has two values one of which is positive and the other negative.

3. For each value of x in the interval $x > 1$, there are only two possible values of $\frac{dy}{dx}$, one of which is postive and the other negative.
4. On the part of the graph for which $y \geq 0$, there will be only one value of y with one (positive) value of $\frac{dy}{dx}$ for each value of x in the interval $x \geq 1$.

   The restricted part of the graph defines the **"principal value"** of the inverse cosine function and is denoted by $\cosh^{-1}x$ using a lower-case c.

   Hence,

$$\frac{d}{dx}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2 - 1}}.$$

10.7.4 THE DERIVATIVE OF AN INVERSE HYPERBOLIC TANGENT

We shall consider the formula

$$y = \text{Tanh}^{-1}x$$

and determine an expression for $\frac{dy}{dx}$.

Note:
The use of the upper-case T in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to

$$x = \tanh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{sech}^2 y \equiv 1 - \tanh^2 y \equiv 1 - x^2.$$

Thus,

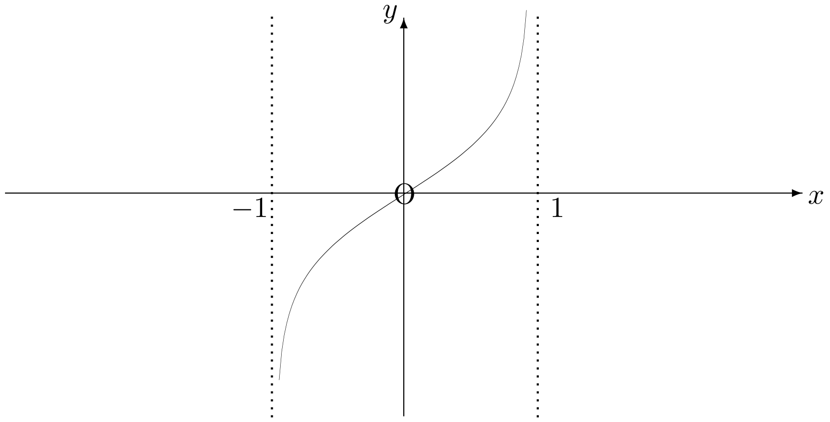
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tanh}^{-1} x,$$

which may be obtained from the graph of $y = \tanh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x must lie in the interval $-1 < x < 1$.
2. For each value of x in the interval $-1 < x < 1$, the variable y has just one value.
3. For each value of x in the interval $-1 < x < 1$, there is only possible value of $\frac{dy}{dx}$, which is positive.
4. As with $\sinh^{-1}x$, there is no need to distinguish between a general value and a principal value of the inverse hyperbolic tangent; but it is customary to denote it by $\tan^{-1}x$ (lower-case t).

$$\text{Hence } \frac{d}{dx}[\tanh^{-1}x] = \frac{1}{1-x^2}.$$

ILLUSTRATIONS

1.

$$\frac{d}{dx}[\sin^{-1}(\tanh x)] = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \text{sech} x.$$

2.

$$\frac{d}{dx}[\cosh^{-1}(5x-4)] = \frac{5}{\sqrt{(5x-4)^2-1}},$$

assuming that $5x - 4 \geq 1$; that is, $x \geq 1$.

“JUST THE MATHS”

SLIDES NUMBER

10.8

DIFFERENTIATION 8
(Higher derivatives)

by

A.J.Hobson

10.8.1 The theory

UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most examples on differentiating a function of x with respect to x, the result obtained is **another** function of x.

The possibilty arises of differentiating again with respect to x.

ILLUSTRATION

Let

$$y = f(x)$$

represent the distance, y, travelled by a moving object at time, x.

(a) The **speed** of the moving object is $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) The **acceleration** is defined as the rate of increase of speed with respect to time.

It is therefore represented by the symbol

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right].$$

The second derivative of y with respect to x is usually written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

and is pronounced "d two y by dx squared".

We could, if necessary, differentiate over and over again to obtain the symbols

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \text{ and } \frac{\mathrm{d}^4y}{\mathrm{d}x^4}.$$

EXAMPLES

1. If $y = \sin 2x$ show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 0.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin 2x = -4y.$$

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3;$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2;$$
$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x;$$
$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 24.$$

We now have a constant function so that all future derivatives will be zero.

Note:

In general, every derivative of $y = x^n$ after the n-th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t.

Solution

Firstly,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

Secondly,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\mathrm{d}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x}.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

This is a general formula

In the present example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\right]}{6t} = \frac{\frac{-1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x), \quad f'''(x), \quad f^{(iv)}(x), \text{ etc.}$$

"JUST THE MATHS"

SLIDES NUMBER

11.1

DIFFERENTIATION APPLICATIONS 1
(Tangents and normals)

by

A.J.Hobson

UNIT 11.1 - APPLICATIONS OF DIFFERENTIATION 1

TANGENTS AND NORMALS

11.1.1 TANGENTS

The derivative of the function $f(x)$ can be interpreted as the gradient of the tangent to the curve $y = f(x)$ at the point (x, y).

This information, together with the geometry of the straight line, determines the equation of the tangent to a given curve at a particular point on it.

EXAMPLES

1. Determine the equation of the tangent at the point $(-1, 2)$ to the curve whose equation is

$$y = 2x^3 + 5x^2 - 2x - 3.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 10x - 2,$$

which takes the value -6 when $x = -1$.

Hence, the tangent is the straight line passing through the point $(-1, 2)$ having gradient -6.

Its equation is therefore

$$y - 2 = -6(x + 1).$$

That is,

$$6x + y + 4 = 0.$$

2. Determine the equation of the tangent at the point $(2, -2)$ to the curve to the curve whose equation is

$$x^2 + y^2 + 3xy + 4 = 0.$$

Solution

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right] = 0.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 3y}{3x + 2y},$$

which takes the value -2 at the point $(2, -2)$.
Hence, the equation of the tangent is

$$y + 2 = -2(x - 2).$$

That is,

$$2x + y - 2 = 0.$$

3. Determine the equation of the tangent at the point where $t = 2$ to the curve given parametrically by

$$x = \frac{3t}{1+t} \text{ and } y = \frac{t^2}{1+t}.$$

Solution

The point at which $t = 2$ has co-ordinates $\left(2, \frac{4}{3}\right)$.

Furthermore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{(1+t)^2} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t+t^2}{(1+t)^2}.$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t+t^2}{3},$$

which takes the value $\frac{8}{3}$ when $t = 2$.

The equation of the tangent is

$$y - \frac{4}{3} = \frac{8}{3}(x-2).$$

That is,

$$3y + 12 = 8x.$$

11.1.2 NORMALS

The normal to a curve at a point on it is defined to be a straight line passing through this point and perpendicular to the tangent there.

If the gradient of the tangent is m, then the gradient of the normal will be $-\frac{1}{m}$.

EXAMPLES

In the examples of section 11.1.1, the normals to each curve at the point given will have equations as follows:

1.

$$y - 2 = \frac{1}{6}(x + 1).$$

That is, $6y = x + 13$.

2.

$$y + 2 = \frac{1}{2}(x - 2).$$

That is, $2y = x - 6$.

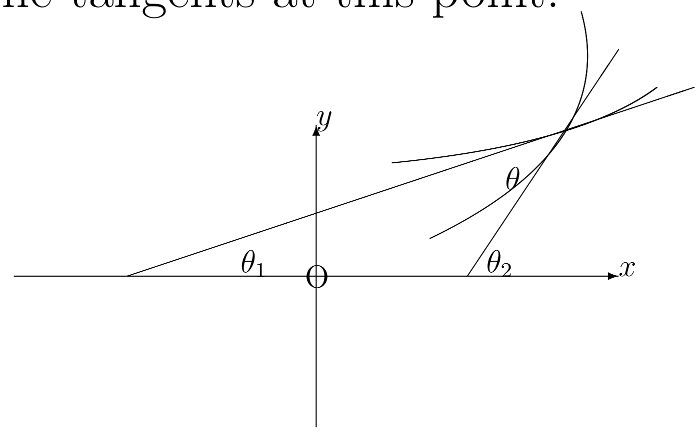
3.

$$y - \frac{4}{3} = -\frac{3}{8}(x - 2).$$

That is, $24y + 9x = 50$

The Angle between two curves.

The angle between two curves is defined to be the angle between the tangents at this point.



If the gradients of the tangents are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, the angle $\theta \equiv \theta_2 - \theta_1$ and is given by

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

That is,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

"JUST THE MATHS"

SLIDES NUMBER

11.2

DIFFERENTIATION APPLICATIONS 2
(Local maxima and local minima)
&
(Points of inflexion)

by

A.J.Hobson

UNIT 11.2 - APPLICATIONS OF DIFFERENTIATION 2

LOCAL MAXIMA, LOCAL MINIMA AND POINTS OF INFLEXION

11.2.1 INTRODUCTION

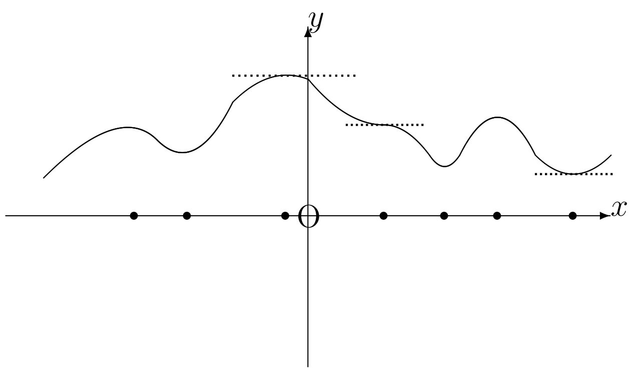
Any relationship,

$$y = f(x),$$

between two variable quantities, x and y, can usually be represented by a graph of y against x.

Any point (x_0, y_0) on the graph at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ takes the value zero is called a **"stationary point"**.

The tangent to the curve at the point (x_0, y_0) will be parallel to x-axis.

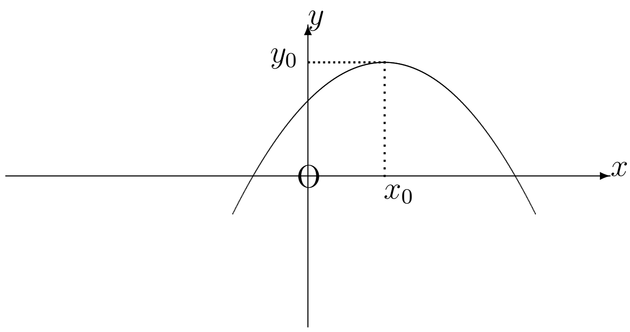


11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



Note:

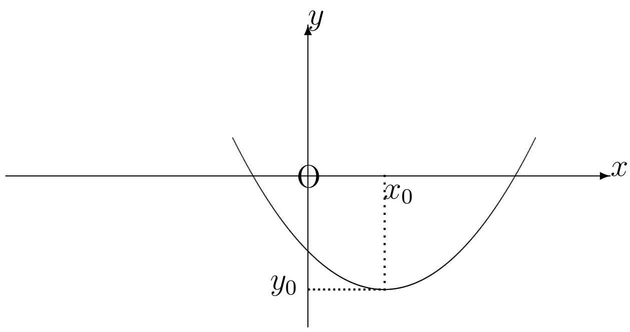
The definition of a local maximum point must refer to the behaviour of y in the **immediate neighbourhood** of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



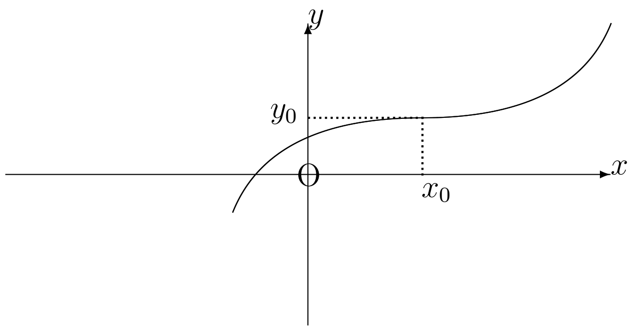
Note:
The definition of a local minimum point must refer to the behaviour of y in the **immediate neighbourhood** of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

First, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we then determine whether it is a local maximum, local minimum, or point of inflexion.

METHOD 1. - The "First Derivative" Method

Let ϵ denote a number which is relatively small compared with x_0.

Examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$.

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

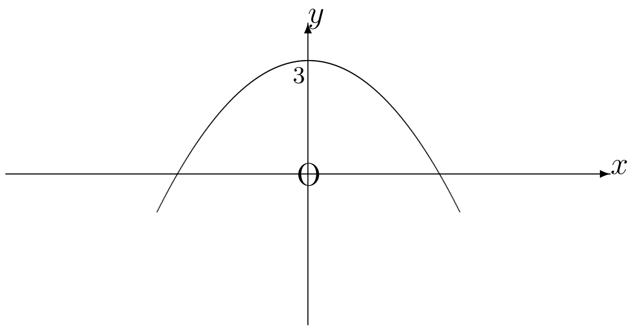
(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

1. Determine the stationary point on the graph whose equation is
$$y = 3 - x^2.$$

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$
which is equal to zero at the point where $x = 0$ and hence, $y = 3$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
Hence, there is a local maximum at the point $(0, 3)$.



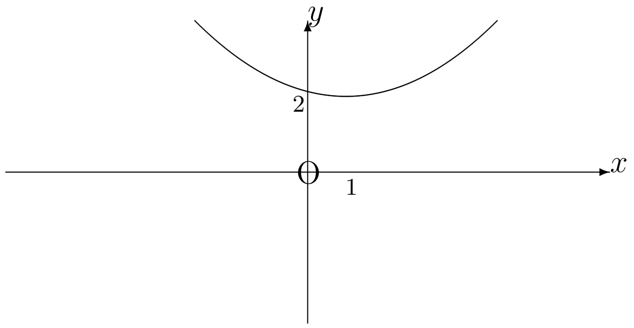
2. Determine the stationary point on the graph whose equation is
$$y = x^2 - 2x + 3.$$

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$
which is equal to zero at the point where $x = 1$ and hence $y = 2$.

If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

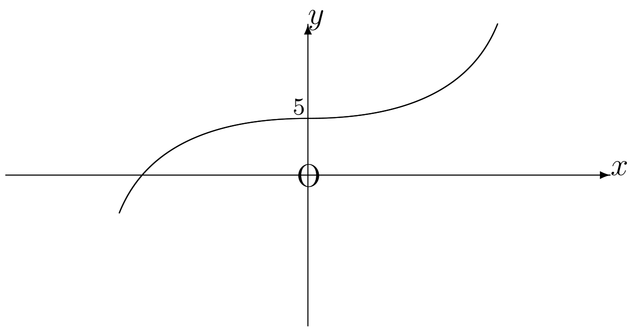
If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

Hence there is a local minimum at the point $(1, 2)$.



3. Determine the stationary point on the graph whose equation is
$$y = 5 + x^3.$$

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$
which is equal to zero at the point where $x = 0$ and hence, $y = 5$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a point of inflexion at $(0, 5)$.

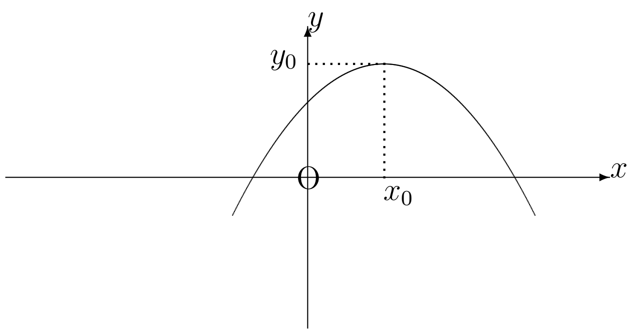


METHOD 2. - The "Second Derivative" Method

The graph of $\frac{dy}{dx}$ against x is called the **"first derived curve"**.

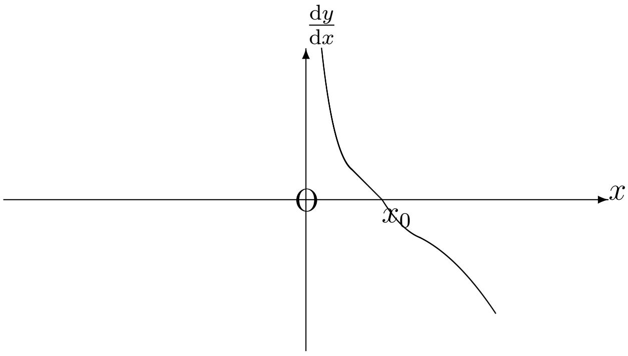
The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is **negative**.

$$\text{TEST} \quad (\text{Max}) : \frac{d^2y}{dx^2} < 0 \quad \text{at} \quad x = x_0.$$

EXAMPLE

For the curve whose equation is

$$y = 3 - x^2,$$

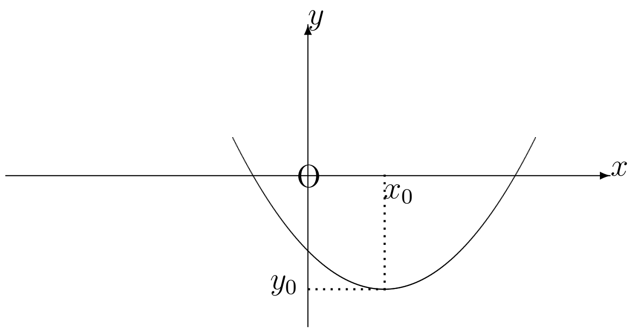
we have

$$\frac{dy}{dx} = -2x \quad \text{and} \quad \frac{d^2y}{dx^2} = -2.$$

The second derivative is negative everywhere.

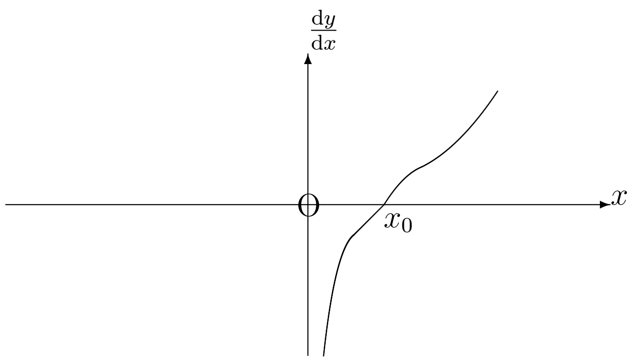
Hence, $(0, 3)$ (obtained earlier) is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0 the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is **positive**.

$$\text{TEST} \quad (\text{Min}): \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0 \quad \text{at} \quad x = x_0.$$

EXAMPLE

For the curve whose equation is

$$y = x^2 - 2x + 3,$$

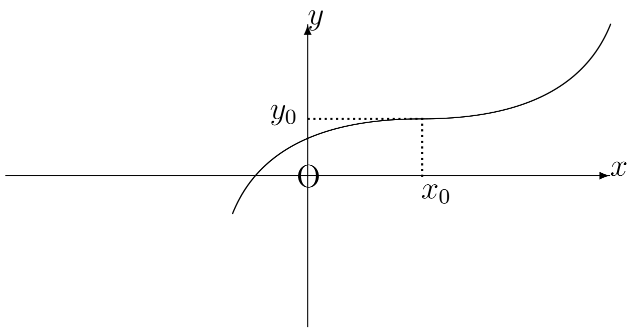
we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2 \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2.$$

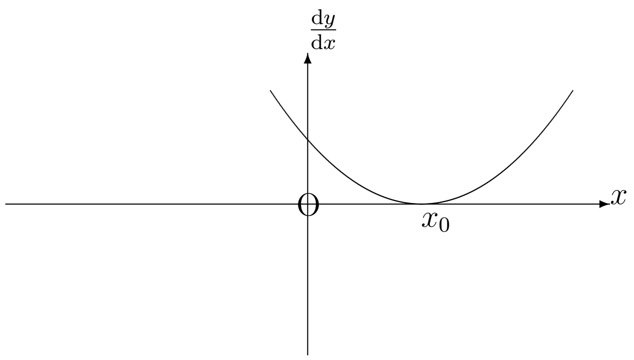
The second derivative is positive everywhere.

Hence, $(1, 2)$ (obtained earlier) is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

TEST (Infl) : $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = x_0$ and changes sign.

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

Hence the stationary point $(0, 5)$ (obtained earlier) is a point of inflexion.

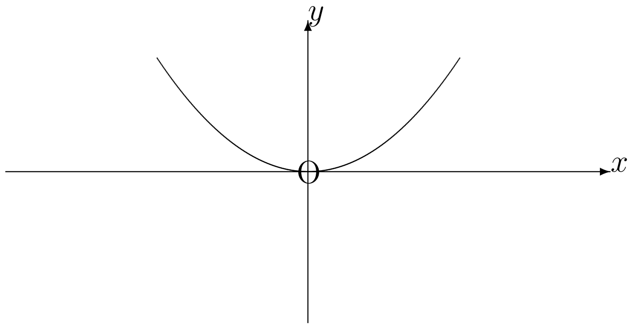
Notes:

(i) For a stationary point of inflexion, it is not enough that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, $y = x^4$ has a local minimum at the point $(0, 0)$; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**.

They are not stationary points and hence, $\frac{dy}{dx} \neq 0$.

But we still use

$$\frac{d^2y}{dx^2} = 0 \text{ and changes sign.}$$

EXAMPLE

For the curve whose equation is

$$y = x^3 + x,$$

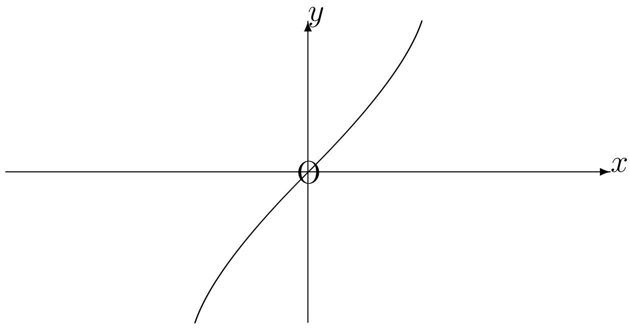
we have

$$\frac{dy}{dx} = 3x^2 + 1 \text{ and } \frac{d^2y}{dx^2} = 6x.$$

Hence, there are no stationary points at all.

But $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, $y = x^3 + x$ has an ordinary point of inflexion at $(0, 0)$.



Note:
In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval.

Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.

"JUST THE MATHS"

SLIDES NUMBER

11.3

DIFFERENTIATION APPLICATIONS 3
(Curvature)

by

A.J.Hobson

UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

CURVATURE

11.3.1 INTRODUCTION

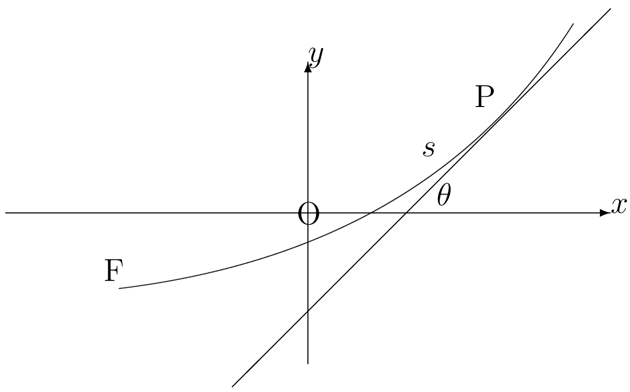
The **"tightness of a bend"** on a curve is called its **"curvature"**.

Tight bends have large curvature.

Curves may be **"concave upwards"** ($\cup$), having positive curvature, or **"concave downwards"** ($\cap$), having negative curvature.

DEFINITION

Let $y = f(x)$ be the equation of a curve.



Let θ be the angle made with the positive x-axis by the tangent to the curve at a point, P(x, y), on it.

Let s be the distance to P, measured along the curve from some fixed point, F, on it.

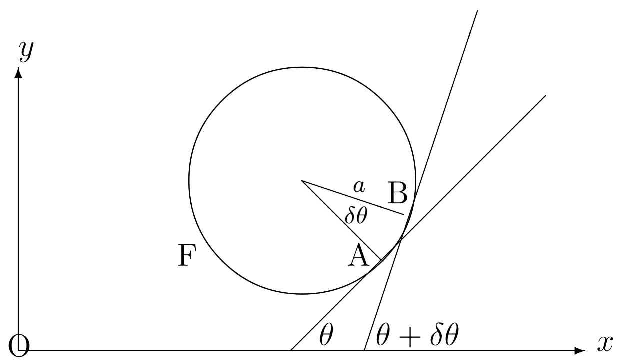
Then the curvature, κ, at P, is defined as the rate of increase of θ with respect to s.

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

EXAMPLE

Determine the curvature at any point of a circle with radius a.

Solution



Let A be a point on the circle at which the tangent is inclined to the positive x-axis at an angle, θ.

Let B be a point (close to A) at which the tangent is inclined to the positive x-axis at an angle, $\theta + \delta\theta$.

Let the length of the arc, AB, be δs, where distances, s, are measured along the circle in a counter-clockwise sense from the fixed point, F.

$\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus,

$$\delta s = a\delta\theta,$$

or

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence δs, to approach zero, we conclude that

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{a}.$$

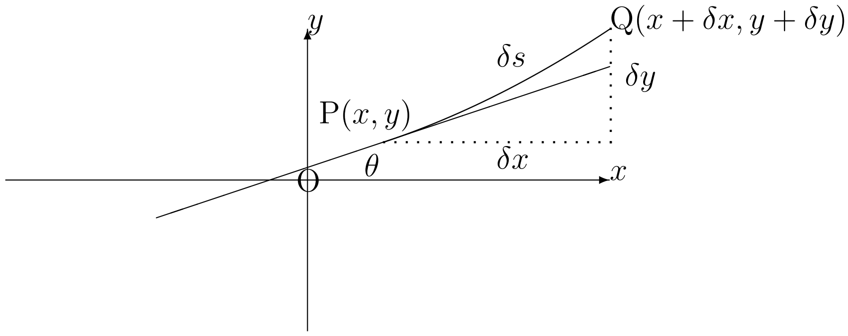
Note:
For the lower half of the circle, θ **increases** as s increases, while, in the upper half of the circle, θ **decreases** as s increases.

The curvature will, therefore, be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

Summary
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, let $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ be two neighbouring points on it (separated by a distance of δs along the curve).



In this diagram,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

Also,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}.$$

Notes:

(i) For a curve which is concave upwards at a particular point, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **increase** as x increases through the point.

Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **decrease** as x increases through the point.

Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be negative at the point.

(ii) We may allow the value of the curvature to take the same sign as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Hence,

$$\kappa = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2\right]^{\frac{3}{2}}}.$$

EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0, 0)$ with radius a.

Solution

The equation of the circle is

$$x^2 + y^2 = a^2.$$

For the upper half,

$$y = \sqrt{a^2 - x^2}.$$

For the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2 - x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}{\left(1+\frac{x^2}{a^2-x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

Considering the lower half,

$$\kappa = \frac{1}{a}.$$

"JUST THE MATHS"

SLIDES NUMBER

11.4

DIFFERENTIATION APPLICATIONS 4
(Circle, radius & centre of curvature)

by

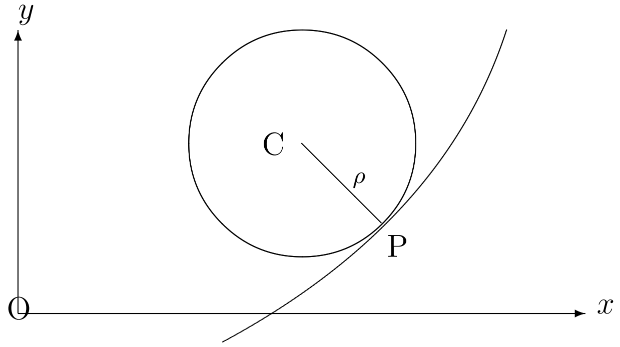
A.J.Hobson

UNIT 11.4 DIFFERENTIATION APPLICATIONS 4

CIRCLE, RADIUS AND CENTRE OF CURVATURE

11.4.1 INTRODUCTION

At a point, P, on a given curve, let a circle be drawn which **just touches** the curve and has the same value of the curvature (including its sign).



This circle is called the
"circle of curvature at P".

Its radius, ρ, is called the
"radius of curvature at P".

Its centre is called the
"centre of curvature at P".

11.4.2 RADIUS OF CURVATURE

If the curvature at P is κ, then $\rho = \frac{1}{\kappa}$.

Hence,

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\theta}.$$

In cartesian co-ordinates,

$$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.$$

Note:
For curves with negative curvature, the length of the radius of curvature is the **numerical** value obtained in the above formula.

Later, it will be necessary to use the appropriate sign for the radius of curvature.

EXAMPLE

Calculate the radius of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}.$$

Also,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^3}.$$

Hence, at the point $(0.5, -1)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1.$$

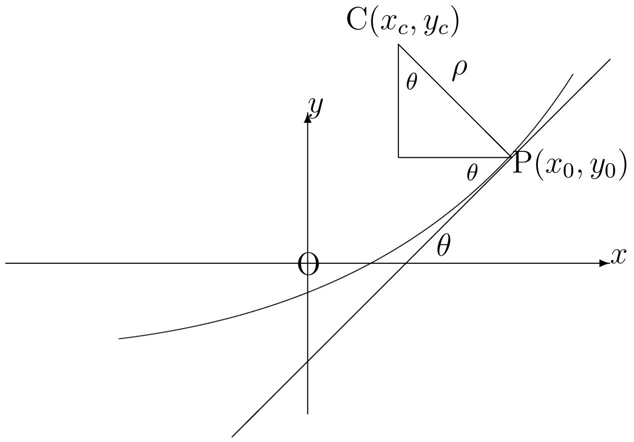
We conclude that

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{1} = 2\sqrt{2}.$$

11.4.3 CENTRE OF CURVATURE

Consider a point, (x_0, y_0), on an arc of a curve $y = f(x)$ for which the curvature is positive, the arc lying in the first quadrant.

It may be shown that the formulae obtained for the co-ordinates, (x_c, y_c), of the centre of curvature apply in any situation, provided that the curvature is associated with its appropriate sign.



From the diagram,

$$
\begin{aligned}
x_c &= x_0 - \rho \sin\theta, \\
y_c &= y_0 + \rho \cos\theta.
\end{aligned}
$$

Note:
It is a good idea to sketch the curve in order estimate, roughly, where the centre of curvature is going to be.

This is especially important where there is uncertainty about the precise value of the angle, θ.

EXAMPLE

Determine the centre of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

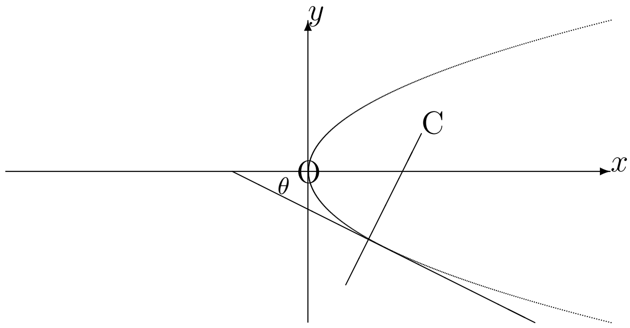
From the earlier example on calculating radius of curvature,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 \quad \text{and} \quad \rho = 2\sqrt{2}$$

at the point $(0.5, -1)$.



The diagram shows that the co-ordinates, (x_c, y_c), of the centre of curvature will be such that $x_c > 0.5$ and $y_c > -1$.

This will be so provided that the angle, θ, is a negative acute angle; (that is, its cosine will be positive and its sine will be negative).

In fact,

$$\theta = \tan^{-1}(-1) = -45^\circ.$$

Hence,

$$\begin{aligned} x_c &= 0.5 - 2\sqrt{2}\sin(-45^\circ), \\ y_c &= -1 + 2\sqrt{2}\cos(-45^\circ). \end{aligned}$$

That is, $x_c = 2.5$ and $y_c = 1$.

"JUST THE MATHS"

SLIDES NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

A.J.Hobson

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

The problem here is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Let $f(x)$ be a given function of x which is not a polynomial.

Assume that $f(x)$ may be expressed as an infinite "power series".

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots$$

To justify this assumption, we must determine the **"coefficients"**, a_0, a_1, a_2, a_3, a_4,.........

This is possible as an application of differentiation.

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$ so that

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

On substituting $x = 0$, $f'(0) = a_1$ so that

$$a_1 = f'(0).$$

(c) Differentiating a second time,

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \ldots$$

On substituting $x = 0$, $f''(0) = 2a_2$ so that

$$a_2 = \frac{1}{2}f''(0).$$

(d) Differentiating a third time,

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \ldots$$

On substituting $x = 0$, $f'''(0) = (3 \times 2)a_3$ so that

$$a_3 = \frac{1}{3!}f'''(0).$$

(e) Continuing this process leads to the general formula

$$a_n = \frac{1}{n!} f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$, of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

This is called the **"Maclaurin's series for** $f(x)$".

Notes:

(i) We assume that all of the derivatives of $f(x)$ exist at $x = 0$; otherwise the above result is invalid.

(ii) The Maclaurin's series for a particular function may not be used when the series diverges.

(iii) If x is small enough to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

The ranges of values of x for which the results are valid will be stated without proof.

1. The Exponential Series

(i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

(ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

(iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

(iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

(v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \dots$$

It may be shown that this series is valid for all values of x.

2. The Sine Series

(i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

(ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

(iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

(iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0$
$= -1$.

(v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

(vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

It may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

It may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$ since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1+x)$

(i) $f(x) \equiv \ln(1+x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2\times 3)$.

Thus,

$$\ln(1+x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2\times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

It may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

When n is a positive integer, the expansion of $(1+x)^n$ in ascending powers of x is a **finite** series obtainable, for example, by Pascal's Triangle.

In all other cases, the series is **infinite** as follows:

(i) $f(x) \equiv (1+x)^n$; hence, $f(0) = 1$

(ii) $f'(x) = n(1+x)^{n-1}$; hence, $f'(0) = n$.

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$;
hence, $f''(0) = n(n-1)$.

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$;
hence, $f'''(0) = n(n-1)(n-2)$.

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$;
hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$.

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 +$$

$$\frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x\to 0} \frac{x+\sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\lim_{x\to 0} \frac{x+x-\frac{x^3}{3!}+\frac{x^5}{5!}- \ \dots}{x^2+x}$$

$$= \lim_{x\to 0} \frac{2x-\frac{x^3}{6}+\frac{x^5}{120}- \ \dots}{x^2+x}$$

$$= \lim_{x\to 0} \frac{2-\frac{x^2}{6}+\frac{x^4}{120}- \ \dots}{x+1} = 2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

Solution

We consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x=0.01$

$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\dots$$

That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ . \ . \ .$$

Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \ . \ . \ .$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ . \ . \ .$$

The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ . \ .\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + . \ .\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ . \ .$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ . \ . \ .$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as **"Taylor's series"**.

One form of Taylor's series is as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ \ldots$$

Proof:

To obtain this result from Maclaurin's series, we let $f(x+h) \equiv F(x)$.

Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ ..$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h.

That is,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ \ldots$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + ...$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$,

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ \ldots\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + ...)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4} + 0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

"JUST THE MATHS"

SLIDES NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

A.J.Hobson

UNIT 11.6
DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

If

$$y = f(x),$$

suppose that x is subject to a small **"increment"**, δx.

"Increment" means that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy, in y is given by

$$\delta y = f(x + \delta x) - f(x).$$

This can often be difficult to evaluate.

However, since δx is small,

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

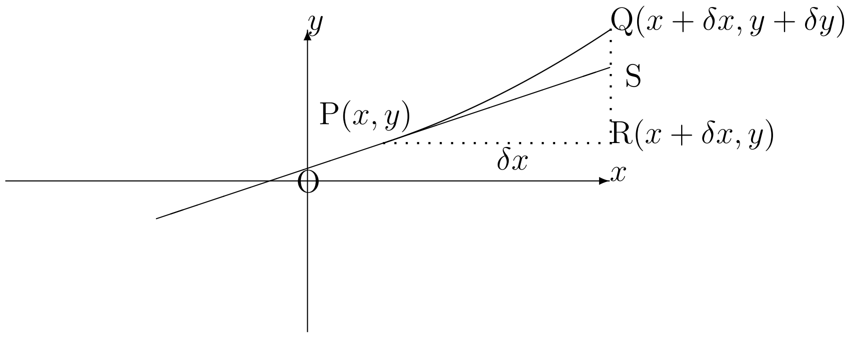
That is,

$$\frac{\delta y}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Thus,

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

Diagramatic approach



PR $= \delta x$, QR $= \delta y$ and the gradient of PS is the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at P.

Taking SR as an approximation to QR,

$$\frac{\mathrm{SR}}{\mathrm{PR}} = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\mathrm{P}}.$$

$$\frac{\text{SR}}{\delta x} = \left[\frac{\text{d}y}{\text{d}x}\right]_{\text{P}}.$$

Hence,

$$\delta y \simeq \left[\frac{\text{d}y}{\text{d}x}\right]_{\text{P}} \delta x.$$

Notes:

(i) $\frac{\text{d}y}{\text{d}x}\delta x$ is the **"total differential of y"** (or simply the "differential of y").

(ii) It is important **not** to use "differential" when referring to a "derivative".

The correct alternative to "derivative" is "differential coefficient".

(iii) A more rigorous calculation of δy comes from the result known as "Taylor's Theorem":

$$f(x+\delta x) = f(x)+f'(x)\delta x+\frac{f''(x)}{2!}(\delta x)^2+\frac{f'''(x)}{3!}(\delta x)^3+\ldots.$$

Hence, if δx is small enough for powers of two and above to be neglected, then

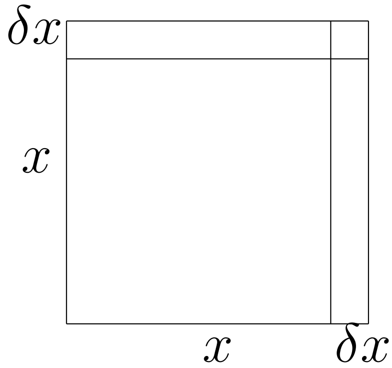
$$f(x+\delta x) - f(x) \simeq f'(x)\delta x.$$

EXAMPLES

1. If a square has side $x$cms., obtain both the exact and the approximate values of the increment in the area Acms^2. when x is increased by δx.

Solution

(a) Exact Method



The area is given by the formula

$$A = x^2.$$

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

to give

$$\delta A \simeq 2x\delta x.$$

From the diagram, that the two results differ only by the area of the small square, with side δx.

2. If

$$y = xe^{-x},$$

calculate approximately the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.3$.

Hence,

$$\delta y \simeq e^{-5}.(1 - 5).(0.3) \simeq -0.00809$$

Thus, y **decreases** by approximately 0.00809

Note:

The exact value is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x, the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1 - x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1 - x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1 - x) \times 0.02x}{xe^{-x}} \times 100 = 2(1 - x).$$

That is, y increases by $2(1-x)$%, which will be positive when $x < 1$ and negative when $x > 1$.

11.6.2 SMALL ERRORS

If

$$y = f(x),$$

suppose that x is known to be subject to an error in measurement.

In particular, suppose x is known to be **too large** by a small amount δx.

The correct value of x could be obtained if we **decreased** it by δx.

That is, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{dy}{dx}\delta x$.

That is, y will **decrease** by approximately $\frac{dy}{dx}\delta x$.

Summary

If x is too large by an amount δx, then y is too large by approximately $\frac{dy}{dx}\delta x$.

Note:

If $\frac{dy}{dx}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If

$$y = x^2 \sin x,$$

calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$

and, hence,

$$\delta y \simeq (x^2 \cos x + 2x \sin x)\delta x,$$

where $x = 3$ and $\delta x = 0.06$.

Thus,

$$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$

That is, y is too small by approximately 0.4838.

2. If

$$y = \frac{x}{1+x},$$

determine approximately, in terms of x, the percentage error in y when x is subject to an error of 5%.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$

so that

$$\delta y \simeq \frac{1}{(1+x)^2}\delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}$% which will be positive when $x > -1$ and negative when $x < -1$.

"JUST THE MATHS"

SLIDES NUMBER

12.1

INTEGRATION 1
(Elementary indefinite integrals)

by

A.J.Hobson

UNIT 12.1 - INTEGRATION 1
ELEMENTARY INDEFINITE INTEGRALS

12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of x and asked to obtain their derivatives.

In Integral Calculus, we are given functions of x and asked what they are the derivatives of.

The process of answering this question is called **"integration"**.

Integration is the reverse of differentiation.

DEFINITION
Given a function $f(x)$, another function ,z, such that

$$\frac{dz}{dx} = f(x)$$

is called an integral of $f(x)$ with respect to x.

Notes:
(i) having found z, such that

$$\frac{dz}{dx} = f(x),$$

$z + C$ is also an integral for any constant value, C.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to x"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) C is an **arbitrary constant** called the **"constant of integration"**.

(iv) The symbol dx is a label, indicating the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the **"integrand"**.

Result:
Two functions z_1 and z_2 are both integrals of the same function $f(x)$ if and only if they differ by a constant.

Proof:
(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where C is a constant.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

(b) Secondly, suppose that z_1 and z_2 are integrals of the same function.

Then,

$$\frac{dz_1}{dx} = \frac{dz_2}{dx}.$$

That is,

$$\frac{dz_1}{dx} - \frac{dz_2}{dx} = 0,$$

or

$$\frac{d}{dx}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C$$

where C may be any constant.

Any result encountered in differentiation could be restated in reverse as a result on integration.

ILLUSTRATIONS

1.

$$\int 3x^2 \mathrm{d}x = x^3 + C.$$

2.

$$\int x^2 \mathrm{d}x = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x} \mathrm{d}x \quad \text{i.e.} \quad \int x^{-1} \mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

Note:
Basic integrals of the above kinds may be quoted from a table of standard integrals in a suitable formula booklet.

More advanced integrals are obtainable using the rules which follow.

12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants.

Then

$$\int [Af(x) + Bg(x)]\mathrm{d}x = A\int f(x)\mathrm{d}x + B\int g(x)\mathrm{d}x.$$

The proof follows because differentiation is already linear.

The result is easily extended to linear combinations of three or more functions.

ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7)\mathrm{d}x = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.

$$\int (3\cos x + 4\sec^2 x)\mathrm{d}x = 3\sin x + 4\tan x + C.$$

(b) Functions of a Linear Function

$$\int f(ax+b)\ \mathrm{d}x.$$

(i) Inspection Method

EXAMPLES

1. Determine the indefinite integral

$$\int (2x+3)^{12}\mathrm{d}x.$$

Solution

To arrive at $(2x+3)^{12}$ by differentiation, we must begin with a function related to $(2x+3)^{13}$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^{13}\right] = 13(2x+3)^{12}.2 = 26(2x+3)^{12}.$$

This is 26 times the function we are trying to integrate.

Hence,

$$\int (2x+3)^{12} = \frac{(2x+3)^{13}}{26} + C.$$

2. Determine the indefinite integral

$$\int \cos(3-5x)\mathrm{d}x.$$

Solution

To arrive at $\cos(3-5x)$ by differentiation, we must begin with a function related to $\sin(3-5x)$.

In fact

$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3-5x)] = \cos(3-5x).-5 = -5\cos(3-5x).$

This is -5 times the function we are trying to integrate.

Hence,

$$\int \cos(3-5x) = -\frac{\sin(3-5x)}{5} + C.$$

3. Determine the indefinite integral

$$\int e^{4x+1}\mathrm{d}x.$$

Solution

To arrive at e^{4x+1} by differentiation, we must begin with a function related to e^{4x+1}.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{4x+1}\right] = e^{4x+1}.4$$

This is 4 times the function we are trying to integrate.

Hence,

$$\int e^{4x+1}\mathrm{d}x = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral

$$\int \frac{1}{7x+3}\mathrm{d}x.$$

Solution

To arrive at $\frac{1}{7x+3}$ by differentiation, we must begin with a function related to $\ln(7x+3)$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(7x+3)] = \frac{1}{7x+3}.7 = \frac{7}{7x+3}$$

This is 7 times the function we are trying to integrate.

Hence,

$$\int \frac{1}{7x+3}\mathrm{d}x = \frac{\ln(7x+3)}{7} + C.$$

Note:

We treat the linear function $ax+b$ like a single x, then divide the result by a.

(ii) Substitution Method

In the integral

$$\int f(ax+b)\mathrm{d}x,$$

we may substitute $u = ax + b$ as follows:

Suppose

$$z = \int f(ax+b)\mathrm{d}x.$$

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(ax+b).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(u).$$

But

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{\mathrm{d}z}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}u} = f(u).\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Hence,

$$z = \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

Note:
The secret is to replace $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}u}.\mathrm{d}u$.

EXAMPLES

1. Determine the indefinite integral

$$z = \int (2x+3)^{12}\mathrm{d}x.$$

Solution

Putting $u = 2x+3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.

Thus,

$$z = \int u^{12}.\frac{1}{2}\mathrm{d}u = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x+3)^{13}}{26} + C$$

as before.

2. Determine the indefinite integral

$$z = \int \cos(3-5x)\mathrm{d}x.$$

Solution

Putting $u = 3-5x$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -5$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{5}$.

Thus,

$$z = \int \cos u. -\frac{1}{5}\mathrm{d}u = -\frac{1}{5}\sin u + C.$$

That is,

$$z = -\frac{1}{5}\sin(3-5x) + C,$$

as before.

3. Determine the indefinite integral

$$z = \int e^{4x+1} \mathrm{d}x.$$

Solution

Putting $u = 4x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 4$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{4}$.

Thus,

$$z = \int e^u . \frac{1}{4} \mathrm{d}u = \frac{e^u}{4} + C.$$

That is,

$$z = \frac{e^{4x+1}}{4} + C,$$

as before

4. Determine the indefinite integral

$$z = \int \frac{1}{7x+3} \mathrm{d}x.$$

Solution

Putting $u = 7x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 7$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{7}$

Thus,

$$z = \int \frac{1}{u} . \frac{1}{7} \mathrm{d}u = \frac{1}{7} \ln u + C.$$

That is,

$$z = \frac{1}{7} \ln(7x+3) + C.$$

as before.

"JUST THE MATHS"

SLIDES NUMBER

12.2

INTEGRATION 2
(Introduction to definite integrals)

by

A.J.Hobson

UNIT 12.2 - INTEGRATION 2
INTRODUCTION TO DEFINITE INTEGRALS

12.2.1 DEFINITION AND EXAMPLES

In Unit 12.1, all the integrals were **"indefinite integrals"**.

Each result contained an arbitrary constant which cannot be assigned a value without further information.

In practical applications , we encounter **"definite integrals"**, which are represented by a numerical value.

DEFINITION

Suppose that

$$\int f(x)\mathrm{d}x = g(x) + C.$$

Then the symbol

$$\int_a^b f(x)\mathrm{d}x$$

is used to mean

(Value of $g(x) + C$ at $x = b$)
minus
(Value of $g(x) + C$ at $x = a$).

C will cancel out; hence,

$$\int_a^b f(x)\mathrm{d}x = g(b) - g(a).$$

The right hand side can also be written

$$[g(x)]_a^b.$$

a is the **"lower limit"** of the definite integral.

b is the **"upper limit"** of the definite integral.

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

2. Evaluate the definite integral

$$\int_1^3 (2x+1)^2\mathrm{d}x.$$

Solution

$$\int_1^3 (2x+1)^2\mathrm{d}x = \left[\frac{(2x+1)^3}{6}\right]_1^3 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

Notes:

(i) Alternatively,

$$\int_1^3 (2x+1)^2 \, \mathrm{d}x = \int_1^3 \left(4x^2+4x+1\right) \, \mathrm{d}x$$

$$= \left[4\frac{x^3}{3} + 2x^2 + x\right]_1^3 .$$

The expression in the brackets differs only from the previous result by the constant value $\frac{1}{6}$.

Hence the numerical result for the definite integral will be the same.

(ii) Another alternative method is to substitute $u = 2x+1$; but the limits of integration should be changed to the appropriate values for u.

Replace $\mathrm{d}x$ by $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ (that is, $\frac{1}{2}\mathrm{d}u$).

Replace $x = 1$ and $x = 3$ by $u = 2 \times 1 + 1 = 3$ and $u = 2 \times 3 + 1 = 7$, respectively.

We obtain

$$\int_3^7 u^2 \frac{1}{2} \mathrm{d}u = \left[\frac{u^3}{6}\right]_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

"JUST THE MATHS"

SLIDES NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

A.J.Hobson

UNIT 12.3 - INTEGRATION 3
THE METHOD OF
COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2 + qx + r} \mathrm{d}x$$

and

$$\int \frac{1}{\sqrt{px^2 + qx + r}} \mathrm{d}x.$$

Note:

These may also be written

$$\int \frac{\mathrm{d}x}{px^2 + qx + r}$$

and

$$\int \frac{\mathrm{d}x}{\sqrt{px^2 + qx + r}}.$$

Standard Results To Use

1.
$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

2.
$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + C.$$

3.
$$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a}+C \text{ or } \ln(x+\sqrt{x^2+a^2})+C.$$

4.
$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a}+C \text{ or } \ln(x+\sqrt{x^2-a^2})+C.$$

5.
$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square,

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable x,

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square,

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2 + 16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$. Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2 + 16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7 = \frac{\pi}{16}.$$

Note:

In cases where $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, we may treat the linear expression within the completed square as if it were a single x, then write the result straight down !

"JUST THE MATHS"

SLIDES NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

A.J.Hobson

UNIT 12.4 - INTEGRATION 4 INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x$$

we may wish to substitute for x in terms of a new variable, u.

From Unit 12.1,

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

This result was originally used for Functions of a Linear Function.

For this Unit, substitutions other than linear ones will be illustrated.

EXAMPLES

1. Use the substitution $x = a\sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

We shall assume that u is the **acute** angle for which $x = a\sin u$.

In effect, we substitute $u = \sin^{-1}\frac{x}{a}$ using the **principal** value of the inverse function.

If $x = a\sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$ so that the integral becomes

$$\int \frac{a\cos u}{\sqrt{a^2 - a^2\sin^2 u}}\mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2\sin^2 u} \equiv a\cos u,$$

both sides being positive when u is an acute angle.

Thus, we have

$$\int 1\mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Writing

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}}. - \frac{1}{u^2}\mathrm{d}u.$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Thus,

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

Two useful results:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are obtained from the substitution

$$u = f(x).$$

In both cases,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x).$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{f'(x)}.$$

This converts the integrals, respectively, into

(a)

$$\int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + C,$$

and

(b)

$$\int \frac{1}{u} \mathrm{d}u = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \sin^3 x . \cos x \ \mathrm{d}x.$$

Solution

In this example, we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.

Thus, by result (a),

$$\int_0^{\frac{\pi}{3}} \sin^3 x . \cos x \mathrm{d}x = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$

using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

2. Integrate the function

$$\frac{2x+1}{x^2+x-11}$$

with respect to x.

Solution

Here, we can identify x^2+x-11 with $f(x)$ and $2x+1$ with $f'(x)$.

Thus, by result (b),

$$\int \frac{2x+1}{x^2+x-11}\mathrm{d}x = \ln(x^2+x-11)+C.$$

"JUST THE MATHS"

SLIDES NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

A.J.Hobson

12.5.1 The standard formula

UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The method described here is for integrating the product of two functions.

It is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{d}{dx}[x\sin x] = x\cos x + \sin x$	$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$
$x\cos x = \frac{d}{dx}[x\sin x] - \sin x$	$u\frac{dv}{dx} = \frac{d}{dx}[uv] - v\frac{du}{dx}$
$\int\ x\cos x\ dx = x\sin x - \int\ \sin x\ dx$	$\int\ u\frac{dv}{dx}\ dx = uv - \int\ v\frac{du}{dx}\ dx$
$= x\sin x + \cos x + C$	

By labelling the product of two given functions as

$$u\frac{dv}{dx},$$

we may express the given integral in terms of another integral (hopefully simpler than the original).

The formula for **"integration by parts"** is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

EXAMPLES

1. Determine the indefinite integral

$$I = \int x^2e^{3x}\,\mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product x^2e^{3x} is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$.

In this case, we shall take

$$u = x^2 \;\text{ and }\; \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x\,\mathrm{d}x.$$

That is,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,\mathrm{d}x.$$

Using integration by parts a second time, we shall set

$$u = x \;\text{ and }\; \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.1 \text{ d}x\right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27}\left[9x^2 - 6x + 2\right] + C.$$

2. Determine the indefinite integral

$$I = \int x \ln x \text{ d}x.$$

Solution

In this case, we choose

$$u = \ln x \quad \text{and} \quad \frac{\text{d}v}{\text{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x} \text{ d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2} \text{ d}x.$$

Hence,

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine the indefinite integral

$$I = \int \ln x \, \mathrm{d}x.$$

Solution

Let

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x \ln x - \int x.\frac{1}{x} \, \mathrm{d}x,$$

giving

$$I = x \ln x - x + C.$$

4. Evaluate the definite integral

$$I = \int_0^1 \sin^{-1} x \, \mathrm{d}x.$$

Solution

Let

$$u = \sin^{-1} x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine the indefinite integral

$$I = \int e^{2x}\cos x\,\mathrm{d}x.$$

Solution

We shall set

$$u = e^{2x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\,\mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\,\mathrm{d}x.$$

Now we integrate by parts again, setting

$$u = e^{2x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\,\mathrm{d}x\right].$$

The original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Priority Order for choosing u

1. LOGARITHMS or INVERSE FUNCTIONS;

2. POWERS OF x;

3. POWERS OF e.

"JUST THE MATHS"

SLIDES NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

A.J.Hobson

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

The following results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.

$$\int \frac{1}{(ax+b)^n}\,\mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \text{ provided } n \neq 1.$$

3.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

Alternatively, if hyperbolic functions have been studied,

$$\int \frac{1}{a^2 - x^2} \, \mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C.$$

5.

$$\int \frac{2ax + b}{ax^2 + bx + c} \, \mathrm{d}x = \ln(ax^2 + bx + c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x + 8}{(2x + 3)(x - 1)} \, \mathrm{d}x = \int \left[\frac{1}{2x + 3} + \frac{3}{x - 1}\right] \, \mathrm{d}x$$

$$= \frac{1}{2}\ln(2x + 3) + 3\ln(x - 1) + C.$$

2.

$$\int_6^8 \frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)} \, \mathrm{d}x = \int_6^8 \left[\frac{2}{x - 5} + \frac{x + 1}{x^2 + 2x + 7}\right] \, \mathrm{d}x$$

$$= \left[2\ln(x - 5) + \frac{1}{2}\ln(x^2 + 2x + 7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right] \mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)} \,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right] \mathrm{d}x.$$

The second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5;

but we simply rearrange as

$$\frac{(2x+4)-3}{x^2+4x+5} \equiv \frac{2x+4}{x^2+4x+5} - \frac{3}{(x+2)^2+1}.$$

By Unit 12.3,

Answer $= 2\ln(x-4)+\ln(x^2+4x+5)-3\tan^{-1}(x+2)+C$.

"JUST THE MATHS"

SLIDES NUMBER

12.7

INTEGRATION 7
(Further trigonometric functions)

by

A.J.Hobson

UNIT 12.7 - INTEGRATION 7
FURTHER TRIGONOMETRIC FUNCTIONS

12.7.1 PRODUCTS OF SINES AND COSINES

To integrate products of sines cosines, we may use one of the following trigonometric identities:

$$\text{sinAcosB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})+\sin(\text{A}-\text{B})\right];$$

$$\text{cosAsinB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})-\sin(\text{A}-\text{B})\right];$$

$$\text{cosAcosB} \equiv \frac{1}{2}\left[\cos(\text{A}+\text{B})+\cos(\text{A}-\text{B})\right];$$

$$\text{sinAsinB} \equiv \frac{1}{2}\left[\cos(\text{A}-\text{B})-\cos(\text{A}+\text{B})\right].$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \, \text{d}x.$$

Solution

$$\int \sin 2x \cos 5x \, \text{d}x = \frac{1}{2}\int [\sin 7x - \sin 3x] \, \text{d}x$$

$$= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C.$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \; \mathrm{d}x.$$

Solution

$$\int \sin 3x \sin x \; \mathrm{d}x = \frac{1}{2}\int [\cos 2x - \cos 4x] \; \mathrm{d}x$$

$$= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C.$$

12.7.2 POWERS OF SINES AND COSINES

We consider the two integrals

$$\int \sin^n x \; \mathrm{d}x \quad \text{and} \quad \int \cos^n \; \mathrm{d}x,$$

where n is a positive integer.

(a) The Complex Number Method

This method requires us to express $\cos^n x$ and $\sin^n x$ as a sum of whole multiples of sines or cosines of whole multiples of x.

EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \; \mathrm{d}x.$$

Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where $z \equiv \cos x + j\sin x$.

That is,

$$16\sin^4 x \equiv z^4 - 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 - 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2\cos 4x - 8\cos 2x + 6.$$

Hence,

$$\cos^4 x \equiv \frac{1}{8}\left(\cos 4x - 4\cos 2x + 3\right).$$

Therefore,

$$\int \sin^4 x \,\mathrm{d}x = \frac{1}{8}\left[\frac{\sin 4x}{4} - 4\frac{\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{32}[\sin 4x - 8\sin 2x + 12x] + C.$$

(b) Odd Powers of Sines and Cosines

The method will be illustrated with examples:

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \,\mathrm{d}x.$$

Solution

$$\int \sin^3 x \,\mathrm{d}x = \int \sin^2 x . \sin x \,\mathrm{d}x.$$

That is,

$$\int \sin^3 x \mathrm{d}x = \int \left(1 - \cos^2 x\right)\sin x \,\mathrm{d}x$$

$$= \int \left(\sin x - \cos^2 x . \sin x\right) \, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

2. Determine the indefinite integral

$$\int \cos^7 x \, dx.$$

Solution

$$\int \cos^7 x \, dx = \int \cos^6 x . \cos x \, dx.$$

That is,

$$\int \cos^7 x \, dx = \int \left(1 - \sin^2 x\right)^3 . \cos x \, dx$$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) . \cos x \, dx$$

$$= \sin x - \sin^3 x + 3.\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(c) Even Powers of Sines and Cosines

This method is tedious if the even power is higher than 4.

In such cases, it is best to use the complex number method.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \, \mathrm{d}x.$$

Solution

$$\int \sin^2 x \, \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \, \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \, \mathrm{d}x$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

3. Determine the indefinite integral

$$\int \cos^4 x \, \mathrm{d}x.$$

Solution

$$\int \cos^4 x \, \mathrm{d}x = \int \left[\cos^2 x\right]^2 \mathrm{d}x = \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 \mathrm{d}x.$$

That is,

$$\int \cos^4 x \, \mathrm{d}x = \int \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \mathrm{d}x$$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}[1+\cos 4x]\right) \mathrm{d}x$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

"JUST THE MATHS"

SLIDES NUMBER

12.8

INTEGRATION 8
(The tangent substitutions)

by

A.J.Hobson

UNIT 12.8 - INTEGRATION 8

THE TANGENT SUBSTITUTIONS

12.8.1 THE SUBSTITUTION $t = \tan x$

This substitution is used for integrals of the form

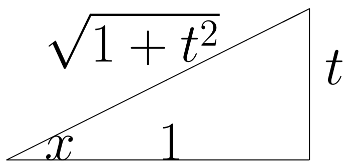
$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x}\,\mathrm{d}x,$$

where a, b and c are constants.

In most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if $t = \tan x$, then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{1}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4-3\sin^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4-3\sin^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{4-\frac{3t^2}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{4+t^2}\,\mathrm{d}t$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\,\mathrm{d}x$$

$$= \int \frac{1}{\frac{t^2}{1+t^2} + \frac{9}{1+t^2}}.\frac{1}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2+9}\,\mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{t}{3} + C$$

$$= \frac{1}{3}\tan^{-1}\left[\frac{\tan x}{3}\right] + C.$$

12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin x + c\cos x}\,\mathrm{d}x,$$

where a, b and c are constants.

In most exercises, one or more of these constants will be zero

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2\sin(x/2).\cos(x/2) \equiv 2\tan(x/2).\cos^2(x/2)$$

$$\equiv \frac{2\tan(x/2)}{\sec^2(x/2)} \equiv \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1+t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2)\left[1-\tan^2(x/2)\right]$$

$$\equiv \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1-t^2}{1+t^2}.$$

(iii)

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \frac{1}{2}\sec^2(x/2)$$

$$\equiv \frac{1}{2}\left[1+\tan^2(x/2)\right] \equiv \frac{1}{2}[1+t^2].$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{2}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}}.\frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2+2t}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t =$$

$$-\frac{2}{1+t} + C = -\frac{2}{1+\tan(x/2)} + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4\cos x - 3\sin x}\, \mathrm{d}x$$

$$= \int \frac{1}{4\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} . \frac{2}{1+t^2}\, \mathrm{d}t$$

$$= \int \frac{2}{4 - 4t^2 - 6t}\, \mathrm{d}t$$

$$= \int -\frac{1}{2t^2 + 3t - 2}\, \mathrm{d}t$$

$$\int -\frac{1}{(2t-1)(t+2)}\, \mathrm{d}t$$

$$= \int \frac{1}{5}\left[\frac{1}{t+2} - \frac{2}{2t-1}\right]\, \mathrm{d}t$$

$$= \frac{1}{5}[\ln(t+2) - \ln(2t-1)] + C$$

$$= \frac{1}{5}\ln\left[\frac{\tan(x/2)+2}{2\tan(x/2)-1}\right] + C.$$

"JUST THE MATHS"

SLIDES NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

A.J.Hobson

UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, the "integrand" consists of a product involving an unspecified integer, say n;
(for example, $x^n e^x$).

Using integration by parts, it is sometimes possible to express such an integral in terms of a similar integral with n replaced by $(n-1)$ or sometimes $(n-2)$.

The relationship between the two integrals is called a **"reduction formula"**.

By repeated application of a reduction formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

EXAMPLES

1. Obtain a reduction formula for the integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x . n x^{n-1} \, \mathrm{d}x.$$

That is,

$$I_n = x^n e^x - n I_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = x e^x - I_0.$$

But

$$I_0 = \int e^x \, \mathrm{d}x = e^x + \text{constant}.$$

Thus,

$$I_3 = x^3 e^x - 3\left[x^2 e^x - 2\left(x e^x - e^x\right)\right] + \text{constant}.$$

That is,

$$I_3 = e^x\left[x^3 - 3x^2 + 6x - 6\right] + C$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . n x^{n-1} \, \mathrm{d}x$$

$$= x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts again, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n \left\{ -x^{n-1} \cos x + \int \cos x . (n-1) x^{n-2} \, \mathrm{d}x \right\}.$$

That is,

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x \cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, \mathrm{d}x = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

EXAMPLES

1. Obtain a reduction formula for the integral

$$I_n = \int_0^1 x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

From the first example of section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = e - 1.$$

Thus,

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the integral

$$I_n = \int_0^\pi x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

From the second example of section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^\pi - n(n-1)I_{n-2}$$

$$= -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^\pi \cos x \, \mathrm{d}x = [\sin x]_0^\pi = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^\pi x \cos x \, \mathrm{d}x = [x \sin x + \cos x]_0^\pi = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$

"JUST THE MATHS"

SLIDES NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

by

A.J.Hobson

UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

There are two definite integrals which are worthy of special consideration.

They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

First, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \, \mathrm{d}x.$$

Writing the integrand as the product of two functions,

$$I_n = \int \sin^{n-1} x \sin x \, \mathrm{d}x.$$

Using integration by parts, with

$$u = \sin^{n-1}x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x,$$

$$I_n = \sin^{n-1}x(-\cos x) + \int (n-1)\sin^{n-2}x\cos^2 x \ \mathrm{d}x.$$

But, $\cos^2 x \equiv 1 - \sin^2 x$.

Hence,

$$I_n = -\sin^{n-1}x\cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[-\sin^{n-1}x\cos x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6 x \ \mathrm{d}x.$$

Solution

$$I_6 = \frac{1}{6}\left[-\sin^5 x\cos x + 5I_4\right].$$

But,

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x + 3I_2\right],$$

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + I_0\right]$$

and

$$I_0 = \int \, \mathrm{d}x = x + A.$$

Hence,

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + x + A\right];$$

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x - \frac{3}{2}\sin x \cos x + \frac{3}{2}x + B\right];$$

$I_6 =$

$$\frac{1}{6}\left[-\sin^5 x \cos x - \frac{5}{4}\sin^3 x \cos x - \frac{15}{8}\sin x \cos x + \frac{15}{8}x + C\right]$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5x}{16} + D.$$

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \, \mathrm{d}x.$$

Writing the integrand as the product of two functions,

$$I_n = \int \cos^{n-1} x \cos x \, \mathrm{d}x.$$

Using integration by parts, with

$$u = \cos^{n-1} x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x,$$

$$I_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, \mathrm{d}x.$$

But, $\sin^2 x \equiv 1 - \cos^2 x$.

Hence,

$$I_n = \cos^{n-1} x \sin x + (n-1)[I_{n-2} - I_n],$$

$$\text{or} \quad I_n = \frac{1}{n}\left[\cos^{n-1} x \sin x + (n-1) I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \, dx.$$

Solution

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + 4I_3\right].$$

But,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2I_1\right]$$

and

$$I_1 = \int \cos x \, dx = \sin x + A.$$

Hence,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2\sin x + B\right];$$

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + \frac{4}{3}\cos^2 x \sin x + \frac{8}{3}\sin x + C\right]$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + D.$$

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

Denoting either of these integrals by I_n, the reduction formula (in both cases) reduces to

$$I_n = \frac{n-1}{n} I_{n-2}.$$

There are two versions of **"Wallis's formulae"**

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} . \ \ldots \ \frac{6}{7} . \frac{4}{5} . \frac{2}{3} . I_1.$$

But,

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x \quad \text{or} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5)\ldots.6.4.2}{n(n-2)(n-4)\ldots.7.5.3}.$$

(b) n is an even number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5)\ldots.5.3.1}{n(n-2)(n-4)\ldots.6.4.2}\frac{\pi}{2}.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x = \frac{4.2}{5.3} = \frac{8}{15}.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x = \frac{3.1}{4.2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Wallis's formulae may be applied to integrals of the form

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, \mathrm{d}x,$$

where either m or n (or both) is an even number.

We use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$.

EXAMPLE

Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \cos^5 x \left(1 - \cos^2 x\right) \, \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^5 x - \cos^7 x\right) \, \mathrm{d}x.$$

This may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

A.J.Hobson

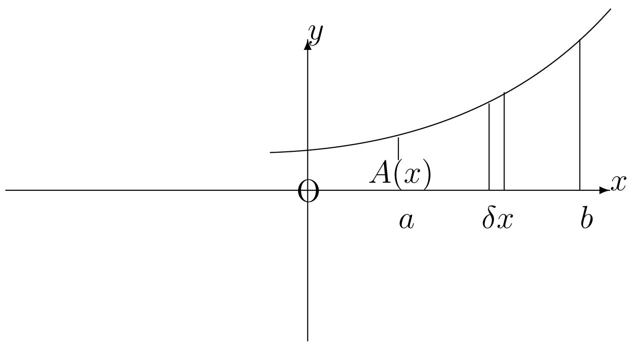
UNIT 13.1 - INTEGRATION APPLICATIONS 1

THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We consider, here, the area contained between the x-axis and the arc, from $x = a$ to $x = b$, of the curve $y = f(x)$.

Let $A(x)$ be the area between the curve, the x-axis, the y-axis and the ordinate at some arbitrary value of x.



A small increase of δx in x will lead to a corresponding increase of δA in A.

The increase in A approximates to a narrow rectangle having width δx and height $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

Letting δx tend to zero,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

Hence, on integrating both sides with respect to x,

$$A(x) = \int \ f(x) \ \mathrm{d}x.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$.

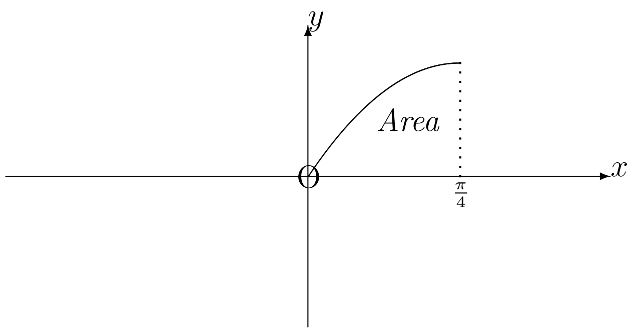
In fact, we do not need to know the constant of integration since

$$A(b) - A(a) = \int_a^b \ f(x) \ \mathrm{d}x.$$

ILLUSTRATIONS

1. The area contained between the x-axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{4}$ is given by

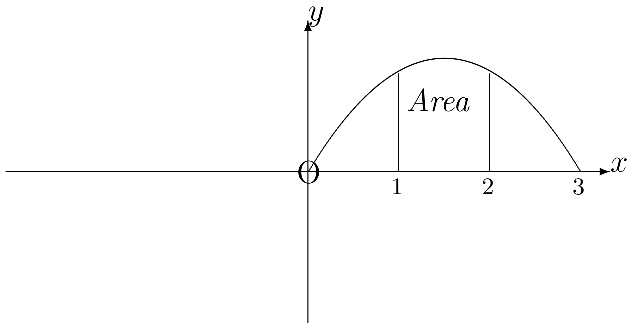
$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. The area contained between the x-axis and the curve $y = 3x - x^2$ from $x = 1$ to $x = 2$ is given by

$$\int_1^2 (3x - x^2) \, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2$$

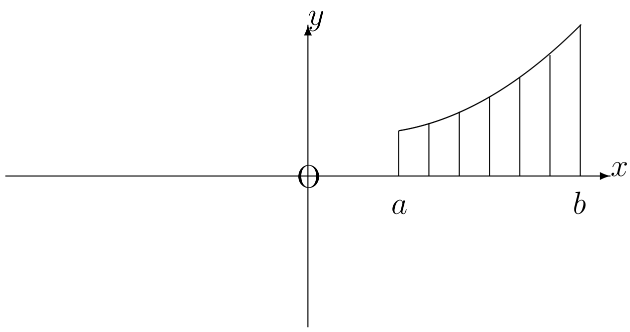
$$= \left(6 - \frac{8}{3}\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{13}{6}.$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider the same area as in the previous section.

We regard the area, now, as the sum (approximately) of a large number of narrow rectangles with typical width δx and typical height $f(x)$.



$$\text{Area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x.$$

We may conclude that

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) An area which lies wholly **below** the x-axis will be **negative**.

(ii) If c is any value of x between $x = a$ and $x = b$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

(iii) To calculate the TOTAL area between the x-axis and a curve which crosses the x-axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is a good idea to sketch the area under consideration.

(v) The formula has a wider field of application than the calculation of areas.

ILLUSTRATIONS

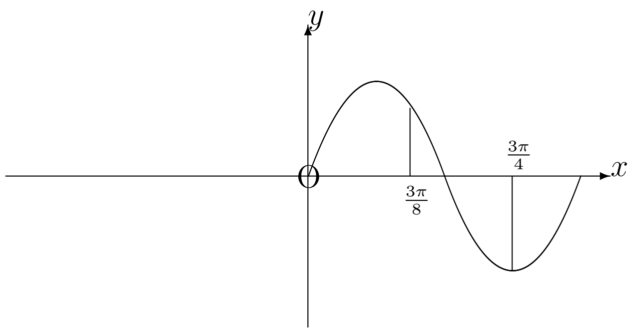
1. The total area between the x-axis and the curve $y = \sin 2x$ from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$ is given by

$$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x \; dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x \; dx.$$

That is,

$$\left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$$

$$\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) - \left(0 - \frac{1}{2}\right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. But, the definite integral

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \; dx = \left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.2

INTEGRATION APPLICATIONS 2
(Mean values)
&
(Root mean square values)

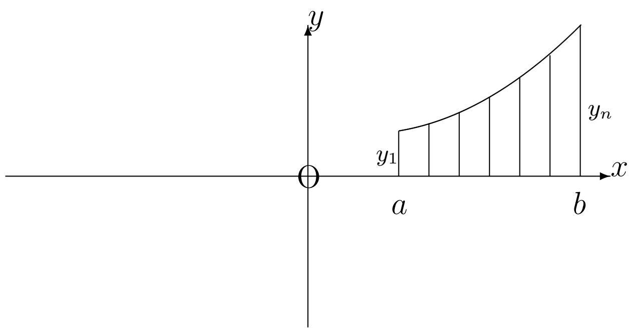
by

A.J.Hobson

UNIT 13.2 - INTEGRATION APPLICATIONS 2

MEAN & ROOT MEAN SQUARE VALUES

13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

let $y_1, y_2, y_3, ..., y_n$ be the y-coordinates at n different x-coordinates,

$$a = x_1, x_2, x_3, ..., x_n = b.$$

The average (that is, the arithmetic mean) of these n y-coordinates is

$$\frac{y_1 + y_2 + y_3 + ... + y_n}{n}.$$

The problem is to determine the average (arithmetic mean) of **all** the y-coordinates, from $x = a$ to $x = b$ on the curve whose equation is $y = f(x)$.

We take a very **large** number, n, of y-coordinates separated in the x-direction by very **small** distances.

If these distances are typically represented by δx then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + \ldots\ldots + y_n\delta x}{n\delta x}.$$

The denominator is equivalent to $(b-a+\delta x)$, since there are only $n-1$ spaces between the n y-coordinates.

Allowing the number of y-coordinates to increase indefinitely, δx will to tend to zero.

Hence, the **"Mean Value"** is given by

$$\text{M.V.} = \frac{1}{b-a}\lim_{\delta x \to 0}\sum_{x=a}^{x=b} y\delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b-a}\int_a^b f(x)\ \mathrm{d}x.$$

Note:
The Mean Value provides the height of a rectangle, with base $b - a$, having the same area as the net area between the curve and the x-axis.

EXAMPLE

Determine the mean value of the function,

$$f(x) \equiv x^2 - 5x,$$

from $x = 1$ to $x = 4$.

Solution

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4-1} \int_1^4 \ (x^2 - 5x) \ \mathrm{d}x$$

$$= \frac{1}{3}\left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_1^4$$

$$= \frac{1}{3}\left[\left(\frac{64}{3} - 40\right) - \left(\frac{1}{3} - \frac{5}{2}\right)\right] = -\frac{33}{2}.$$

13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$

The **"Root Mean Square Value"** provides a measure of "central tendency" for the **numerical** values of $f(x)$.

The Root Mean Square Value is defined to be the square root of the mean value of $f(x)$ from $x = a$ to $x = b$.

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a}\int_a^b \ [f(x)]^2 \ \mathrm{d}x}.$$

EXAMPLE

Determine the Root Mean Square Value of the function,

$$f(x) \equiv x^2 - 5,$$

from $x = 1$ to $x = 3$.

Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1}\int_1^3 (x^2-5)^2 \, \mathrm{d}x}.$$

First, we determine the **"Mean Square Value"**.

$$\text{M.S.V.} = \frac{1}{2}\int_1^3 (x^4 - 10x^2 + 25) \, \mathrm{d}x.$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right]_1^3$$

$$= \frac{1}{2}\left[\left(\frac{243}{5} - \frac{270}{3} + 75\right) - \left(\frac{1}{5} - \frac{10}{3} + 25\right)\right]$$

$$= \frac{1}{30}\left[(729 - 1350 + 1125) - (3 - 50 + 375)\right]$$

$$= \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

“JUST THE MATHS”

SLIDES NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

A.J.Hobson

UNIT 13.3
INTEGRATION APLICATIONS 3

VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

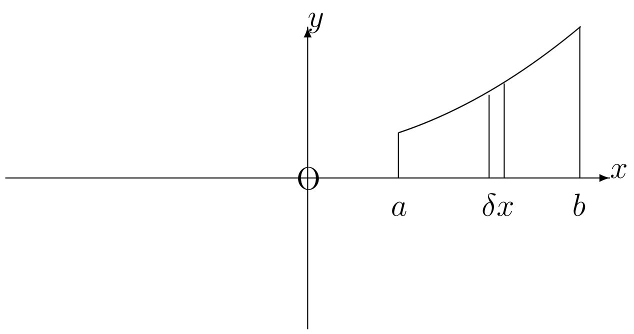
Suppose that the area between a curve

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$ lies wholly above the x-axis.

Let this area be rotated through 2π radians about the x-axis.

Then, a solid figure is obtained whose volume may be determined as an application of definite integration.



Rotating a narrow strip of width δx and height y through 2π radians about the x-axis gives a disc.

The volume, δV, of the disc is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

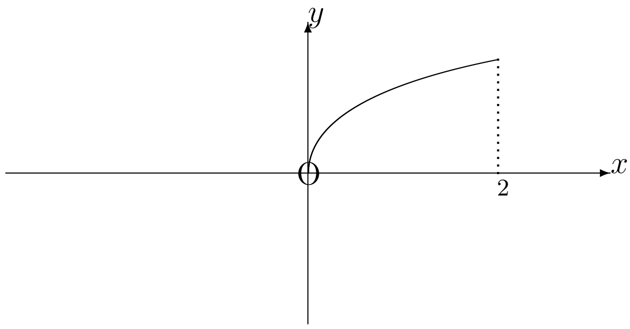
That is,

$$V = \int_a^b \pi y^2 \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the x-axis.

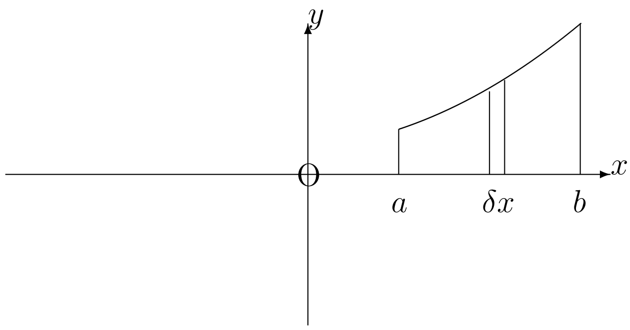
Solution



$$V = \int_0^2 \pi \times 8x \, \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

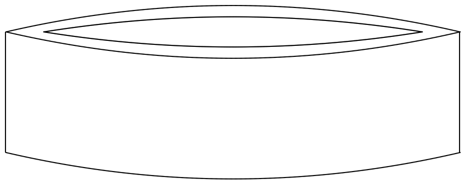
Consider the previous diagram:



Rotating the narrow strip of width δx through 2π radians about the y-axis gives a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is given by

$$\delta V \simeq 2\pi xy\delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\delta x.$$

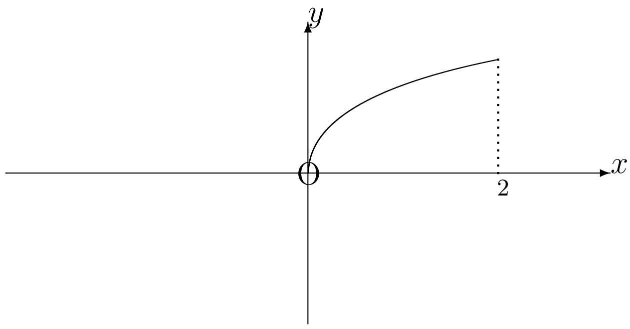
That is,

$$V = \int_a^b 2\pi xy \ \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



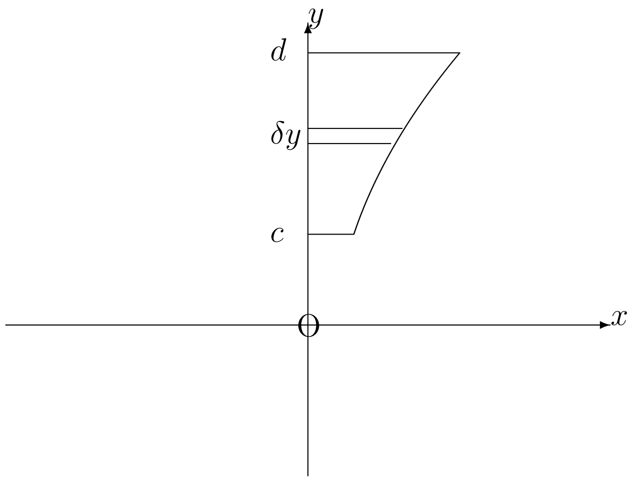
$$V = \int_0^2 2\pi x \times \sqrt{8x}\, \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2}\int_0^2 x^{\frac{3}{2}}\mathrm{d}x = \pi 4\sqrt{2}\left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



Here, we interchange the roles of x and y in the original formula for rotation about the x-axis.

$$V = \int_c^d \pi x^2 \, \mathrm{d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi yx \, \mathrm{d}y.$$

"JUST THE MATHS"

SLIDES NUMBER

13.4

INTEGRATION APPLICATIONS 4
(Lengths of curves)

by

A.J.Hobson

13.4.1 The standard formulae

UNIT 13.4 - INTEGRATION APPLICATIONS 4

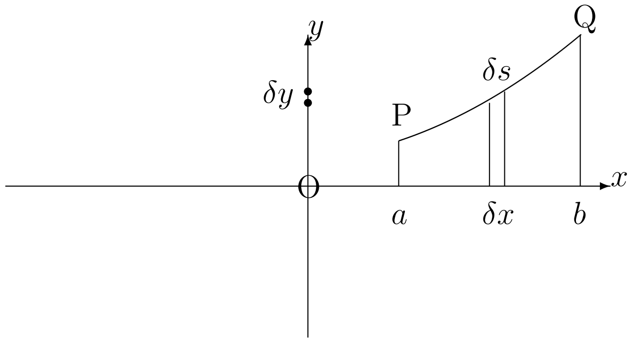
LENGTHS OF CURVES

13.4.1 THE STANDARD FORMULAE

The problem is to calculate the length of the arc of the curve with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$.



For two neighbouring points along the curve, the arc joining them may be considered, approximately, as a straight line segment.
Let these neighbouring points be separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively.

The length, δs, of arc between two neighbouring points is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

The total length, s, of arc is given by

$$s = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

That is,

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Notes:
(i) If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the above formula needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$s = \pm\int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(ii) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y, so that the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

EXAMPLES

1. A curve has equation

$$9y^2 = 16x^3.$$

Determine the length of the arc of the curve between the point $\left(1, \frac{4}{3}\right)$ and the point $\left(4, \frac{32}{3}\right)$.

Solution

The equation of the curve can be written

$$y = \frac{4x^{\frac{3}{2}}}{3};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{\frac{1}{2}}.$$

Hence,

$$s = \int_1^4 \sqrt{1+4x}\ \mathrm{d}x$$

$$= \left[\frac{(1+4x)^{\frac{3}{2}}}{6}\right]_1^4$$

$$= \frac{17^{\frac{3}{2}}}{6} - \frac{5^{\frac{3}{2}}}{6} \simeq 13.55$$

2. A curve is given parametrically by

$$x = t^2 - 1, \quad y = t^3 + 1.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 1$.

Solution

Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2,$$

we have

$$s = \int_0^1 \sqrt{4t^2 + 6t^4}\ \mathrm{d}t$$

$$= \int_0^1 t\sqrt{4 + 6t^2}\ \mathrm{d}t$$

$$= \left[\frac{1}{18}(4 + 6t^2)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{18}\left(10^{\frac{3}{2}} - 8\right) \simeq 1.31$$

"JUST THE MATHS"

SLIDES NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

A.J.Hobson

UNIT 13.5 - INTEGRATION APPLICATIONS 5

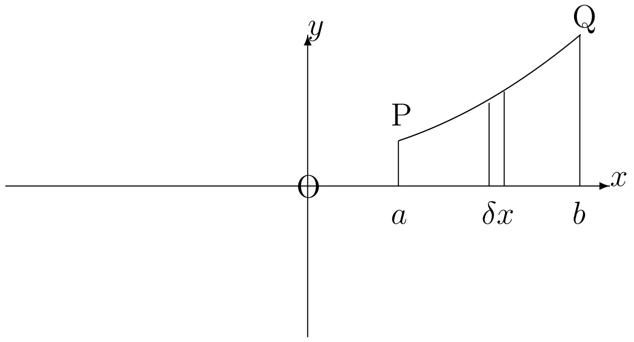
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$ respectively, is rotated through 2π radians about the x-axis or the y-axis.



For two neighbouring points along the curve, the arc joining them may be considered, approximately as a straight line segment.

If the two neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs is rotated through 2π radians about the x-axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x.$$

The total surface area, S, is thus given by

$$S = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x.$$

That is,

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the above result needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point (2, 2)) and the point (8, 4) is rotated through 2π radians about the x-axis.

Solution

The equation of the arc may be written

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1 + \frac{1}{2x}} \ \mathrm{d}x$$

$$= \int_2^8 \sqrt{2x+1}\,\mathrm{d}x$$

$$= \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x-axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively.

Also,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sqrt{2}\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta.$$

Thus,

$$S = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi\sin\theta\sqrt{2\sin^2\theta + 2\cos^2\theta}\,\mathrm{d}\theta$$

$$= - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi \sin\theta \ \mathrm{d}\theta.$$

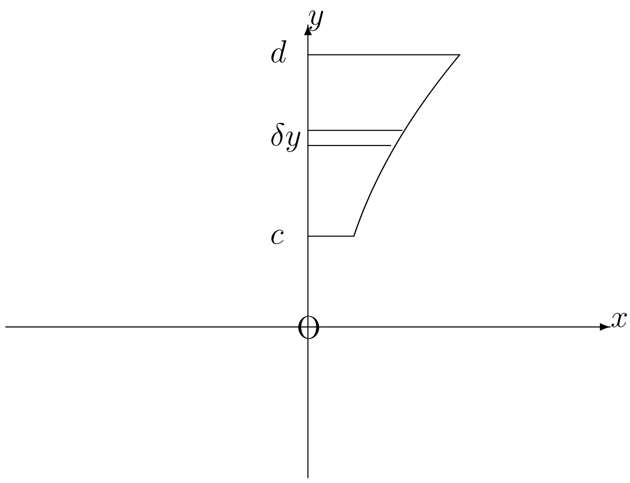
That is,

$$S = -\left[-4\pi\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y-axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \ \mathrm{d}y.$$



We simply reverse the previous roles of x and y.

If the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y-axis, determine the surface area obtained.

Solution

Using a result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}} \ \mathrm{d}y \simeq 19.64$$

"JUST THE MATHS"

SLIDES NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

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UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

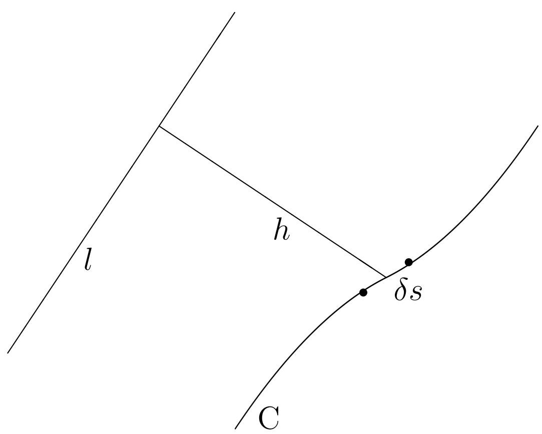
13.6.1 INTRODUCTION

Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates, and let δs be the length of a small element of this arc.

Then, the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\text{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

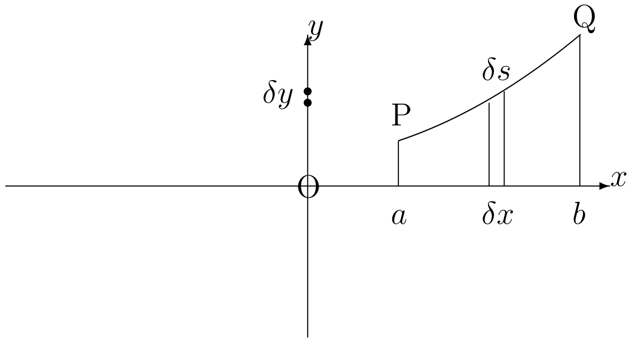


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Consider an arc of the curve, with equation

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is $x\delta s$.

Hence, the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x \delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

Thus, the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x$$

$$= \int_a^b x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the above formula needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b \, x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \, x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

Thus, the first moment of the arc about the y-axis is given by

$$\pm\int_{t_1}^{t_2} \, x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

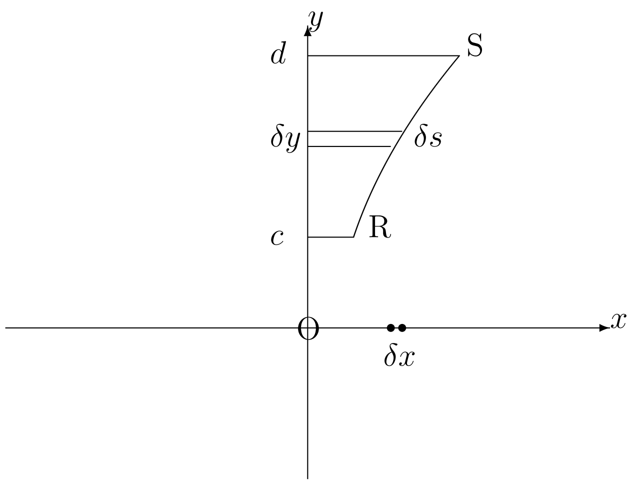
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment about the x-axis is given by

$$\int_c^d y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where
$t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

   $$x^2 + y^2 = a^2,$$

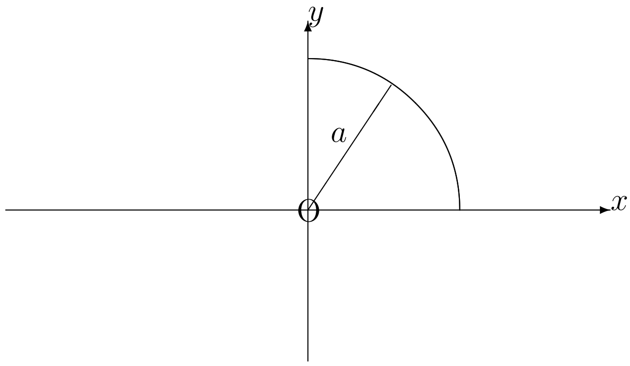
   lying in the first quadrant.

   Solution

   Using implicit differentiation

   $$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

$$\text{Hence,} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$



The first moment about the y-axis is given by

$$\int_0^a x\sqrt{1+\frac{x^2}{y^2}}\,\mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2+y^2}\,\mathrm{d}x.$$

But

$$x^2+y^2=a^2 \ \text{ and } \ y=\sqrt{a^2-x^2}.$$

Hence,

$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,\mathrm{d}x$$

$$= \left[-a\sqrt{(a^2-x^2)}\right]_0^a = a^2.$$

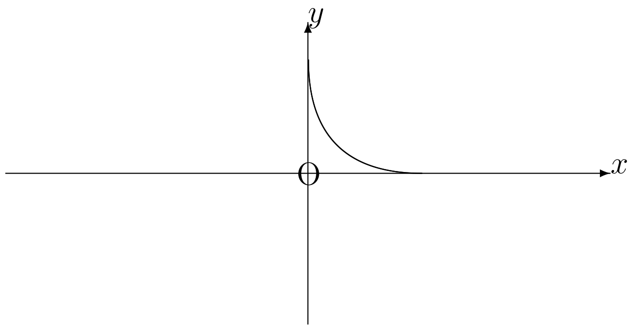
By symmetry, the first moment about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$



The first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

Using $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3a^2}{5}.$$

Note:

This second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc.

For an arc of the curve, with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b \ x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b \ \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \ \text{ and } \ \overline{y} = \frac{\int_a^b \ y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \ \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero.

In particular, let the y-axis be parallel to the given axis.

Let x be the perpendicular distance from an element, δs, to the y-axis.

The first moment about the given axis will be

$$\sum_{\mathrm{C}} (x - \overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}} \delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments.

In practice, the centroid corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

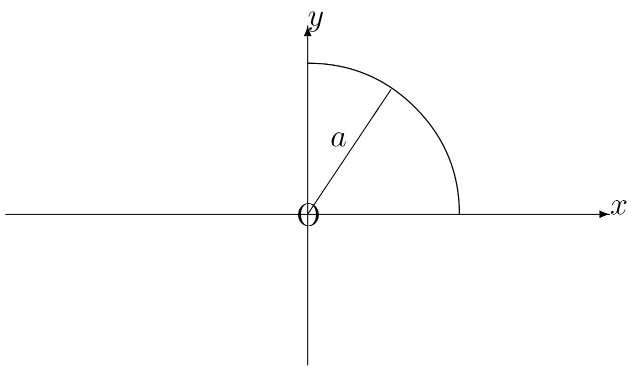
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant

Solution



From an earlier example in this unit, the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

Also, the length of the arc is $\frac{\pi a}{2}$.

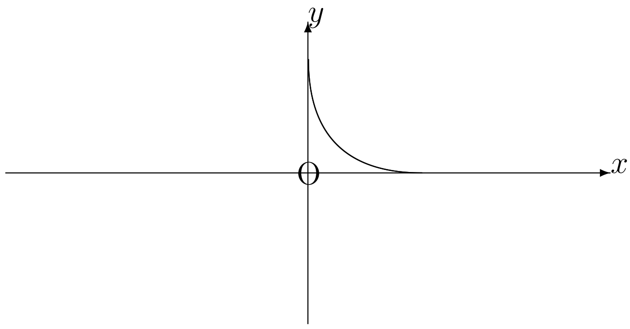
Hence,

$$\overline{x} = \frac{2a}{\pi} \text{ and } \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\text{cos}^3\theta, \quad y = a\text{sin}^3\theta.$$

Solution



From an earlier example in this unit,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\text{cos}^2\theta \sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\text{sin}^2\theta \cos\theta.$$

The first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9a^2\text{cos}^4\theta\text{sin}^2\theta + 9a^2\text{sin}^4\theta\text{cos}^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \ \text{ and } \ \overline{y} = \frac{2a}{5}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

A.J.Hobson

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

13.7.1 INTRODUCTION

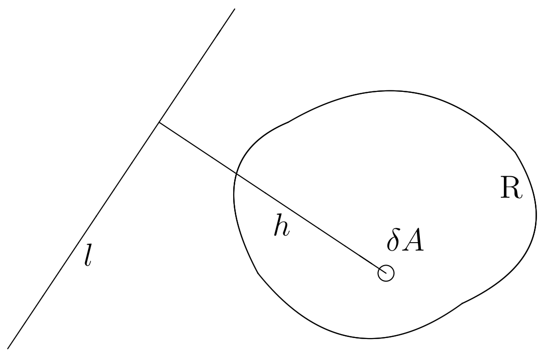
Let R denote a region (with area A) of the xy-plane of cartesian co-ordinates.

Let δA denote the area of a small element of this region.

Then, the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h\delta A,$$

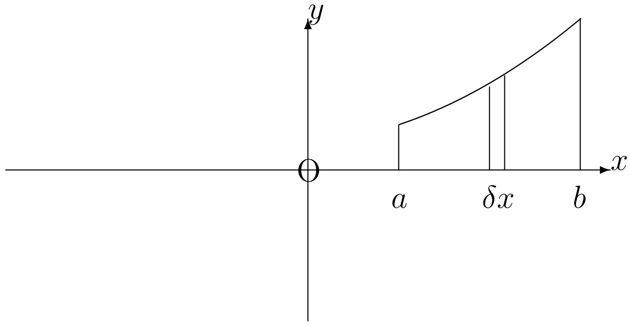
where h is the perpendicular distance, from l, of the element with area δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may divided up into small elements by using a network, of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

All of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x $x(y\delta x)$.

Thus, the total first moment of the region about the y-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, \mathrm{d}x.$$

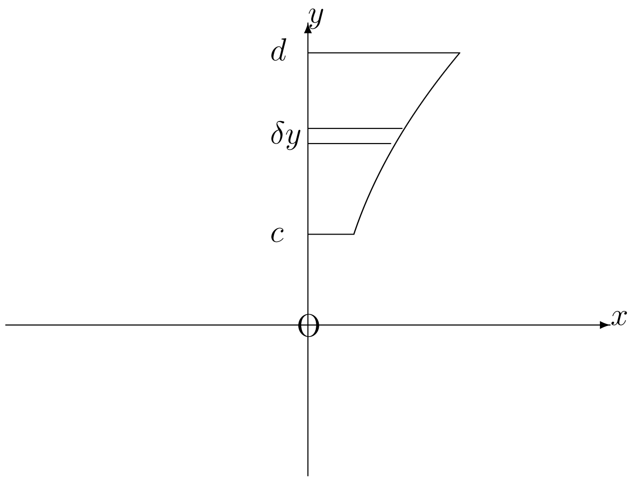
Note:

For a region of the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

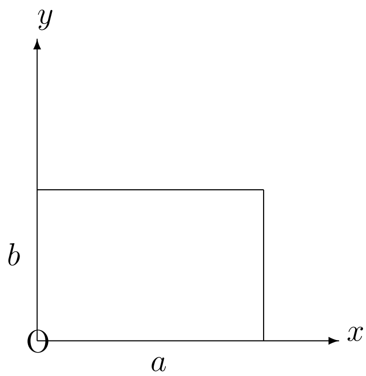
$$\int_c^d yx \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b, about the side of length b.

   Solution



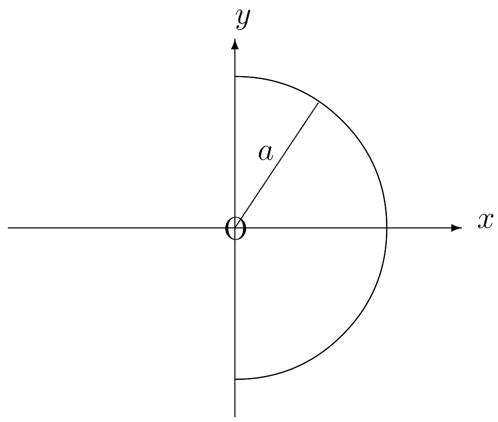
The first moment about the y-axis is given by

$$\int_0^a xb \, \mathrm{d}x = \left[\frac{x^2 b}{2}\right]_0^a = \frac{1}{2}a^2 b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

$$2\int_0^a x\sqrt{a^2 - x^2}\, \mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:

The symmetry of the above region shows that its first moment about the x-axis would be zero.

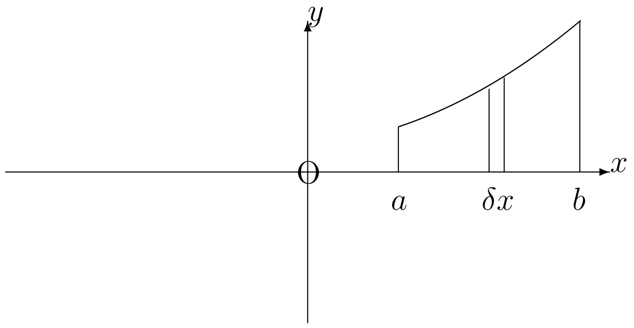
This is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In Example 1 of Section 13.7.2, a formula was established for the first moment of a rectangular region about one of its sides.

This result may be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip of width δx and height y is regarded as approximately a rectangle, its first moment about the x-axis is

$$\frac{1}{2}y^2\delta x.$$

Hence, the first moment of the whole region about the x-axis is given by

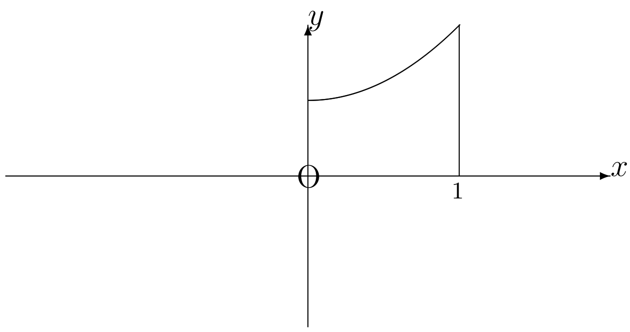
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2} y^2 \delta x = \int_a^b \frac{1}{2} y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



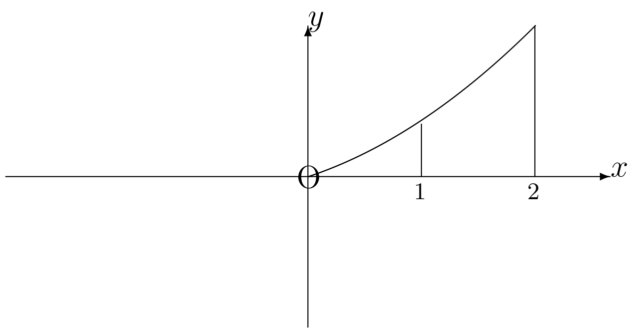
$$\text{First moment} = \int_0^1 \frac{1}{2}(x^2 + 1)^2 \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^1 (x^4 + 2x^2 + 1) \, \mathrm{d}x = \frac{1}{2} \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2e^{2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\,\mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region.

In the case of a region bounded in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b \ xy \ \mathrm{d}x}{\int_a^b \ y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b \ \frac{1}{2}y^2 \ \mathrm{d}x}{\int_a^b \ y \ \mathrm{d}x}.$$

Notes:

(i) The first moment of an area about an axis through its centroid will, by definition, be zero.

In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

$$\sum_{\text{R}} (x - \overline{x})\delta A = \sum_{\text{R}} x\delta A - \overline{x}\sum_{\text{R}} \delta A = A\overline{x} - A\overline{x} = 0.$$

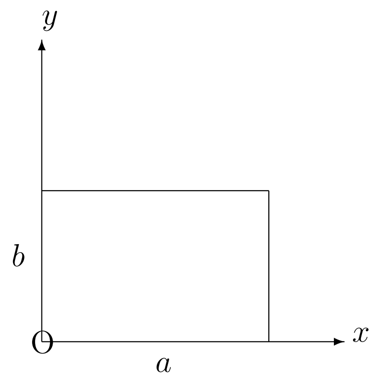
(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments.

In practice, the centroid corresponds to the position of the centre of mass for a thin plate with uniform density whose shape is that of the region considered.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

Solution



The area of the rectangle is ab and the first moments about the y-axis and x-axis are

$$\frac{1}{2}a^2b \text{ and } \frac{1}{2}b^2a, \text{ respectively}$$

Hence,

$$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

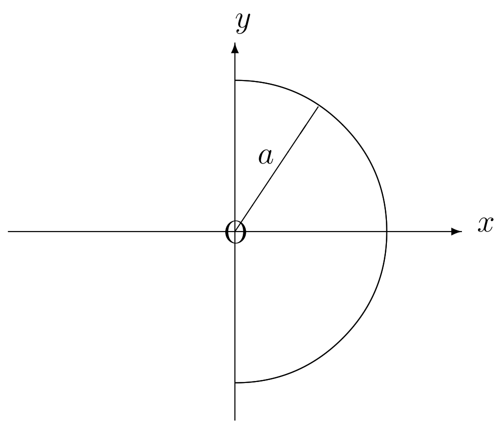
and

$$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b.$$

2. Determine the position of the centroid of the semi-circular region bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



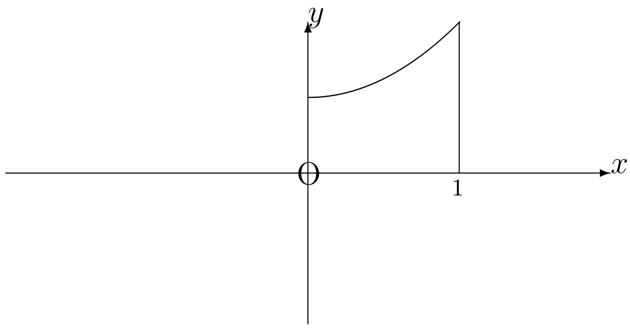
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2 in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \quad \text{and} \quad \overline{y} = 0$$

3. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\,\mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\,\mathrm{d}x = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

The first moment about the x-axis is $\frac{28}{15}$ from Example 1 in Section 13.7.3.

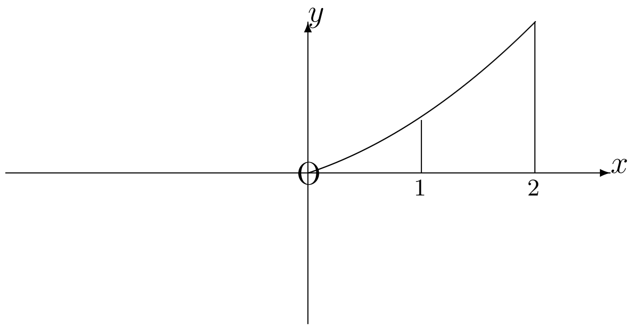
Therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2 e^x \, \mathrm{d}x = \left[x^2 e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06$$

$$\text{The area} = \int_1^2 xe^x \, \mathrm{d}x = \left[xe^x - e^x\right]_1^2 \simeq 7.39$$

$$\text{Hence } \overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in Section 13.7.3.

$$\text{Thus } \overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

"JUST THE MATHS"

SLIDES NUMBER

13.8

INTEGRATION APPLICATIONS 8
(First moments of a volume)

by

A.J.Hobson

UNIT 13.8 - INTEGRATION APPLICATIONS 8

FIRST MOMENTS OF A VOLUME

13.8.1 INTRODUCTION

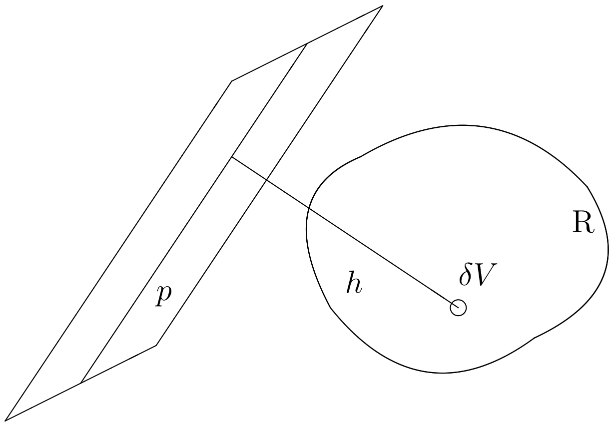
Let R denote a region of space (with volume V).

Let δV denote the volume of a small element of this region.

Then the **"first moment"** of R about a fixed plane, p, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h\delta V,$$

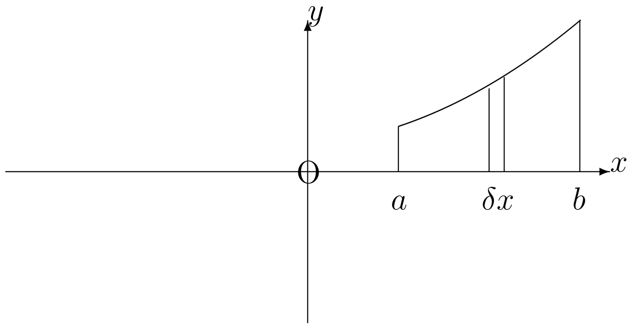
where h is the perpendicular distance, from p, of the element with volume, δV.



13.8.2 FIRST MOMENT OF A VOLUME OF REVOLUTION ABOUT A PLANE THROUGH THE ORIGIN, PERPENDICULAR TO THE X-AXIS

Consider the volume of revolution about the x-axis of a region, in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



For a narrow 'strip' of width δx and height y (parallel to the y-axis), the volume of revolution will be a thin disc, with volume $\pi y^2 \delta x$.

All the elements of volume within the disc have the same perpendicular distance, x, from the plane about which moments are being taken.

Hence, the first moment of this disc about the given plane is

$$x(\pi y^2 \delta x).$$

The total first moment is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi x y^2 \delta x$$

$$= \int_a^b \pi x y^2 \, \mathrm{d}x.$$

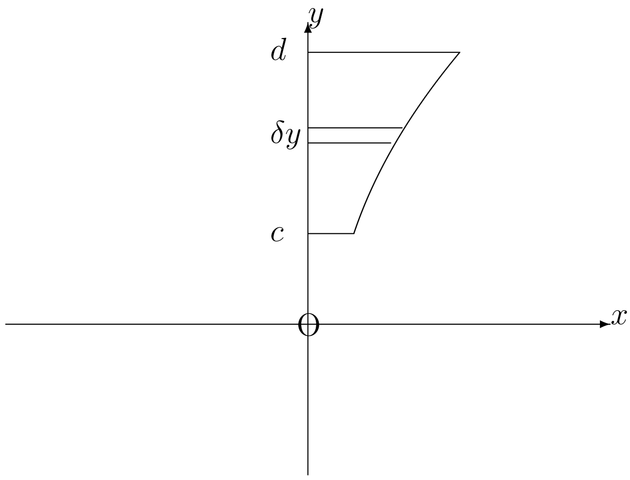
Note:

For the volume of revolution about the y-axis of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment of the volume about a plane through the origin, perpendicular to the y-axis, is given by

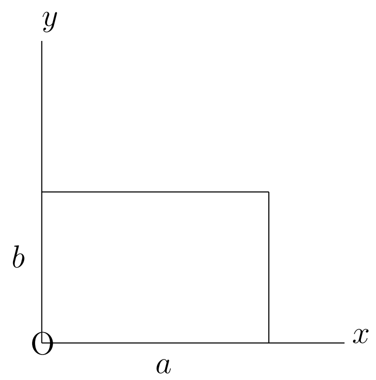
$$\int_c^d \pi y x^2 \, \mathrm{d}y$$



EXAMPLES

1. Determine the first moment of a solid right-circular cylinder with height, a and radius b, about one end.

   Solution



Consider the volume of revolution about the x-axis of the region, bounded in the first quadrant of the xy-plane by the x-axis, the y-axis and the lines $x = a$, $y = b$.

The first moment of the volume about a plane through the origin, perpendicular to the x-axis is given by

$$\int_0^a \pi x b^2 \, \mathrm{d}x$$

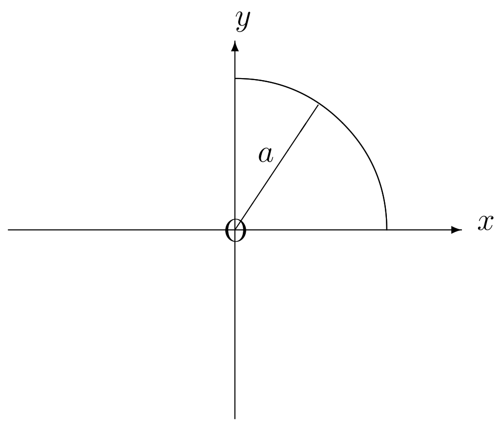
$$= \left[\frac{\pi x^2 b^2}{2}\right]_0^a = \frac{\pi a^2 b^2}{2}.$$

2. Determine the first moment of volume, about its plane base, of a solid hemisphere, with radius a.

Solution

Consider the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$



The first moment of volume about a plane through the origin, perpendicular to the x-axis is given by

$$\int_0^a \pi x(a^2 - x^2)\,\mathrm{d}x$$

$$= \left[\pi(\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)\right]_0^a$$

$$= \pi\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi a^4}{4}.$$

Note:

The symmetry of the solid figures in the above two examples shows that their first moments about a plane through the origin, perpendicular to the y-axis, would be zero.

This is because, for each $y\delta V$ in the calculation of the total first moment, there will be a corresponding $-y\delta V$.

In much the same way, the first moments of volume about the xy-plane (or <u>any</u> plane of symmetry) would also be zero.

13.8.3 THE CENTROID OF A VOLUME

Let R denote a volume of revolution about the x-axis of a region of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$

Having calculated the first moment of R about a plane through the origin, perpendicular to the x-axis (not a plane of symmetry), it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $V\overline{x}$, where V is the total volume of revolution about the x-axis.

The point is called the **"centroid"** or the **"geometric centre"** of the volume, and $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \pi x y^2 \, \mathrm{d}x}{\int_a^b \pi y^2 \mathrm{d}x} = \frac{\int_a^b x y^2 \, \mathrm{d}x}{\int_a^b y^2 \mathrm{d}x}.$$

Notes:

(i) The centroid effectively tries to concentrate the whole volume at a single point for the purposes of considering first moments. It will always lie on the line of intersection of any two planes of symmetry.

In practice, it corresponds to the position of the centre of mass for a solid with uniform density, whose shape is that of the volume of revolution considered.

(ii) For a volume of revolution about the y-axis, from $y = c$ to $y = d$, the centroid will lie on the y-axis and its distance, $\overline{y}$, from the origin will be given by

$$\overline{y} = \frac{\int_c^d \pi y x^2 \, \mathrm{d}y}{\int_c^d \pi x^2 \mathrm{d}y} = \frac{\int_c^d y x^2 \, \mathrm{d}y}{\int_c^d x^2 \mathrm{d}y}.$$

(iii) The first moment of a volume about a plane through its centroid will, by definition, be zero.

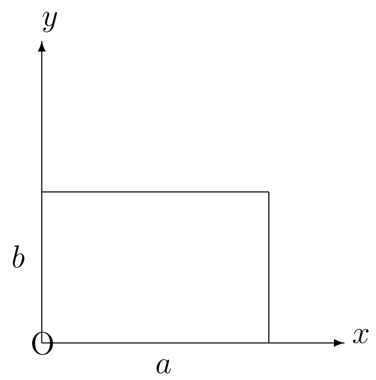
In particular, if we take the plane through the y-axis, perpendicular to the x-axis to be parallel to the plane through the centroid, with x as the perpendicular distance from an element, δV, to the plane through the y-axis, the first moment about the plane through the centroid will be

$$\sum_{\mathrm{R}} (x - \overline{x})\delta V = \sum_{\mathrm{R}} x\delta V - \overline{x}\sum_{\mathrm{R}} \delta V = V\overline{x} - V\overline{x} = 0.$$

EXAMPLES

1. Determine the position of the centroid of a solid right-circular cylinder with height, a, and radius, b.

   Solution



   The centroid of the volume of revolution will lie on the x-axis.

Using Example 1 in Section 13.8.2, the first moment about a plane through the origin, perpendicular to the x-axis is $\left(\pi a^2 b^2\right)/2$.

The volume is $\pi b^2 a$.

Hence,

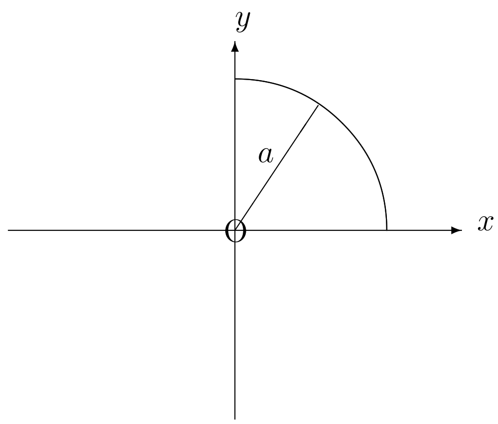
$$\overline{x} = \frac{\left(\pi a^2 b^2\right)/2}{\pi b^2 a} = \frac{a}{2}.$$

2. Determine the position of the centroid of a solid hemisphere with base-radius, a.

Solution

Consider the volume of revolution about the x-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$



The centroid of the volume of revolution will lie on the x-axis.

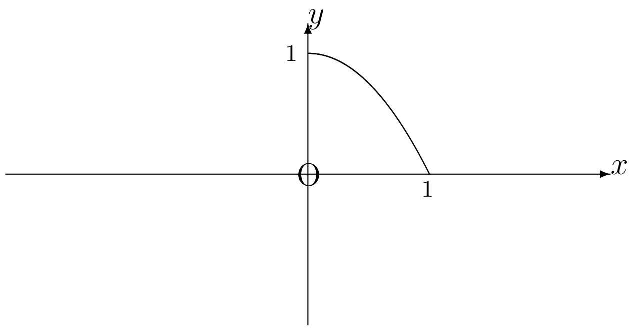
Using Example 2 in Section 13.8.2, the first moment of volume about a plane through the origin, perpendicular to the x-axis is $\left(\pi a^4\right)/4$.
The volume of the hemisphere is $\frac{2}{3}\pi a^3$.
Hence,

$$\overline{x} = \frac{\frac{2}{3}\pi a^3}{\left(\pi a^4\right)/4} = \frac{3a}{8}.$$

3. Determine the position of the centroid of the volume of revolution about the y-axis of region, bounded in the first quadrant, by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - x^2.$$

Solution



The centroid of the volume of revolution will lie on the y-axis.

The first moment about a plane through the origin, perpendicular to the y-axis, is given by

$$\int_0^1 \pi y(1-y)\ \mathrm{d}y = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

The volume is given by

$$\int_0^1 \pi(1-y)\ \mathrm{d}y = \left[y - \frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

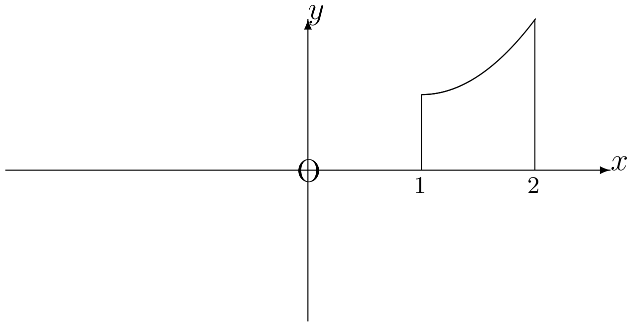
Hence,

$$\overline{y} = \frac{\pi}{6} \div \frac{\pi}{2} = \frac{1}{3}.$$

4. Determine the position of the centroid of the volume of revolution about the x-axis of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = e^x.$$

Solution



The centroid of the volume of revolution will lie on the x axis.

The first moment about a plane through the origin, perpendicular to the x-axis is given by

$$\int_1^2 \pi x e^{2x}\ \mathrm{d}x$$

$$= \pi\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_1^2 \simeq 122.84$$

The volume is given by

$$\int_1^2 \pi e^{2x}\ \mathrm{d}x$$

$$= \pi\left[\frac{e^{2x}}{2}\right]_1^2 \simeq 74.15$$

Hence,

$$\overline{x} \simeq 122.84 \div 74.15 \simeq 1.66$$

"JUST THE MATHS"

SLIDES NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates.

Let δs denote the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} 2\pi x y \delta s,$$

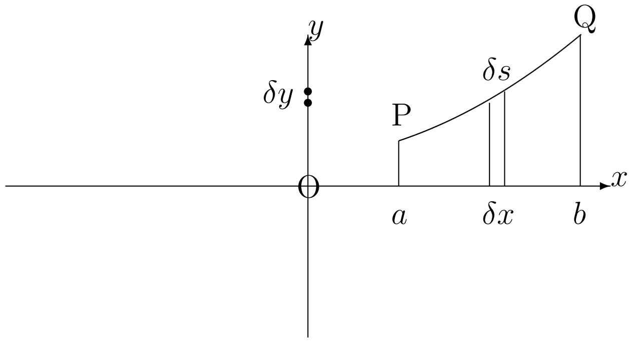
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y \delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

For the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\delta x$$

$$= \int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t)$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the previous line needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

The first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

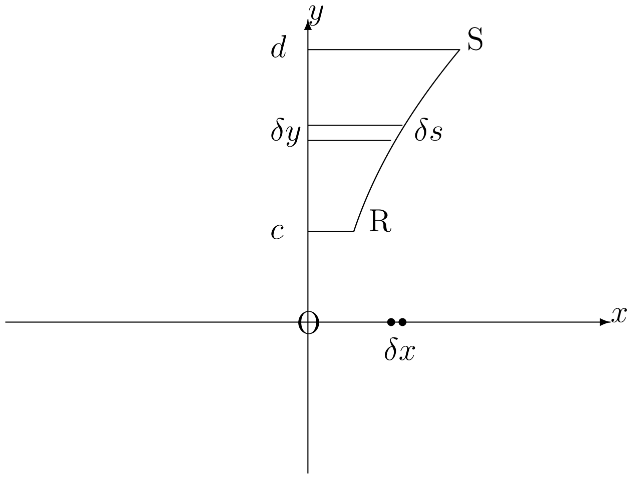
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is as follows:

$$\text{First moment} = \int_c^d 2\pi y x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\text{First moment} = \pm\int_{t_1}^{t_2} 2\pi y x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

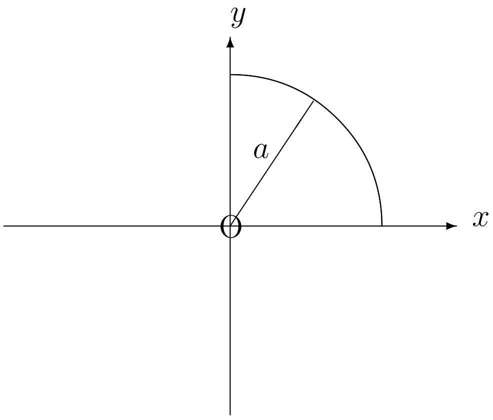
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant

Solution



$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

$$\text{But}\ \ x^2+y^2=a^2.$$

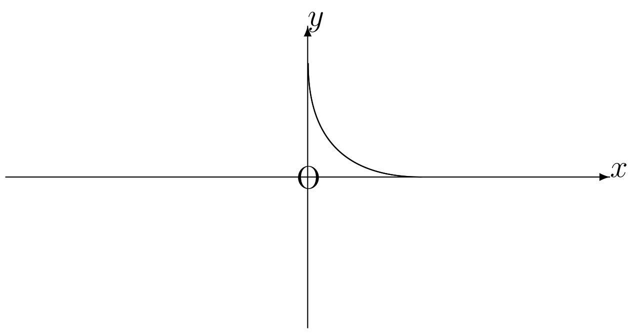
Thus, the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta,\quad y = a\sin^3\theta.$$

Solution



$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta\ \ \text{and}\ \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

On using $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta$$

$$= \frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta$$

$$= \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also

$$\frac{9\pi a^3}{128}.$$

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \ 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}{\int_a^b \ 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}$$

$$= \frac{\int_a^b \ xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \ y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}.$$

Note: The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments.

In practice, the centroid of a surface corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

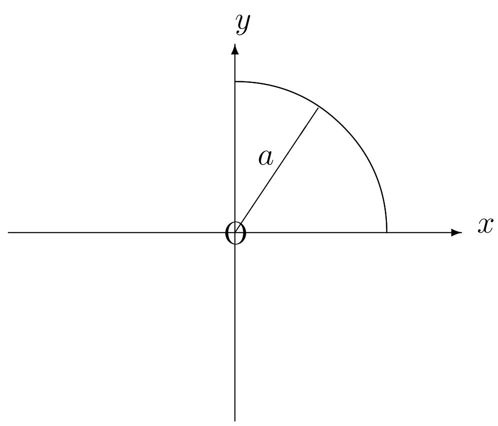
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.9.2, the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

The total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x.$$

Using $x^2+y^2=a^2$,

$$\text{surface area} = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2.$$

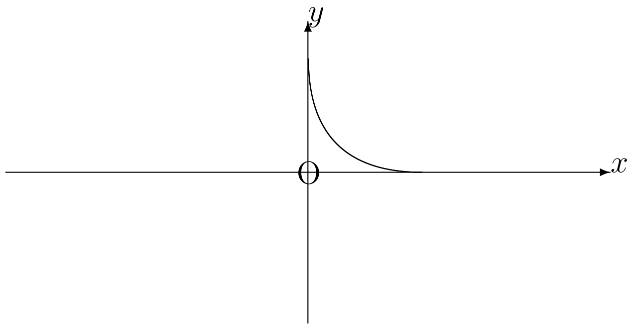
Hence,

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.9.2, the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to

$$\frac{9\pi a^3}{128}.$$

The total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta \ \mathrm{d}\theta$$

$$= \int_{0}^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta \ \mathrm{d}\theta$$

$$= 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{9\pi a^3}{128} \div \frac{3\pi a^2}{5}$$

or

$$\overline{x} = \frac{15a}{128}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

A.J.Hobson

UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

13.10.1 INTRODUCTION

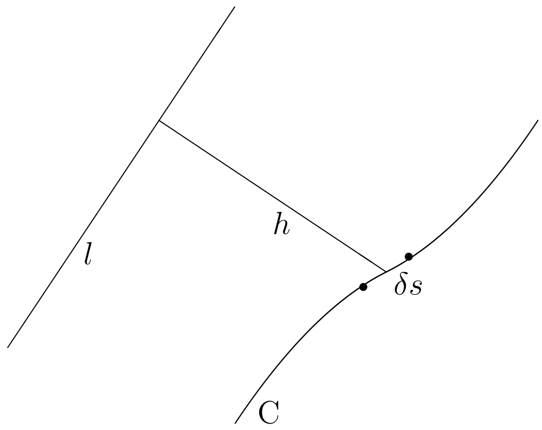
Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates.

Let δs denote the length of a small element of this arc.

Then the **"second moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{C} h^2 \delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

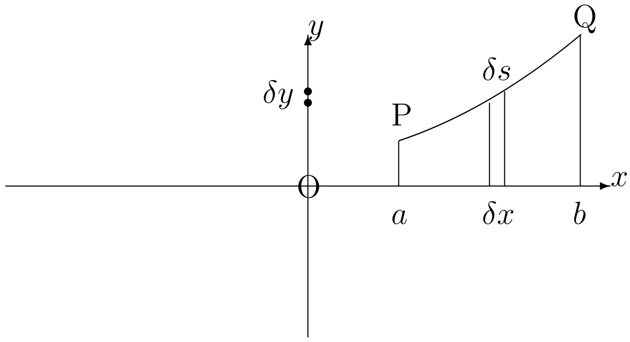


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The second moment of each element about the y-axis is $x^2\delta s$.

The total second moment of the arc about the y-axis is given by

$$\lim_{\delta s\to 0} \sum_{\mathrm{C}} x^2\delta s.$$

But, by Pythagoras' Theorem

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x.$$

Thus, the second moment of the arc becomes

$$\lim_{\delta x\to 0} \sum_{x=a}^{x=b} x^2\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x$$

$$= \int_a^b x^2\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the above formula needs to be prefixed by a negative sign.

Thus, the second moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc of the curve whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x-axis will be

$$\int_a^b y^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t,$$

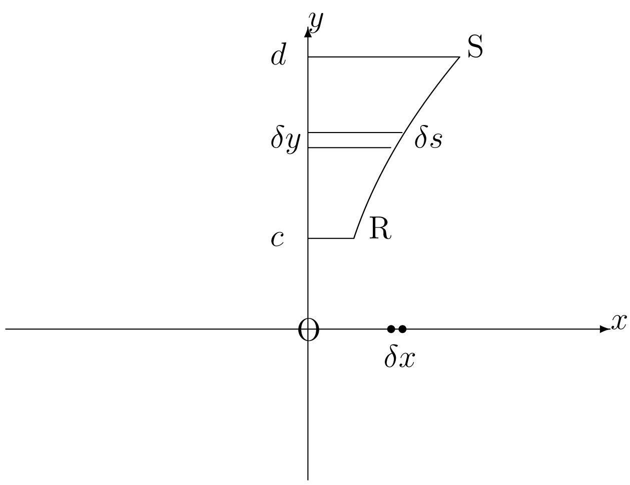
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc of the curve whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x-axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

according as $\frac{dy}{dt}$ is positive or negative and where
$t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

EXAMPLES

1. Determine the second moments about the x-axis and the
   y-axis of the arc of the circle whose equation is

   $$x^2 + y^2 = a^2,$$

   lying in the first quadrant.

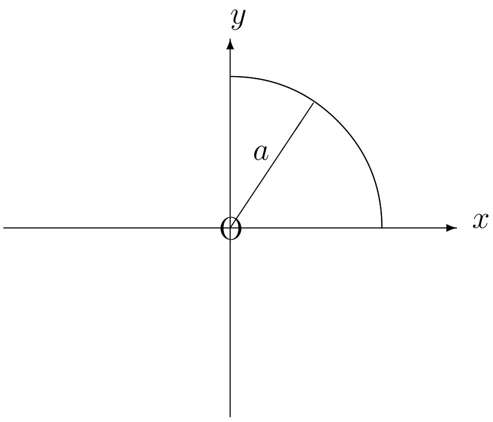
   Solution

   Using implicit differentiation,

   $$2x + 2y\frac{dy}{dx} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$



The second moment about the y-axis is therefore given by

$$\int_0^a x^2\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x$$

$$= \int_0^a \frac{x^2}{y}\sqrt{x^2+y^2}\ \mathrm{d}x.$$

But

$$x^2 + y^2 = a^2.$$

Therefore,

$$\text{Second} \quad \text{moment} = \int_0^a \frac{ax^2}{y}\mathrm{d}x.$$

Making the substitution $x = a\sin u$,

$$\text{Second moment} = \int_0^{\frac{\pi}{2}} a^3 \sin^2 u \, \mathrm{d}u$$

$$= a^3 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2u}{2} \, \mathrm{d}u$$

$$= a^3 \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.$$

By symmetry, the second moment about the x-axis will also be

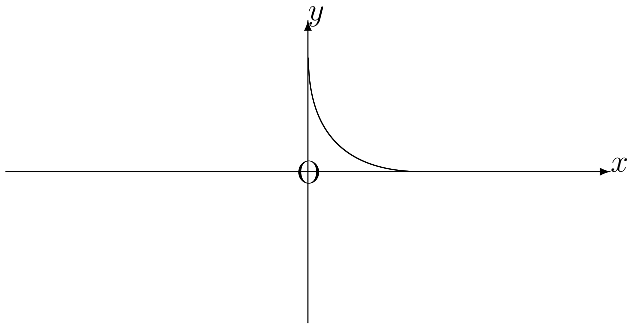
$$\frac{\pi a^3}{4}.$$

2. Determine the second moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta \cos\theta.$$



The second moment about the y-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} x^2\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

On using $\cos^2\theta+\sin^2\theta\equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} a^2\cos^6\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$=3a^3\int_0^{\frac{\pi}{2}} \cos^7\theta\sin\theta\ \mathrm{d}\theta$$

$$=3a^2\left[-\frac{\cos^8\theta}{8}\right]_0^{\frac{\pi}{2}}=\frac{3a^3}{8}.$$

Similarly, the second moment about the x-axis is given by

$$\int_0^{\frac{\pi}{2}} y^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$\int_0^{\frac{\pi}{2}} a^2\sin^6\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos\theta \ d\theta$$

$$= 3a^3 \left[\frac{\sin^8\theta}{8}\right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

This second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by sk^2, where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

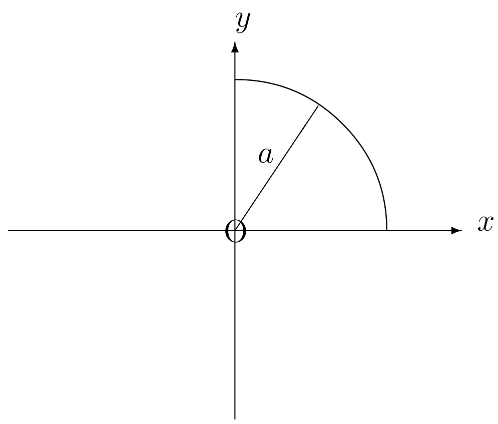
EXAMPLES

1. Determine the radius of gyration, about the y-axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.10.3, the second moment of the arc about the y-axis is equal to

$$\frac{\pi a^3}{4}.$$

Also, the length of the arc is

$$\frac{\pi a}{2}.$$

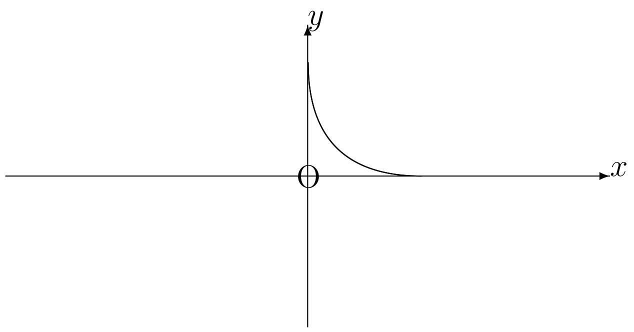
Hence, the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Also, the second moment of the arc about the y-axis is equal to

$$\frac{3a^3}{8}.$$

The length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8} \times \frac{2}{3a}} = \frac{a}{2}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

A.J.Hobson

UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

13.11.1 INTRODUCTION

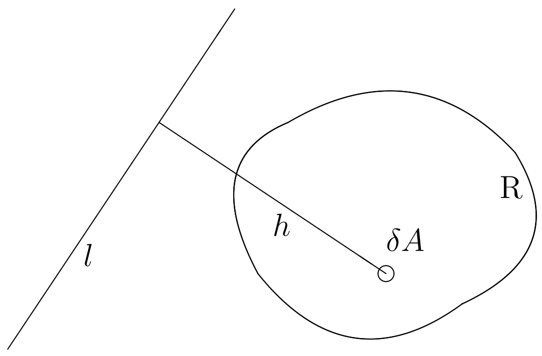
Let R denote a region (with area A) of the xy-plane in cartesian co-ordinates.

Let δA denote the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h^2 \delta A,$$

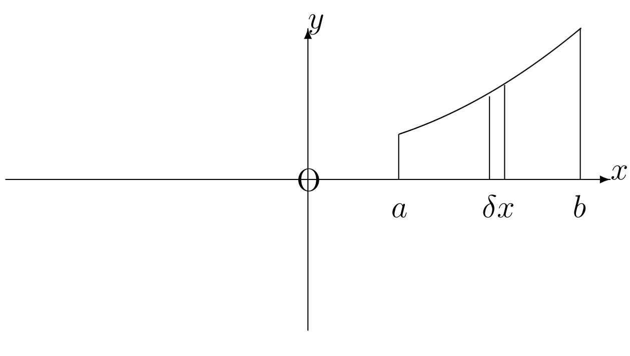
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

All of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence, the second moment of this strip about the y-axis is $x^2(y\delta x)$.

The total second moment of the region about the y-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \, \mathrm{d}x.$$

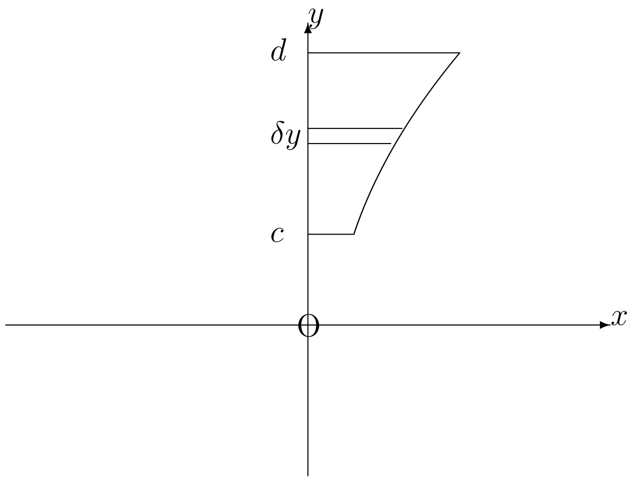
Note:

For a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

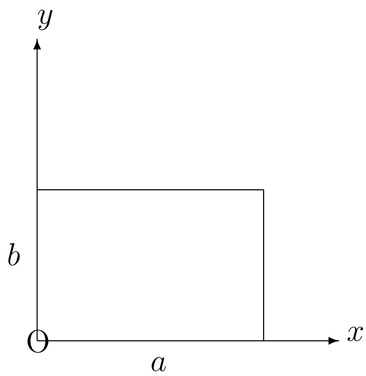
$$\int_c^d y^2 x \, \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



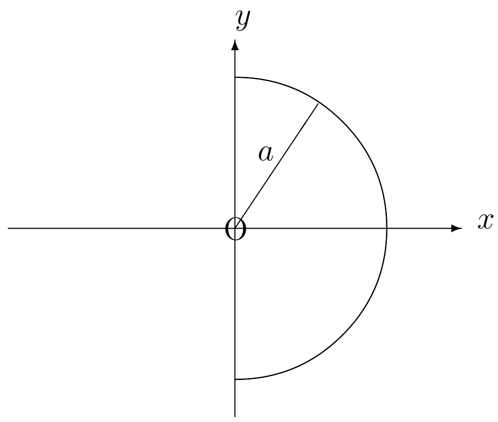
The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



There will be equal contributions from the upper and lower halves of the region.

Hence, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\, \mathrm{d}x$$

$$= 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta.a\cos\theta.a\cos\theta \mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$2a^4 \int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4} \, d\theta$$

$$= \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} \, d\theta$$

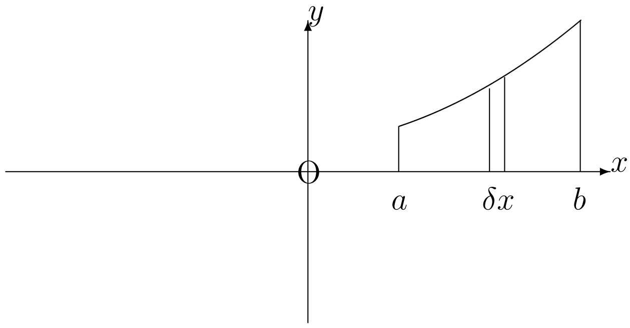
$$= \frac{a^4}{4} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides.

This result may now be used to determine the second moment about the x-axis of a region, enclosed in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow 'strip', of width δx and height y, is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$.

Hence, the second moment of the whole region about the x-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x$$

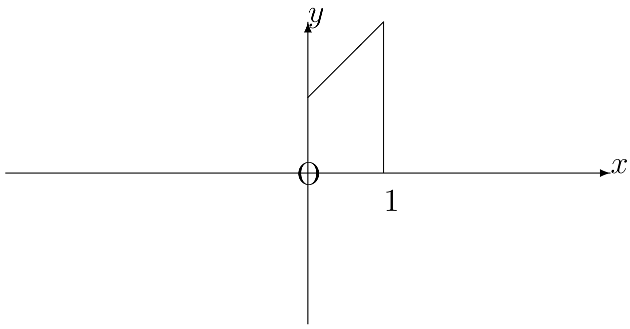
$$= \int_a^b \frac{1}{3}y^3 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

$$= \frac{1}{3}\int_0^1 (x^3 + 3x^2 + 3x + 1) \, \mathrm{d}x$$

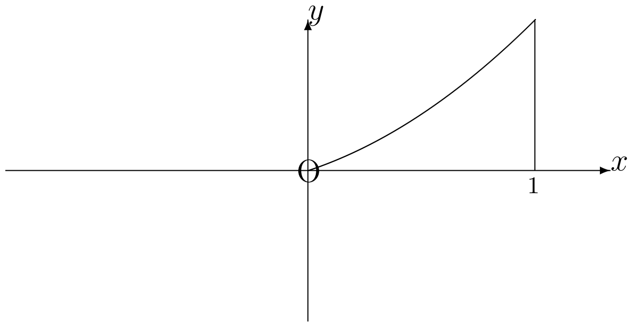
$$= \frac{1}{3}\left[\frac{x^4}{4} + x^3 + + \frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{3}{2} + 1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3e^{3x}\ \mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2e^{3x}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1$$

$$= \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The second moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis.

In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area since multiplication by ρ, of elements of area, converts them into elements of mass.

"JUST THE MATHS"

SLIDES NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

A.J.Hobson

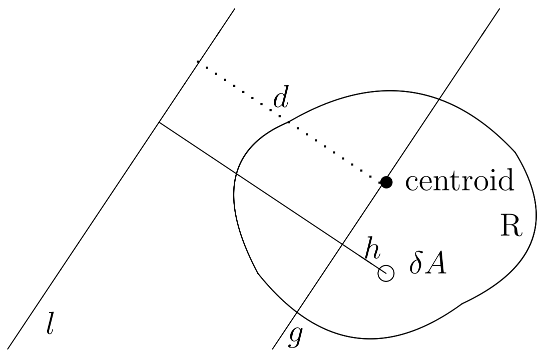
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Let M_g denote the second moment of a given region, R, about an axis, g, through its centroid.

Let M_l denote the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\mathrm{R}} (h+d)^2 \delta A = \sum_{\mathrm{R}} (h^2 + 2hd + d^2)$$

That is,

$$M_l = \sum_{\mathrm{R}} h^2 \delta A + 2d \sum_{\mathrm{R}} h\delta A + d^2 \sum_{\mathrm{R}} \delta A = M_g + Ad^2$$

Note:
The summation, $\Sigma_{\mathrm{R}}\ h\delta A$, is the first moment about the an axis through the centroid and, therefore, zero.

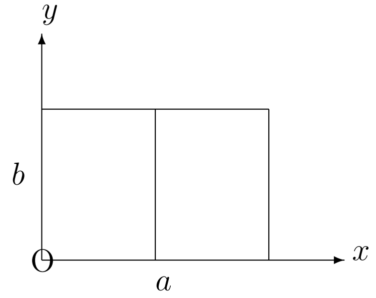
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution



For a rectangular region with sides of length a and b, the second moment about the side of length b is

$$\frac{a^3b}{3}.$$

The perpendicular distance between the two axes is $\frac{a}{2}$. Hence,

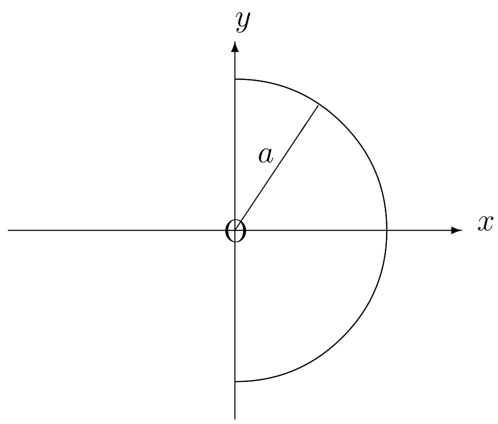
$$\frac{a^3b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3b}{4},$$

giving

$$M_g = \frac{a^3b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is

$$\frac{\pi a^4}{8}.$$

The position of the centroid is a distance of

$$\frac{4a}{3\pi}$$

from the diameter along the radius which perpendicular to it.

Hence,

$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}$$

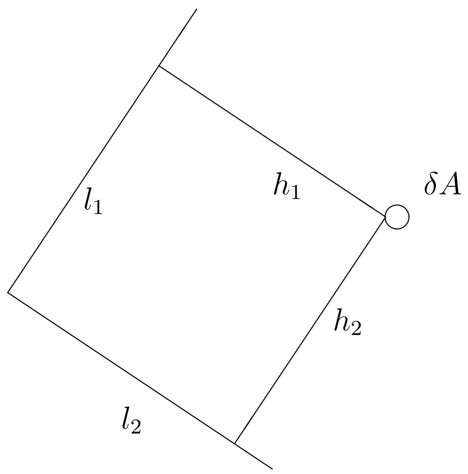
That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Let l_1 and l_2 denote two straight lines, at right-angles to each other, in the plane of a region R with area A.

Let h_1 and h_2 be the perpendicular distances from these two lines respectively of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A.$$

The second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

Adding these two together gives the second moment about an axis perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2.

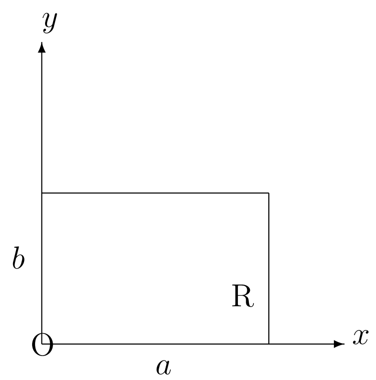
This is because the square of the perpendicular distance, h_3 of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b about an axis through one corner, perpendicular to the plane of R.

   Solution

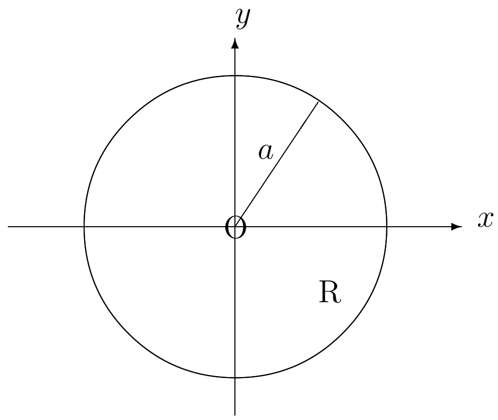


The required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is

$$\frac{\pi a^4}{4}.$$

That is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two-dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

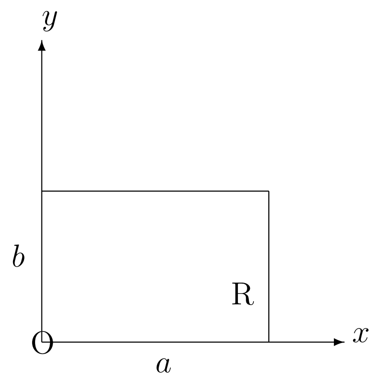
Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments.

Unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

Solution



The second moment is

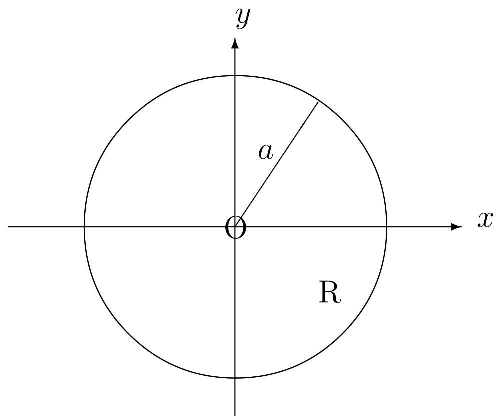
$$\frac{1}{3}ab(a^2 + b^2).$$

Since the area itself is ab, we obtain

$$k = \sqrt{a^2 + b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment about the given axis is

$$\frac{\pi a^4}{2}.$$

Since the area itself is πa^2, we obtain

$$k = \frac{a}{\sqrt{2}}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.13

INTEGRATION APPLICATIONS 13
(Second moments of a volume (A))

by

A.J.Hobson

UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

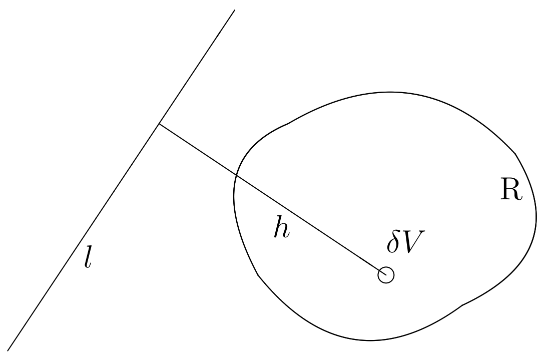
13.13.1 INTRODUCTION

Let R denote a region (with volume V) in space and suppose that δV is the volume of a small element of this region

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

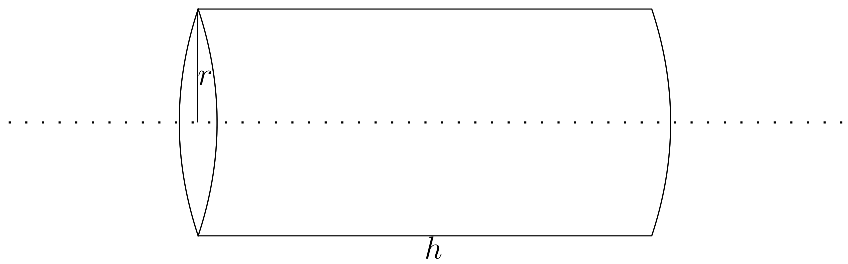
where h is the perpendicular distance from l of the element with volume, δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell is

$$r^2(2\pi r h \delta r).$$

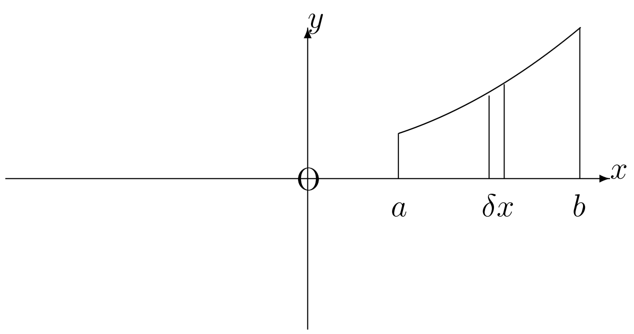
The total second moment is therefore given by

$$\lim_{\delta r \to 0} \sum_{r=0}^{r=a} r^2(2\pi r h \delta r) = \int_0^a 2\pi h r^3 \, \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is

$$2\pi x^3 y \delta x.$$

Thus, the total second moment about the y-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x^3 y \delta x$$

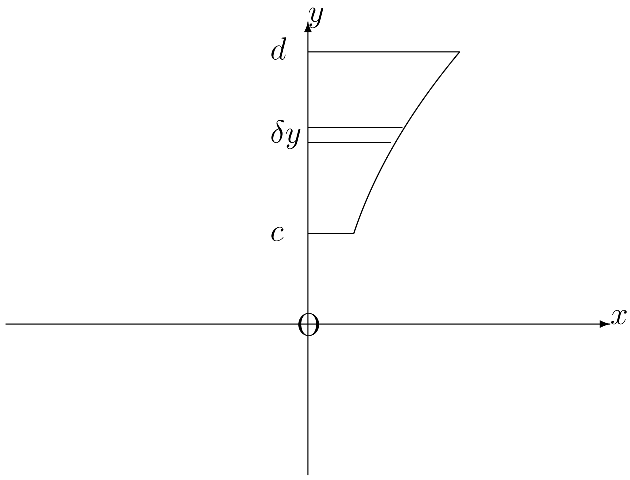
$$= \int_a^b 2\pi x^3 y \, \mathrm{d}x.$$

Note:
For the volume of revolution, about the x-axis, of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \, \mathrm{d}y.$$



EXAMPLE

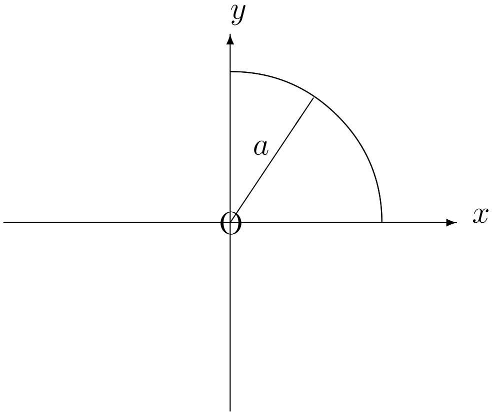
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a 2\pi x^3\sqrt{a^2-x^2}\,\mathrm{d}x$$

$$=4\pi\int_0^{\frac{\pi}{2}} a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \sin^3\theta\ \cos^2\theta\mathrm{d}\theta$$

$$=4\pi\int_0^{\frac{\pi}{2}} \left(\cos^2\theta-\cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta\equiv 1-\cos^2\theta.$$

The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

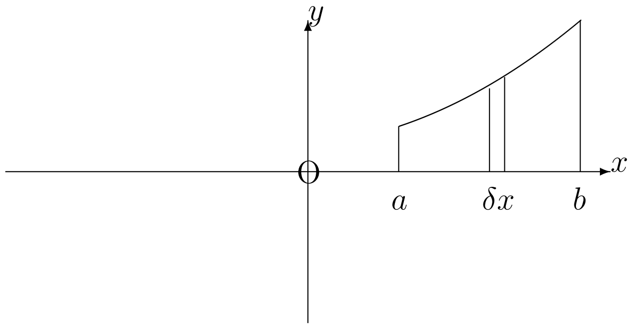
$$= 4\pi a^5\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis.

This result may now be used to determine the second moment, about the x-axis, for the volume of revolution about this axis, of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is

$$\frac{\pi y^4 \delta x}{2}.$$

Hence, the second moment of the whole region about the x-axis is given by

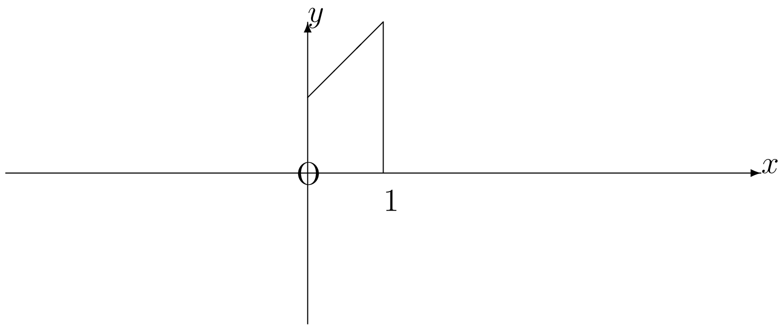
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis, for the volume of revolution about this axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\,\mathrm{d}x$$

$$= \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis

In fact, for a solid with uniform density, ρ, the moment of inertia is ρ times the second moment of volume, since multiplication by ρ, of elements of volume, converts them into elements of mass

"JUST THE MATHS"

SLIDES NUMBER

13.14

INTEGRATION APPLICATIONS 14
(Second moments of a volume (B))

by

A.J.Hobson

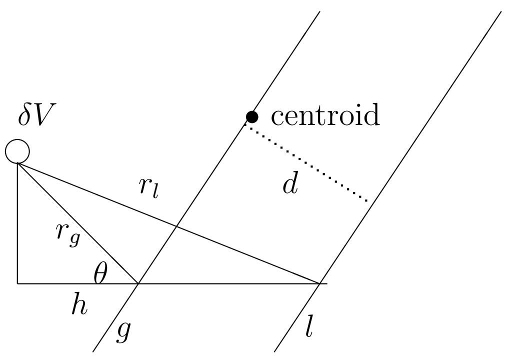
UNIT 13.14 - INTEGRATION APPLICATIONS 14

SECOND MOMENTS OF A VOLUME (B)

13.14.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis and has a perpendicular distance of d from the first axis.



In the above **three dimensional** diagram, we have

$$M_l = \sum_{\mathrm{R}} r_l^2 \delta V \text{ and } M_g = \sum_{\mathrm{R}} r_g^2 \delta V.$$

But, from the Cosine Rule,

$$r_l^2 = r_g^2 + d^2 - 2r_g d\cos(180^\circ - \theta) = r_g^2 + d^2 + 2r_g d\cos\theta.$$

Hence,

$$r_l^2 = r_g^2 + d^2 + 2dh;$$

and so

$$\sum_R r_l^2 \delta V = \sum_R r_g^2 \delta V + \sum_R d^2 \delta V + 2d \sum_R h\delta V$$

Finally, the expression

$$\sum_R h\delta V$$

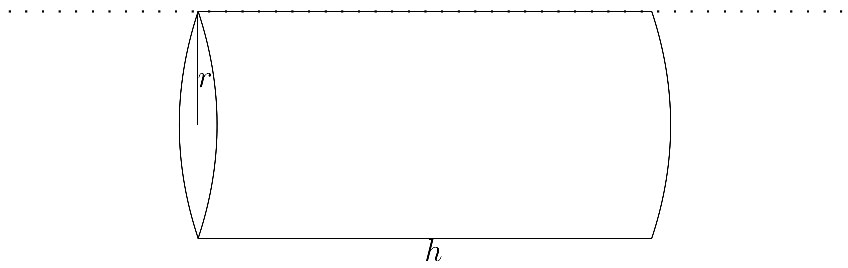
represents the first moment of R about a plane through the centroid which is perpendicular to the plane containing l and g. Such a first moment will be zero and hence,

$$M_l = M_g + Vd^2.$$

EXAMPLE

Determine the second moment of a solid right-circular cylinder about one of its generators (that is, a line in the surface, parallel to the central axis).

Solution



The second moment of the cylinder about the central axis was shown in Unit 13.13, section 13.13.2, to be

$$\frac{\pi a^4 h}{2}.$$

Since the central axis and the generator are a distance a apart, the required second moment is given by

$$\frac{\pi a^4 h}{2} + (\pi a^2 h)a^2 = \frac{3\pi a^4 h}{2}.$$

13.14.2 THE RADIUS OF GYRATION OF A VOLUME

Having calculated the second moment of a three-dimensional region about a certain axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Vk^2, where V is the total volume of the region.

We simply divide the value of the second moment by V in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole volume at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

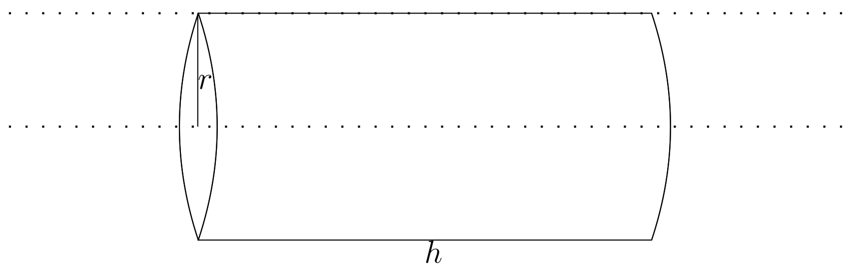
1. Determine the radius of gyration of a solid right-circular cylinder with height, h, and radius, a, about

   (a) its own axis

   and

   (b) one of its generators.

   Solution

   Using earlier examples, together with the volume, $V = \pi a^2 h$, the required radii of gyration are

   (a)

$$\sqrt{\frac{\pi a^4 h}{2} \div \pi a^2 h} = \frac{a}{\sqrt{2}}$$

   and

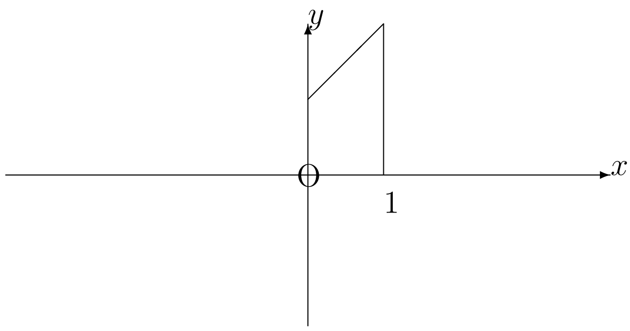
   (b)

$$\sqrt{\frac{3\pi a^4 h}{2} \div \pi a^2 h} = a\sqrt{\frac{3}{2}}.$$

2. Determine the radius of gyration of the volume of revolution about the x-axis, of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



From Unit 13.13, section 13.13.3, the second moment about the given axis is

$$\frac{31\pi}{10}.$$

The volume itself is given by

$$\int_0^1 \pi(x+1)^2 \, \mathrm{d}x = \left[\pi\frac{(x+1)^3}{3}\right]_0^1 = \frac{7\pi}{3}.$$

Hence,

$$k^2 = \frac{31\pi}{10} \times \frac{3}{7\pi} = \frac{93}{70} \text{ and so } k = \sqrt{\frac{93}{70}} \simeq 1.15$$

"JUST THE MATHS"

SLIDES NUMBER

13.15

INTEGRATION APPLICATIONS 15
(Second moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.15 - INTEGRATION APPLICATIONS 15

SECOND MOMENTS OF A SURFACE OF REVOLUTION

13.15.1 INTRODUCTION

Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates.

Let δs denote the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"second moment"** about the x-axis, is given by

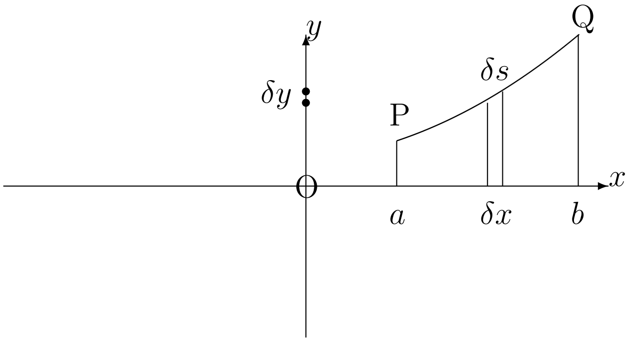
$$\lim_{\delta s \to 0} \sum_{\text{C}} y^2.2\pi y \delta s.$$

13.15.2 INTEGRATION FORMULAE FOR SECOND MOMENTS

(a) Consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

Hence, for the surface of revolution of the arc about the x-axis, the second moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y^3 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

$$= \int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, then the last result needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

$$= \int_{t_1}^{t_2} 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} . \frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

Thus, the second moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi y^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

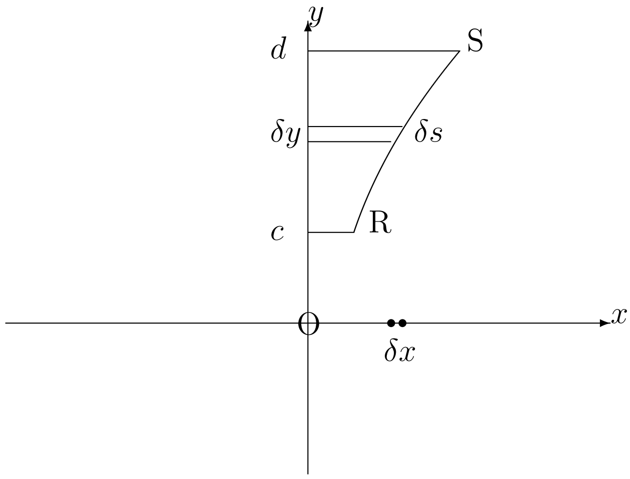
(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section.

In this case, the second moment about the y-axis is given by

$$\int_c^d 2\pi x^3 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the second moment about the y-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi x^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

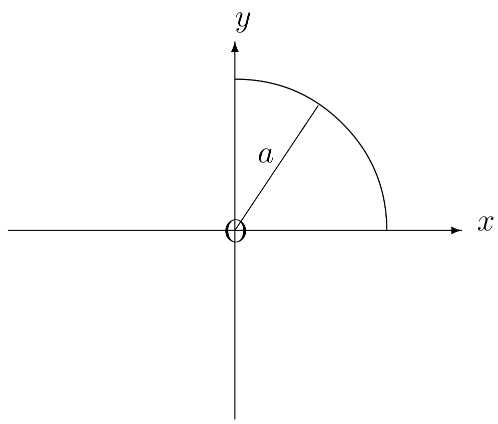
EXAMPLES

1. Determine the second moment about the x-axis of the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

The second moment about the x-axis is therefore given by

$$\int_0^a 2\pi y^3\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x$$

$$= \int_0^a 2\pi y^3 \sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

But

$$x^2 + y^2 = a^2.$$

Thus, the second moment becomes

$$\int_0^a 2\pi a(a^2 - x^2)\ \mathrm{d}x$$

$$= 2\pi a \left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{4\pi a^4}{3}.$$

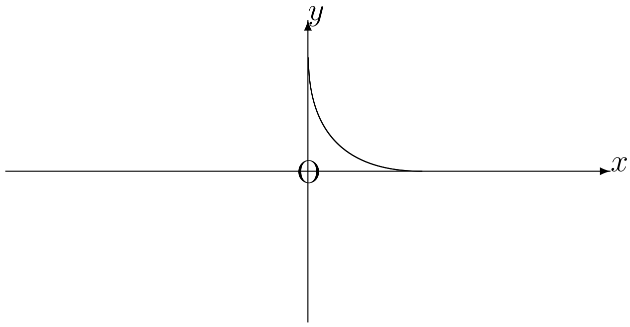
2. Determine the second moment about the axis of revolution, when the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

is rotated through 2π radians about (a) the x-axis and (b) the y-axis.

Solution

(a)



$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

The second moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi y^3\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

On using $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^3\sin^{27}\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 6\pi a^4\sin^{28}\theta\cos\theta\ \mathrm{d}\theta$$

$$= 6\pi a^4\int_0^{\frac{\pi}{2}} \sin^{28}\theta\cos\theta\ \mathrm{d}\theta$$

$$= 6\pi a^4\left[\frac{\sin^{29}\theta}{29}\right]_0^{\frac{\pi}{2}} = \frac{6\pi a^4}{29}.$$

(b) By symmetry, or by direct integration, the second moment about the y-axis is also

$$\frac{6\pi a^4}{29}.$$

13.15.4 THE RADIUS OF GYRATION OF A SURFACE OF REVOLUTION

Having calculated the second moment of a surface of revolution about a specified axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Sk^2, where S is the total surface area of revolution.

We divide the value of the second moment by S in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole surface at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

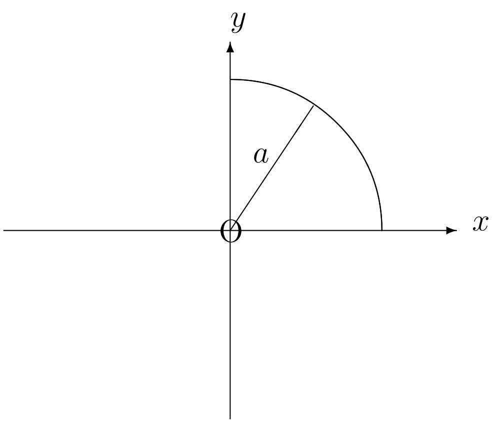
EXAMPLES

1. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant

Solution



From Example 1 in section 13.15.2, we know that the second moment of the surface about the x-axis is equal to

$$\frac{4\pi a^4}{3}.$$

Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2.$$

Hence,

$$k^2 = \frac{4\pi a^4}{3} \times \frac{1}{2\pi a^2} \quad = \frac{2a^2}{3}.$$

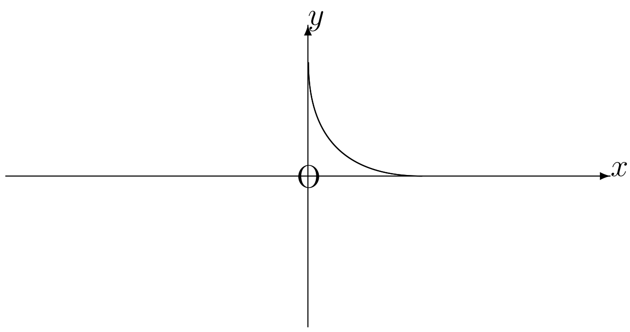
The radius of gyration is thus given by

$$k = a\sqrt{\frac{2}{3}}.$$

2. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in section 13.15.2, we know that that the second moment of the surface about the x-axis is equal to $\frac{6\pi a^4}{29}$.

Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0}\ 2\pi a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\ 3a^2\sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$=3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}=\frac{3\pi a^2}{5}.$$

Thus,

$$k^2=\frac{6\pi a^4}{29}\times\frac{5}{3\pi a^2}=\frac{10a^2}{29}.$$

"JUST THE MATHS"

SLIDES NUMBER

13.16

INTEGRATION APPLICATIONS 16
(Centres of pressure)

by

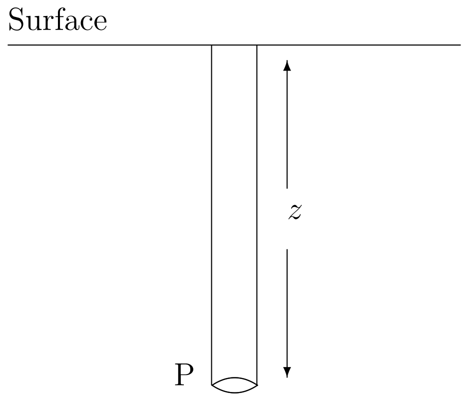
A.J.Hobson

UNIT 13.16 - INTEGRATION APPLICATIONS 16

CENTRES OF PRESSURE

13.16.1 THE PRESSURE AT A POINT IN A LIQUID

In the following diagram, we consider the pressure in a liquid at a point, P, whose depth below the surface of the liquid is z.



Ignoring atmospheric pressure, the pressure, p, at P is measured as the thrust acting upon unit area and is due to the weight of the column of liquid with height z above it.

Hence,

$$p = wz,$$

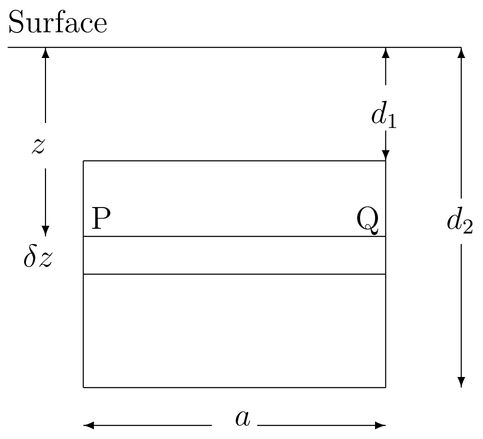
where w is the weight, per unit volume, of the liquid.

Note:
The pressure at P is directly proportional to the depth of P below the surface.

We assume that the pressure acts equally in all directions at P.

13.16.2 THE PRESSURE ON AN IMMERSED PLATE

Consider a rectangular plate, with dimensions a and $(d_2 - d_1)$, immersed vertically in a liquid as shown below:



For a thin strip, PQ, of width δz at a depth z below the surface of the liquid, the thrust on PQ will be the pressure at P multiplied by the area of the strip.

That is,

$$\text{Thrust on PQ} = wz \times a\delta z.$$

The total thrust on the whole plate will therefore be

$$\sum_{z=d_1}^{z=d_2} waz\delta z.$$

Allowing δz to tend to zero, the total thrust becomes

$$\int_{d_1}^{d_2} waz \, \mathrm{d}z$$

$$= \left[\frac{waz^2}{2}\right]_{d_1}^{d_2} = \frac{wa}{2}(d_2^2 - d_1^2).$$

This may be written

$$wa(d_2 - d_1)\left(\frac{d_2 + d_1}{2}\right),$$

where $a(d_2 - d_1)$ is the area of the plate and $(d_2 + d_1)/2$ is the depth of the centroid of the plate.

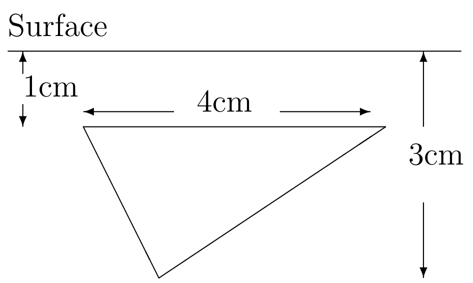
Total thrust =

$$\text{area of plate} \times \text{pressure at the centroid.}$$

Note:
It may be shown that this result holds whatever the shape of the plate is; and even when the plate is not vertical.

EXAMPLES

1. A triangular plate is immersed vertically in a liquid for which the weight per unit volume is w. The dimensions of the plate and its position in the liquid is shown in the following diagram:



Determine the total thrust on the plate as a multiple of w.

Solution

The area of the plate is given by

$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4\text{cm}^2.$$

The centroid of the plate is at a distance from its horizontal side equal to one third of its perpendicular height.

The centroid will therefore lie at a depth of

$$\left(1 + \frac{1}{3} \times 2\right) \text{cms} = \frac{5}{3}\text{cms}.$$

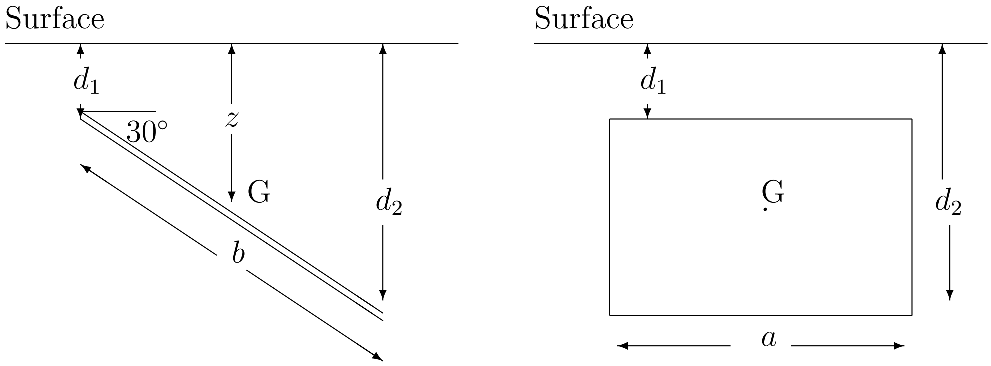
Hence, the pressure at the centroid is

$$\frac{5w}{3}.$$

We conclude that

$$\text{Total \ \ thrust} = 4 \times \frac{5w}{3} = \frac{20w}{3}.$$

2. The following diagram shows a rectangular plate immersed in a liquid for which the weight per unit volume is w; and the plate is inclined at $30°$ to the horizontal:



Determine the total thrust on the plate as a multiple of w.

Solution

The depth, z, of the centroid, G, of the plate is given by

$$z = d_1 + \frac{b}{2}\sin 30° = d_1 + \frac{b}{4}.$$

Hence, the pressure, p, at G is given by

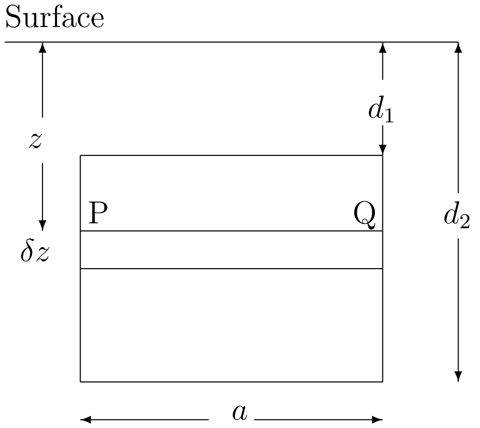
$$p = \left(d_1 + \frac{b}{4}\right) w.$$

Since the area of the plate is ab, we obtain

$$\text{Total thrust} = ab\left(d_1 + \frac{b}{4}\right)w.$$

13.16.3 THE DEPTH OF THE CENTRE OF PRESSURE

Consider again a rectangular plate, immersed vertically in a liquid whose weight per unit volume is w.



The total thrust on the plate is

$$\int_{d_1}^{d_2} waz \, \mathrm{d}z = w\int_{d_1}^{d_2} az \, \mathrm{d}z.$$

This is the resultant of varying thrusts, acting according to depth, at each level of the plate.

By taking first moments of these thrusts about the line in which the plane of the plate intersects the surface of the liquid, we may determine a particular depth at which the total thrust may be considered to act.

This depth is called **"the depth of the centre of pressure"**.

The Calculation

In the diagram, the thrust on the strip PQ is

$$waz\delta z.$$

The first moment of the strip about the line in the surface is

$$waz^2\delta z.$$

Hence, the sum of the first moments on all such strips is given by

$$\sum_{z=d_1}^{z=d_2} waz^2\delta z = w\int_{d_1}^{d_2} az^2 \, \mathrm{d}z.$$

The definite integral is, in fact, the <u>second</u> moment of the plate about the line in the surface.

Next, we define the depth, C_p, of the centre of pressure to be such that

Total thrust $\times C_p =$

sum of first moments of strips like PQ.

That is,

$$w \int_{d_1}^{d_2} az \, \mathrm{d}z \times C_p = w \int_{d_1}^{d_2} az^2 \, \mathrm{d}z.$$

Hence,

$$C_p = \frac{\int_{d_1}^{d_2} az^2 \, \mathrm{d}z}{\int_{d_1}^{d_2} az \, \mathrm{d}z}.$$

This may be interpreted as

$$C_p = \frac{Ak^2}{A\overline{z}} = \frac{k^2}{\overline{z}},$$

where A is the area of the plate, k is the radius of gyration of the plate about the line in the surface of the liquid and $\overline{z}$ is the depth of the centroid of the plate.

Notes:

(i) It may be shown that the formula

$$C_p = \frac{k^2}{\overline{z}}$$

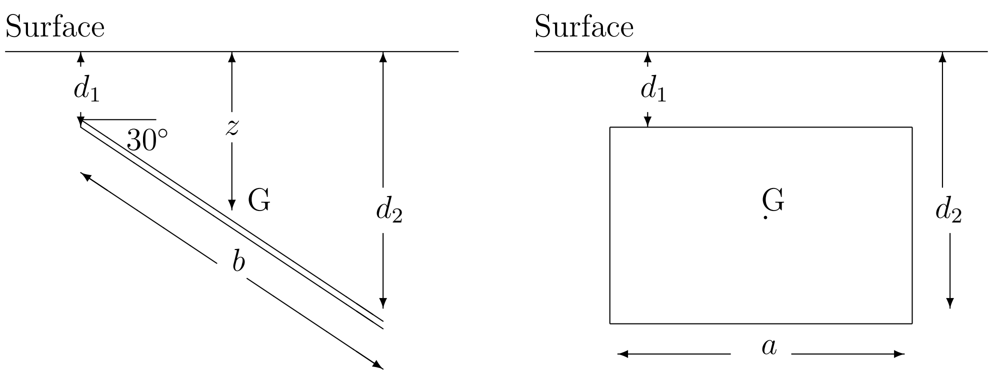
holds for any shape of plate immersed at any angle.

(ii) The phrase, "centre of pressure" suggests a particular point at which the total thrust is considered to act; but this is simply for convenience. The calculation is only for the depth of the centre of pressure.

EXAMPLE

Determine the depth of the centre of pressure for the second example of the previous section.

Solution



The depth of the centroid is

$$d_1 + \frac{b}{4}.$$

The square of the radius of gyration of the plate about an axis through the centroid, parallel to the side with length a is $\frac{a^2}{12}$.

The perpendicular distance between this axis and the line of intersection of the plane of the plate with the surface of the liquid is

$$\frac{b}{2} + \frac{d_1}{\sin 30^\circ} = \frac{b}{2} + 2d_1.$$

Hence, the square of the radius of gyration of the plate about the line in the surface is

$$\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2,$$

using the Theorem of Parallel Axes.

Finally, the depth of the centre of pressure is given by

$$C_p = \frac{\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2}{d_1 + \frac{b}{4}}.$$

"JUST THE MATHS"

SLIDES NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

A.J.Hobson

UNIT 14.1
PARTIAL DIFFERENTIATION 1

PARTIAL DERIVATIVES OF THE FIRST ORDER

14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1.

$$z = f(x, y).$$

2.

$$w = F(x, y, z).$$

Normally, the variables on the right-hand side are called the **"independent variables"**.

The variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

ILLUSTRATION

$$x^2 + y^2 + z^2 = 16.$$

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, may not always be independent of one another

ILLUSTRATION

In the formula

$$z = xy^2 + \sin(x - y),$$

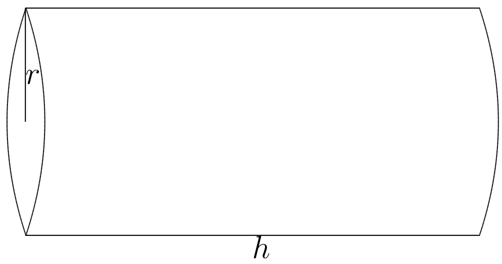
suppose that $x = t - 1$ and $y = 3t + 2$;

Then the variables, x and y, are not independent of each other.

In fact, $y = 3(x + 1) + 2 = 3x + 5.$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION



The volume, V, and the surface area, S, of a solid right-circular cylinder with radius r and height h are given by

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h.$$

V and S are functions of r and h.

Suppose r is held constant while h is allowed to vary.

Then,

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r \text{ const.}} = 2\pi r.$$

These are the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose h is held constant while r is allowed to vary.

Then,

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h \text{ const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These are the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

This is indicated by

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

   (a)

$$z = (x^2 + 3y)^5.$$

Solution

$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4.2x = 10x(x^2 + 3y)^4$$

and

$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4.3 = 15(x^2 + 3y)^4.$$

   (b)

$$w = ze^{3x-7y}.$$

Solution

$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

and

$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

(c)

$$z = x \sin(2x^2 + 5y).$$

Solution

$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2 \cos(2x^2 + 5y)$$

and

$$\frac{\partial z}{\partial y} = 5x \cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. If

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

(where z is the dependent variable), determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

$$-\sin(x + 2z). \left(1 + 2\frac{\partial z}{\partial x}\right) + 2y \left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x \left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$

"JUST THE MATHS"

SLIDES NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

A.J.Hobson

UNIT 14.2
PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF ORDER HIGHER THAN ONE

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

A partial derivative will, in general contain **all** of the independent variables.

We may need to differentiate again with respect to **any** of the independent variables.

If z is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two.

ILLUSTRATIONS

$$\frac{\partial^3 z}{\partial x \partial y^2} \quad \text{means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2} \quad \text{means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.

$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\frac{\partial z}{\partial x} = 21x^2 - 10xy.$$

$$\frac{\partial z}{\partial y} = -5x^2 + 18y^2.$$

$$\frac{\partial^2 z}{\partial x^2} = 42x - 10y.$$

$$\frac{\partial^2 z}{\partial y^2} = 36y.$$

$$\frac{\partial^2 z}{\partial y \partial x} = -10x.$$

$$\frac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.

$$z = y \sin x + x \cos y.$$

Solution

$$\frac{\partial z}{\partial x} = y \cos x + \cos y.$$

$$\frac{\partial z}{\partial y} = \sin x - x \sin y.$$

$$\frac{\partial^2 z}{\partial x^2} = -y \sin x.$$

$$\frac{\partial^2 z}{\partial y^2} = -x \cos y.$$

$$\frac{\partial^2 z}{\partial y \partial x} = \cos x - \sin y.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\frac{\partial z}{\partial x} = e^{xy}[y(2x - y) + 2] = e^{xy}[2xy - y^2 + 2].$$

$$\frac{\partial z}{\partial y} = e^{xy}[x(2x - y) - 1] = e^{xy}[2x^2 - xy - 1].$$

$$\frac{\partial^2 z}{\partial x^2} = e^{xy}[y(2xy - y^2 + 2) + 2y]$$

$$= e^{xy}[2xy^2 - y^3 + 4y].$$

$$\frac{\partial^2 z}{\partial y^2} = e^{xy}[x(2x^2 - xy - 1) - x]$$

$$= e^{xy}[2x^3 - x^2 y - 2x].$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^{xy}[x(2xy - y^2 + 2) + 2x - 2y]$$

$$= e^{xy}[2x^2 y - xy^2 + 4x - 2y].$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{xy}[y(2x^2 - xy - 1) + 4x - y]$$

$$= e^{xy}[2x^2 y - xy^2 + 4x - 2y].$$

"JUST THE MATHS"

SLIDES NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

A.J.Hobson

UNIT 14.3
PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

If

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let

$$z = f(x, y),$$

where x and y are independent variables.

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It can be shown that, for increments (or errors) in both x and y,

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

Notes:

(i) From **"Taylor's Theorem"**

$$f(x+\delta x, y+\delta y) = f(x,y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right)$$

$$+ \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x+\delta x, y+\delta y) - f(x,y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y,$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms. Calculate, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy respectively.

   Solution

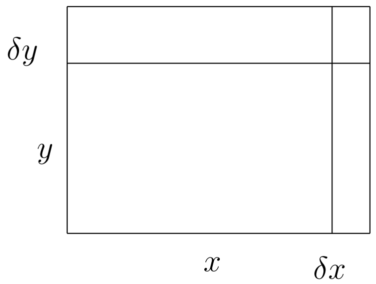
   The area, A, is given by

   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

Note:
The exact value of δA may be seen in the following diagram:



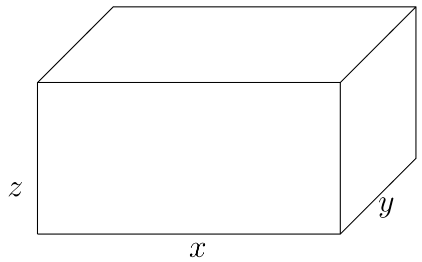
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

Solution

First, denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10$cms., $y = 12$cms. and $z = 20$cms.

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y + z)\delta x + 2(x + z)\delta y + 2(y + x)\delta z.$$

On substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12{+}20)(0.05) \pm 2(10{+}20)(0.05) \pm 2(12{+}10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign.

Hence, the greatest error is given by

$$\delta S_{\text{max}} \simeq \pm 8.4\text{cms.}^2$$

This represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \ \text{ and } \ \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

For percentage errors or percentage increments, we may use logarithms if the right hand side of the formula involves a product, a quotient, or a combination of these two in which the independent variables are separated.

A formula of this type would be,

$$w = \frac{x^3 z}{y^4}.$$

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives.

GENERAL THEORY (two variables)

Suppose that

$$z = f(x, y).$$

Then,

$$\ln z = \ln f(x, y).$$

If we let $w = \ln z$, then

$$w = \ln f(x, y),$$

giving

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x, y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x, y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x, y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

Hence,

$$\delta w \simeq \frac{\delta z}{z}.$$

CONCLUSION

The fractional increment (or error) in z approximates to the actual increment (or error) in $\ln z$.

Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:

The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If
$$w = \frac{x^3 z}{y^4},$$
calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

$$\ln w = 3\ln x + \ln z - 4\ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \text{ and } \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

$$\ln w = \frac{1}{2}[3\ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$
We require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06$$

Hence, y must be approximately 6% too large.

"JUST THE MATHS"

SLIDES NUMBER

14.4

PARTIAL DIFFERENTIATION 4 (Exact differentials)

by

A.J.Hobson

UNIT 14.4
PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

The expression,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

is an approximation for the increment (or error), δf in the function $f(x, y, ...)$ when x, y etc. are subject to increments (or errors) of δx, δy etc., respectively

The expression may be called the **"total differential"** of $f(x, y, ...)$, and may be denoted by $\mathrm{d}f$, giving

$$\mathrm{d}f \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ \ldots$$

(a) If $f(x, y, ...) \equiv x$, then $\mathrm{d}f = \delta x$.

Hence,

$$\mathrm{d}x = \delta x.$$

(b) If $f(x, y, ...) \equiv y$, then $\mathrm{d}f = \delta y$.

Hence,

$$\mathrm{d}y = \delta y.$$

(c) The total differential of each **independent** variable is the same as the small increment (or error) in that variable;

but the total differential of the **dependent** variable is only approximately equal to the increment (or error) in that variable.

(d) The observations may be summarised by the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \ldots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form,

$$P(x,y,...)\mathrm{d}x + Q(x,y,...)\mathrm{d}y + \ldots,$$

will not be the total differential of a function $f(x,y,...)$ unless $P(x,y,...)$, $Q(x,y,...)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this IS possible, then the expression is known as an **"exact differential"**.

RESULTS

(i) The expression,

$$P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y,$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

is an exact differential, $\mathrm{d}f$, then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \;\text{ and }\; \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence,

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x, y) \equiv \frac{\partial u}{\partial x}$$

for <u>some</u> function $u(x, y)$, since $P(x, y)$ could be integrated partially with respect to x.

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x}.$$

On integrating partially with respect to x,

$$Q(x, y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y.

Thus,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \left(\frac{\partial u}{\partial y} + A(y)\right)\mathrm{d}y.$$

The right-hand side is the exact differential of

$$u(x, y) + \int A(y)\ \mathrm{d}y.$$

(ii) By similar reasoning, it may be shown that

$$P(x, y, z)\mathrm{d}x + Q(x, y, z)\mathrm{d}y + R(x, y, z)\mathrm{d}z$$

is an exact differential when

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left[\frac{1}{2}\left(x^2 + y^2\right)\right].$$

2.

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}[xy].$$

3.

$$y\mathrm{d}x - x\mathrm{d}y$$

is not an exact differential, since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2\ln y\mathrm{d}x + (x+z)\mathrm{d}y + z^2\mathrm{d}z$$

is not an exact differential, since

$$\frac{\partial(2\ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x+z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y\cos x)\mathrm{d}x + (1 + \sin x)\mathrm{d}y,$$

is an exact differential and obtain the function of which it is the total differential

Solution

Firstly,

$$\frac{\partial}{\partial y}(x + y\cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x$$

and, hence, the expression is an exact differential.

Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.

Then,

$$\frac{\partial f}{\partial x} \equiv x + y\cos x \quad - - - - - - - - - - - (1)$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad - - - - - - - - - - - -(2)$$

Integrating (1) partially with respect to x gives

$$f(x,y) \equiv \frac{x^2}{2} + y\sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1 + \sin x.$$

That is,

$$\frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1,$$

Hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x,y) \equiv \frac{x^2}{2} + y\sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz+2)\mathrm{d}x + (xz+6y)\mathrm{d}y + (xy+3z^2)\mathrm{d}z,$$

is an exact differential and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(yz+2) \equiv \frac{\partial}{\partial x}(xz+6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz+6y) \equiv \frac{\partial}{\partial y}(xy+3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy+3z^2) \equiv \frac{\partial}{\partial z}(yz+2) \equiv y;$$

so that the given expression is an exact differential. Suppose it is the total differential of the function, $F(x, y, z)$. Then,

$$\frac{\partial F}{\partial x} \equiv yz+2, \quad ----------(1)$$
$$\frac{\partial F}{\partial y} \equiv xz+6y, \quad ---------(2)$$
$$\frac{\partial F}{\partial z} \equiv xy+3z^2. \quad ---------(3)$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.

Substituting into both (2) and (3) gives

$$
\begin{aligned}
xz + \frac{\partial A}{\partial y} &\equiv xz + 6y, \\
xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\frac{\partial A}{\partial y} &\equiv 6y, \quad ------(4) \\
\frac{\partial A}{\partial z} &\equiv 3z^2. \quad ------(5)
\end{aligned}
$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.
Substituting into (5) gives

$$\frac{\mathrm{d}B}{\mathrm{d}z} \equiv 3z^2,$$

which implies that $B(z) \equiv z^3 + \text{constant}$ and, hence,

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant}.$$

"JUST THE MATHS"

SLIDES NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

A.J.Hobson

UNIT 14.5
PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

We shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let t be subject to a small increment of δt, so that the variables, $x, y...$, are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively.

The corresponding increment, δf, in $f(x, y...)$, is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots$$

Note: It is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a Paraboloid)}$$

   and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a Cylinder)}.$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

From the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{2y}{9}.$$

From the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right)$$

$$= \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$u = xy + yz + zx, \;\; x = e^t, \;\; y = e^{-t}$ and $z = x + y$.

Solution

We use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \;\; \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \;\; \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the previous work to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function $f(x, y..)$ is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution

We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r},$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right).$$

$$\frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta\sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta \quad \text{and} \quad z = u\cos\phi.$$

Solution

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}$$

These give

$$\frac{\partial w}{\partial u} = 2x \sin\phi \cos\theta + 4y \sin\phi \sin\theta + 4z \cos\phi$$

$$= 2u\sin^2\phi\cos^2\theta + 4u\sin^2\phi\sin^2\theta + 4u\cos^2\phi,$$

$$\frac{\partial w}{\partial \theta} = -2xu \sin\phi \sin\theta + 4yu \sin\phi \cos\theta$$

$$= -2u^2\sin^2\phi \sin\theta \cos\theta + 4u^2\sin^2\phi \sin\theta \cos\theta$$

$$= 2u^2\sin^2\phi \sin\theta \cos\theta$$

$$\frac{\partial w}{\partial \phi} = 2xu \cos\phi \cos\theta + 4yu \cos\phi \sin\theta - 4zu \sin\phi$$

$$= 2u^2 \sin\phi \cos\phi\cos^2\theta + 4u^2 \sin\phi \cos\phi\sin^2\theta$$

$$-4u^2 \sin\phi \cos\phi$$

$$= 2u^2 \sin\phi \cos\phi \left(\cos^2\theta + 2\sin^2\theta - 2\right).$$

"JUST THE MATHS"

SLIDES NUMBER

14.6

PARTIAL DIFFERENTIATION 6 (Implicit functions)

by

A.J.Hobson

UNIT 14.6
PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule can be applied to implicit relationships of the form,

$$f(x, y) = \text{constant},$$

between two independent variables, x and y.

We may determine the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, $f(x, y)$ is a function of x and y in which both x and y are functions of x.

Differentiating $f(x, y) =$ constant with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If
$$f(x,y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$
determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If

$$f(x, y) \equiv x \sin(2x - 3y) + y \cos(2x - 3y),$$

determine an expression for $\frac{dy}{dx}$.

Solution

$$\frac{\partial f}{\partial x} = \sin(2x-3y)+2x\cos(2x-3y)-2y\sin(2x-3y)$$

and

$$\frac{\partial f}{\partial y} = -3x\cos(2x-3y)+\cos(2x-3y)+3y\sin(2x-3y).$$

Hence,

$$\frac{dy}{dx} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x, y, z) = \text{constant},$$

let x and y be independent of each other.

Then, $f(x, y, z)$ is a function of x, y and z, where x, y and z are **all** functions of x and y.

The chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0.$$

But,

$$\frac{\partial x}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0.$$

Hence,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

and, similarly,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If
$$f(x, y, z) \equiv z^2xy + zy^2x + x^2 + y^2 = 5,$$
determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = z^2y + zy^2 + 2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and
$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,
$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and
$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x, y, z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$

"JUST THE MATHS"

SLIDES NUMBER

14.7

PARTIAL DIFFERENTIATION 7
(Change of independent variable)

by

A.J.Hobson

UNIT 14.7
PARTIAL DIFFERENTIATON 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

The following technique would be necessary, for example, in changing from one geometrical reference system to another, especially with
"partial differential equations".

The method is an application of the chain rule for partial derivatives and is illustrated with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ, the following partial differential equations:

   (a)

$$\frac{\partial V}{\partial x} + 5\frac{\partial V}{\partial y} = 1;$$

   (b)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Solution

Both differential equations involve a function, $V(x,y)$, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Hence,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial r},$$

or

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta$$

and

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x}r\sin\theta + \frac{\partial V}{\partial y}r\cos\theta.$$

We eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

Hence, the differential equation, (a), becomes

$$(\cos\theta + 5\sin\theta)\frac{\partial V}{\partial r}$$

$$+\left(\frac{5\cos\theta}{r} - \sin\theta\right)\frac{\partial V}{\partial\theta} = 1.$$

To find the <u>second</u>-order derivatives of V with respect to x and y, we write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)[V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)[V].$$

From these,

$$\frac{\partial^2 V}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right),$$

which gives

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

$$-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}$$

$$+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r\cos\theta, \quad y = r\sin\theta \;\text{ and }\; z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta, \quad \text{and} \quad z = u\cos\phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u\cos\phi, \quad r = u\sin\phi, \quad \theta = \theta.$$

Hence, by using the previous example with z, r, θ in place of x, y, z, respectively, and u, ϕ in place of r, θ, respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$.

By using r, u, ϕ in place of y, r, θ, respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r} = \sin\phi\frac{\partial V}{\partial u} + \frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{1}{u^2 \sin^2 \phi}\frac{\partial^2 V}{\partial \theta^2}$$

$$+\frac{1}{u \sin\phi}\left[\sin\phi \frac{\partial V}{\partial u} + \frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}\right] = 0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2} + \frac{2}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}$$

$$+\frac{\cot\phi}{u^2}\frac{\partial V}{\partial \phi} + \frac{1}{u^2 \sin^2 \phi}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

“JUST THE MATHS”

SLIDES NUMBER

14.8

PARTIAL DIFFERENTIATION 8 (Dependent and independent functions)

by

A.J.Hobson

UNIT 14.8
PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x, y) \quad \text{and} \quad v \equiv v(x, y)$$

are two functions of two independent variables,
x and y.

Then, it is not normally possible to express u solely in terms of v, nor v solely in terms of u.

However, it may sometimes be possible

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then,

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1.$$

This gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then,

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**

THEOREM

Two functions, $u(x,y)$ and $v(x,y)$ are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove that $u(x,y)$ and $v(x,y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that

$$v \equiv v(u).$$

Then,

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\partial v}{\partial x} \div \frac{\partial u}{\partial x} \equiv \frac{\partial v}{\partial y} \div \frac{\partial u}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial y} \equiv 0.$$

That is,

$$J \equiv 0.$$

(b) Secondly, suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

In theory, we could express v in terms of u and x by eliminating y between $u(x,y)$ and $v(x,y)$.

We assume that

$$v \equiv A(u,x)$$

and show that $A(u,x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x .$$

Hence, if $J \equiv 0$,

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0.$$

On expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x.

Hence, x could be expressed in terms of u giving v as a function of u only.

If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u,v)}{\partial(x,y)}.$$

It is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x, y, z), \quad v \equiv v(x, y, z) \text{ and } w \equiv w(x, y, z)$$

are independent if and only if

$$J \equiv \frac{\partial(u, v, w)}{\partial(x, y, z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x + y}{x} \text{ and } v \equiv \frac{x - y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0.$$

2.

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x + 2y & 2x + 2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2 + 2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2 - 2y \not\equiv 0.$$

4.

$$u \equiv x^2 - 2y + z, \quad v \equiv x + 3y^2 - 2z, \quad \text{and} \quad w \equiv 5x + y + z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz + 4x - 30y + 4z + 25 \not\equiv 0.$$

"JUST THE MATHS"

SLIDES NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor's series)
for
(Functions of several variables)

by

A.J.Hobson

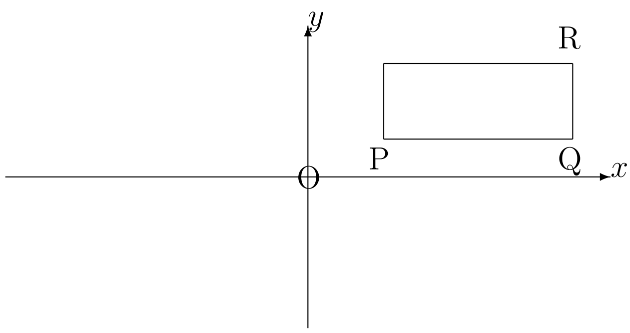
UNIT 14.9
PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

First, we obtain a formula for $f(x+h, y+k)$ in terms of $f(x, y)$ and its partial derivatives.

Let P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x+h, y)$ and $(x+h, y+k)$,
respectively.



(a) On the straight line from P to Q, y remains constant, so $f(x, y)$ behaves as a function of x only.

By Taylor's theorem for one independent variable,

$$f(x+h,y) = f(x,y) + f_x(x,y) + \frac{h^2}{2!} f_{xx}(x,y) + \ldots$$

Notes:

(i) $f_x(x,y)$ and $f_{xx}(x,y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$, respectively

(ii) In abbreviated notation,

$$f(\mathrm{Q}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + \frac{h^2}{2!} f_{xx}(\mathrm{P}) + \ . \ .$$

(b) On the straight line from Q to R, x remains constant, so $f(x,y)$ behaves as a function of y only.

Hence, $f(x+h,y+k) =$

$$f(x+h,y) + kf_x(x+h,y) + \frac{k^2}{2!} f_{xx}(x+h,y) + \ldots$$

Note:

In abbreviated notation,

$$f(\mathrm{R}) = f(\mathrm{Q}) + kf_y(\mathrm{Q}) + \frac{k^2}{2!} f_{yy}(\mathrm{Q}) + \ . \ .$$

(c) From the result in (a),

$$f_y(\mathrm{Q}) = f_y(\mathrm{P}) + hf_{yx}(\mathrm{P}) + \frac{h^2}{2!}f_{yxx}(\mathrm{P}) + \ldots$$

and

$$f_{yy}(\mathrm{Q}) = f_{yy}(\mathrm{P}) + hf_{yyx}(\mathrm{P}) + \frac{h^2}{2!}f_{yyxx}(\mathrm{Q}) + \ldots$$

(d) Substituting into (b) gives

$$f(\mathrm{R}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + kf_y(\mathrm{P}) +$$

$$\frac{1}{2!}\left[h^2 f_{xx}(\mathrm{P}) + 2hkf_{yx}(\mathrm{P}) + k^2 f_{yy}(\mathrm{P})\right] + \ldots$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) +$$

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \ \ldots$$

Notes:

(i) The equivalent result for a function of three variables is

$$f(x+h, y+k, z+l) =$$

$$f(x, y, z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(x, y, z) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x, y, z) +$$

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^3 f(x, y, z) + \ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h, y+k) = f(h, k) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(h, k) +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(h, k) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(h, k) + \ \ldots$$

(iii) Replacing x with $x-h$ and y with $y-k$ in (ii) gives

$$f(x,y) = f(h,k) + \left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^2 f(h,k) +$$

$$\frac{1}{3!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

This is the **"Taylor expansion of $f(x,y)$ about the point (a,b)"**

(iv) A special case of Taylor's series (for two independent variables) with $h=0$ and $k=0$ is

$$f(x,y) =$$

$$f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

This is called a **"MacLaurin's series"**

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x+1, y+\frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x, y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

$$f\left(x+1, y+\frac{\pi}{3}\right) = f\left(1, \frac{\pi}{3}\right) +$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1, \frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1, \frac{\pi}{3}\right) + \ldots$$

The first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y\cos xy = -\frac{\pi}{6} \text{ at } x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x\cos xy = \frac{1}{2} \text{ at } x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2\sin xy = -\frac{\pi^2\sqrt{3}}{18} \text{ at } x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy = \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \text{ at } x = 1, \ y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy = -\frac{\sqrt{3}}{2} \text{ at } x = 1, \quad y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two,

$$\sin xy =$$

$$\frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$

"JUST THE MATHS"

SLIDES NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

A.J.Hobson

UNIT 14.10
PARTIAL DIFFERENTIATION 10

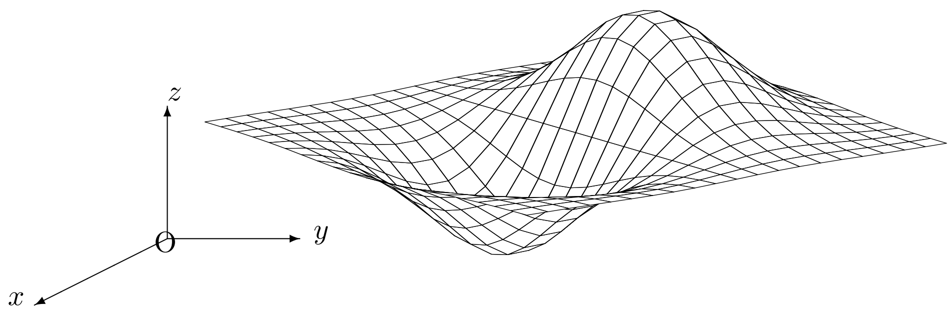
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

The equation,

$$z = f(x, y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x, y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x, y)$, is said to have a **"local maximum"** at a point, P(x_0, y_0, z_0), if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

DEFINITION 3

The function, $z = f(x, y)$, is said to have a **"local minimum"** at a point, P(x_0, y_0, z_0), if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

Note:

At a stationary point, P(x_0, y_0, z_0), on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

The following conditions are stated without proof:

(a) Sufficient conditions for a local maximum

A point, P(x_0, y_0, z_0), on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \text{ or } \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, P(x_0, y_0, z_0), on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \text{ or } \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are.

(iii) The values of z at the local maxima and local minima of the function $z = f(x, y)$ may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \text{ and } \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, - - - - - - - - - - - -(1) \\ -x + 2y &= 0. - - - - - - - - - - - - (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \ \text{ or } \ 3x(2x+1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0, \quad z = 4$ and $y = -\frac{1}{4}, \quad z = -\frac{65}{16}$, respectively. The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \ \frac{\partial^2 z}{\partial y^2} > 0, \ \text{and} \ \frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 3 > 0$$

and, therefore, the point $(0, 0, 4)$ is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -3 < 0$$

and, therefore, the point $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$ is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{and} \quad \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, we solve the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - - (2) \end{aligned}$$

From equation (2), $y = 0$ or $4x = 3y$.

Putting $y = 0$ in equation (1) gives $x = \pm 5$.

Stationary points occur at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into (1) gives $y = \pm 4$, $x = \pm 3$.
Stationary points occur at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points,

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y.$$

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-pt.
$(-3, -4, 300)$	-36	36	-48	negative	saddle-pt.

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion.

Outline Proof of the Sufficient Conditions

From Taylor's theorem for two variables,

$$f(a+h,b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) +$$

$$\frac{1}{2}\left(h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right) + \ \ldots ,$$

where h and k are small compared with a and b.

f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, the conditions for a local minimum at the point $(a, b, f(a,b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form,

$$Lh^2 + 2Mhk + Nk^2,$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If $LN - M^2 < 0$, it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a, b)$, $f_{yy}(a, b)$ and $f_{xy}(a, b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.

"JUST THE MATHS"

SLIDES NUMBER

14.11

PARTIAL DIFFERENTIATION 11 (Constrained maxima and minima)

by

A.J.Hobson

UNIT 14.11
PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

We consider the determination of local maxima and local minima for a function, $f(x, y, ...)$, subject to an additional constraint in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique for elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

   $$f(x, y) \equiv 3x^2 + 2y^2,$$

   subject to the constraint that $x + 2y - 1 = 0$.

   Solution

   Here, it is possible to eliminate either x or y by using the constraint.

If we eliminate x, we may write $f(x,y)$ as a function, $F(y)$, of y only.

$$f(x,y) \equiv F(y) \equiv 3(1-2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$$

Using the principles of maxima and minima for functions of a single independent variable,

$$F'(y) \equiv 28y - 12 \text{ and } F'' \equiv 28.$$

A local minimum occurs when $y = 3/7$ and, hence, $x = 1/7$.

The corresponding local minimum value of $f(x,y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x,y,z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x,y,z)$ as a function, $F(y,z)$, of y and z only.

$$f(x,y,z) \equiv F(y,z) \equiv (1-2y-3z)^2 + y^2 + z^2.$$

That is,

$$F(y,z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \ \text{ and } \ \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z.$$

A stationary value will occur when these are both equal to zero.

Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$.

The corresponding value of x is $1/14$, which gives a stationary value for $f(x,y,z)$ of $14/(14)^2 = \frac{1}{14}$.

Also,

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \quad \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial z} \equiv 12.$$

Thus,

$$\frac{\partial^2 F}{\partial y^2}.\frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2 + y^2 + z^2$, subject to the constraint that $x + 2y + 3z = 1$, at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad \text{and} \quad z = \frac{3}{14}.$$

Note:

Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x + 2y + 3z = 1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x, y, ...)$ subject to the constraint that $g(x, y, ...) = 0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x, y, ...$

The following example illustrates an alternative method for a function of two independent variables:

(a) Suppose that the function $z \equiv f(x, y)$ is subject to the constraint that $g(x, y) = 0$.

Then, z is effectively a function of x only.

The stationary values of z will be determined by the equation,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0.$$

(b) The total derivative of $z \equiv f(x, y)$ with respect to x, when x and y are not independent of each other, is given by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From $g(x, y) = 0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence, for all points on the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right) \frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will occur when

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x} = 0.$$

This may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then, $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0, \quad \text{and} \quad g(x, y) = 0.$$

Conclusion

The stationary values of the function $z \equiv f(x, y)$, subject to the constraint that $g(x, y) = 0$, occur at the points for which the function,

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y),$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**

Notes:

(i) To determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function, $z \equiv 3x^2 + 2y^2$, subject to the constraint that $x + 2y - 1 = 0$.

   Solution

   Firstly, we write

   $$\phi(x, y, \lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

   Then,

   $$\frac{\partial \phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 4y + 2\lambda \text{ and } \frac{\partial \phi}{\partial \lambda} \equiv x + 2y - 1.$$

   The third of these is already equal to zero; but we equate the first two to zero, giving

   $$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

   Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$.

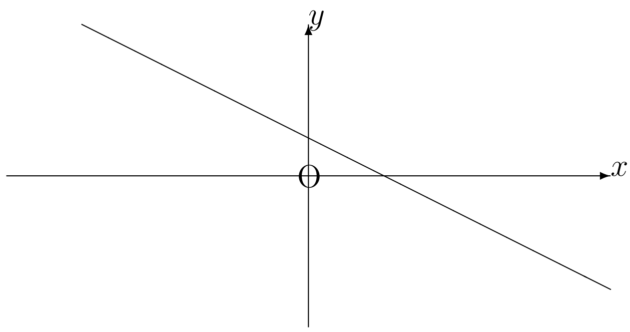
   Substituting this into the constraint, $7x - 1 = 0$.

   $$\text{Hence, } \; x = \frac{1}{7}, \; y = \frac{3}{7} \text{ and } \lambda = -\frac{6}{7}.$$

A single stationary point occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \text{ and } z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

The geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line with equation, $x + 2y - 1 = 0$.

For example, at the points $0.12, 0.44)$ and $(0.16, 0.42)$ on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

Solution

Firstly, we write

$$\phi(x,y,\lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial\phi}{\partial x} \equiv 3+2\lambda x, \quad \frac{\partial\phi}{\partial y} \equiv 4+2\lambda y \text{ and } \frac{\partial\phi}{\partial\lambda} \equiv x^2+y^2-1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \text{ and } y = \frac{2}{\lambda}.$$

Substituting into the constraint gives

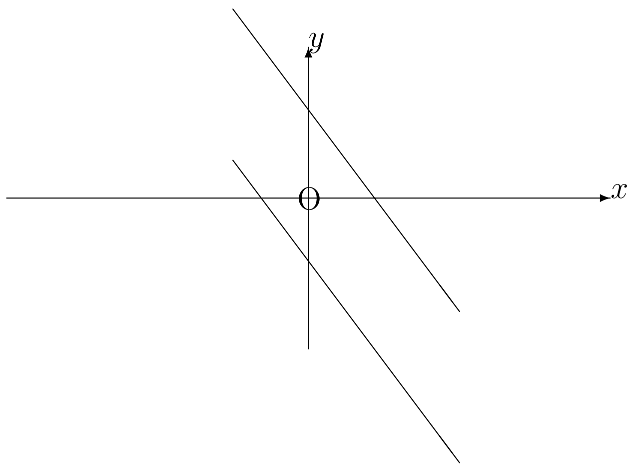
$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \text{ and hence } \lambda = \pm\frac{5}{2}.$$

Hence, $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.

Finally, the geometrical conditions suggest that we consider a straight line with equation,
$3x+4y=c$, (a constant) moving across the circle with equation, $x^2+y^2=1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$, will occur where the straight line touches the circle.

We have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2+y^2+z^2,$$

subject to the constraint that $x+2y+3z=1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda$$

and

$$\frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$.

Substituting these into the constraint gives
$14x = 1$.

Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } \lambda = -\frac{1}{7}.$$

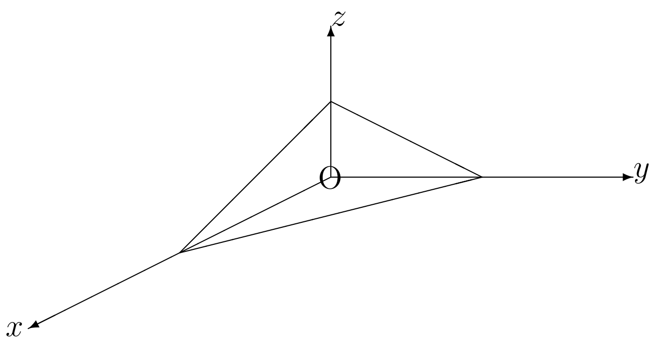
A single stationary point occurs, therefore, at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14}$$

and

$$w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2+y^2+z^2$, represents the square of the distance of a point (x, y, z) from the origin.

If the point is constrained to lie on a plane, this distance is bound to have a minimum value.

"JUST THE MATHS"

SLIDES NUMBER

14.12

PARTIAL DIFFERENTIATION 12
(The principle of least squares)

by

A.J.Hobson

UNIT 14.12
PARTIAL DIFFERENTIATION 12

THE PRINCIPLE OF LEAST SQUARES

14.12.1 THE NORMAL EQUATIONS

Suppose x and y, are known to obey a **"straight line law"** of the form $y = a + bx$, where a and b are constants to be found.

In an experiment to test this law, let n pairs of values be (x_i, y_i), where i = 1,2,3,...,n.

If the values, x_i, are **assigned** values, they are likely to be free from error.

The **observed** values, y_i, will be subject to experimental error

For the straight line of **"best fit"**, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

The Calculation

The y-deviation, ϵ_i, of the point, (x_i, y_i), is given by

$$\epsilon_i = y_i - (a + bx_i).$$

Hence,

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [y_i - (a + bx_i]^2 = P \text{ say.}$$

Regarding P as a function of a and b, it will be a minimum when

$$\frac{\partial P}{\partial a} = 0, \quad \frac{\partial P}{\partial b} = 0, \quad \frac{\partial^2 P}{\partial a^2} > 0 \text{ or } \frac{\partial^2 P}{\partial b^2} > 0,$$

and

$$\frac{\partial^2 P}{\partial a^2} . \frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b} \right)^2 > 0.$$

For these conditions,

$$\frac{\partial P}{\partial a} = -2 \sum_{i=1}^{n} [y_i - (a + bx_i] \text{ and } \frac{\partial P}{\partial b} = -2 \sum_{i=1}^{n} x_i [y_i + bx_i].$$

These will be zero when

$$\sum_{i=1}^{n} [y_i - (a + bx_i] = 0 \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i [y_i + bx_i] = 0 \quad - - - (2)$$

From (1),

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i = 0.$$

That is,

$$\sum_{i=1}^{n} y_i = na + b \sum_{i=1}^{n} x_i \quad - - - (3).$$

From (2),

$$\sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 \quad - - - (4).$$

Statements (3) and (4) (which must be solved for a and b) are called the **"normal equations"**.

A simpler notation for the normal equations is

$$\Sigma\, y = na + b\Sigma\, x;$$

$$\Sigma xy = a\Sigma x + b\Sigma\, x^2.$$

Eliminating a and b in turn,

$$a = \frac{\Sigma\, x^2 . \Sigma\, y - \Sigma x . \Sigma\, xy}{n\Sigma\, x^2 - (\Sigma\, x)^2} \quad \text{and} \quad b = \frac{n\Sigma\, xy - \Sigma\, x . \Sigma\, y}{n\Sigma\, x^2 - (\Sigma\, x)^2}.$$

The straight line, with equation $y = a + bx$, is called the **"regression line of y on x"**.

Note:
We also need the results that

$$\frac{\partial^2 P}{\partial a^2} = \sum_{i=1}^{n} 2 = 2n, \quad \frac{\partial^2 P}{\partial b^2} = \sum_{i=1}^{n} 2x_i^2, \text{ and } \frac{\partial^2 P}{\partial a \partial b} = \sum_{i=1}^{n} 2x_i.$$

The first two of these are clearly positive.

It may also be shown that

$$\frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

EXAMPLE

Determine the equation of the regression line of y on x for the following data, which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x has equation $y = a + bx$, where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus,

$$y = 0.176x - 0.645$$

14.12.2 SIMPLIFIED CALCULATION OF REGRESSION LINES

We consider a temporary change of origin to the point $(\overline{x}, \overline{y})$ where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\, y}{n} = a + b\frac{\Sigma\, x}{n}$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y is associated with the formulae

$$X = x - \overline{x} \ \text{ and } \ Y = y - \overline{y}.$$

In this system of reference, the regression line will pass through the origin.

The equation of the regression line is

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\ X = \Sigma\ (x - \overline{x}) = \Sigma\ x - \Sigma\ \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\ Y = \Sigma\ (y - \overline{y}) = \Sigma\ y - \Sigma\ \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\ XY}{\Sigma\ X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x}).$$

There may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5)$$

That is,

$$y = 0.176x - 0.638$$

"JUST THE MATHS"

SLIDES NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

A.J.Hobson

UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more (ordinary) derivatives of y with respect to x.

   Partial differential equations, which involve partial derivatives (see Units 14), are not discussed here.
   In what follows, we shall refer simply to "differential equations".
   For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x} \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the

dependent and independent variables which satisfies it.

Such a solution will not contain any derivatives.

The solution will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation).

The solution need not be an explicit formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution.

   A boundary condition requires the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

is an elementary example of an **"exact differential equation"**.

To find its solution, it is necessary only to integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples:

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3.$$

By direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2}, \quad \text{giving} \quad C = 4.$$

Hence,

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x \sin y$.

Hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification.

An explicit formula for y in terms of x may be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates differential equations which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x,

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

This simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat dx and dy as separate numbers.

We then rearrange the equation so that one side contains only y while the other side contains only x.

The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

given that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

Hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C.$$

Thus, $\quad C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$

Hence, $\quad \ln y = \ln x + \ln 3 \;\text{ or }\; y = 3x.$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C.

In the above example, we would write

$$\ln y = \ln x + \ln A \text{ simplifying to } y = Ax.$$

On applying the boundary condition, $6 = 2A$.

Therefore, $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x(4-x)}\,\mathrm{d}x.$$

Using partial fractions,

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\,\mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

"JUST THE MATHS"

SLIDES NUMBER

15.2

ORDINARY DIFFERENTIAL EQUATIONS 2 (First order equations (B))

by

A.J.Hobson

UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{\mathrm{d}y}{\mathrm{d}x}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0$$

and

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

From the first of these, a factor of λ could be cancelled.

From the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

We make the substitution

$$\boxed{y = vx},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}.$$

This always converts a homogeneous differential equation into one in which the variables can be separated.

The method will be illustrated by examples:

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2y,$$

given that $y = 6$ when $x = 6$.

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that

$$x\left(v + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x + 2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x,$$

giving

$$\ln(1+v) = \ln x + \ln A,$$

where A is an arbitrary constant.

An alternative form of this solution, without logarithms, is

$$Ax = 1 + v.$$

Substituting back $v = \frac{y}{x}$,

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and hence $A = 7$, giving $y = 7x^2 - x$.

2. Determine the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+(4x-y)=0.$$

Solution

If $y=vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that

$$(x+vx)\left(v+x\frac{\mathrm{d}v}{\mathrm{d}x}\right)+(4x-vx)=0.$$

That is,

$$(1+v)\left(v+x\frac{\mathrm{d}v}{\mathrm{d}x}\right)+(4-v)=0$$

or

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v-4}{v+1}.$$

On further rearrangement,

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v-4}{v+1}-v=\frac{-4-v^2}{v+1}.$$

On separating the variables,

$$\int\frac{v+1}{4+v^2}\,\mathrm{d}v=-\int\frac{1}{x}\,\mathrm{d}x$$

or

$$\frac{1}{2}\int\left[\frac{2v}{4+v^2}+\frac{2}{4+v^2}\right]\,\mathrm{d}v=-\int\frac{1}{x}\,\mathrm{d}x.$$

Hence,

$$\frac{1}{2}\left[\ln(4+v^2)+\tan^{-1}\frac{v}{2}\right]=-\ln x+C,$$

where C is an arbitrary constant.
Substituting back $v=\frac{y}{x}$,

$$\frac{1}{2}\left[\ln\left(4+\frac{y^2}{x^2}\right)+\tan^{-1}\left(\frac{y}{2x}\right)\right]=-\ln x+C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2+y^2)=0.$$

Solution

If $y=vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that

$$2vx^2\left(v+x\frac{\mathrm{d}v}{\mathrm{d}x}\right)+(x^2+v^2x^2)=0.$$

That is,

$$2v\left(v+x\frac{\mathrm{d}v}{\mathrm{d}x}\right)+(1+v^2)=0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x}=-(1+3v^2).$$

On separating the variables,

$$\int \frac{2v}{1+3v^2}\,\mathrm{d}x = -\int \frac{1}{x}\,\mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1+3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.

Hence,

$$(1+3v^2)^{\frac{1}{3}} = \frac{A}{x}.$$

On substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2+3y^2}{x^2}\right)^{\frac{1}{3}} = Ax.$$

This can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

"JUST THE MATHS"

SLIDES NUMBER

15.3

ORDINARY DIFFERENTIAL EQUATIONS 3 (First order equations (C))

by

A.J.Hobson

UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

EXAMPLE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3.$$

This equation has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **“integrating factor”**, but such factors cannot always be found by inspection.

(i) We shall now develop a formula for determining integrating factors for what are known as **“linear differential equations”**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **“linear”**.

RESULT

(a) Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x)\ \mathrm{d}x}$$

is always an integrating factor;

(b) On multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x)\ \mathrm{d}x}\right].$$

Proof

Suppose that the function $R(x)$ is an integrating factor.

Then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

We can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

This requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)} \ \mathrm{d}R(x) = \int \ P(x) \ \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int \ P(x) \ \mathrm{d}x.$$

That is,

$$R(x) = e^{\int \ P(x) \ \mathrm{d}x}.$$

The solution of the linear differential equation is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated;

C would introduce a constant factor of e^C in the above result, which would then cancel out on multiplying by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution

An integrating factor is

$$e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln x} = x.$$

On multiplying by the integrating factor,

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3.$$

Hence,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution

An integrating factor is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

This type of differential equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence, the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{-x^2}.$$

Solution
The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

$$\text{Hence,} \qquad \frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.

Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

Substituting $z = y^{-1}$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Thus,

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\, \mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

“JUST THE MATHS”

SLIDES NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

A.J.Hobson

UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

A second order ordinary linear differential equation (with constant coefficients) has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the present context means that $f(x) \equiv 0$.

That is, we consider

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants.

We should therefore expect two arbitrary constants in the solution of any second order linear differential equation with constant coefficients.

THE STANDARD GENERAL SOLUTION

The equivalent of

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of first order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0.$$

This could have been solved using an integrating factor of

$$e^{\int \frac{c}{b}\ \mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

We therefore make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \ \text{ and } \ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence,

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0.$$

In other words,

$$am^2 + bm + c = 0.$$

This quadratic equation is called the
"auxiliary equation", having the same (constant) coefficients as the orginal differential equation.

In general, the auxiliary equation will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions,

$$y = Ae^{m_1 x} \text{ and } y = Be^{m_2 x},$$

of the differential equation.

The linearity of the differential equation implies that the sum of any two solutions is also a solution.

Thus,

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution.

Since this contains two arbitrary constants, we take it to be the <u>general</u> solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form, though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified.

(iii) Boundary conditions usually consist of a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0,$$

which can be factorised as

$$(m + 2)(m + 3) = 0.$$

Its solutions are therefore $m = -2$ amd $m = -3$.

Hence, the differential equation has general solution

$$y = Ae^{-2x} + Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2Ae^{-2x} - 3Be^{-3x}.$$

Hence, $2 = A + B$ and $-5 = -2A - 3B$,

giving $A = 1$, $B = 1$.

The required particular solution is

$$y = e^{-2x} + e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, $am^2 + bm + c$ is a **"perfect square"** and hence $\equiv a(m - m_1)^2$

Apparently, the general solution of the differential equation is

$$y = Ae^{m_1 x} + Be^{m_1 x}.$$

But this does not genuinely contain two arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \quad \text{where} \quad C = A + B.$$

It will not, therefore, count as the general solution.

The fault seems to lie with the constants A and B rather than with m_1.

We now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

Substituting these into the differential equation, we obtain

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side is zero since m_1 is already a solution of the auxiliary equation.

The second term on the left hand side is also zero since the auxiliary equation is equivalent to

$$a(m - m_1)^2 = 0.$$

That is,

$$am^2 - 2am_1m + am_1^2 = 0.$$

Thus $b = -2am_1$.

We conclude that

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0,$$

with the result that

$$z = Ax + B,$$

where A and B are arbitrary constants.

Summary

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation

$$y = (Ax + B)e^{m_1 x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m+1)^2 = 0.$$

It has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using $m = \alpha \pm j\beta$ instead of $m = m_1$ and $m = m_2$, the general solution of the differential equation will be

$$y = Pe^{(\alpha + j\beta)x} + Qe^{(\alpha - j\beta)x},$$

where P and Q are arbitrary constants.

By properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos\beta x + j\sin\beta x) + Q(\cos\beta x - j\sin\beta x)]$$

or

$$y = e^{\alpha x}[(P + Q)\cos\beta x + j(P - Q)\sin\beta x].$$

Replacing $P + Q$ and $j(P - Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions.

The general solution is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 13 \times 1}}{2 \times 1}$$

$$= \frac{6 \pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x].$$

"JUST THE MATHS"

SLIDES NUMBER

15.5

ORDINARY DIFFERENTIAL EQUATIONS 5 (Second order equations (B))

by

A.J.Hobson

UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

Here, we examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

in which a, b and c are constants, but $f(x)$ is not identically equal to zero.

THE PARTICULAR INTEGRAL AND THE COMPLEMENTARY FUNCTION

(i) Let $y = u(x)$ be any particular solution of the differential equation; that is, it contains no arbitrary constants.

In the present context, we shall refer to such particular solutions as **"particular integrals"**.

Systematic methods of finding particular integrals will be discussed later.

Since $y = u(x)$ is a particular solution,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = f(x).$$

(ii) Let the substitution $y = u(x) + v(x)$ be made in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2} + b\frac{\mathrm{d}(u+v)}{\mathrm{d}x} + c(u+v) = f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu + a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = f(x).$$

Hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = 0.$$

Thus, $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation.

The complementary function complements the particular integral to provide the general solution.

Summary

$$\boxed{\text{General Solution} = \text{Partic. Int.} + \text{Comp. Functn.}}$$

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0$$

or

$$(m + 2)(m + 5) = 0,$$

having solutions, $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px+q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=px+q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side.

In this case, the particular integral is another expression consisting of a multiple of x and constant term.

The method is illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-11\frac{\mathrm{d}y}{\mathrm{d}x}+28y=84x-5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta.$$

This implies that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Substituting into the differential equation,

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0$$

or

$$(m - 4)(m - 7) = 0.$$

The solutions of the auxiliary equation are $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function must not be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

"JUST THE MATHS"

SLIDES NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6 (Second order equations (C))

by

A.J.Hobson

UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is

$$y = \frac{k}{c}$$

and

(b) when $f(x) \equiv px + q$, a **linear** function in which p and q are given constants, we obtain a particular integral by making a **"trial solution"**,

$$y = \alpha x + \beta.$$

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

In future, it will be best to determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

$$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

Note:

This is the trial solution even if q or r (or both) are zero.

EXAMPLE

Determine the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

Solution

The auxilary equation is

$$2m^2 - 7m - 4 = 0$$

or

$$(2m + 1)(m - 4) = 0.$$

The auxiliary equation has solutions

$$m = 4 \quad \text{and} \quad m = -\frac{1}{2}.$$

Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha x^2 + \beta x + \gamma,$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha.$$

We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing coefficents on both sides,

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10, \quad 4\alpha - 7\beta - 4\gamma = -23.$$

Hence,

$$\alpha = -1, \quad \beta = 1 \ \text{ and } \ \gamma = 3.$$

The particular integral is therefore

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p \sin kx + q \cos kx$, a **trigonometric** function in which p, q and k are given constants

$$\text{Trial solution :} \quad y = \alpha \sin kx + \beta \cos kx.$$

Note:

This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0.$$

The auxiliary equation has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x.$$

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x$$

$$+ 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x$$

$$\equiv 8\cos 3x - 19\sin 3x.$$

Comparing coefficients on both sides,

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by

$$\alpha = 1 \ \text{ and } \ \beta = -2.$$

The particular integral is therefore

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^{x}(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0$$

or

$$(3m + 1)^2 = 0.$$

The auxiliary equation has coincident solutions at $m = -\frac{1}{3}$.

The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x}.$$

This is gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}.$$

Substituting into the differential equation,

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which $\alpha = \frac{1}{2}$.
The particular integral is therefore

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p\sinh kx + q\cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sinh kx + \beta\cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93\cosh 5x - 75\sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

or

$$(m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$.

The complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha \sinh 5x + \beta \cosh 5x,$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha \cosh 5x + 5\beta \sinh 5x$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 25\alpha \sinh 5x + 25\beta \cosh 5x.$$

The left-hand side of the differential equation becomes,

$$25\alpha \sinh 5x + 25\beta \cosh 5x - 25\alpha \cosh 5x$$

$$-25\beta \sinh 5x + 6\alpha \sinh 5x + 6\beta \cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta) \sinh 5x + (31\beta - 25\alpha) \cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93. \end{aligned}$$

These equations are satisfied by $\alpha = 0$ and $\beta = 3$. The particular integral is therefore

$$y = 3 \cosh 5x.$$

Finally, the general solution is

$$y = 3 \cosh 5x + Ae^{2x} + Be^{3x}.$$

5. Combinations of Different Types of Function

In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

"JUST THE MATHS"

SLIDES NUMBER

15.7

ORDINARY DIFFERENTIAL EQUATIONS 7 (Second order equations (D))

by

A.J.Hobson

15.7.1 Problematic cases of particular integrals

UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function

We illustrate with examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$.

Hence, the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of

$$y = \alpha e^{2x}$$

gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}.$$

Substituting these into the differential equation,

$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$

That is, $0 \equiv e^{2x}$, which is impossible
Since $y = \alpha e^{2x}$ is unsatisfactory, we investigate, instead,

$$y = F(x)e^{2x},$$

where $F(x)$ is a function of x instead of a constant.
We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

Substituting these into the differential equation,

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x))\, e^{2x}$$

$$+ (-6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

$$\text{That is} \;\; F''(x) + F'(x) = 1.$$

This is satisfied by the function

$$F(x) \equiv x.$$

Thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown in other cases that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution

The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$. Hence, the complementary function is $A\sin x + B\cos x$, where A and B are arbitrary constants.

A trial solution of

$$y = \alpha\sin x + \beta\cos x$$

gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha\sin x - \beta\cos x.$$

Substituting into the differential equation, $0 \equiv \sin x$, which is impossible.

Here, we may try

$$y = x(\alpha\sin x + \beta\cos x),$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha\sin x + \beta\cos x + x(\alpha\cos x - \beta\sin x)$$

$$= (\alpha - \beta x)\sin x + (\beta + \alpha x)\cos x.$$

Therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha - \beta x)\cos x - \beta\sin x - (\beta + \alpha x)\sin x + \alpha\cos x$$

$$= (2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x.$$

Substituting into the differential equation, $\sin x \equiv$

$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x)$.

That is,

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Hence, $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2} x \cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions, $m = -\frac{1}{3}$.

Hence, the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both

$$e^{-\frac{1}{3}x} \ \textbf{and} \ xe^{-\frac{1}{3}x}$$

are contained in the complementary function.

Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, $50e^{-\frac{1}{3}x} \equiv$

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x}.$$

Hence $18\alpha = 50$ or $\alpha = \frac{25}{9}$.

An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$, which has solutions $m = 2$ and $m = 3$.

Hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However,

$$\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x}).$$

Thus, **part** of $\sinh 2x$ is contained in the complementary function and we must find a particular integral for each part separately.

(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation,

$$(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha)\, e^{2x} \equiv \frac{1}{2}e^{2x}.$$

This gives $\alpha = -\frac{1}{2}$.
(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation,

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.
The overall particular integral is thus,

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

"JUST THE MATHS"

SLIDES NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

A.J.Hobson

15.7.1 The substitution method

UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

We consider, here, cases of two <u>first</u> order differential equations which must be satisfied simultaneously. The technique will be illustrated by examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$\begin{aligned} 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, ------(1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. -----(2) \end{aligned}$$

Solution

First, we eliminate one of the dependent variables from the two equations.

In this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right).$$

Substituting this into equation (1),

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right)$$

$$+4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is,

$$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0.$$

The complementary function is $Ae^x + Be^{-2x}$, where A and B are arbitrary constants.

A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$.

Thus,

$$y = 2e^{-x} + Ae^x + Be^{-2x}.$$

From the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^{x} - 2Be^{-2x}\right)$$

$$+\frac{1}{3}\left(16e^{-x} + 8Ae^{x} + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been more difficult if the second differential equation had contained a term in $\frac{dz}{dx}$.
In such a case, we could eliminate $\frac{dz}{dx}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned}\frac{dz}{dx} + 2y &= e^{x}, - - - - - -(1)\\ \frac{dy}{dx} - 2z &= 1 + x, - - - - -(2)\end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.

Solution

From equation (2),

$$z = \frac{1}{2}\left[\frac{\mathrm{d}y}{\mathrm{d}x} - 1 - x\right].$$

Substituting into the first differential equation

$$\frac{1}{2}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2+4=0$, having solutions $m = \pm j2$.

The complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.

The particular integral will be of the form

$$y = pe^x + q,$$

where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require that $5p = 2$ and $4q = 1$.

The general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z,

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right]$$

$$= B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \quad \text{giving} \quad A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \quad \text{giving} \quad B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

"JUST THE MATHS"

SLIDES NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

A.J.Hobson

UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a convenient method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation.

The discussion is limited to the simpler kinds of example.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

We consider two simultaneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2.\end{aligned}$$

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\mathrm{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \mathrm{K}e^{\lambda t} = \mathrm{MK}e^{\lambda t} \quad \text{or} \quad \lambda\mathrm{K} = \mathrm{MK}.$$

We may recognise $\lambda\mathrm{K} = \mathrm{MK}$ as the condition that λ is an eigenvalue of the matrix, M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation",

$$|\mathrm{M} - \lambda \mathrm{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0.$$

This leads to a quadratic equation having either:

real and distinct solutions, ($\lambda = \lambda_1$ and $\lambda = \lambda_2$),

real and coincident solutions, (λ only),

or conjugate complex solutions, ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving two sets of homogeneous linear equations as follows:

$$\begin{aligned} (a - \lambda_1 k_1 + bk_2 &= 0, \\ ck_1 + (d - \lambda_1)k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say).

$$
\begin{aligned}
(a-\lambda_2)k_1 + bk_2 &= 0,\\
ck_1 + (d-\lambda_2)k_2 &= 0,
\end{aligned}
$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the type of solutions to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$
A\begin{bmatrix}1\\ \alpha\end{bmatrix}e^{\lambda_1 t} + B\begin{bmatrix}1\\ \beta\end{bmatrix}e^{\lambda_2 t};
$$

(b)

$$
\left\{(At+B)\begin{bmatrix}1\\ \alpha\end{bmatrix} + \frac{A}{b}\begin{bmatrix}0\\ 1\end{bmatrix}\right\}e^{\lambda t};
$$

or

(c)

$$
e^{lt}\left\{\begin{bmatrix}A\\ pA+qB\end{bmatrix}\cos mt + \begin{bmatrix}B\\ pB-qA\end{bmatrix}\sin mt\right\},
$$

where, in (c), $1 : \alpha = 1 : p + jq$ and $1 : \beta = 1 : p - jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 5x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1 + 2x_2. \end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda + 3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.

When $\lambda = -3$, we need to solve the homogeneous equations

$$
\begin{aligned}
-k_1 + 5k_2 &= 0, \\
-k_1 + 5k_2 &= 0,
\end{aligned}
$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.

The general solution is therefore

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}
$$

or, alternatively,

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},
$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - x_2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 + 3x_2.
\end{aligned}
$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \quad \text{or} \quad (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 - k_2 &= 0, \\ k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ (At + B) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - 5x_2,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 2x_1 + 3x_2.$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.

When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$(-1 - j3)k_1 - 5k_2 = 0,$$
$$2k_1 + (1 - j3)k_2 = 0,$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.

When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned}(-1 + j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1 + j3)k_2 &= 0,\end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.
The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A + 3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B - 3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.

Note:

From any set of simultaneous differential equations of the form

$$\begin{aligned}a\frac{dx_1}{dt} + b\frac{dx_2}{dt} + cx_1 + dx_2 = 0, \\ a'\frac{dx_1}{dt} + b'\frac{dx_2}{dt} + b'x_1 + c'x_2 = 0,\end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

"JUST THE MATHS"

SLIDES NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

A.J.Hobson

UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

The general solution of a single linear differential equation with constant coefficients is made up of a particular integral and a complementary function.

The complementary function is the general solution of the corresponding homogeneous differential equation.

A similar principle is now applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t).\end{aligned}$$

The method will be illustrated by an example.

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, - - - - - - - - - - - - - (1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -4x_1 - 5x_2 + g(t), - - - - (2)$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t).$$

This may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

where

$$\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \text{ and } \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding homogeneous system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX}.$$

The characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

and gives

$$\lambda(5+\lambda)+4=0,$$

or

$$\lambda^2+5\lambda+4=0,$$

or

$$(\lambda+1)(\lambda+4)=0.$$

Hence, $\lambda = -1$ or $\lambda = -4$.

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0. \end{aligned}$$

These are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0. \end{aligned}$$

These are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form.

The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution } \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t.$$

Equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \quad \text{and} \quad \mathrm{Q} = \mathrm{MP}.$$

Thus,

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}e^{2t}.$$

We require that

$$2\mathrm{P}e^{2t} = \mathrm{MP}e^{2t} + \mathrm{N}e^{2t}.$$

That is,

$$2\mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The matrix P may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N}.$$

In more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix} .\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18} \begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix} e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution} \quad X = P\sin t + Q\cos t.$$

We require that

$$P\cos t - Q\sin t = M(P\sin t + Q\cos t) + N\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$P = MQ \quad \text{and} \quad -Q = MP + N.$$

This means that

$$-Q = M^2Q + N \quad \text{or} \quad (M^2 + I)Q = -N.$$

Thus,

$$Q = -(M^2 + I)^{-1}N.$$

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

Hence,

$$Q = -\frac{1}{34}\begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Also,

$$P = MQ = \frac{1}{34}\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t}.$$

Equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1.

From earlier work,

$$\mathrm{Q} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Thus, $k = \frac{k+1}{4}$ giving $k = \frac{1}{3}$; and the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is, therefore,

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

The general solution is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

"JUST THE MATHS"

SLIDES NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

A.J.Hobson

UNIT 16.1 - LAPLACE TRANSFORMS 1 DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** is used to solve certain kinds of **"differential equation"**.

ILLUSTRATIONS

(a) A **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of $x(0)$.

We obtain a formula for x in terms of t which does not include any derivatives.

(b) A **"second order linear differential equation with constant coefficients"**,

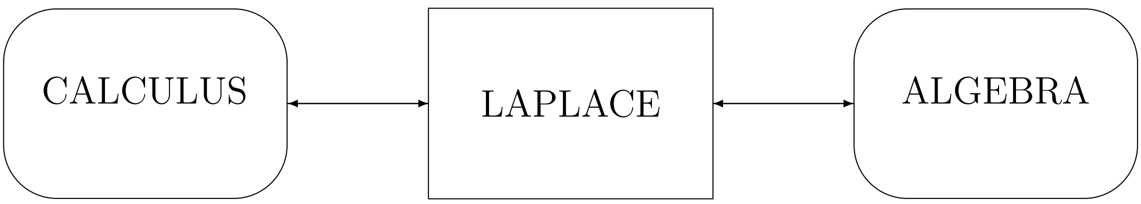
$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of $x(0)$ and $x'(0)$.

We obtain a formula for x in terms of t which does not include any derivatives

The method of Laplace Transforms converts a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



DEFINITION

The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes

(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations.

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n \, \mathrm{d}t = I_n \text{ say.}$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} \, \mathrm{d}t = \frac{n}{s}.I_{n-1}.$$

Note:

A **"decaying exponential"** will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st} \, \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:

This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st} e^{-at} \, \mathrm{d}t = \int_0^\infty e^{-(s+a)t} \, \mathrm{d}t$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:

$$L[e^{bt}] = \frac{1}{s-b}, \quad \text{assuming that} \quad s > b.$$

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st} \cos at \, \mathrm{d}t$$

$$= \left[\frac{e^{-st} \sin at}{a} \right]_0^\infty + \frac{s}{a} \int_0^\infty e^{-st} \sin at \, \mathrm{d}t$$

$$F(s) = 0 + \frac{s}{a} \left\{ \left[- \frac{e^{-st} \cos at}{a} \right]_0^\infty - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, \mathrm{d}t \right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2} . F(s).$$

That is,

$$F(s) = \frac{s}{s^2 + a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2 + a^2}.$$

4. $f(t) \equiv \sin at$.

The method is similar to that for $\cos at$, and we obtain

$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

1. LINEARITY

If A and B are constants, then

$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,

$$2t^5 - 7\cos 4t - 1.$$

Solution

$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2+4^2} - \frac{1}{s}$$

$$= \frac{240}{s^6} + \frac{7s}{s^2+16} - \frac{1}{s}.$$

2. THE LAPLACE TRANSFORM OF A DERIVATIVE

(a)

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:

$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

using Integration by Parts.

Thus,

$$L[f'(t)] = -f(0) + sL[f(t)].$$

as required.

(b)

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0).$$

Proof:

Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0).$$

This gives the required result on substituting the expression for $L[f'(t)]$.

Alternative Forms (Using $L[x(t)] = X(s)$)**:**

(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\,\mathrm{d}t.$

Note:

$$L[e^{bt}f(t)] = F(s-b).$$

EXAMPLE

Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.

Solution

First, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t$$

$$= \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right]$$

$$= -\frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\text{sinh}at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\text{cosh}at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at \cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.
$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.
$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0)$$

or

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$

"JUST THE MATHS"

SLIDES NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

A.J.Hobson

UNIT 16.2 - LAPLACE TRANSFORMS 2

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORM

A function of t, whose Laplace Transform is $F(s)$, is called the **"Inverse Laplace Transform"** of $F(s)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:

(i) Two functions which coincide for $t > 0$ will have the same Laplace Transform, so we can determine $L^{-1}[F(s)]$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear**.

Proof:

$L^{-1}\left[A.F(s) + B.G(s)\right]$ is a function of t whose Laplace Transform is $A.F(s) + B.G(s)$.

By the linearity of Laplace Transforms, such a function is

$$A.L^{-1}[F(s)] + B.L^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

We consider problems where the Laplace Transforms are **"rational functions of** s**"**.

Partial fractions will be used.

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Using partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s + 3 \equiv A(s+3) + Bs.$$

Note:
Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B.$$

Thus,

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}.$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2 + 9}.$$

Solution

$$f(t) = \frac{1}{3} \sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2 + 5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}} \sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Using partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs + C)(s+1).$$

Substituting $s = -1$,

$$5 = 5A \text{ implying that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24} t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t} \sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**.

This gives

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

To use the First Shifting Theorem, we must include $s+2$ in the numerator.

Thus, we write

$$F(s) = \frac{(s+2)-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+3^2} - \frac{2}{3}.\frac{3}{(s+2)^2+3^2}.$$

Hence, for $t > 0$,

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t = \frac{1}{3}e^{-2t}\left[3\cos 3t - 2\sin 3t\right].$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2+4s+8)}.$$

Solution

Using partial fractions,

$$\frac{8(s+1)}{s(s^2+4s+8)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2+4s+8) + (Bs+C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s+4}{s^2+4s+8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square.

This gives

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

On rearrangement,

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

From the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$.

Partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s+10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s=-2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s=6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

Note:

If we did not factorise the denominator,

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\cosh 4t + 3\sinh 4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform could certainly be obtained by using partial fractions.

But also, it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\,\mathrm{d}T = \int_0^t e^{(3T-2t)}\,\mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

"JUST THE MATHS"

SLIDES NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

A.J.Hobson

UNIT 16.3 - LAPLACE TRANSFORMS 3

DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking Laplace Transforms,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square.

$$X(s) \equiv \frac{3s + 12}{(s+2)^2 + 9} \equiv \frac{3(s+2) + 6}{(s+2)^2 + 9}.$$

$$X(s) \equiv 3.\frac{s+2}{(s+2)^2+9} + 2.\frac{3}{(s+2)^2+9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking Laplace Transforms,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2+1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2+1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

$$X(s) \equiv \frac{50}{(s^2+6s+9)(s^2+1)} + \frac{s+10}{s^2+6s+9}$$

or

$$X(s) \equiv \frac{50}{(s+3)^2(s^2+1)} + \frac{s+10}{(s+3)^2}.$$

Using partial fractions,

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Hence,

$50 \equiv A(s^2+1)+B(s+3)(s^2+1)+(Cs+D)(s+3)^2.$

Substituting $s = -3$,

$$50 = 10A, \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides,

$$0 = B + 9C + 6D. \quad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \quad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0 \quad (4).$$

These give $B = 3$ and $D = 4$, so that $C = -3$.
We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition,

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ when $t = 0$.

Solution

Taking Laplace Transforms,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1},$$

giving

$$(s^2 + 4s - 3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2 + 4s - 3) + (Bs + C)(s - 1).$$

Substituting $s = 1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may be given no boundary conditions at all.

Also, the boundary conditions given may not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we let $x(0) = A$ and $x'(0) = B$.

We obtain a solution in terms of A and B called the **General Solution**.

If non-standard boundary conditions are provided, we substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking Laplace Transforms,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0,$$

which may be written as

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0.$$

To apply the boundary conditions, we need

$$x'(t) = -\,2A\sin 2t + 2B\cos 2t.$$

Hence, $-\,3 = -\,A$ and $10 = -\,2B$ giving $A = 3$ and B $=$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

"JUST THE MATHS"

SLIDES NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

A.J.Hobson

16.4.1 An example of solving simultaneous linear differential equations

UNIT 16.4 - LAPLACE TRANSFORMS 4

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

We consider, here, a pair of differential equations of the form

$$a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y = f_1(t),$$
$$a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y = f_2(t).$$

To solve these equations simultaneously, we take the Laplace Transform of each equation.

This leads to two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t,\\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t,\end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations

$$\begin{aligned}sY(s) - 2 + 2X(s) &= \frac{1}{s-1},\\ sX(s) - 1 - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2}.\end{aligned}$$

That is,

$$\begin{aligned}2X(s) + sY(s) &= \frac{1}{s-1} + 2, \qquad (1)\\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. \qquad (2)\end{aligned}$$

Using (1) $\times\, 2 +$ (2) $\times\, s$, we obtain

$$(4+s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Using partial fractions, it may be shown that

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y.

However, in this example,

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t; \quad \text{and so.}$$

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

"JUST THE MATHS"

SLIDES NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

A.J.Hobson

UNIT 16.5 - LAPLACE TRANSFORMS 5

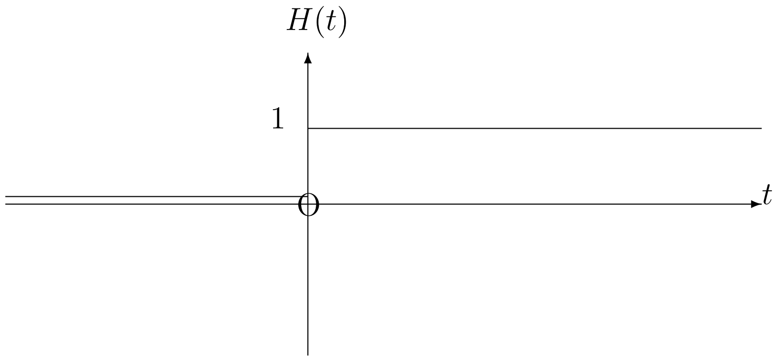
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The **"Heaviside step function"**, $H(t)$, is defined by the statements,

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note: $H(t)$ is undefined when $t = 0$.

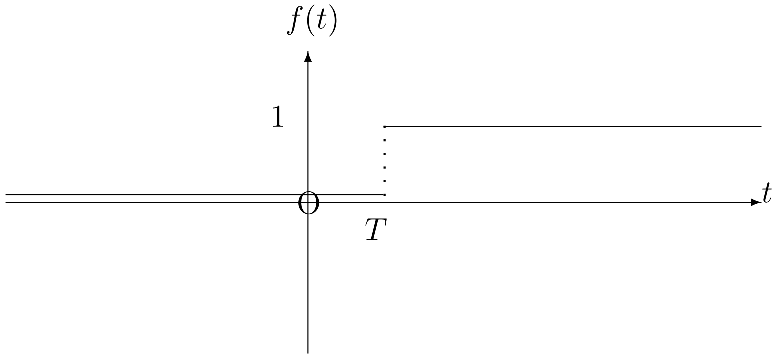


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



$f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$.

Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function, $H(t - T)$, is of importance in constructing what are known as **"pulse functions"**.

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t-T)] = \int_0^\infty \ e^{-st} H(t-T) \ \mathrm{d}t$$

$$= \int_0^T \ e^{-st}.0 \ \mathrm{d}t + \int_T^\infty \ e^{-st}.1 \ \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$,

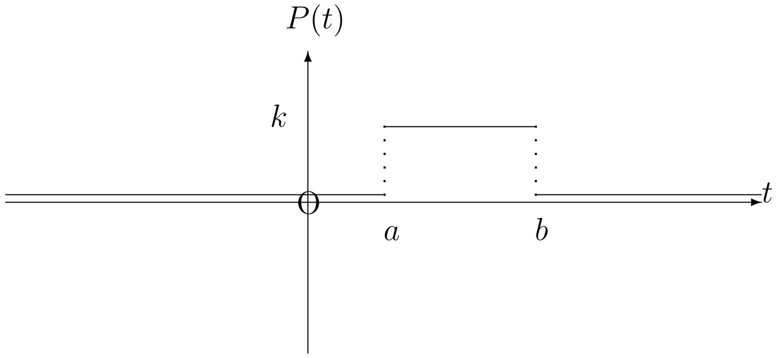
$$L[H(t)] = \frac{1}{s}.$$

This can be expected, since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration $b - a$ and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



In terms of Heaviside functions,

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$.

Hence, the above right-hand side = 0.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$.

Hence, the above right-hand side = 0.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$.

Hence, the above right-hand side $= k$.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right]$$

$$= k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse described above is defined as the area of the rectangle with base, $b - a$, and height, k.

That is,

$$\text{strength} = k(b - a).$$

(ii) The expression,

$$[H(t-a) - H(t-b)]f(t),$$

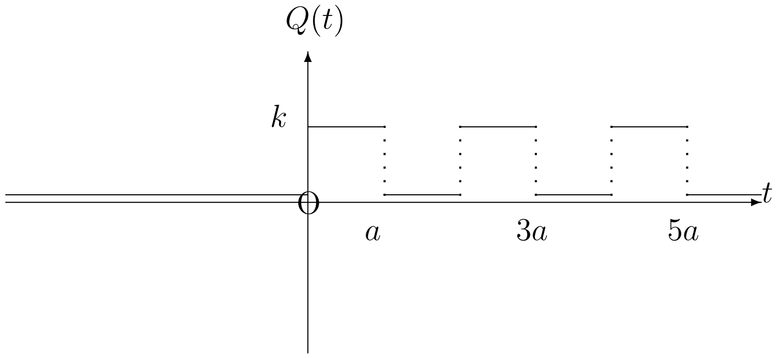
can be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) The expression,

$$H(t-a)f(t),$$

can be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, consider the train of rectangular pulses, $Q(t)$, in the following diagram:



The graph can be represented by the function

$$k\{[H(t) - H(t-a)] + [H(t-2a) - H(t-3a)]$$

$$+[H(t-4a) - H(t-5a)] + ...\}$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$

$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$

$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution, $u = t - T$, gives

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$

$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1 \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we can write

$$f(t) = (t-1)^2 H(t-1).$$

Using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s} L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression,

$$\frac{e^{-7s}}{s^2 + 4s + 5}.$$

Solution

First, we find the inverse Laplace Transform of the expression

$$\frac{1}{s^2 + 4s + 5} \equiv \frac{1}{(s+2)^2 + 1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t} \sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)} \sin(t-7), \quad t > 0.$$

"JUST THE MATHS"

SLIDES NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

UNIT 16.6 - LAPLACE TRANSFORMS 6

THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

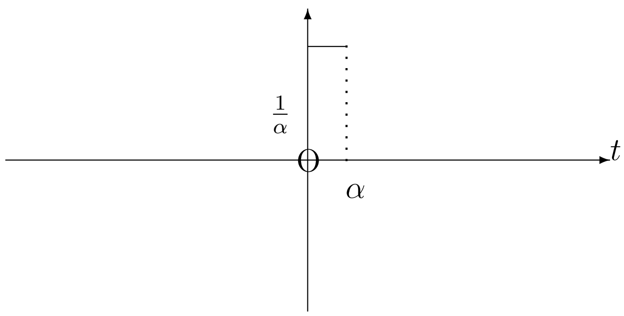
DEFINITION 1

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**.

A **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude $\frac{1}{\alpha}$. The strength of the pulse is then 1.



Using Heaviside step functions, this pulse is given by

$$\frac{H(t) - H(t - \alpha)}{\alpha}.$$

Allowing α to tend to zero, we obtain a unit impulse located at $t = 0$.

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$, is defined to be an impulse of unit strength, located at $t = 0$.

It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

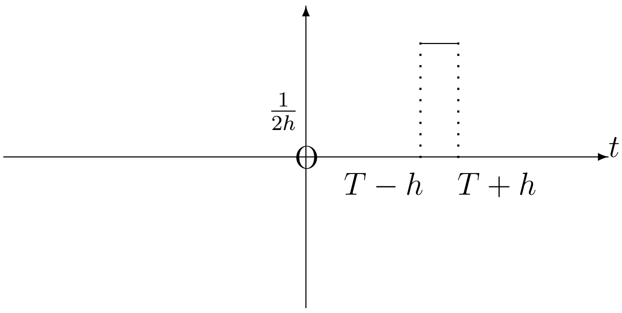
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t-T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t-T)\ \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t-T)\ \mathrm{d}t = f(T) \ \text{ if } \ a < T < b.$$

Proof:

Since $\delta(t-T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h\to 0}\int_{T-h}^{T+h} f(t)\delta(t-T)\ \mathrm{d}t$$

But in the <u>small</u> interval from $T-h$ to $T+h$, $f(t)$ is approximately constant and equal to $f(T)$.

Hence, the left-hand side may be written

$$f(T)\left[\lim_{h\to 0}\int_{T-h}^{T+h}\delta(T-T)\ \mathrm{d}t\right].$$

This reduces to $f(T)$, using note (ii) earlier

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular, $L[\delta(t)] = 1$.

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem just discussed,
with $f(t) = e^{-st}$,

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

given that $x = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$3sX(s) + 4X(s) = 1.$$

That is,

$$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

Hence,

$$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$.

The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function $f(t)$"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

We may refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, when $t = 0$

That is, we shall assume zero initial conditions.

Impulse response and transfer function

Consider, first, the differential equation

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

We refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c}.$$

This is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1.$$

Hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System Response for any Input

Assuming zero initial conditions, the Laplace Transform of the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t),$$

is given by

$$(as^2 + bs + c)X(s) = F(s).$$

Thus,

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$.

This may possibly be found using partial fractions; but it may, if necessary, be found by using the Convolution Theorem (Unit 16.1)

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s).$$

In other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be

$$u(t) = \frac{10e^{-t}}{3}.$$

Determine the response, $x(t)$, of the system to an input of

$$f(t) \equiv \sin 3t.$$

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9}.$$

Thus,

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \ \sin 3T.\frac{10e^{-(t-T)}}{3}\ \mathrm{d}T.$$

The integration here may be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \ \frac{10}{3}e^{-t}.e^{(1+j3)T}\ \mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
In this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

Consider the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

Suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots.

That is, the impulse response, $u(t)$, contains negative powers of e and, hence, tends to zero as t tends to infinity.

Suppose also that $f(t)$ is either $\cos\omega t$ or $\sin\omega t$.

These may be regarded, respectively, as the real and imaginary parts of the function $e^{j\omega t}$.

It may be shown that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)}$$

$$= \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using partial fractions,

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2}$$

$$+\frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t}$$

$$+\frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity.

The final term represents the steady state response.

We need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

In this example

$$U(j7) = \frac{1}{-47 + j21} = \frac{-47}{2650} - j\frac{21}{2650}.$$

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

"JUST THE MATHS"

SLIDES NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

by

A.J.Hobson

One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) The problem is to solve a second order linear differential equation with constant coefficients,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

(ii) We assume that the equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

has already been studied.

We examine the following example:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t},$$

given that $x = 0$ when $t = 0$.

Solution

The **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation

The integrating factor in the current example is e^{3t}, since the coefficient of x is 3.

We obtain,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right] = e^{5t}.$$

This is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] = e^{5t}.$$

Integrating both sides with respect to t,

$$xe^{3t} = \frac{e^{5t}}{5} + C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$, and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) We shall now examine a different way of setting out the above working in which the boundary condition is substituted earlier.

We multiply both sides of the differential equation by e^{3t} as before, then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] \mathrm{d}t = \int_0^t e^{5t} \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5} \text{ since } x = 0 \text{ when } t = 0.$$

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before

(iv) We consider, next, whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

Solution

We assume that an integrating factor for this equation is e^{st}, where s, at present, is unknown, but is assumed to be positive

Hence, we multiply throughout by e^{st} and integrate from 0 to t.

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right] \mathrm{d}t = \int_0^t e^{(s+9)t} \mathrm{d}t$$

$$= \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Using **"integration by parts"**, and the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = e^{st}x - s\int_0^t e^{st}x \mathrm{d}t.$$

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$$

$$= e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x \mathrm{d}t.$$

Substituting these results into the differential equation, we may collect together terms which involve

$$\int_0^t e^{st}x \mathrm{d}t \quad \text{and} \quad e^{st}$$

as follows:

$$(s^2+10s+21)\int_0^t e^{st}x \mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation.

That is,

$$(s^2 - 10s + 21).$$

(b) Using e^{-st} with $s > 0$, suppose we had integrated from 0 to ∞ instead of 0 to t.

The term,

$$e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right],$$

would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞.

We obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x\ \mathrm{d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty.$$

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Note:
This works only if $s > 9$, but we can easily assume that it is so.

Using $s^2 - 10s + 21 \equiv (s-3)(s-7)$,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{1}{(s-9)(s-3)(s-7)}.$$

Using partial fractions,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) Finally, it can be shown, by an independent method of solution, that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

We may conclude that the solution of the differential equation is closely linked to the integral,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.

"JUST THE MATHS"

SLIDES NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

A.J.Hobson

UNIT 16.8 - Z TRANSFORMS 1

DEFINITION AND RULES

16.8.1 INTRODUCTION

Linear Difference Equations

We consider "linear difference equations with constant coefficients".

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1 u_{n+1} + a_0 u_n = f(n);$$

a_0, a_1 are constants;

n is a positive integer;

$f(n)$ is a given function of n (possibly zero);

u_n is the general term of an infinite sequence of numbers,

$$\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots$$

DEFINITION 2

A second order linear difference equation with constant coefficients has the general form

$$a_2 u_{n+2} + a_1 u_{n+1} + a_0 u_n = f(n);$$

a_0, a_1, a_2 are constants;

n is an integer;

$f(n)$ is a given function of n (possibly zero);

u_n is the general term of an infinite sequence of numbers,

$$\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots$$

Notes:

(i) We shall assume that $u_n = 0$ whenever $n < 0$.

(ii) **"Boundary conditions"** will be given as follows:

The value of u_0 for a first-order equation;

The values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain simple difference equations may be solved by very elementary methods.

For example, to solve

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$,

we may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n.$$

By using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2,$$

$$u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \quad . \quad . \quad .$$

Hence,

$$u_n = n!$$

16.8.2 STANDARD Z-TRANSFORM DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \dots.$ is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

$$\{u_n\} \equiv \{a^n\},$$

where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ \dots. = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a},$$

by properties of infinite geometric series.

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} rz^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ \ldots .$$

or

$$\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ \ldots .\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ \ldots .\right) +$$

$$\left(\frac{1}{z^3} + \frac{1}{z^4} + \ \ldots .\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1-\frac{1}{z}} + \frac{\frac{1}{z^2}}{1-\frac{1}{z}} + \frac{\frac{1}{z^3}}{1-\frac{1}{z}} + \ldots . = \frac{1}{1-\frac{1}{z}}\left[\frac{\frac{1}{z}}{1-\frac{1}{z}}\right],$$

by properties of infinite geometric series.

Thus,

$$Z\{n\} = \frac{z}{(1-z)^2} = \frac{z}{(z-1)^2}$$

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region
$\{1\}$	$\frac{z}{z-1}$	$\|z\| > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\{e^{-nT}\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\|z\| > e^{-T}$
$\sin nT$ (T constant)	$\frac{z \sin T}{z^2-2z\cos T+1}$	$\|z\| > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\|z\| > 1$
1 for $n = 0$ 0 for $n > 0$ (Unit Pulse sequence)	1	All z
0 for $n = 0$ $\{a^{n-1}\}$ for $n > 0$	$\frac{1}{z-a}$	$\|z\| > \|a\|$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} (Au_r + Bv_r)z^{-r} \equiv A \sum_{r=0}^{\infty} u_r z^{-r} + B \sum_{r=0}^{\infty} v_r z^{-r}.$$

This is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1, \ 2, \ 3, \ 4, \ \ldots.$, are the terms $u_0, \ u_1, \ u_3, \ u_4, \ \ \ldots$ of the original sequence.

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1}z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ \ldots.,$$

since $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ \ldots.\right].$$

This is equivalent to the right-hand side.

Note:
A more general form of the First Shifting Theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to
$n = 0,\ 1,\ 2,\ \ldots\ldots, k-1$, are taken as zero and whose subsequent terms, corresponding to
$n = k,\ k+1,\ k+2,\ \ldots\ldots$ are the terms $u_0,\ u_1,\ u_2,\ \ldots\ldots$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
$\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ \ldots$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5} \ldots\ldots$$

$$\text{Hence,}\quad Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

That is,

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

This is equivalent to the right-hand side

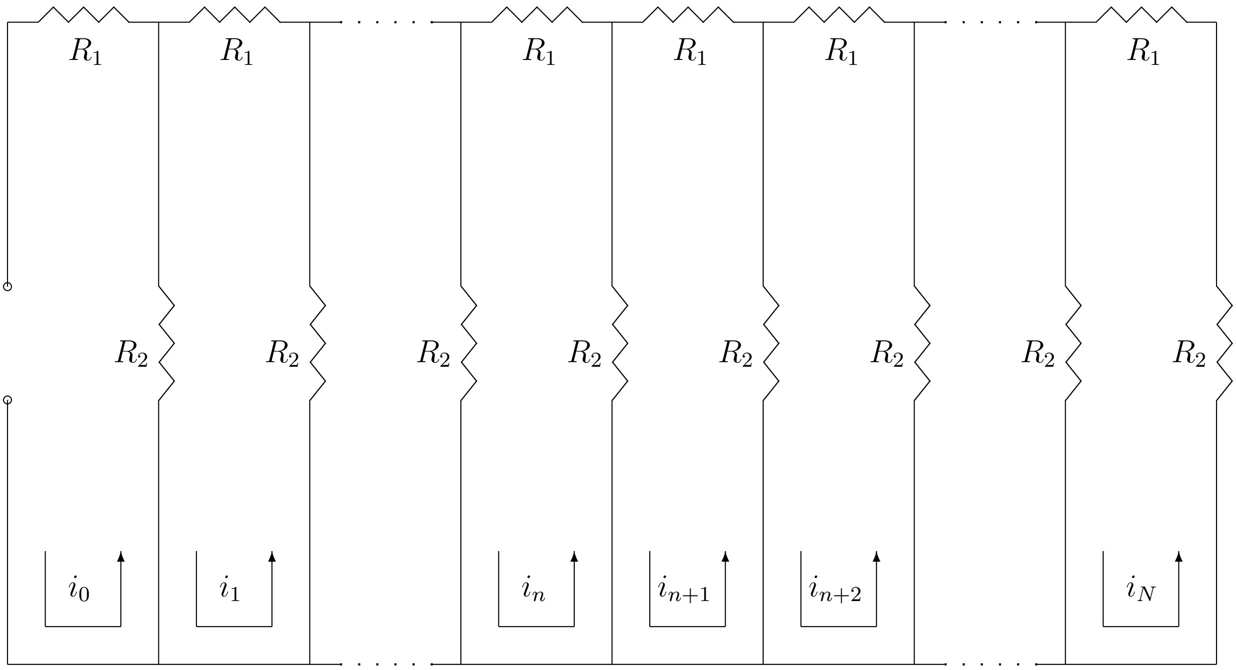
Note:

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

Z-TRANSFORMS
(AN ENGINEERING INTRODUCTION)

The mathematics of certain engineering problems leads to what are know as **"difference equations"**

For example, consider the following electrical **"ladder network"**.



It may be shown that

$$R_1 i_{n+1} + R_2(i_{n+1} - i_n) + R_2(i_{n+1} - i_{n+2}) = 0$$

where $0 \leq n \leq N - 2$.

The question which arises is "how can we determine a formula for i_n in terms of n" ?

"JUST THE MATHS"

SLIDES NUMBER

16.9

Z-TRANSFORMS 2
(Inverse Z-Transforms)

by

A.J.Hobson

UNIT 16.9 - Z-TRANSFORMS 2

INVERSE Z-TRANSFORMS

16.9.1 THE USE OF PARTIAL FRACTIONS

Here, we determine a sequence, $\{u_n\}$, of numbers whose Z-Transform is a known function, $F(z)$, of z.

Such a sequence is called the **"inverse Z-Transform of $F(z)$"** and may be denoted by

$$\mathrm{Z}^{-1}[F(z)].$$

For simple difference equations, $F(z)$ is usually a rational function of z.

EXAMPLES

1. Determine the inverse Z-Transform of the function,

$$F(z) \equiv \frac{10z(z+5)}{(z-1)(z-2)(z+3)}.$$

Solution

First, we recall that

$$Z\{a^n\} = \frac{z}{z-a}.$$

Then, we write

$$F(z) \equiv z.\left[\frac{10(z+5)}{(z-1)(z-2)(z+3)}\right].$$

This gives

$$F(z) \equiv z.\left[\frac{-15}{z-1} + \frac{14}{z-2} + \frac{1}{z+3}\right]$$

or

$$F(z) \equiv \frac{z}{z+3} + 14\frac{z}{z-2} - 15\frac{z}{z-1}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \left\{(-3)^n + 14(2)^n - 15\right\}.$$

2. Determine the inverse Z-Transform of the function,

$$F(z) \equiv \frac{1}{z-a}.$$

Solution

In this example, there is no factor, z, in the function, $F(z)$, and we shall see that it is necessary to make use of the first shifting theorem.

First, we may write

$$F(z) \equiv \frac{1}{z}\left[\frac{z}{z-a}\right]$$

and, since the inverse Z-Transform of the expression inside the brackets is a^n, the first shifting theorem tells us that

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ a^{n-1} & \text{when } n > 0. \end{cases}$$

Note:

This may now be taken as a standard result

3. Determine the inverse Z-Transform of the function,

$$F(z) \equiv \frac{4(2z+1)}{(z+1)(z-3)}.$$

Solution

Expressing $F(z)$ in partial fractions, we obtain

$$F(z) \equiv \frac{1}{z+1} + \frac{7}{(z-3)}.$$

Hence,

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ (-1)^{n-1} + 7.(3)^{n-1} & \text{when } n > 0. \end{cases}$$

"JUST THE MATHS"

SLIDES NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

A.J.Hobson

UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear Difference Equations may be solved by constructing the Z-Transform of both sides of the equation.

16.10.1 FIRST ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}}.$$

On rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z-\frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z-\frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z,

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1.$$

On rearrangement,

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}.$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation,

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\}.$$

On rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1}.$$

This may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)}.$$

From the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation,

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation,

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2}.$$

On rearrangement,

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2}.$$

Hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions,

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution to the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation,

$$u_{n+2} + 2u_n = 0,$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation,

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0.$$

On rearrangement,

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z-j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z+j\sqrt{2})}\right]$$

or

$$Z\{u_n\} \equiv z.\left[\frac{(1-j)}{2(z-j\sqrt{2})} + \frac{(1+j)}{2(z+j\sqrt{2})}\right].$$

Hence,

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}\left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$

"JUST THE MATHS"

SLIDES NUMBER

17.1

NUMERICAL MATHEMATICS 1
(Approximate solution of equations)

by

A.J.Hobson

UNIT 17.1
NUMERICAL MATHEMATICS 1

THE APPROXIMATE SOLUTION OF ALGEBRAIC EQUATIONS

17.1.1 INTRODUCTION

In the work which follows, we shall consider the solution of the equation $f(x) = 0$, where $f(x)$ is a given function of x.

The equation $f(x) = 0$ cannot, in general, be solved algebraically to give **exact** solutions.

It is often possible to find approximate solutions which are correct to any specified degree of accuracy.

Graphical methods of solving the equation,
$f(x) = 0$, use a graph of the equation $y = f(x)$ to determine where the graph crosses the x-axis.

"Iterative" methods involve **repeated** use of a technique to improve the accuracy of an approximate solution already obtained.

17.1.2 THE BISECTION METHOD

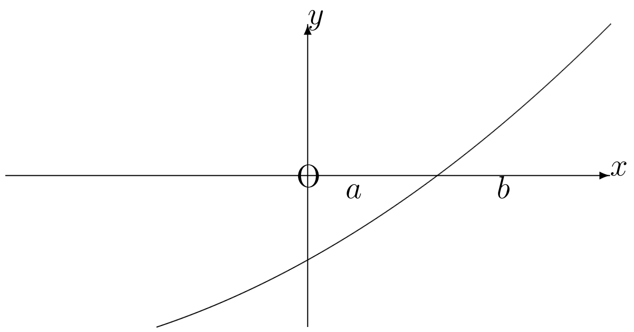
Suppose a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$.

We may obtain these,

(a) by trial and error;

(b) by sketching, roughly, the graph of the equation

$$y = f(x).$$



Whole number values of a and b will usually suffice.

If we let $c = (a + b)/2$, there are three possibilities;

(i) $f(c) = 0$, in which case we have solved the equation;

(ii) $f(c) < 0$, in which case there is a solution between b and c; so repeat the procedure with
b and c;

(iii) $f(c) > 0$, in which case there is a solution between a and c; so repeat the procedure with
a and c.

Each time we apply the method, we bisect the interval between the two numbers being used.

Eventually, the two numbers used will be very close together.

The method stops when two consecutive values of the mid-point agree with each other to the required number of decimal places or significant figures.

Convenient labels for the three numbers used at each stage (or iteration) are

$$a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, ..., a_n, b_n, c_n, ...$$

EXAMPLE

Determine, correct to three decimal places, the positive solution of the equation

$$e^x = x + 2.$$

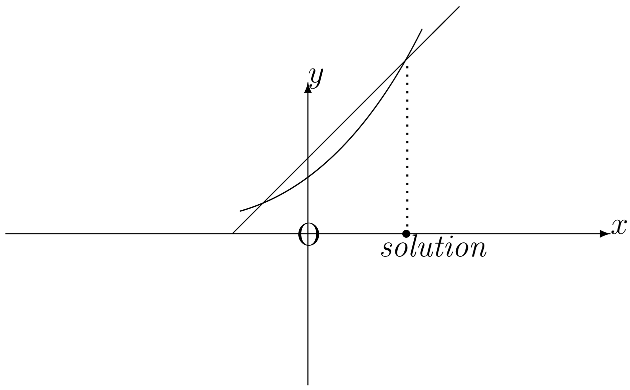
Solution

The graphs of

$$y = e^x \quad \text{and} \quad y = x + 2$$

intersect each other at a positive value of x.

This confirms that there is a positive solution to our equation.



But now let

$$f(x) = e^x - x - 2$$

and look for two numbers between which $f(x)$ changes sign, from positive to negative.

By trial and error, suitable numbers are 1 and 2, since

$$f(1) = e - 3 < 0 \quad \text{and} \quad f(2) = e^2 - 5 > 0.$$

The rest of the solution may be set out in the form of a table as follows:

n	a_n	b_n	c_n	$f(c_n)$
0	1.00000	2.00000	1.50000	0.98169
1	1.00000	1.50000	1.25000	0.24034
2	1.00000	1.25000	1.12500	- 0.04478
3	1.12500	1.25000	1.18750	0.09137
4	1.12500	1.18750	1.15625	0.02174
5	1.12500	1.15625	1.14062	- 0.01191
6	1.14063	1.15625	1.14844	0.00483
7	1.14063	1.14844	1.14454	- 0.00354
8	1.14454	1.14844	1.14649	0.00064
9	1.14454	1.14649	1.14552	- 0.00144

As a general rule, it is appropriate to work to two more places of decimals than that of the required accuracy and so, in this case, we work to five.

We can stop at stage 9 since c_8 and c_9 are the same value when rounded to three places of decimals.

The required solution is therefore $\boxed{x = 1.146}$

17.1.3 THE RULE OF FALSE POSITION

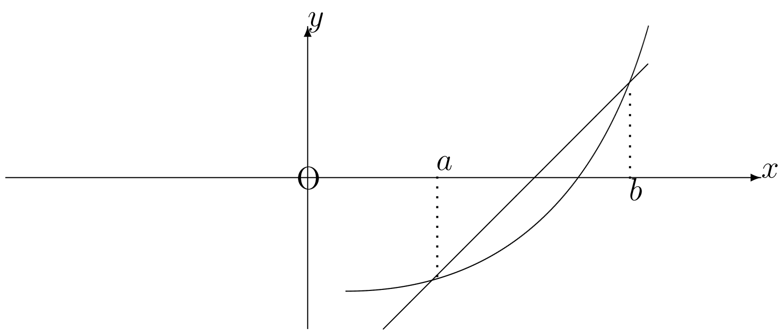
This method is commonly known by its Latin name **"Regula Falsi"**.

We consider that the two points, $(a, f(a))$ and $(b, f(b))$, on the graph of

$$y = f(x)$$

are joined by a straight line.

The point at which this straight line crosses the x-axis is taken as c.



From elementary co-ordinate geometry, the equation of the straight line is given by

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}.$$

Hence, when $y = 0$, we obtain

$$x = a - \frac{(b - a)f(a)}{f(b) - f(a)}.$$

That is,

$$x = \frac{a[f(b) - f(a)] - (b - a)f(a)}{f(b) - f(a)}.$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

In setting out the tabular form of a Regula Falsi solution, the c_n column uses the general formula

$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}.$$

EXAMPLE

For the equation

$$f(x) \equiv x^3 + 2x - 1 = 0,$$

use the Regula Falsi method, with $a_0 = 0$ and $b_0 = 1$, to obtain the first approximation, c_0, to the solution between $x = 0$ and $x = 1$.

Solution

We have $f(0) = -1$ and $f(1) = 2$, so that there is a solution between $x = 0$ and $x = 1$.

From the general formula,

$$c_0 = \frac{0 \times 2 - 1 \times (-1)}{2 - (-1)} = \frac{1}{3}.$$

Continuing with the method, we would observe that $f(1/3) < 0$ so that $a_1 = 1/3$ and $b_1 = 1$.

Note:
The Bisection Method would've given $c_0 = \frac{1}{2}$.

17.1.4 THE NEWTON-RAPHSON METHOD

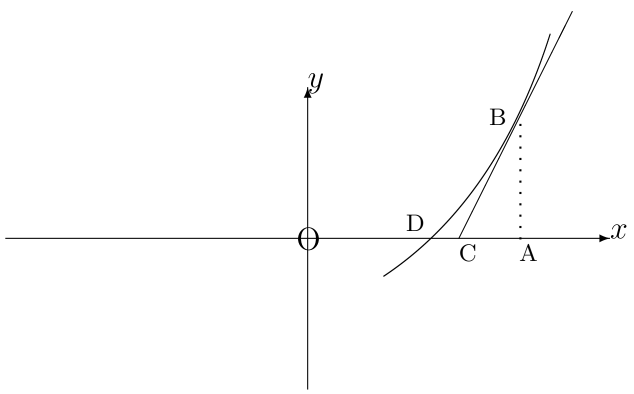
This method is based on first guessing an approximate solution $x = x_0$ to the equation $f(x) = 0$.

We then draw the tangent to the curve, whose equation is

$$y = f(x),$$

at the point, $x_0, f(x_0)$, to find out where this tangent crosses the x-axis.

The point obtained is normally a better approximation, x_1, to the solution.



In the diagram,

$$f'(x_0) = \frac{\text{AB}}{\text{AC}} = \frac{f(x_0)}{h}.$$

Hence,

$$h = \frac{f(x_0)}{f'(x_0)}.$$

Thus, a better approximation to the exact solution at the point, D, is given by

$$x_1 = x_0 - h.$$

Repeating the process, gives rise to the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Notes:

(i) To guess the starting approximation, x_0, it is normally sufficient to find a pair of whole numbers, a and b, such that $f(a) < 0$ and $f(b) > 0$; then we take $x_0 = (a+b)/2$.

In some exercises, an alternative starting approximation may be suggested in order to speed up the rate of convergence to the final solution.

(ii) There are situations where the Newton-Raphson Method fails to give a better approximation:

When the tangent to the curve has a very small gradient, it meets the x-axis at a relatively great distance from the previous approximation.

EXAMPLE

Use the Newton-Raphson method to calculate $\sqrt{5}$ correct to three places of decimals.

Solution

We are required to solve the equation

$$f(x) \equiv x^2 - 5 = 0.$$

By trial and error, a solution exists between $x = 2$ and $x = 3$, since $f(2) = -1 < 0$ and $f(3) = 4 > 0$.

Hence, we use $x_0 = 2.5$

Furthermore,

$$f'(x) = 2x,$$

so that

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Thus,

$$x_1 = 2.5 - \frac{1.25}{5} = 2.250,$$

$$x_2 = 2.250 - \frac{0.0625}{4.5} \simeq 2.236,$$

$$x_3 = 2.236 - \frac{-0.000304}{4.472} \simeq 2.236$$

At each stage, we round off the result to the required number of decimal places and use the rounded figure in the next iteration.

The last two iterations give the same result to three places of decimals and this is therefore the required result.

"JUST THE MATHS"

SLIDES NUMBER

17.2

NUMERICAL MATHEMATICS 2
(Approximate integration (A))

by

A.J.Hobson

UNIT 17.2 - NUMERICAL MATHEMATICS 2

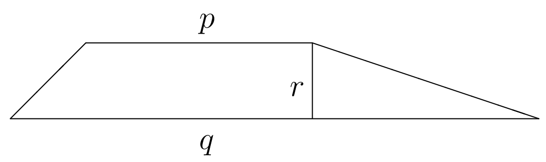
APPROXIMATE INTEGRATION (A)

17.2.1 THE TRAPEZOIDAL RULE

The Trapezoidal Rule is based on the formula for the area of a trapezium.

If the parallel sides of a trapezium are of length p and q, while the perpendicular distance between them is r, then the area A is given by

$$A = \frac{r(p+q)}{2}.$$



Suppose that the curve

$$y = f(x)$$

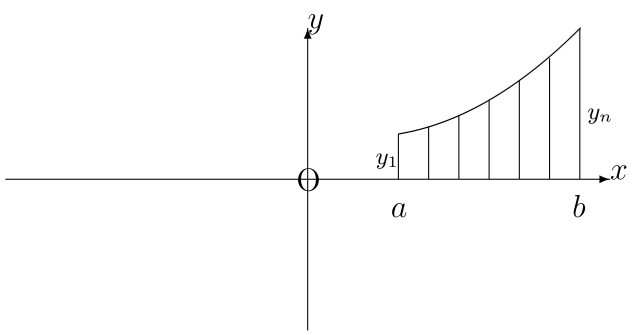
lies wholly above the x-axis between
$x = a$ and $x = b$.

The definite integral,

$$\int_a^b f(x)\,\mathrm{d}x,$$

can be regarded as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

Let this area be divided into several narrow strips of equal width h by marking the values $x_1, x_2, x_3, \ldots\ldots, x_n$ along the x-axis (where $x_1 = a$ and $x_n = b$) and drawing in the corresponding lines of length $y_1, y_2, y_3, \ldots\ldots, y_n$ parallel to the y-axis



Each narrow strip of width h may be considered approximately as a trapezium whose parallel sides are of lengths y_i and y_{i+1} where $i = 1, 2, 3, \ldots\ldots, n-1$.

Thus, the area under the curve, and hence the value of the definite integral, approximates to

$$\frac{h}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + \ldots\ldots + (y_{n-1} + y_n)].$$

That is,

$$\int_a^b f(x)\ \mathrm{d}x \simeq \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})].$$

Alternatively,

$$\int_a^b f(x)\ \mathrm{d}x = \frac{h}{2}[\text{First} + \text{Last} + 2 \times \ \text{The Rest}].$$

Note:
Care must be taken at the beginning to ascertain whether or not the curve $y = f(x)$ crosses the x-axis between $x = a$ and $x = b$.

If it does, then allowance must be made for the fact that areas below the x-axis are negative and should be calculated separately from those above the x-axis.

EXAMPLE

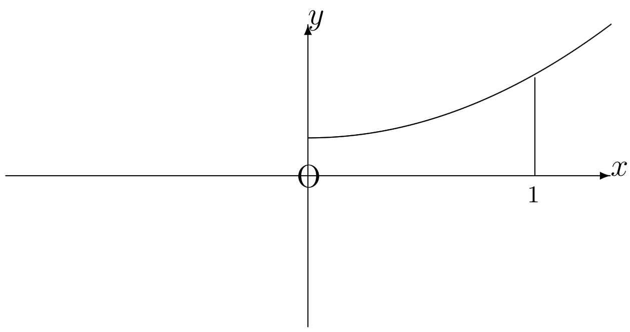
Use the trapezoidal rule with five divisions of the x-axis in order to evaluate, approximately, the definite integral:

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

First we make up a table of values as follows:

x	0	0.2	0.4	0.6	0.8	1.0
e^{x^2}	1	1.041	1.174	1.433	1.896	2.718



Then, using $h = 0.2$,

$$\int_0^1 e^{x^2}\, \mathrm{d}x$$

$$\simeq \frac{0.2}{2}[1 + 2.718 + 2(1.041 + 1.174 + 1.433 + 1.896)]$$

$$\simeq 1.481$$

"JUST THE MATHS"

SLIDES NUMBER

17.3

NUMERICAL MATHEMATICS 3
(Approximate integration (B))

by

A.J.Hobson

17.3.1 Simpson's rule

UNIT 17.3
NUMERICAL MATHEMATICS 3

APPROXIMATE INTEGRATION (B)

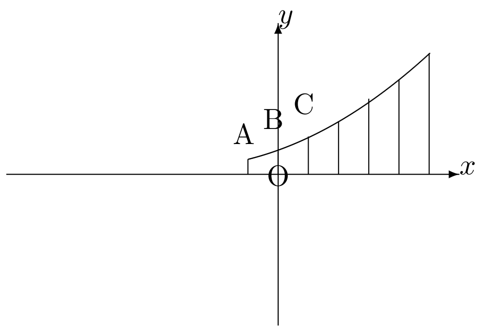
17.3.1 SIMPSON'S RULE

A better approximation to

$$\int_a^b f(x)\mathrm{d}x$$

than that provided by the Trapezoidal rule (Unit 17.2) may be obtained by using an **even** number of narrow strips of width, h, and considering them in pairs.

First, we examine a **special** case as in the following diagram:



A, B and C have co-ordinates $(-h, y_1)$, $(0, y_2)$ and (h, y_3) respectively.

The arc of the curve passing through the points A$(-h, y_1)$, B$(0, y_2)$ and C(h, y_3) may be regarded as an arc of a parabola whose equation is

$$y = Lx^2 + Mx + N.$$

L, M and N must satisfy the following equations:

$$\begin{aligned} y_1 &= Lh^2 - Mh + N, \\ y_2 &= N, \\ y_3 &= Lh^2 + Mh + N. \end{aligned}$$

Also, the area of the first pair of strips is given by

$$\text{Area} = \int_{-h}^{h} (Lx^2 + Mx + N)\,\mathrm{d}x$$

$$= \left[L\frac{x^3}{3} + M\frac{x^2}{2} + Nx\right]_{-h}^{h}$$

$$= \frac{2Lh^3}{3} + 2Nh$$

$$= \frac{h}{3}[2Lh^2 + 6N].$$

From the earlier simultaneous equations,

$$\text{Area} = \frac{h}{3}[y_1 + y_3 + 4y_2].$$

But the area of **every** pair of strips will be dependent only on the three corresponding y co-ordinates, together with the value of h.

Hence, the area of the next pair of strips will be

$$\frac{h}{3}[y_3 + y_5 + 4y_2],$$

and the area of the pair after that will be

$$\frac{h}{3}[y_5 + y_7 + 4y_4].$$

Thus, the total area is given by

$$\frac{h}{3}[y_1 + y_n + 4(y_2 + y_4 + y_6 + \ldots) + 2(y_3 + y_5 + y_7 + \ldots)$$

This is usually interpreted as

$$\frac{h}{3}[\text{First}+\text{Last}+4\times \text{ even numbered } y'\text{s} + 2\times \text{ remaining } \ y \ 's],$$

or

$$\text{Area} = \frac{h}{3}[\text{F} + \text{L} + 4\text{E} + 2\text{R}]$$

This result is known as **"Simpson's rule"**.

Notes:

(i) Simpson's rule provides an approximate value of the definite integral

$$\int_a^b f(x)\, \mathrm{d}x$$

provided the curve does not cross the x-axis between $x = a$ and $x = b$;

(ii) If the curve **does** cross the x-axis between $x = a$ and $x = b$, it is necessary to consider separately the positive parts of the area above the x-axis and the negative parts below the x-axis.

EXAMPLES

1. Working to a maximum of three places of decimals throughout, use Simpson's rule with ten divisions to evaluate, approximately, the definite integral

$$\int_0^1 e^{x^2}\, \mathrm{d}x.$$

Solution

x_i	$y_i = e^{x_i^2}$	F & L	E	R
0	1	1		
0.1	1.010		1.010	
0.2	1.041			1.041
0.3	1.094		1.094	
0.4	1.174			1.174
0.5	1.284		1.284	
0.6	1.433			1.433
0.7	1.632		1.632	
0.8	1.896			1.896
0.9	2.248		2.248	
1.0	2.718	2.718		
	F + L $\rightarrow$	3.718	7.268	5.544
	4E $\rightarrow$	29.072	$\times 4$	$\times 2$
	2R $\rightarrow$	11.088	29.072	11.088
	(F + L) + 4E + 2R $\rightarrow$	43.878	//////	//////

Hence,

$$\int_0^1 e^{x^2}\, \mathrm{d}x \simeq \frac{0.1}{3} \times 43.878 \simeq 1.463$$

2. Working to a maximum of three places of decimals throughout, use Simpson's rule with eight divisions between $x = -1$ and $x = 1$ and four divisions between $x = 1$ and $x = 2$ in order to evaluate, approximately, the area between the curve whose equation is

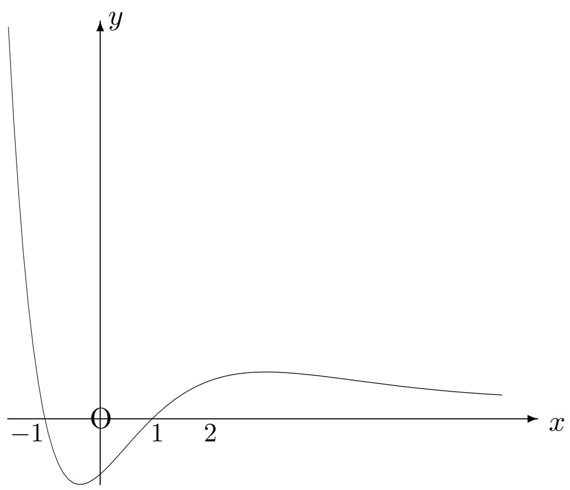
$$y = (x^2 - 1)e^{-x}$$

and the x-axis from $x = -1$ to $x = 2$.

Solution

The curve crosses the x-axis when $x = -1$ and $x = 1$.

y is negative between $x = -1$ and $x = 1$ and positive between $x = 1$ and $x = 2$.



(a) The Negative Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
-1	0	0		
-0.75	-0.926		-0.926	
-0.5	-1.237			-1.237
-0.25	-1.204		-1.204	
0	-1			-1
0.25	-0.730		-0.730	
0.50	-0.455			-0.455
0.75	-0.207		-0.207	
1	0	0		
F + L $\rightarrow$		0	-2.860	-2.692
4E $\rightarrow$		-11.440	$\times 4$	$\times 2$
2R $\rightarrow$		-5.384	-11.440	-5.384
(F + L) + 4E + 2R $\rightarrow$		-16.824	//////	//////

(b) The Positive Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
1	0	0		
1.25	0.161		0.161	
1.5	0.279			0.279
1.75	0.358		0.358	
2	0.406	0.406		
	F + L →	0.406	0.519	0.279
	4E →	2.076	×4	×2
	2R →	0.558	2.076	0.558
	(F + L) + 4E + 2R →	3.040	//////	//////

The total area is thus

$$\frac{0.25}{3} \times (16.824 + 3.040) \simeq 1.655$$

"JUST THE MATHS"

SLIDES NUMBER

17.4

NUMERICAL MATHEMATICS 4
(Further Gaussian elimination)

by

A.J.Hobson

UNIT 17.4
NUMERICAL MATHEMATICS 4
FURTHER GAUSSIAN ELIMINATION

The **elementary** method of Gaussian Elimination for simultaneous linear equations was discussed in Unit 9.4.

We introduce, here, a more **general** method, suitable for use with sets of equations having **decimal** coefficients.

17.4.1 GAUSSIAN ELIMINATION BY "PARTIAL PIVOTING" WITH A CHECK COLUMN

First, we consider an example in which the coefficients are **integers**.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We may set out the solution, in the form of a **table** indicating each of the **"pivot elements"** in a box:

	x	y	z	constant	Σ
	$\boxed{2}$	1	1	3	7
$\frac{1}{2}$	1	-2	-1	2	0
$\frac{3}{2}$	3	-1	1	8	11
		$\boxed{\frac{-5}{2}}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{-7}{2}$
1		$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
			1	3	4

INSTRUCTIONS

(i) Divide the coefficients of x in lines 2 and 3 by the coefficient of x in line 1 and write the respective results at the side of lines 2 and 3; (that is, $\frac{1}{2}$ and $\frac{3}{2}$ in this case).

(ii) Eliminate x by subtracting $\frac{1}{2}$ times line 1 from line 2 and $\frac{3}{2}$ times line 1 from line 3.

(iii) Repeat the process starting with lines 4 and 5.

(iv) line 6 implies that $z = 3$ and by substitution back into earlier lines, we obtain the values $y = -2$ and $x = 1$.

OBSERVATIONS

Difficulties can arise for pivot elements which are very small compared with the other quantities in the same column; the errors involved in dividing by small numbers are likely to be large.

A better choice of pivot element would be the one with the **largest** numerical value in its column.

In the next example, the working will be carried out using fractional quantities though, in practice,
decimals would normally be used instead.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 5, \\ 2x + y - z &= 1, \\ x + 3y - z &= 4. \end{aligned}$$

Solution

	x	y	z	constant	Σ
$\frac{1}{2}$	1	-1	2	5	7
	$\boxed{2}$	1	-1	1	3
$\frac{1}{2}$	1	3	-1	4	7

On eliminating x, we obtain the new table:

	y	z	constant	Σ
$\frac{-3}{5}$	$\frac{-3}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
	$\boxed{\frac{5}{2}}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Eliminating y takes us to the final table as follows:

z	constant	Σ
$\frac{11}{5}$	$\frac{33}{5}$	$\frac{44}{5}$

We conclude that
$11z = 33$ and, hence, $\boxed{z = 3}$.

Substituting into the second table (either line will do), we have
$5y - 3 = 7$ and, hence, $\boxed{y = 2}$.

Substituting into the original table (any line will do), we have
$x - 2 + 6 = 5$ so that $\boxed{x = 1}$.

Notes:

(i) In questions which involve decimal quantities stated to n decimal places, the calculations should be carried out to $n+2$ decimal places to allow for rounding up.

(ii) A final check on accuracy is obtained by adding the original three equations together and verifying that the solution obtained also satisfies this further equation.

In the recent example, this would be

$$4x + 3y = 10,$$

and is satisfied by $x = 1$, $y = 2$.

(iii) It is not essential to set out the solution in the form of separate tables (at each step) with their own headings. A continuation of the first table is acceptable.

"JUST THE MATHS"

SLIDES NUMBER

17.5

NUMERICAL MATHEMATICS 5
(Iterative methods)
for solving
(simultaneous linear equations)

by

A.J.Hobson

UNIT 17.5 - NUMERICAL MATHEMATICS 5

ITERATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS

17.5.1 INTRODUCTION

An **"iterative method"** is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy.

For example, if results are required to five places of decimals, the number of **"iterations"** is continued until two consecutive iterations give the same result when rounded off to that number of decimal places.

It is usually enough for the calculations themselves to be carried out to **two extra** places of decimals.

A similar interpretation holds for accuracy which requires a certain number of **significant figures**.

We shall discuss sets of simultaneous linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned}$$

The system must be **"diagonally dominant"**, which, in this case, means that

$$\begin{aligned} |a_1| &> |b_1| + |c_1|, \\ |b_2| &> |a_2| + |c_2|, \\ |c_3| &> |a_3| + |b_3|. \end{aligned}$$

The methods would be adaptable to a different number of simultaneous equations.

17.5.2 THE GAUSS-JACOBI ITERATION

This method begins by making x the subject of the first equation, y the subject of the second equation and z the subject of the third equation.

An initial approximation $x_0 = 1, y_0 = 1, z_0 = 1$ is substituted on the new right-hand sides to give values $x = x_1, y = y_1$ and $z = z_1$ on the new left-hand sides.

The results of the $(n+1)$-th iteration are as follows:

$$\begin{aligned} x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\ y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_n \qquad - c_2 z_n\right), \\ z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_n - b_3 y_n \qquad\right). \end{aligned}$$

EXAMPLES

1. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} 5x + y - z &= 4, \\ x + 4y + 2z &= 15, \\ x - 2y + 5z &= 12, \end{aligned}$$

obtaining x, y and z correct to the nearest integer.

Solution

$$
\begin{aligned}
x_{n+1} &= 0.8 \qquad\quad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n.
\end{aligned}
$$

Using

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1,$$

we obtain

$$x_1 = 0.8, \quad y_1 = 3.0, \quad z_1 = 2.6,$$

$$x_2 = 0.72, \quad y_2 = 2.25, \quad z_2 = 3.44,$$

$$x_3 = 1.038, \quad y_3 = 1.85, \quad z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, \quad y = 2, \quad z = 3,$$

when rounded to the nearest integer.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

2. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} x + 7y - z &= 3, \\ 5x + y + z &= 9, \\ -3x + 2y + 7z &= 17, \end{aligned}$$

obtaining x, y and z correct to the nearest integer.

Solution

This set of equations is not diagonally dominant; but they can be rewritten as

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17. \end{aligned}$$

Note:

We could also interchange the first two of the original equations.

Thus,

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_n - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_n - 0.29y_n. \end{aligned}$$

Using

$$y_0 = 1, \quad x_0 = 1, \quad z_0 = 1,$$

we obtain

$$y_1 = 0.43, \quad x_1 = 1.4, \quad z_1 = 2.57,$$

$$y_2 = 0.59, \quad x_2 = 1.2, \quad z_2 = 2.91,$$

$$y_3 = 0.67, \quad x_3 = 1.1, \quad z_3 = 2.78$$

Hence, $x = 1, \quad y = 1, \quad z = 3$ to the nearest integer.

17.5.3 THE GAUSS-SEIDEL ITERATION

This method differs from the Gauss-Jacobi Iteration in that successive approximations are used within each step **as soon as they become available**.

The rate of convergence of this method is usually faster than that of the Gauss-Jacobi method.

The scheme of the calculations is according to the following pattern:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_{n+1} \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_{n+1} - b_3 y_{n+1} \qquad \right).
\end{aligned}
$$

EXAMPLES

1. Use the Gauss-Seidel method to solve the simultaneous linear equations

$$\begin{aligned} 5x + y - z &= 4, \\ x + 4y + 2z &= 15, \\ x - 2y + 5z &= 12. \end{aligned}$$

Solution

$$\begin{aligned} x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\ y_{n+1} &= 3.75 - 0.25x_{n+1} \qquad - 0.5z_n, \\ z_{n+1} &= 2.4 - 0.2x_{n+1} + 0.4y_{n+1}. \end{aligned}$$

The sequence of successive results is as follows:

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1,$$

$$x_1 = 0.8, \quad y_1 = 3.05, \quad z_1 = 3.46,$$

$$x_2 = 0.88, \quad y_2 = 1.80, \quad z_2 = 2.94,$$

$$x_3 = 1.03, \quad y_3 = 2.02, \quad z_3 = 3.00$$

Hence, $x = 1, \quad y = 2, \quad z = 3$ to the nearest integer.

2. Use the Gauss-Seidel method to solve the simultaneous linear equations:

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17. \end{aligned}$$

Solution

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_{n+1} - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_{n+1} - 0.29y_{n+1}. \end{aligned}$$

The sequence of successive results is:

$$y_0 = 1, \quad x_0 = 1, \quad z_0 = 1,$$

$$y_1 = 0.43, \quad x_1 = 1.51, \quad z_1 = 2.96,$$

$$y_2 = 0.63, \quad x_2 = 1.08, \quad z_2 = 2.71,$$

$$y_3 = 0.66, \quad x_3 = 1.13, \quad z_3 = 2.73$$

The solutions are $x = 1, y = 1, z = 3$ to the nearest integer.

"JUST THE MATHS"

SLIDES NUMBER

17.6

NUMERICAL MATHEMATICS 6
(Numerical solution)
of
(ordinary differential equations (A))

by

A.J.Hobson

UNIT 17.6
NUMERICAL MATHEMATICS 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (A)

17.6.1 EULER'S UNMODIFIED METHOD

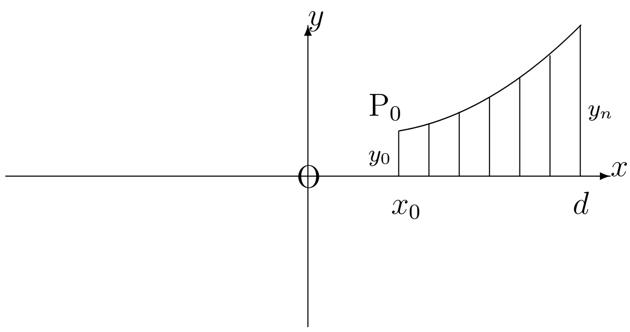
Every first order ordinary differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

If $y = y_0$ when $x = x_0$, then the solution for y in terms of x represents some curve through the point $\mathrm{P}_0(x_0, y_0)$.

Suppose we need y at $x = d$, where $d > x_0$.

Sub-divide the interval from $x = x_0$ to $x = d$ into n equal parts of width δx.



Let $x_1, x_2, x_3, ...$ be the points of subdivision.

$$x_1 = x_0 + \delta x,$$
$$x_2 = x_0 + 2\delta x,$$
$$x_3 = x_0 + 3\delta x,$$
$$...,$$
$$...,$$
$$d = x_n = x_0 + n\delta x.$$

If $y_1, y_2, y_3, ...$ are the y co-ordinates of $x_1, x_2, x_3, ...$, we must find y_n.

The increase in y, when x increases by $\delta x \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x$.

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$,

$$
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\delta x, \\
y_2 &= y_1 + f(x_1, y_1)\delta x, \\
y_3 &= y_2 + f(x_2, y_2)\delta x, \\
&\ldots, \\
&\ldots, \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\delta x.
\end{aligned}
$$

Each stage uses the previously calculated y value.

Note:
The method will be the same if $d < x_0$ except that δx will be negative.

In general,

$$y_{i+1} = y_i + f(x_i, y_i)\delta x.$$

EXAMPLE

Use Euler's method with 5 sub-intervals to continue to $x = 0.5$ the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

i	x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + f(x_i, y_i)\delta x$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.061106	0.4244424	1.1035524
5	0.5	1.1035524	-	-

Accuracy

Here, we may compare the exact result with the approximation by Eulers' method.

$$\int \frac{\mathrm{d}y}{y} = \int x\mathrm{d}x.$$

$$\ln y = \frac{x^2}{2} + C.$$

$$y = Ae^{\frac{x^2}{2}}.$$

At $x = 0$, $y = 1$ and, hence, $A = 1$.

$$y = e^{\frac{x^2}{2}}.$$

But a table of values of x against y reveals the following:

x	$e^{\frac{x^2}{2}}$
0	1
0.1	1.00501
0.2	1.0202
0.3	1.04603
0.4	1.08329
0.5	1.13315

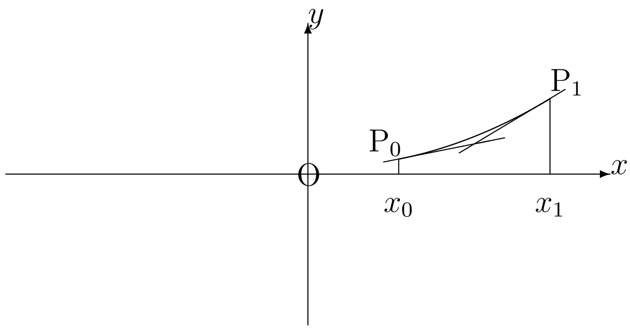
There is an error in our approximate value of 0.0296, which is about 2.6%.

Attempts to determine y for values of x which are greater than 0.5 would result in a very rapid growth of error.

17.6.2 EULER'S MODIFIED METHOD

In the previous method, we used the gradient at P_0 in order to find P_1, and so on up to P_n.

But the approximation is better if we use the **average** of the two gradients at P_0 and P_1.



The gradient, m_0 at P_0 is given by

$$m_0 = f(x_0, y_0).$$

The gradient, m_1 at P_1 is given approximately by

$$m_1 = f(x_0 + \delta x, y_0 + \delta y_0),$$

where $\delta y_0 = f(x_0, y_0)\delta x$.

The average gradient between P_0 and P_1 is given by

$$m_0^* = \frac{1}{2}(m_0 + m_1).$$

The modified approximation to y at the point P_1 is given by

$$y_1 = y_0 + m_0^* \delta x.$$

Similarly, we proceed from y_1 to y_2 and so on until we reach y_n.

In general,

$$y_{i+1} = y_i + m_i^* \delta x.$$

EXAMPLE

Use Euler's modified method with 5 sub-intervals to continue to $x = 0.5$ the solution to the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

i	x_i	y_i	$m_i = f(x_i, y_i)$	$\delta y_i = f(x_i, y_i)\delta x$
0	0	1	0	0
1	0.1	1.005	0.1005	0.0101
2	0.2	1.0202	0.2040	0.0204
3	0.3	1.0460	0.3138	0.0314
4	0.4	1.0832	0.4333	0.0433
5	0.5	1.1330	——	——

i	$m_{i+1} = f(x_i + \delta x, y_i + \delta y_i$	$m_i^* = \frac{1}{2}(m_i + m_{i+1})$	$y_{i+1} = y_i + m_i^* \delta x$
0	0.1	0.05	1.005
1	0.2030	0.1518	1.0202
2	0.3122	0.2581	1.0460
3	0.4310	0.3724	1.0832
4	0.5633	0.4983	1.1330
5	——	——	——

"JUST THE MATHS"

SLIDES NUMBER

17.7

NUMERICAL MATHEMATICS 7
(Numerical solution)
of
(ordinary differential equations (B))

by

A.J.Hobson

UNIT 17.7
NUMERICAL MATHEMATICS 7

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (B)

17.7.1 PICARD'S METHOD

This method of solution is one of successive approximation. It is an **iterative** method in which the numerical results become more and more accurate, the more times it is used.

An approximation for y in terms of x is substituted into the right hand side of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

The equation is then integrated with respect to x, giving y in terms of x as a second approximation.

Given numerical values are substituted into the second approximation and the result rounded off to an assigned number of decimal places or significant figures.

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

A hint on notation

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

given that $y = y_0 = 7$ when $x = x_0 = 2$.

This can be solved exactly to give

$$y = x^3 + C,$$

which requires that

$$7 = 2^3 + C.$$

Hence,

$$y - 7 = x^3 - 2^3;$$

or, in more general terms,

$$y - y_0 = x^3 - x_0^3.$$

Thus,

$$\int_{y_0}^{y} \mathrm{d}y = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

That is,

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

In future, we shall integrate both sides of the given differential equation with respect to x, from x_0 to x.

EXAMPLES

1. Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2,$$

and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

Solution

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} (x + y^2)\mathrm{d}x,$$

where $x_0 = 0$

Hence,

$$y - y_0 = \int_{x_0}^{x} (x + y^2)\mathrm{d}x,$$

where $y_0 = 0$.

That is,

$$y = \int_0^x (x + y^2)\mathrm{d}x.$$

(a) First Iteration

Replace y by y_0 in the function to be integrated.

$$y_1 = \int_0^x x\mathrm{d}x = \frac{x^2}{2} \simeq 0.0450 \quad \text{at} \quad x = 0.3$$

(b) Second Iteration

Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_1^2 = x + \frac{x^4}{4}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20}.$$

$$y_2 = \frac{x^2}{2} + \frac{x^5}{20} \simeq 0.0451 \quad \text{at} \quad x = 0.3$$

(c) Third Iteration

Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_2^2$$
$$= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \simeq 0.0451 \quad \text{at} \quad x = 0.3$$

2. If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y}{x},$$

and $y = 2$ when $x = 1$, perform three iterations of Picard's method to estimate a value for y when $x = 1.2$. Work to four places of decimals throughout and state how accurate is the result of the third iteration.

(a) First Iteration

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $x_0 = 1$.

That is,

$$y - y_0 = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $y_0 = 2$.

Hence,

$$y - 2 = \int_{1}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x.$$

Replacing y by $y_0 = 2$ in the function being integrated, we have

$$y - 2 = \int_{1}^{x} \left(2 - \frac{2}{x}\right) \mathrm{d}x.$$

Therefore,

$$y = 2 + [2x - 2\ln x]_1^x$$

$$= 2 + 2x - 2\ln x - 2 + 2\ln 1 = 2(x - \ln x).$$

$$y_1 = 2(x - \ln x) \simeq 2.0354 \quad \text{when} \quad x = 1.2$$

(b) Second Iteration

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_1}{x} = 2 - \frac{2(x - \ln x)}{x} = \frac{2\ln x}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \frac{2lnx}{x}\mathrm{d}x.$$

That is,

$$y - 2 = \left[(\ln x)^2\right]_1^x = (\ln x)^2.$$

$$y_2 = 2 + (\ln x)^2 \simeq 2.0332 \quad \text{when} \quad x = 1.2$$

(c) Third Iteration

Finally, we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_2}{x} = 2 - \frac{2}{x} - \frac{(\ln x)^2}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \left[2 - \frac{2}{x} - \frac{(lnx)^2}{x}\right]\mathrm{d}x.$$

That is,

$$y - 2 = \left[2x - 2\ln x - \frac{(\ln x)^3}{3}\right]_1^x.$$

$$y - 2 = 2x - 2\ln x - \frac{(\ln x)^3}{3} - 2.$$

$$y_3 = 2x - 2\ln x - \frac{(\ln x)^3}{3} \simeq 2.0293 \quad \text{when} \quad x = 1.2$$

The results of the last two iterations are identical when rounded off to two places of decimals, namely 2.03.

"JUST THE MATHS"

SLIDES NUMBER

17.8

NUMERICAL MATHEMATICS 8
(Numerical solution)
of
(ordinary differential equations (C))

by

A.J.Hobson

17.8.1 Runge's method

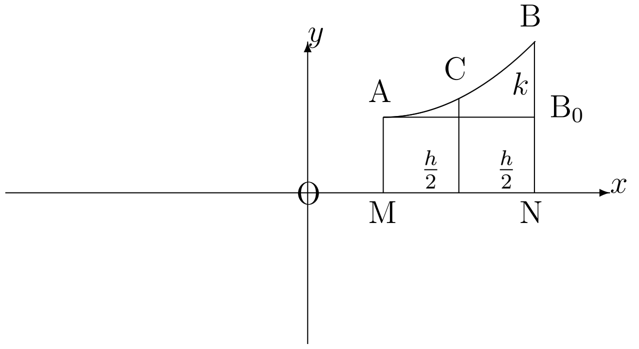
UNIT 17.8
NUMERICAL MATHEMATICS 8

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (C)

17.8.1 Runge's Method

We solve the differential equation, $\frac{dy}{dx} = f(x,y)$, subject to the condition that $y = y_0$ when $x = x_0$.

Consider the **graph** of the solution passing through the two points, A(x_0, y_0) and B$(x_0 + h, y_0 + k)$.



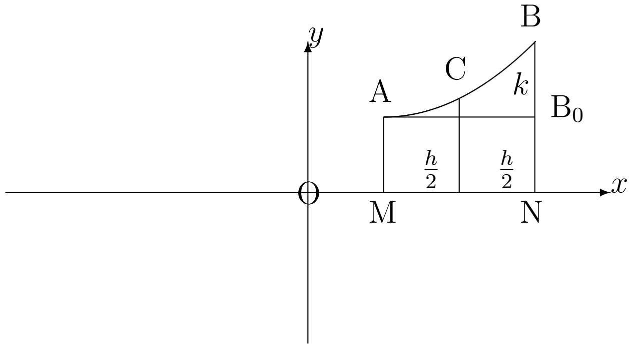
We can say that

$$\int_{x_0}^{x_0+h} \frac{dy}{dx} dx = \int_{x_0}^{x_0+h} f(x,y) dx.$$

That is,

$$y_B - y_A = \int_{x_0}^{x_0+h} f(x,y) dx.$$

Reminder: $f(x,y)$ is the gradient at points on the solution curve.



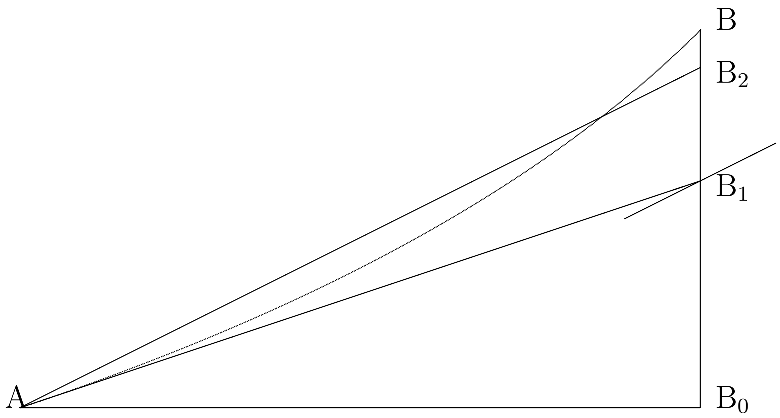
Suppose we knew the values of $f(x,y)$ at A,B and C, where C is the intersection with the curve of the perpendicular bisector of MN.

Then, by Simpson's Rule for approximate integration,

$$\int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x = \frac{h/2}{3}\left[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}\right].$$

(i) The value of $f(\mathrm{A})$
This is already given, namely, $f(x_0, y_0)$.

(ii) The Value of $f(\mathrm{B})$



If the tangent at A meets $\mathrm{B_0B}$ in $\mathrm{B_1}$, then the gradient at A is given by

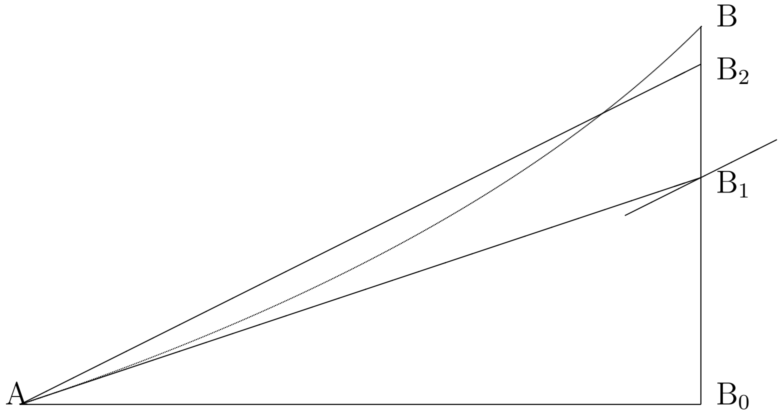
$$\frac{\mathrm{B_1B_0}}{\mathrm{AB_0}} = f(x_0, y_0).$$

Therefore,

$$\mathrm{B_1B_0} = \mathrm{AB_0} f(x_0, y_0) = h f(x_0, y_0).$$

Calling this value k_1 as an initial approximation to k,

$$k_1 = h f(x_0, y_0).$$



As a rough approximation to the gradient of the solution curve passing through B, we now take the gradient of the solution curve passing through B_1.

Its value is

$$f(x_0 + h, y_0 + k_1).$$

For a better approximation, assume that a straight line of gradient $f(x_0 + h, y_0 + k_1)$, drawn at A, meets B_0B in B_2 a point nearer to B than B_1.

Letting $B_0B_2 = k_2$,

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

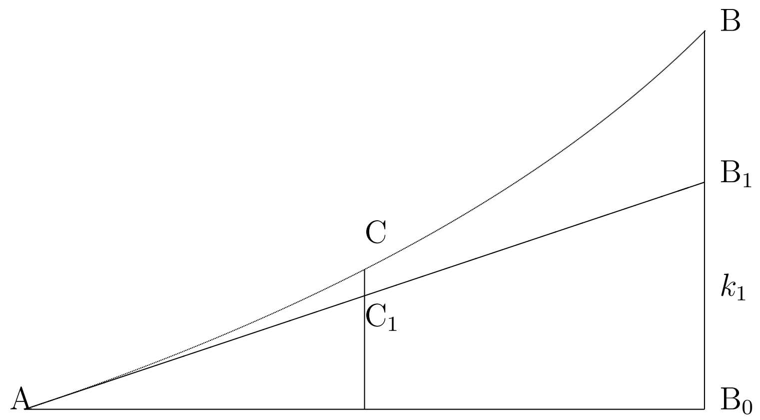
The co-ordinates of B_2 are $(x_0 + h, y_0 + k_2)$.

The gradient of the solution curve through B_2 is taken as a closer approximation than before to the gradient of the solution curve through B.

The gradient of the solution curve through B_2 is

$$f(x_0 + h, y_0 + k_2).$$

(iii) The Value of f(C)



Let C_1 be the intersection of the ordinate through C and the tangent at A.

Then C_1 is the point

$$\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

The gradient at C_1 of the solution curve through C_1 is

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

We take this to be an approximation to the gradient at C for the arc, AB.

We saw earlier that

$$y_{\mathrm{B}} - y_{\mathrm{A}} = \int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

Therefore,

$$y_{\mathrm{B}} - y_{\mathrm{A}} = \frac{h}{6}[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}].$$

That is, $y =$

$$y_0 + \frac{h}{6}\left[f(x_0, y_0) + f(x_0 + h, y_0 + k_2) + 4f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\right].$$

PRACTICAL LAYOUT

If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

and $y = y_0$ when $x = x_0$, then the value of y when $x = x_0 + h$ is determined by the following sequence of calculations:

1. $k_1 = hf(x_0, y_0)$.
2. $k_2 = hf(x_0 + h, y_0 + k_1)$.
3. $k_3 = hf(x_0 + h, y_0 + k_2)$.
4. $k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.
5. $k = \frac{1}{6}(k_1 + k_3 + 4k_4)$.
6. $y \simeq y_0 + k$.

EXAMPLE

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 3y$$

at $x = 0.1$ given that $y = 1$ when $x = 0$

Solution

We use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

1. $k_1 = 0.1(5 - 3) = 0.2$
2. $k_2 = 0.1(5 - 3[1.2]) = 0.14$
3. $k_3 = 0.1(5 - 3[1.14]) = 0.158$
4. $k_4 = 0.1(5 - 3[1.1]) = 0.17$
5. $k = \frac{1}{6}(0.2 + 0.158 + 4[0.17]) = 0.173$
6. $y \simeq 1.173$ at $x = 0.1$

Note: It can be shown that the error in the result is of the order h^5; that is, the error is equivalent to some constant multiplied by h^5.

"JUST THE MATHS"

SLIDES NUMBER

18.1

STATISTICS 1
(The presentation of data)

by

A.J.Hobson

UNIT 18.1 - STATISTICS 1
THE PRESENTATION OF DATA

18.1.1 INTRODUCTION

(i) The collection of numerical information often leads to large masses of data

If they are to be understood, or presented effectively, they must be summarised and analysed in some way.

This is the purpose of the subject of **"Statistics"**.

(ii) The source from which a set of data is collected is called a **"population"**.

For example, a population of 1000 ball-bearings could provide data relating to their diameters.

(iii) Statistical problems may be either:

"descriptive problems" (all the data is known and can be analysed)

or

"inference problems" (data collected from a **"sample"** population is used to infer properties of a larger population).

For example, the annual pattern of rainfall over several years in a particular place could be used to estimate the rainfall pattern in other years.

(iv) The variables measured in a statistical problem may be either:

"discrete" (taking only certain values)

or

"continuous" (taking any values within the limits of the problem itself).

For example, the number of students passing an examination from a particular class of students is a discrete variable; but the diameter of ball-bearings from a stock of 1000 is a continuous variable.

(v) Various methods are seen in the commercial presentation of data but, in this series of Units, we shall be concerned with just two methods - one of which is tabular and the other graphical.

18.1.2 THE TABULATION OF DATA

(a) Ungrouped Data

Suppose we have a collection of measurements given by numbers. Some may occur only once, while others may be repeated several times.

If we write down the numbers as they appear, the processing of them is likely to be cumbersome. This is known as **"ungrouped (or raw) data"**.

For example, the following table shows rainfall figures (in inches) for a certain location in specified months over a 90 year period:

TABLE 1 - Ungrouped (or Raw) Data

18.6	13.8	10.4	15.0	16.0	22.1	16.2	36.1	11.6	7.8
22.6	17.9	25.3	32.8	16.6	13.6	8.5	23.7	14.2	22.9
17.7	26.3	9.2	24.9	17.9	26.5	26.6	16.5	18.1	24.8
16.6	32.3	14.0	11.6	20.0	33.8	15.8	15.2	24.0	16.4
24.1	23.2	17.3	10.5	15.0	20.2	20.2	17.3	16.6	16.9
22.0	23.9	24.0	12.2	21.8	12.2	22.0	9.6	8.0	20.4
17.2	18.3	13.0	10.6	17.2	8.9	16.8	14.2	15.7	8.0
17.7	16.1	17.8	11.6	10.4	13.6	8.4	12.6	8.1	11.6
21.1	20.5	19.8	24.8	9.7	25.1	31.8	24.9	20.0	17.6

(b) Ranked Data

A slightly more convenient method of tabulating a collection of data would be to arrange them in rank order, so making it easier to see how many times each number appears. This is known as **"ranked data"**.

The next table shows the previous rainfall figures in this form.

TABLE 2 - Ranked Data

7.8	8.0	8.0	8.1	8.4	8.5	8.9	9.2	9.6	9.7
10.4	10.4	10.5	10.6	11.6	11.6	11.6	11.6	12.2	12.2
12.6	13.0	13.6	13.6	13.8	14.0	14.2	14.2	15.0	15.0
15.2	15.7	15.8	16.0	16.1	16.2	16.4	16.5	16.6	16.6
16.6	16.8	16.9	17.2	17.2	17.3	17.3	17.6	17.7	17.7
17.8	17.9	17.9	18.1	18.3	18.6	19.8	20.0	20.0	20.2
20.2	20.4	20.5	21.1	21.8	22.0	22.0	22.1	22.6	22.9
23.2	23.7	23.9	24.0	24.0	24.1	24.8	24.8	24.9	24.9
25.1	25.3	26.3	26.5	26.6	31.8	32.3	32.8	33.8	36.1

(c) Frequency Distribution Tables

Thirdly, it is possible to save a little space by making a table in which each individual item of the ranked data is written down once only, but paired with the number of times it occurs. The data is then presented as a **"frequency distribution table"**.

TABLE 3 - Frequency Distribution Table

Value	Frequency	Value	Frequency	Value	Frequency
7.8	1	15.8	1	21.1	1
8.0	2	16.0	1	21.8	1
8.1	1	16.1	1	22.0	2
8.4	1	16.2	1	22.1	1
8.5	1	16.4	1	22.6	1
8.9	1	16.5	1	22.9	1
9.2	1	16.6	3	23.2	1
9.6	1	16.8	1	23.7	1
9.7	1	16.9	1	23.9	1
10.4	2	17.2	2	24.0	2
10.5	1	17.3	2	24.1	1
10.6	1	17.6	1	24.8	2
11.6	4	17.7	2	24.9	2
12.2	2	17.8	1	25.1	1
12.6	1	17.9	2	25.3	1
13.0	1	18.1	1	26.3	1
13.6	2	18.3	1	26.5	1
13.8	1	18.6	1	26.6	1
14.0	1	19.8	1	31.8	1
14.2	2	20.0	2	32.3	1
15.0	2	20.2	2	32.8	1
15.2	1	20.4	1	33.8	1
15.7	1	20.5	1	36.1	1

(d) Grouped Frequency Distribution Tables
For about forty or more items in a set of numerical data, it usually most convenient to group them together into between 10 and 25 **"classes"** of values, each covering a specified range or "class interval"(eg. $7.5 - 10.5$, $10.5 - 13.5$, $13.5 - 16.5$,......)

Each item is counted every time it appears in order to obtain the **"class frequency"** and each class interval has the same **"class width"**.

Too few classes means that the data is over-summarised while too many classes means that there is little advantage in summarising at all.

Here, we use the convention that the lower boundary of the class is included while the upper boundary is excluded.

Each item in a particular class is considered to be approximately equal to the **"class mid-point"**; that is, the average of the two **"class boundaries"**.

A **"grouped frequency distribution table"** normally has columns which show the class intervals, class mid-points, class frequencies, and cumulative frequencies, the last of these being a running total of the frequencies themselves.

There may also be a column of **"tallied frequencies"** if the table is being constructed from the raw data.

TABLE 4 - Grouped Frequency Distribution

Class Intvl.	Class Md pt	Tallied Freq	Freq	Cumtv Freq
7.5 – 10.5	9	////| ////| //	12	12
10.5 – 13.5	12	////| ////|	10	22
13.5 – 16.5	15	////| ////| ////|	15	37
16.5 – 19.5	18	////| ////| ////| ////	19	56
19.5 – 22.5	21	////| ////| //	12	68
22.5 – 25.5	24	////| ////| ////	14	82
25.5 – 28.5	27	///	3	85
28.5 – 31.5	30		0	85
31.5 – 34.5	33	////	4	89
34.5 – 37.5	36	/	1	90

Notes:

(i) The cumulative frequency shows, at a glance, how many items in the data are less than a specified value. In Table 4, 82 items are less than 25.5

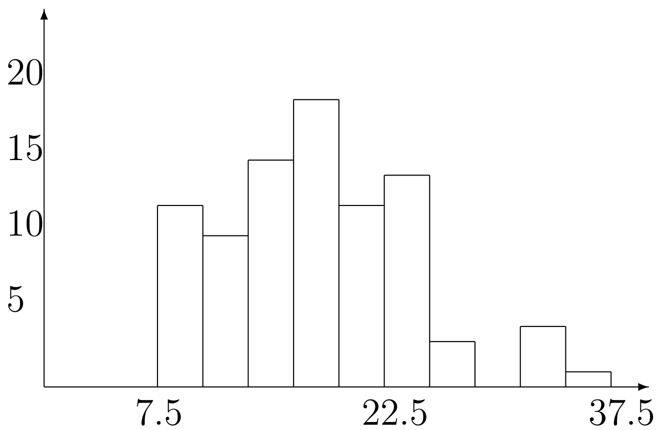
(ii) It is sometimes more useful to use the ratio of the cumulative frequency to the total number of observations (**"relative cumulative frequency"**); and, in Table 4, the percentage of items which are less than 25.5 is

$$\frac{82}{90} \times 100 \simeq 91\%.$$

18.1.3 THE GRAPHICAL REPRESENTATION OF DATA

(a) The Histogram

A **"histogram"** is a diagram which is directly related to a grouped frequency distribution table and consists of a collection of rectangles whose height represents the class frequency (to some suitable scale) and whose breadth represents the class width.

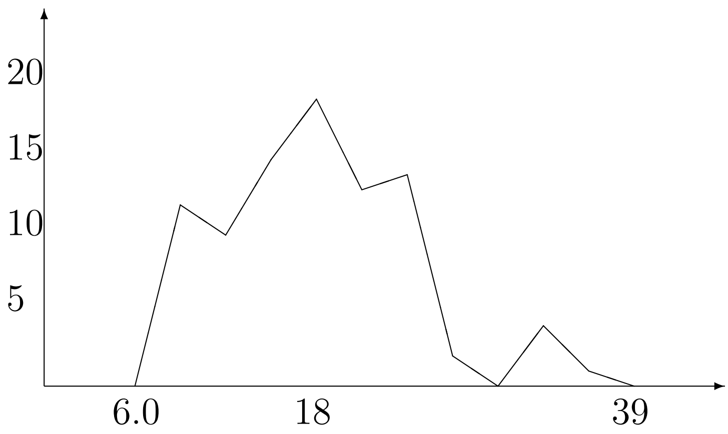


The histogram shows, at a glance, not just the class intervals with the highest and lowest frequencies but also how the frequencies are distributed.

In the case of examination results, for example, there is usually a group of high frequencies around the central class intervals and lower ones at the ends. Such an ideal situation would be called a **"Normal Distribution"**.

(b) The Frequency Polygon

Using the fact that each class interval may be represented, on average, by its class mid-point, we may plot the class mid-points against the class frequencies to obtain a display of single points. By joining up these points with straight line segments and including two extra class mid-points, we obtain a **"frequency polygon"**.



Notes:

(i) Although the frequency polygon officially plots only the class mid-points against their frequencies, it is sometimes convenient to read-off intermediate points in order to estimate additional data. For example, we might estimate that the value 11.0 occurred 11 times when, in fact, it did not occur at all.
We may use this technique only for continuous variables.

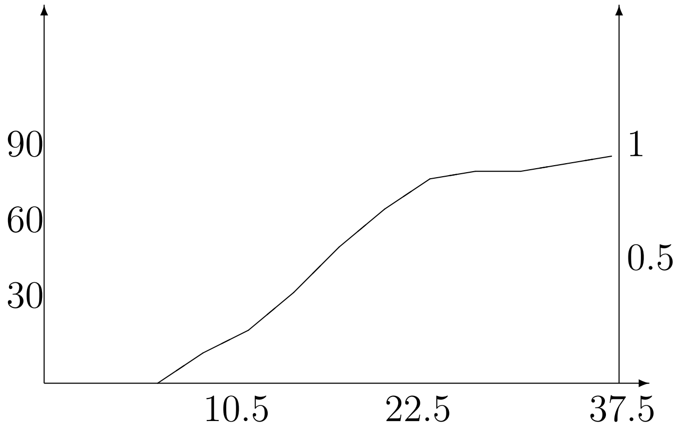
(ii) Frequency polygons are more useful than histograms if we wish to compare two or more frequency distributions. A clearer picture is obtained if we plot them on the same diagram.

(iii) If the class intervals are made smaller and smaller while, at the same time, the total number of items in the data is increased more and more, the points of the frequency polygon will be very close together. The smooth curve joining them is called the **"frequency curve"** and is of greater use for estimating intermediate values.

(c) The Cumulative Frequency Polygon (or Ogive)
The earlier use of the cumulative frequency to estimate the number (or proportion) of values less than a certain amount may be applied graphically by plotting the upper class-boundary against cumulative frequency; then joining up the points plotted with straight line segments. The graph obtained is called the **"cumulative frequency polygon"** or **"ogive"**.

We may also use a second vertical axis at the right-hand end of the diagram showing the relative cumulative frequency. The range of this axis will always be 0 to 1.

The Ogive



"JUST THE MATHS"

SLIDES NUMBER

18.2

STATISTICS 2
(Measures of central tendency)

by

A.J.Hobson

UNIT 18.2 - STATISTICS 2

MEASURES OF CENTRAL TENDENCY

18.2.1 INTRODUCTION

We shall be concerned, here with the methods of analysing the data in order to obtain the maximum amount of information from it.

In both "descriptive" and "inference" types of problem, it is useful to be able to measure some value around which all items in the data may be considered to cluster.

This is called **"A measure of central tendency"**.

We find it by using several types of average value as follows:

18.2.2 THE ARITHMETIC MEAN (BY CODING)

To obtain the arithmetic mean of a finite collection of n numbers, we may simply add all the numbers together and then divide by n.

This elementary rule applies even if some of the numbers occur more than once and even if some of the numbers are negative.

The purpose of this section is to introduce some short-cuts (called **"coding"**) in the calculation of the arithmetic mean of large collections of data.

The methods will be illustrated by the following example in which the number of items of data is not over-large:

EXAMPLE

The solid contents, x, of water (in parts per million) was measured in eleven samples and the following data was obtained:

4520	4490	4500	4500
4570	4540	4520	4590
4520	4570	4520	

Determine the arithmetic mean, $\overline{x}$, of the data.

Solution

(i) Direct Calculation

By adding together the eleven numbers, then dividing by 11, we obtain

$$\overline{x} = 49840 \div 11 \simeq 4530.91$$

(ii) Using Frequencies

We could first make a frequency table having a column of distinct values x_i, $(i = 1, 2, 3, ..., 11)$, a column of frequencies f_i, $(i = 1, 2, 3......, 11)$ and a column of corresponding values $f_i x_i$.

The arithmetic mean is then calculated from the formula

$$\overline{x} = \frac{1}{11} \sum_{i=1}^{11} f_i x_i.$$

In the present example, the table would be

x_i	f_i	$f_i x_i$
4490	1	4490
4500	2	9000
4520	4	18080
4540	1	4540
4570	2	9140
4590	1	4590
	Total	49840

The arithmetic mean is then

$$\overline{x} = 49840 \div 11 \simeq 4530.91,$$

as before.

(iii) Reduction by a constant

With large data-values it can be convenient to reduce all of the values by a constant, k, before calculating the arithmetic mean.

Result
By adding the constant, k, to the arithmetic mean of the reduced data, we obtain the arithmetic mean of the original data.

Proof:

For n values, $x_1, x_2, x_3, ..., x_n$, suppose each value is reduced by a constant, k.

Then the arithmetic mean of the reduced data is

$$\frac{(x_1 - k) + (x_2 - k) + (x_3 - k) + ... + (x_n - k)}{n}$$

$$= \frac{x_1 + x_2 + x_3 + ... + x_n}{n} - \frac{nk}{n} = \overline{x} - k.$$

(iv) Division by a constant

In a similar way to the previous paragraph, each value in a collection of data could by divided by a constant, k, before calculating the arithmetic mean.

Result
The arithmetic mean of the original data is obtained on

multiplying the arithmetic mean of the reduced data by k.

Proof:

$$\frac{\frac{x_1}{k}+\frac{x_2}{k}+\frac{x_3}{k}+......+\frac{x_n}{k}}{n}=\frac{\overline{x}}{k}.$$

To summarise the shortcuts used in the present example, the following table shows a combination of the use of frequencies and of the two types of reduction made to the data:

x_i	$x_i - 4490$	$x_i' = (x_i - 4490)/10$	f_i	$f_i x_i'$
4490	0	0	1	0
4500	10	1	2	2
4520	30	3	4	12
4540	50	5	1	5
4570	80	8	2	16
4590	100	10	1	10
			Total	45

The Fictitious arithmetic mean, $\overline{x}' = \dfrac{45}{11} \simeq 4.0909$

Arithm. mean, $\overline{x} \simeq (4.0909 \times 10) + 4490 \simeq 4530.91$

(v) The approximate arithmetic mean for a grouped distribution

For a large number of items of data, we may take all items within a class interval to be equal to the class mid-point. We then reduce each mid-point by the first mid-point and divide by the class width (or other convenient number).

EXAMPLE

Calculate, approximately, the arithmetic mean of the data in the following table (Table 4 in Unit 18.1).

Cls. Intvl.	Cls. Md. pt. x_i	$x_i - 9$	$(x_i - 9)/3 = x'_i$	Freq. f_i	$f_i x'_i$
7.5 – 10.5	9	0	0	12	0
10.5 – 13.5	12	3	1	10	10
13.5 – 16.5	15	6	2	15	30
16.5 – 19.5	18	9	3	19	57
19.5 – 22.5	21	12	4	12	48
22.5 – 25.5	24	15	5	14	70
25.5 – 28.5	27	18	6	3	18
28.5 – 31.5	30	21	7	0	0
31.5 – 34.5	33	24	8	4	32
34.5 – 37.5	36	27	9	1	9
			Totals	90	274

Solution

$$\text{Fictitious arithmetic mean} \quad \overline{x}' = \frac{274}{90} \simeq 3.0444$$

$$\text{Actual arithmetic mean} = 3.044 \times 3 + 9 \simeq 18.13$$

Notes:

(i) By direct calculation from Table 1 in Unit 18.1, it may be shown that the arithmetic mean is 17.86 correct to two places of decimals; and this indicates an error of about 1.5%.

(ii) The arithmetic mean is widely used where samples are taken of a larger population.

It usually turns out that two samples of the same population have arithmetic means which are close in value.

18.2.3 THE MEDIAN

Collections of data often include one or more values which are widely out of character with the rest; and the arithmetic mean can be significantly affected by such extreme values.

For example, the values 8,12,13,15,21,23 have an arithmetic mean of $\frac{92}{6} \simeq 15.33$; but the values 5,12,13,15,21,36 have an arithmetic mean of $\frac{102}{6} \simeq 17.00$

A second type of average, not so much affected, is defined as follows:

DEFINITION

The **"median"** of a collection of data is the middle value when the data is arranged in rank order.

For an even number of values in the collection of data, the median is the arithmetic mean of the centre two values.

EXAMPLES

1. For both 8,12,13,15,21,23 and 5,12,13,15,21,36, the median is given by
$$\frac{13 + 15}{2} = 14.$$
2. For a grouped distribution, the problem is more complex since we no longer have access to the individual values from the data.

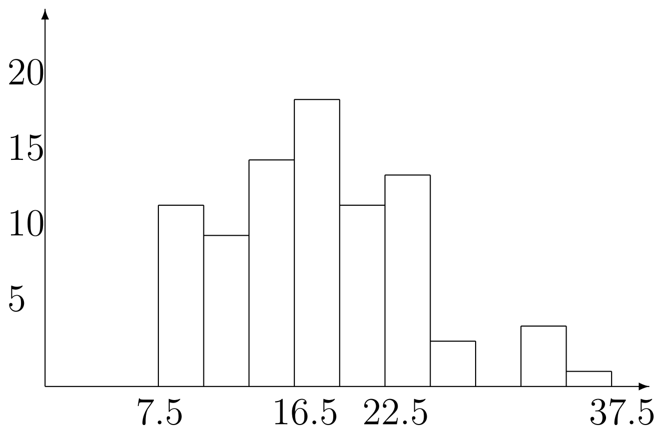
For a grouped distribution, the area of a histogram is directly proportional to the total number of values which it represents since the base of all the rectangles are the same width and each height represents a frequency.

We may thus take the median to be the value for which the vertical line through it divides the histogram into two equal areas.

For non-symmetrical histograms, the median is often a better measure of central tendency than the arithmetic mean.

ILLUSTRATION

Consider the histogram from Unit 18.1, representing rainfall figures over a 90 year period.



The total area of the histogram $= 90 \times 3 = 270$.

Half the area of the histogram $= 135$.

The area up as far as $16.5 = 3 \times 37 = 111$, while the area up as far as $19.5 = 3 \times 56 = 168$; hence the median must lie between 16.5 and 19.5

The median $= 16.5 + x$, where $18x = 135 - 111 = 24$, since 18 is the frequency of the class interval $16.5 - 19.5$

That is,

$$x = \frac{24}{18} = \frac{4}{3} \simeq 1.33,$$

giving a median of 17.83

Notes:

(i) The median, in this case, is close to the arithmetic mean since the distribution is fairly symmetrical.

(ii) If a sequence of zero frequencies occurs, it may be necessary to take the arithmetic mean of two class mid-points which are not consecutive to each other.

(iii) Another example of the advantage of median over arithmetic mean would be the average life of 100 electric lamps. To find the arithmetic mean, all 100 must be tested; but to find the median, the testing may stop after the 51st.

18.2.4 THE MODE

DEFINITIONS

1. For a collection of individual items of data, the mode is the value having the highest frequency
2. In a grouped frequency distribution, the mid-point of the class interval with the highest frequency is called the **“crude mode”** and the class interval itself is called the **“modal class”**.

Note:
Like the median, the mode is not much affected by changes in the extreme values of the data. However, some distributions may have several different modes, which is a disadvantage of this measure of central tendency.

EXAMPLE

For the histogram discussed earlier, the mode is 18.0; but if the class interval $22.5-25.5$ had 5 more members, then 24.0 would be a mode as well.

18.2.5 QUANTILES

To conclude this Unit, we shall define three more standard measurements which, in fact, extend the idea of a median.

We may recall that a median divides a collection of values in such a way that half of them fall on either side of it.

Collectively, these three new measurements are called **"Quantiles"** but may be considered separately by their own names as follows:

(a) Quartiles

These are the three numbers dividing a ranked collection of values (or the area of a histogram) into 4 equal parts.

(b) Deciles

These are the nine numbers dividing a ranked collection of values (or the area of a histogram) into 10 equal parts.

(c) Percentiles

These are the ninety nine numbers dividing a ranked collection of values (or the area of a histogram) into 100 equal parts.

Note:
For collections of individual values, quartiles may need to be calculated as the arithmetic mean of two consecutive values.

EXAMPLES

1. (a) The 25th percentile = The 1st Quartile.
   (b) The 5th Decile = The median.
   (c) The 85th Percentile = the point at which 85% of the values fall below it and 15% above it.
2. For the collection of values

$$5, 12, 13, 19, 25, 26, 30, 33,$$

the quartiles are 12.5, 22 and 28.

3. For the collection of values

$$5, 12, 13, 19, 25, 26, 30,$$

the quartiles are 12.5, 19 and 25.5

"JUST THE MATHS"

SLIDES NUMBER

18.3

STATISTICS 3
(Measures of dispersion (or scatter))

by

A.J.Hobson

UNIT 18.3 - STATISTICS 3

MEASURES OF DISPERSION (OR SCATTER)

18.3.1 INTRODUCTION

Averages typify a whole collection of values but give little information about how the values are distributed within the whole collection.

For example, 99.9, 100.0, 100.1 is a collection which has an arithmetic mean of 100.0 and so is 99.0,100.0,101.0; but the second collection is more widely dispersed than the first.

In this Unit, we examine two types of quantity which typify the distance of all the values in a collection from their arithmetic mean.

They are known as **"measures of dispersion (or scatter)"** and the smaller these quantities are, the more clustered are the values around the arithmetic mean.

18.3.2 THE MEAN DEVIATION

If the n values x_1, x_2, x_3, ...,x_n have an arithmetic mean of $\overline{x}$, then $x_1 - \overline{x}$, $x_2 - \overline{x}$, $x_3 - \overline{x}$,......,$x_n - \overline{x}$ are called the deviations of x_1, x_2, x_3, ..., x_n from the arithmetic mean.

Note:
These deviations add up to zero since

$$\sum_{i=1}^{n} (x_i - \overline{x})$$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x}$$

$$= n\overline{x} - n\overline{x} = 0.$$

DEFINITION
The mean deviation (*or, more accurately, the mean absolute deviation*) is defined by the formula

$$\text{M.D.} = \frac{1}{n} \sum_{i=1}^{n} |x_i - \overline{x}|.$$

18.3.3 PRACTICAL CALCULATION OF THE MEAN DEVIATION

In calculating a mean deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values:

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x} - k$, in the formula, then the mean deviation is unaffected.

Proof:

$$\frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} |(x_i - k) - (\overline{x} - k)|.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a positive constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then the mean deviation will be divided by l.

Proof:

$$\frac{1}{ln}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} \left|\frac{x_i}{l} - \frac{\overline{x}}{l}\right|.$$

Summary
If we code the data using both a subtraction by k and a division by l, the value obtained from the mean deviation formula needs to multiplied by l to give the correct value.

18.3.4 THE ROOT MEAN SQUARE (OR STANDARD) DEVIATION

A more common method of measuring dispersion, which ensures that negative deviations from the arithmetic mean do not tend to cancel out positive deviations, is to determine the arithmetic mean of their squares and then take the square root.

DEFINITION
The **"root mean square deviation"** (*or "standard deviation"*) is defined by the formula

$$\text{R.M.S.D.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}.$$

Notes:

(i) The root mean square deviation is usually denoted by the symbol, σ.

(ii) The quantity σ^2 is called the **"variance"**.

18.3.5 PRACTICAL CALCULATION OF

THE STANDARD DEVIATION

In calculating a standard deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values.

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x} - k$, in the formula, then σ is unaffected.

Proof:

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\ (x_i - \overline{x})^2}$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\ [(x_i - k) - (\overline{x} - k)]^2}.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then σ will be divided by l.

Proof:

$$\frac{1}{l}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\ (x_i - \overline{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\ \left(\frac{x_i}{l} - \frac{\overline{x}}{l}\right)^2}.$$

Summary

If we code the data using both a subtraction by k and a division by l, the value obtained from the standard deviation formula needs to multiplied by l to give the correct value, σ.

(c) For the calculation of the standard deviation, whether by coding or not, a more convenient formula may be obtained by expanding out the expression $(x_i - \overline{x})^2$ as follows:

$$\sigma^2 = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\overline{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \overline{x}^2\right].$$

That is,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\overline{x}^2 + \overline{x}^2.$$

This gives the formula

$$\sigma = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \overline{x}^2}.$$

Note:
In advanced statistical work, the above formulae for standard deviation are used only for descriptive problems in which we know every member of a collection of observations.

For inference problems, it may be shown that the standard deviation of a sample is always smaller than that of a total population; and the basic formula used for a sample is

$$\sigma = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

18.3.6 OTHER MEASURES OF DISPERSION

We mention here, briefly, two other measures of dispersion:

(i) The Range

This is the difference between the highest and the smallest members of a collection of values.

(ii) The Coefficient of Variation

This is a quantity which expresses the standard deviation as a percentage of the arithmetic mean.

It is given by the formula

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100.$$

EXAMPLE

The following grouped frequency distribution table shows the diameter of 98 rivets:

Class Intvl.	Cls. Mid Pt. x_i	Freq f_i	Cum Freq.	$(x_i - 6.61)/0.02 = x_i'$	$f_i x_i'$	$x_i'^2$	$f_i x_i'^2$	$f_i \lvert x_i' - \overline{x}' \rvert$
6.60 – 6.62	6.61	1	1	0	0	0	0	0.58
6.62 – 6.64	6.63	4	5	1	4	1	4	2.40
6.64 – 6.66	6.65	6	11	2	12	4	24	3.72
6.66 – 6.68	6.67	12	23	3	36	9	108	7.68
6.68 – 6.70	6.69	5	28	4	20	16	80	3.30
6.70 – 6.72	6.71	10	38	5	50	25	250	6.80
6.72 – 6.74	6.73	17	55	6	102	36	612	11.90
6.74 – 6.76	6.75	10	65	7	70	49	490	7.20
6.76 – 6.78	6.77	14	79	8	112	64	896	10.36
6.78 – 6.80	6.79	9	88	9	81	81	729	6.84
6.80 – 6.82	6.81	7	95	10	70	100	700	5.46
6.82 – 6.84	6.83	2	97	11	22	121	242	1.60
6.84 – 6.86	6.85	1	98	12	12	144	144	0.82
Totals		98			591		4279	68.66

Estimate the arithmetic mean, the standard deviation and the mean (absolute) deviation of these diameters.

Solution

$$\text{Fictitious arithmetic mean} = \frac{591}{98} \simeq 6.03$$

$$\text{Actual arithmetic mean} = 6.03 \times 0.02 + 6.61 \simeq 6.73$$

$$\text{Fictitious standard deviation} = \sqrt{\frac{4279}{98} - 6.03^2} \simeq 2.70$$

$$\text{Actual standard deviation} = 2.70 \times 0.02 \simeq 0.054$$

$$\text{Fictitious mean deviation} = \frac{68.66}{98} \simeq 0.70$$

$$\text{Actual mean deviation} \simeq 0.70 \times 0.02 \simeq 0.014$$

"JUST THE MATHS"

SLIDES NUMBER

18.4

STATISTICS 4
(The principle of least squares)

by

A.J.Hobson

UNIT 18.4 - STATISTICS 4

THE PRINCIPLE OF LEAST SQUARES

18.4.1 THE NORMAL EQUATIONS

Suppose x and y, are known to obey a **"straight line law"** of the form $y = a+bx$, where a and b are constants to be found.

In an experiment to test this law, let n pairs of values be (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error.

The **observed** values, y_i will be subject to experimental error.

For the straight line of **"best fit"**, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

Using partial differentiation, it may be shown that

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad --- (1)$$

and

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2. \quad --- (2)$$

Statements (1) and (2) (which must be solved for a and b) are called the **"normal equations"**.

A simpler notation for the normal equations is

$$\Sigma\ y = na + b\Sigma\ x$$

and

$$\Sigma xy = a\Sigma x + b\Sigma\ x^2.$$

Eliminating a and b in turn,

$$a = \frac{\Sigma\ x^2.\Sigma\ y - \Sigma x.\Sigma\ xy}{n\Sigma\ x^2 - (\Sigma\ x)^2} \ \text{ and } \ b = \frac{n\Sigma\ xy - \Sigma\ x.\Sigma\ y}{n\Sigma\ x^2 - (\Sigma\ x)^2}.$$

The straight line $y = a + bx$ is called the **"regression line of y on x"**.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x has equation $y = a + bx$, where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus,

$$y = 0.176x - 0.645$$

18.4.2 SIMPLIFIED CALCULATION OF REGRESSION LINES

We consider a temporary change of origin to the point $(\overline{x}, \overline{y})$ where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\, y}{n} = a + b\frac{\Sigma\, x}{n}$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y is associated with the formulae

$$X = x - \overline{x} \ \text{ and } \ Y = y - \overline{y}.$$

In this system of reference, the regression line will pass through the origin.

The equation of the regression line is

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\ X = \Sigma\ (x - \overline{x}) = \Sigma\ x - \Sigma\ \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\ Y = \Sigma\ (y - \overline{y}) = \Sigma\ y - \Sigma\ \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\ XY}{\Sigma\ X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x}).$$

There may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	−0.5	−0.84	0.42	0.25
42	6.30	−3.5	−1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	−3.5	−0.38	1.33	12.25
35	5.90	−10.5	−1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	−5.5	−1.17	6.435	30.25
39	6.55	−6.5	−0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5)$$

That is,

$$y = 0.176x - 0.638$$

"JUST THE MATHS"

SLIDES NUMBER

19.1

PROBABILITY 1 (Definitions and rules)

by

A.J.Hobson

UNIT 19.1 - PROBABILITY 1

DEFINITONS AND RULES

19.1.1 INTRODUCTION

Suppose 30 high-strength bolts became mixed with 25 ordinary bolts by mistake, all of the bolts being identical in appearance.

How sure can we be that, in choosing a bolt, it will be a high-strength one ?

Phrases like "quite sure" or "fairly sure" are useless, mathematically.

Hence, we define a way of measuring the certainty.

In 55 simultaneous choices, 30 will be of high strength and 25 will be ordinary.

We say that, in one choice, there is a $\frac{30}{55}$ chance of success.

That is, approximately, a 0.55 chance of success.

Just over half the choices will most likely give a high-strength bolt.

Such predictions can be used, for example, to
estimate the cost of mistakes on a production line.

DEFINITION 1

The various occurrences which are possible in a statistical problem are called **"events"**.

If we are interested in one particular event, it is termed **"successful"** when it occurs and **"unsuccessful"** when it does not.

ILLUSTRATION

If, in a collection of 100 bolts, there are 30 high-strength, 25 ordinary and 45 low-strength, we can make 100 **"trials"**.

In each trial, one of three events will occur (high, ordinary or low strength).

DEFINITION 2

If, in n possible trials, a successful event occurs s times, then the number $\frac{s}{n}$ is called the **"probability of success in a single trial"**.

It is also known as the **"relative frequency of success"**.

ILLUSTRATIONS

1. From a bag containing 7 black balls and 4 white balls, the probability of drawing a white ball is $\frac{4}{11}$.
2. In tossing a perfectly balanced coin, the probability of obtaining a head is $\frac{1}{2}$.
3. In throwing a die, the probability of getting a six is $\frac{1}{6}$.
4. If 50 chocolates are identical in appearance, but consist of 15 soft-centres and 35 hard-centres, the probability of choosing a soft-centre is $\frac{15}{50} = 0.3$

19.1.2 APPLICATION OF PROBABILITY TO GAMES OF CHANCE

If a competitor in a game of chance has a probability, p, of winning, and the prize money is £m, then £mp is considered to be a fair price for entry to the game.

The quantity mp is known as the **"expectation"** of the competitor.

19.1.3 EMPIRICAL PROBABILITY

So far, all the problems discussed on probability have been **"descriptive"**; that is, we know all the possible events, the number of successes and the number of failures

In other problems, called **"inference"** problems, it is necessary either

(a) to take **"samples"** in order to infer facts about a total **"population"**;

for example, a public census or an investigation of moon-rock.

or

(b) to rely on past experience;

for example past records of heart deaths, road accidents, component failure.

If the probability of success, used in a problem, has been inferred by samples or previous experience, it is called **"empirical probability"**.

However, once the probability has been calculated, the calculations are carried out in the same way as for descriptive problems.

19.1.4 TYPES OF EVENT

DEFINITION 3

If two or more events are such that not more than one of them can occur in a single trial, they are called **"mutually exclusive"**.

ILLUSTRATION

Drawing an Ace or drawing a King from a pack of cards are mutually exclusive events; but drawing an Ace and drawing a Spade are not mutually exclusive events.

DEFINITION 4

If two or more events are such that the probability of any one of them occurring is not affected by the occurrence of another, they are called **"independent"** events.

ILLUSTRATION

From a pack of 52 cards (i.e. Jokers removed), the event of drawing and immediately replacing a red card will have a probability of $\frac{26}{52} = 0.5$; and the probability of this occurring a second time will be exactly the same. They are independent events.

However, two successive events of drawing a red card **without** replacing it are **not** independent. If the first card drawn is red, the probability that the second is red will be $\frac{25}{51}$; but, if the first card drawn is black, the probability that the second is red will be $\frac{26}{51}$.

19.1.5 RULES OF PROBABILITY

1. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r mutually exclusive events, then the probability that some **one** of the r events will occur is

   $$p_1 + p_2 + p_3 + ... + p_r.$$

   ILLUSTRATION

   Suppose a bag contains 100 balls of which 1 is red, 2 are blue and 3 are black.

   The probability of choosing any one of these three colours will be

   $$0.06 = 0.01 + 0.02 + 0.03$$

   However, the probability of drawing a spade or an ace from a pack of 52 cards will not be $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$ but $\frac{16}{52}$ since there are just 16 cards which are either a spade or an ace.

2. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r independent events, then the probability that **all** will occur in a single trial is

$$p_1.p_2.p_3......p_r.$$

ILLUSTRATION

Suppose there are three bags, each containing white, red and blue balls.

Suppose also that the probabilities of drawing a white ball from the first bag, a red ball from the second bag and a blue ball from the third bag are, respectively, p_1, p_2 and p_3.

The probability of making these three choices in succession is $p_1.p_2.p_3$ because they are independent events.

However, if three cards are drawn, without replacing, from a pack of 52 cards, the probability of drawing a 3, followed by an ace, followed by a red card will not be $\frac{4}{52}.\frac{4}{52}.\frac{26}{52}$.

19.1.6 CONDITIONAL PROBABILITIES

For **dependent** events, the multiplication rule requires a knowledge of the **new** probabilities of successive events in the trial, after the previous ones have been dealt with. These are called **"conditional probabilities"**.

EXAMPLE

From a box, containing 6 white balls and 4 black balls, 3 balls are drawn at random without replacing them. What is the probability that there will be 2 white and 1 black ?

Solution

The cases to consider, together with their probabilities are as follows:

(a) White, White, Black

Probability $= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.

(b) Black, White, White

Probability $= \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

(c) White, Black, White

Probability $= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

The probability of any one of these three outcomes is therefore

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

“JUST THE MATHS”

SLIDES NUMBER

19.2

PROBABILITY 2
(Permutations and combinations)

by

A.J.Hobson

UNIT 19.2 - PROBABILITY 2

PERMUTATIONS AND COMBINATIONS

19.2.1 INTRODUCTION

In “descriptive” problems, we can work out the probability that an event will occur by counting up the total number of possible trials and the number of successful ones amongst them. But this can often be a tedious process without the results of the work which follows:

DEFINITION 1

Each different arrangement of all or part of a set of objects is called a **“permutation”**.

DEFINITION 2

Any set which can be made by using all or part of a given collection of objects, without regard to order, is called a **“combination”**.

EXAMPLES

1. Nine balls, numbered 1 to 9, are put into a bag, then emptied into a channel which guides them into a line of pockets. What is the probability of obtaining a particular nine digit number ?

   Solution

We require the total number of arrangements of the nine digits.

There are nine ways in which a digit can appear in the first pocket and, for each of these ways, there are then eight choices for the second pocket.

Hence, the first two pockets can be filled in $9 \times 8 = 72$ ways.

Continuing in this manner, the total number of arrangements will be

$$9 \times 8 \times 7 \times 6 \times ...... \times 3 \times 2 \times 1 = 362880 = T, \text{ (say)}.$$

This is the number of permutations of the nine digits and the required probability is therefore $\frac{1}{T}$.

2. A box contains five components of identical appearance but different qualities. What is the probability of choosing a pair of components from the highest two qualities ?

Solution

Method 1

Let the components be A, B, C, D, E in order of descending quality.

The choices are

AB AC AD AE

BC BD BE

CD CE

DE,

giving ten choices.

These are the various combinations of five objects, two at a time; hence, the probability for AB $= \frac{1}{10}$.

Method 2.

We could also use the ideas of conditional probability as follows:

The probability of drawing A is $\frac{1}{5}$.

The probability of drawing B without replacing A is $\frac{1}{4}$.

The probability of drawing A and B **in either order** is

$$2 \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}.$$

19.2.2 RULES OF PERMUTATIONS AND COMBINATIONS

1. The number of permutations of all n objects in a set of n is

$$n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots 3.2.1$$

   This is denoted for short by the symbol n! It is called **"n factorial"**.

   This rule was demonstrated in
   Example 1, earlier.

2. The number of permutations of n objects r at a time is given by

$$n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

   EXPLANATION

   The first object can be chosen in any one of n different ways.

   For each of these, the second object can then be chosen in $n-1$ ways.

   For each of these, the third object can then be chosen in $n-2$ ways.

   ...

   For each of these, the r-th object can be chosen in $n-(r-1) = n-r+1$ ways.

Note:

In Example 2, earlier, the number of permutations of five components two at a time is given by

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5.4.3.2.1}{3.2.1} = 20.$$

This is double the number of choices we obtained for any two components out of five because, in a permutation, the order matters.

3. The number of combinations of n objects r at a time is given by

$$\frac{n!}{(n-r)!r!}$$

EXPLANATION

This is very much the same problem as the number of permutations of n objects r at a time; but, as permutations, a particular set of objects will be counted $r!$ times

In the case of combinations, such a set will be counted only once, which reduces the number of possibilities by a factor of $r!$

In Example 2, earlier, it is precisely the number of combinations of five objects two at a time which is being calculated. That is,

$$\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5.4.3.2.1}{3.2.1.2.1} = 10$$

Note:

A traditional notation for the number of permutations of n objects r at a time is ${}^nP_r$.

$$\text{That is } {}^nP_r = \frac{n!}{(n-r)!}$$

A traditional notation for the number of combinations of n objects r at a time is ${}^nC_r$.

$$\text{That is } {}^nC_r = \frac{n!}{(n-r)!r!}$$

EXAMPLES

1. How many four digit numbers can be formed from the numbers 1,2,3,4,5,6,7,8,9 if no digit can be repeated ?

   Solution

   This is the number of permutations of 9 objects four at a time.

   $$\text{That is } \frac{9!}{5!} = 9.8.7.6. = 3,024.$$

2. In how many ways can a team of nine people be selected from twelve ?

   Solution

   The required number is

   $${}^{12}C_9 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

3. In how many ways can we select a group of three men and two women from five men and four women ?

Solution

The number of ways of selecting three men from five men is

$$^{5}C_3 = \frac{5!}{2!3!} = \frac{5.4.}{2.1} = 10.$$

For each of these ways, the number of ways of selecting two women from four women is

$$^{4}C_2 - \frac{4!}{2!2!} = \frac{4.3}{2.1} = 6.$$

The total number of ways is therefore $10 \times 6 = 60$.

4. What is the probability that one of four bridge players will obtain a thirteen card suit ?

Solution

The number of possible suits for each player is

$$N = {}^{52}C_{13} = \frac{52!}{39!13!}$$

The probabilty that any one of the four players will obtain a thirteen card suit is thus

$$4 \times \frac{1}{N} = \frac{4.(39!)(13!)}{52!} \simeq 6.29 \times 10^{-10}.$$

5. A coin is tossed six times. Find the probablity of obtaining exactly four heads.

Solution

In a single throw of the coin, the probability of a head (and of a tail) is $\frac{1}{2}$.

Secondly, the probability that a particular four out of six throws will be heads, **and** the other two tails, will be

$$\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\left[\frac{1}{2}.\frac{1}{2}\right] = \frac{1}{2^6} = \frac{1}{64}.$$

Finally, the number of choices of four throws from six throws is

$$^6C_4 = \frac{6!}{2!4!} = 15.$$

Hence the required probability of exactly four heads is

$$\frac{15}{64}.$$

19.2.3 PERMUTATIONS OF SETS WITH SOME OBJECTS ALIKE

INTRODUCTORY EXAMPLE

Suppose twelve switch buttons are to be arranged in a row, and there are two red buttons, three yellow and seven green. How many possible distinct patterns can be formed ?

Solution

If all twelve buttons were of a <u>different</u> colour, there would be 12! possible arrangements.

If we now colour two switches red, there will be only half the number of arrangements since every pair of positions previously held by them would have counted 2! times, that is, twice.

If we then colour another three switches yellow, the positions previously occupied by them would have counted 3! times; that is, 6 times, so we reduce the number of arrangements further by a factor of 6.

Similarly, by colouring another seven switches green, we reduce the number of arrangements further by a factor of 7!

Hence the final number of arrangements will be

$$\frac{12!}{2!3!7!} = 7920.$$

This example illustrates another standard rule that, if we have n objects of which r_1 are alike of one kind, r_2 are alike of another, r_3 are alike of another ....and r_k are alike of another, then the number of permutations of these n objects is given by

$$\frac{n!}{r_1!r_2!r_3!...r_k!}$$

"JUST THE MATHS"

SLIDES NUMBER

19.3

PROBABILITY 3
(Random variable)

by

A.J.Hobson

UNIT 19.3 - RANDOM VARIABLES

19.3.1 DEFINING RANDOM VARIABLES

(i) The theory of probability usually discusses **"random experiments"**.

For example, in throwing an unbiased die, it is just as likely to show one face as any other.

Similarly, drawing 6 lottery numbers out of 45 is a random experiment provided it is just as likely for one number to be drawn as any other.

An experiment is a random experiment if there is more than one possible outcome (or event) and any one of those possible outcomes may occur.

We assume that the outcomes are **"mutually exclusive"**.

The probabilities of the possible outcomes of a random experiment form a collection called the **"probability distribution"** of the experiment.

These probabilities need not be the same as one another.

The complete list of possible outcomes is called the **"sample space"** of the experiment.

(ii) In a random experiment, each outcome may be associated with a numerical value called a
"random variable"

This variable, x, makes it possible to refer to an outcome without having to use a complete description of it.

In tossing a coin, we might associate a head with the number 1 and a tail with the number 0.

Then the probabilities of either a head or a tail are given by the formulae

$$P(x = 1) = 0.5 \text{ and } P(x = 1) = 0.5$$

Note:
There is no restriction on the way we define the values of x; it would have been just as correct to associate a head with -1 and a tail with 1.

It is customary to assign the values of random variables as logically as possible.

For example, in discussing the probability that two 6's would be obtained in 5 throws of a dice, we could sensibly use $x = 1, 2, 3, 4, 5$ and 6 respectively for the results that a 1,2,3,4,5 and 6 would be thrown.

(iii) It is necessary to distinguish between
"discrete" and **"continuous"** random variables.

Discrete random variables may take only certain specified values.

Continuous random variables may take any value with in a certain specified range.

Examples of discrete random variables include those associated with the tossing of coins, the throwing of dice and numbers of defective components in a batch from a production line.

Examples of continuous variables include those associated with persons' height or weight, lifetimes of manufactured components and rainfall figures.

Note:
For a random variable, x, the associated probabilities form a function, $P(x)$, called the
"probability function"

19.3.2 PROBABILITY DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS

(a) Probability Distribution Functions

A **"probability distribution function"**, denoted here by $F(x)$, is the relationship between a random variable, x, and the probability of obtaining any outcome for which the random variable can take values up to and including x.

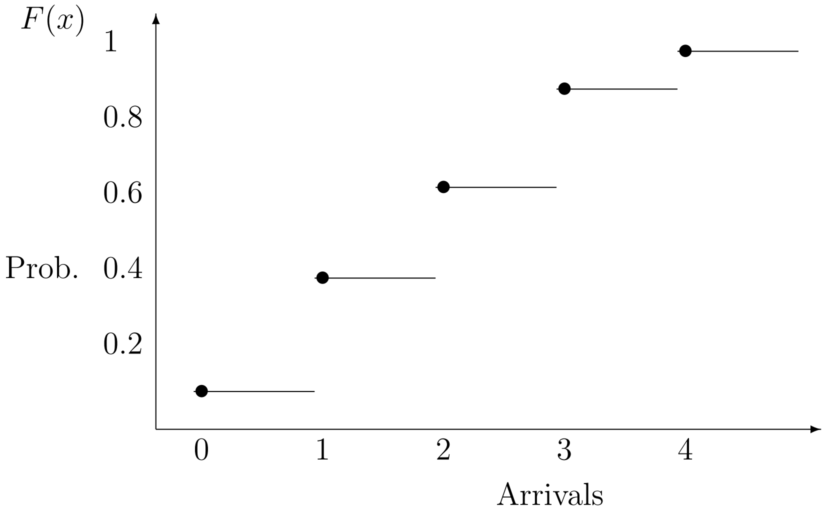
It is the probability, $P(\leq x)$, that the random variable for the outcome is less than or equal to x.

(i) Probability distribution functions for discrete variables

Suppose that the number of ships arriving at a container terminal during any one day can be 0,1,2,3 or 4, with respective probabilites 0.1, 0.3, 0.35, 0.2 and 0.05

The probabilites for other outcomes is taken to be zero.

The graph of the probability distribution function is as follows:



The value of the probability distribution function at a value, x, of the random variable is the sum of the probabilities to the left of, and including, x.

The "bullet" marks to indicate which end of each horizontal line belongs to the graph.

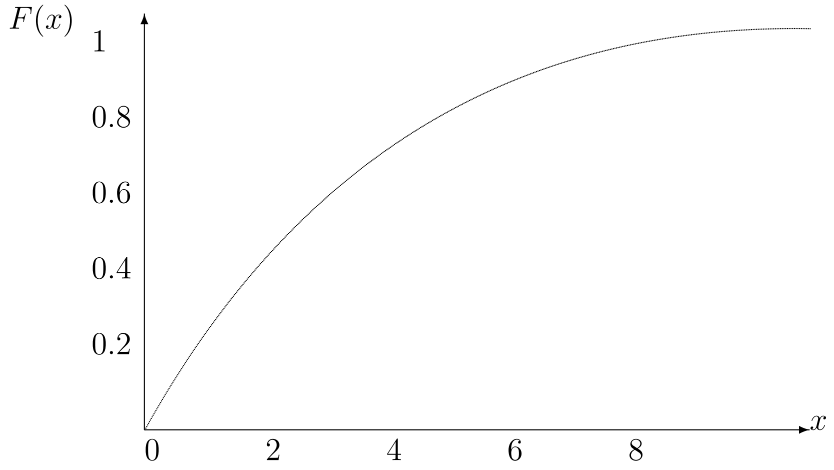
(ii) Probability distribution functions for continuous variables

For a continuous random variable, the probability distribution function is also measures the probability that the value of the random variable is less than or equal to x.

We illustrate with the an **"exponential distribution"**, in which the lifetime of a certain electronic component (in thousands of hours) is represented by a probability distribution function.

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

The graph of the probability distribution function is as follows:



(b) Probability Density Functions

For continuous random variables, the **"probability density function"**, $f(x)$, is defined by

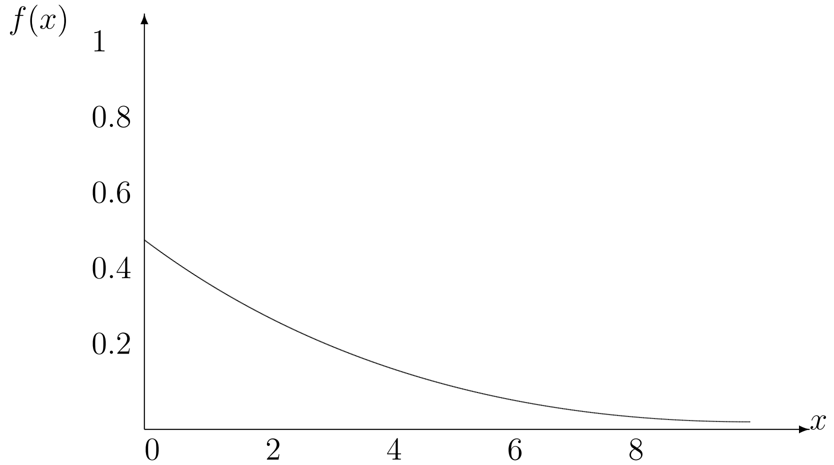
$$f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)].$$

The probability density function measures the **concentration** of possible values of x.

In the previous example, the probability density function is given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0 \end{cases}$$

The graph of the probability density function is as follows:



Here, most components have short lifetimes, while a small number can survive for much longer.

(c) Properties of probability distribution and probability density functions

(i)

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

Proof:

It is impossible for a random variable to have a value less than $-\infty$ and it is certain to have a value less than ∞.

(ii) If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

Proof:

The outcomes of an experiment with random variable values up to and including x_2 includes those outcomes with random variable values up to and including x_1 so that $F(x_2)$ is as least as great as $F(x_1)$.

Note:
Results **(i)** and **(ii)** imply that, for any value of x, the probability distribution function is either constant or increasing between 0 and 1.

(iii) The probability that an outcome will have a random variable value, x, within the range
$x_1 < x \leq x_2$ is given by the expression

$$F(x_2) - F(x_1)$$

Proof:

From, the outcomes of an experiment with random variable values up to and including x_2, suppose we exclude those outcomes with random variable values up to and including x_1.

The residue will be those outcomes with random variable values which lie within the range
$x_1 < x \leq x_2$.

Note:
For a continuous random variable, this is the area under the graph of the probability density function between the two given points because $f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)]$.

That is,

$$\int_{x_1}^{x_2} f(x) \, \mathrm{d}x.$$

(iv)

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 1.$$

Proof:

The total area under the probability density function must be 1 since the random variable must have a value somewhere.

EXAMPLE

For the distribution of component lifetimes (in thousands of hours) given by

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0, \end{cases}$$

determine the proportion of components which last longer than 3000 hours but less than or equal to 6000 hours.

Solution

The probability that components have lifetimes up to and including 3000 hours is $F(3) = 1 - e^{-\frac{3}{2}}$.

The probability that components have lifetimes up to and including 600 hours is $F(6) = 1 - e^{-\frac{6}{2}}$.

The probability that components last longer than 3000 hours but less than or equal to 6000 hours is
$F(6) - F(3) = e^{-\frac{3}{2}} - e^{-3} \simeq 0.173$

"JUST THE MATHS"

SLIDES NUMBER

19.4

PROBABILITY 4
(Measures of location and dispersion)

by

A.J.Hobson

19.4.1 Common types of measure

UNIT 19.4 - PROBABILITY 4

MEASURES OF LOCATION AND DISPERSION

19.4.1 COMMON TYPES OF MEASURE

We include three common measures of location (or central tendency) used in the discussion of probability distributions and one common measure of dispersion (or scatter).

They are as follows:

(a) The Mean

(i) For Discrete Random Variables

If the values $x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_n$ of a discrete random variable, x, have probabilities $P_1,\ P_2,\ P_3,\ \ldots\ ,P_n$ respectively, then P_i represents the expected frequency of x_i divided by the total number of possible outcomes.

For example, if the probability of a certain value of x is 0.25, then there is a one in four chance of its occurring.

The arithmetic mean, μ, of the distribution may therefore be given by the formula

$$\mu = \sum_{i=1}^{n} x_i P_i.$$

(ii) For Continuous Random Variables

In this case, we use the probability density function, $f(x)$, for the distribution, which is the rate of increase of the probability distribution function, $F(x)$.

For a small interval, δx of x-values, the probability that any of these values occurs is approximately $f(x)\delta x$, which leads to the formula

$$\mu = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x.$$

(b) The Median

(i) For Discrete Random Variables

The median provides an estimate of the middle value of x, taking into account the frequency at which each value occurs.

More precisely, the median is a value, m, of the random variable, x, for which

$$P(x \leq m) \geq \frac{1}{2} \ \textbf{and} \ P(x \geq m) \geq \frac{1}{2}.$$

The median for a discrete random variable may not be unique (see Example 1, following).

(ii) For Continuous Random Variables

The median for a continuous random variable is a value of the random variable, x, for which there are equal chances of x being greater than or less than the median itself.

More precisely, it may be defined as the value, m, for which $P(x \leq m) = F(m) = \frac{1}{2}$.

Note:
Other measures of location are sometimes used, such as **"quartiles"**, **"deciles"** and **"percentiles"**, which divide the range of x values into four, ten and one hundred equal parts respectively.

For example, the third quartile of a distribution function, $F(x)$, may be defined as a value, q_3, of the random variable, x, such that

$$F(q_3) = \frac{3}{4}.$$

(c) The Mode

The mode is a measure of the most likely value occurring of the random variable, x.

(i) For Discrete Random Variables

In this case, the mode is any value of x with the highest probability, and, again, it may not be unique (see Example 1, following).

(ii) For Continuous Random Variables

In this case, we require a value of x for which the probability density function (measuring the concentration of x values) has a maximum.

(d) The Standard Deviation

The most common meaure of dispersion (or scatter) for a probability distribution is the **"standard deviation"**, σ.

(i) For Discrete Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 P(x)}.$$

(ii) For Continuous Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, \mathrm{d}x},$$

where $f(x)$ denotes the probability density function.

Each measures the dispersion of the x values around the mean, μ.

Note:
σ^2 is known as the **"variance"** of the probability distribution.

EXAMPLES

1. Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviation for a simple toss of an unbiased die.

   Solution

   (a) The mean is given by

   $$\mu = \sum_{i=1}^{6} i \times \frac{1}{6} = \frac{22}{6} = 3.5$$

   (b) Both 3 and 4 on the die fit the definition of a median since

   $$P(x \leq 3) = \frac{1}{2}, \quad P(x \geq 3) = \frac{2}{3}$$

   and

   $$P(x \leq 4) = \frac{2}{3}, \quad P(x \geq 4) = \frac{1}{2}.$$

   (c) All six outcomes count as a mode since they all have a probability of $\frac{1}{6}$.

(d) The standard deviation is given by

$$\sigma = \sqrt{\sum_{i=1}^{6} \frac{1}{6}(i - 3.5)^2} \simeq 2.917$$

2. Determine (a) the mean, (b) median and (c) the mode and (d) the standard deviation for the distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

Solution

First, we need the probability density function, $f(x)$, which is given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ \\ 0 & \text{when } x < 0 \end{cases}$$

Hence,

(a)

$$\mu = \int_0^\infty \frac{1}{2} x e^{-\frac{x}{2}} \, \mathrm{d}x.$$

On integration by parts, this gives

$$\mu = \left[-x e^{-\frac{x}{2}} \right]_0^\infty + \int_0^\infty e^{-\frac{x}{2}} \, \mathrm{d}x = \left[-2 e^{-\frac{x}{2}} \right]_0^\infty = 2.$$

(b) The median is the value, m, for which

$$F(m) = \frac{1}{2}.$$

That is,

$$1 - e^{-\frac{m}{2}} = \frac{1}{2},$$

giving

$$-\frac{m}{2} = \ln \left[\frac{1}{2} \right].$$

Hence, $m \simeq 1.386$.

(c) The mode is zero since the maximum value of the probability density function occurs when $x = 0$.

(d) The standard deviation is given by

$$\sigma^2 = \int_0^{\infty} \frac{1}{2}(x-2)^2 e^{-\frac{x}{2}}\, \mathrm{d}x.$$

On integration by parts, this gives

$$\sigma^2 = -\left[(x-2)^2 e^{-\frac{x}{2}}\right]_0^{\infty} + \int_0^{\infty} 2(x-2)e^{-\frac{x}{2}}\, \mathrm{d}x$$

$$= 4 - \left[4(x-2)e^{-\frac{x}{2}}\right]_0^{\infty} \ 4e^{-\frac{x}{2}}\, \mathrm{d}x = 4$$

Thus $\sigma = 2$.

“JUST THE MATHS”

SLIDES NUMBER

19.5

PROBABILITY 5
(The binomial distribution)

by

A.J.Hobson

UNIT 19.5 - PROBABILITY 5

THE BINOMIAL DISTRIBUTION

19.5.1 INTRODUCTION AND THEORY

In this Unit, we consider, first, probability problems having only **two** events (mutually exclusive and independent), although many trials may be possible.

For example, the pairs of events could be "up and down", "black and white", "good and bad", and, in general, "successful and unsuccessful".

Statement of the problem

If the probability of success in a single trial is unaffected when successive trials are carried out (independent events), then what is the probability that, in n successive trials, **exactly** r will be successful ?

General Analysis of the problem

We build up the solution in simple stages:

(a) If p is the probability of success in a single trial, then the probability of failure is $1 - p = q$, say.

(b) In the following table, let S stand for success and let F stand for failure:

The table shows the possible results of one, two or three trials and their corresponding probabilities:

Trials	Possible Results	Respective Probabilities
1	F,S	q, p
2	FF,FS,SF,SS	q^2, qp, pq, p^2
3	FFF,FFS,FSF,FSS, SFF,SFS,SSF,SSS	$q^3, q^2p, q^2p, qp^2,$ q^2p, qp^2, qp^2, p^3

(c) **Summary**

(i) In **one** trial, the probabilities that there will be exactly 0 or exactly 1 successes are the respective terms of the expression

$$q + p.$$

(ii) In **two** trials, the probabilities that there will be exactly 0, exactly 1 or exactly 2 successes are the respective terms of the expression

$$q^2 + 2qp + p^2; \quad \text{that is,} \quad (q + p)^2.$$

(iii) In **three** trials, the probabilities that there will be exactly 0, exactly 1, exactly 2 or exactly 3 successes are the respective terms of the expression

$$q^3 + 3q^2p + 3qp^2 + p^3; \quad \text{that is,} \quad (q + p)^3.$$

(iv) In **any number** of trials, n the probabilities that there will be exactly 0, exactly 1, exactly 2, exactly 3, ... or exactly n successes are the respective terms in the binomial expansion of the expression

$$(q+p)^n.$$

(d) **MAIN RESULT:**

The probability that, in n trials, there will be exactly r successes, is the term containing p^r in the binomial expansion of $(q+p)^n$.

It can be shown that this is the value of

$$^nC_r q^{n-r} p^r.$$

EXAMPLES

1. Determine the probability that, in 6 tosses of a coin, there will be exactly 4 heads.

   Solution

   $$q = 0.5, \quad p = 0.5, \quad n = 6, \quad r = 4.$$

   Hence, the required probability is given by

   $$^6C_4.(0.5)^2.(0.5)^4 = \frac{6!}{2!4!}.\frac{1}{4}.\frac{1}{16} = \frac{15}{64}.$$

2. Determine the probability of obtaining the most probable number of heads in 6 tosses of a coin.

   Solution

   The most probable number of heads is given by

$$\frac{1}{2} \times 6 = 3.$$

   The probability of obtaining exactly 3 heads is given by

$$^6C_3.(0.5)^3(0.5)^3 = \frac{6!}{3!3!}.\frac{1}{8}.\frac{1}{8} = \frac{20}{64} \simeq 0.31$$

3. Determine the probability of obtaining exactly 2 fives in 7 throws of a die.

   Solution

$$q = \frac{5}{6}, \quad p = \frac{1}{6}, \quad n = 7, \quad r = 2.$$

   Hence, the required probability is given by

$$^7C_2.\left(\frac{5}{6}\right)^5.\left(\frac{1}{6}\right)^2 = \frac{7!}{5!2!}.\left(\frac{5}{6}\right)^5.\left(\frac{1}{6}\right)^2 \simeq 0.234$$

4. Determine the probability of throwing at most 2 sixes in 6 throws of a die.

   Solution

   The phrase "at most 2 sixes" means exactly 0, or exactly 1, or exactly 2 sixes.

Hence, we add together the first three terms in the expansion of $(q+p)^6$, where $q = \frac{5}{6}$ and $p = \frac{1}{6}$.

It can be shown that

$$(q+p)^6 = q^6 + 6q^5p + 15q^4p^2 + \ldots\ldots$$

By substituting for q and p, the sum of the first three terms turns out to be

$$\frac{21875}{23328} \simeq 0.938$$

5. It is known that 10% of certain components manufactured are defective. If a random sample of 12 such components is taken, what is the probability that at least 9 are defective ?

Solution

The information suggests that removal of components for examination does not affect the probability of 10%.

This is reasonable since our sample is almost certainly very small compared with all components in existence.

The probability of <u>success</u> in this exercise is 0.1, even though it refers to <u>defective</u> items, and hence the probability of failure is 0.9

Using $p = 0.1$, $q = 0.9$, $n = 12$, we require the probabilities (added together) of exactly 9, 10, 11 or 12 defective items.

These are the <u>last</u> four terms in the expansion of $(q+p)^n$.

That is,

$$^{12}C_9.(0.9)^3.(0.1)^9 + {}^{12}C_{10}.(0.9)^2.(0.1)^{10}$$

$$+^{12}C_{11}.(0.9).(0.1)^{11} + (0.1)^{12}$$

$$\simeq 1.658 \times 10^{-7}.$$

Note:
The use of the **"binomial distribution"** becomes very tedious when the number of trials is large; and two other standard distributions may be used.

They are called the

" normal distribution"

and the

"Poisson distribution".

"JUST THE MATHS"

SLIDES NUMBER

19.6

PROBABILITY 6
(Statistics for the binomial distribution)

by

A.J.Hobson

UNIT 19.6 - PROBABILITY 6

STATISTICS FOR THE BINOMIAL DISTRIBUTION

19.6.1 CONSTRUCTION OF HISTOGRAMS

Frequency tables, histograms etc. usually involve experiments which are actually carried out.

Here, we illustrate how the binomial distribution may be used to estimate the results of a certain kind of experiment before it is performed.

EXAMPLE

For four coins, tossed 32 times, construct a histogram showing the expected number of occurrences of 0,1,2,3,4.....heads.

Solution

Firstly, in a single toss of the four coins, the probability of head (or tail) for each coin is $\frac{1}{2}$.

The terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^4$ give the probabilities of exactly 0,1,2,3 and 4 heads, respectively.

The expansion is

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4.$$

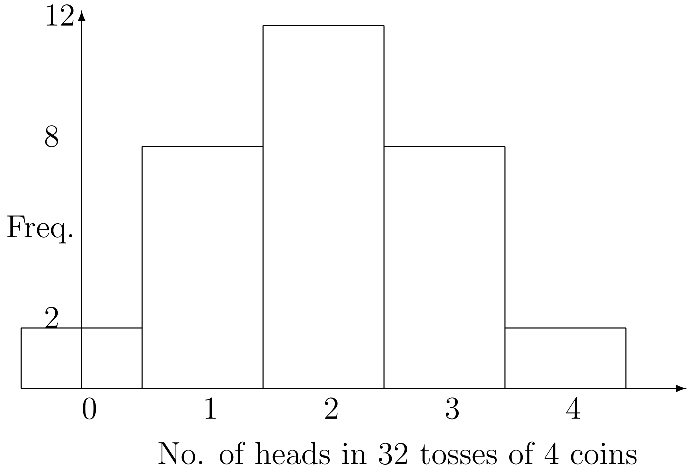
That is,

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4 (1+4+6+4+1).$$

This shows that the probabilities of 0,1,2,3 and 4 heads in a single toss of four coins are $\frac{1}{16}$, $\frac{1}{4}$, $\frac{6}{16}$, $\frac{1}{4}$, and $\frac{1}{16}$, respectively

Therefore, in 32 tosses of four coins, we may expect 0 heads, twice; 1 head, 8 times; 2 heads, 12 times; 3 heads, 8 times and 4 heads, twice.

The following histogram uses class-intervals for which each member is situated at the mid-point:



Notes:

(i) The histogram is symmetrical in shape since the probability of success and failure are equal to each other (the binomial expansion itself is symmetrical).

(ii) Since the widths of the class-intervals in the above histogram are 1, the areas of the rectangles are equal to their heights.

Thus, for example, the total area of the first three rectangles represents the expected number of times of obtaining <u>at most</u> 2 heads in 32 tosses of 4 coins.

19.6.2 MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

THEOREM

If p is the probability of success of an event in a single trial and q is the probability of its failure, then the binomial distribution, giving the expected frequencies of 0,1,2,3,.....n successes in n trials, has a Mean of np and a Standard Deviation of $\sqrt{npq}$ irrespective of the number of times the experiment is to be carried out.

Proof (Optional):

(a) Mean

From the binomial theorem,

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$\frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ldots + nqp^{n-1} + p^n.$$

Hence, if the n trials are made N times, the average number of successes is equal to the following expression, multiplied by N, then divided by N:

$$0 \times q^n + 1 \times nq^{n-1}p + 2 \times \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$3 \times \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ ..(n-1) \times nqp^{n-1} + np^n.$$

That is,

$$np[q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2}q^{n-3}p^2 + \ \ldots$$

$$+(n-1)qp^{n-2} + p^{n-1}]$$

$$= np(q+p)^{n-1} = np \ \text{ since } \ q + p = 1.$$

(b) Standard Deviation

For the standard deviation, we observe that, if f_r is the frequency of r successes when the n trials are conducted N times, then

$$f_r = N \frac{n!}{(n-r)!r!} q^{n-r} p^r.$$

We use this, first, to establish a result for

$$\sum_{r=0}^{n} r^2 f_r.$$

For example,

$$0^2 f_0 = 0.Nq^n = 0.f_0 \text{ and } 1^2 f_1 = 1.Nnq^{n-1}p = 1.f_1;$$

$$2^2 f_2 = 2Nn(n-1)q^{n-2}p^2$$

$$= Nn(n-1)q^{n-2}p^2 + Nn(n-1)p^2 q^{n-2}$$

$$= 2f_2 + Nn(n-1)p^2 q^{n-2};$$

$$3^2 f_3 = 3N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3$$

$$= N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 + Nn(n-1)p^2(n-2)q^{n-3}p$$

$$3^2 f_3 = 3f_3 + Nn(n-1)p^2(n-2)q^{n-3}p;$$

$$4^2 f_4 = 4N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$$

$$= N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$$

$$+Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2$$

$$= 4f_4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2.$$

In general, when $r \geq 2$, $r^2 f_r =$

$$N \frac{n(n-1)(n-2)....(n-r+1)}{(r-1)!} + Nn(n-1)p^2 q^{n-r} p^r =$$

$$rf_r + Nn(n-1)p^2 \frac{(n-2)!}{(n-r)!(r-2)!} q^{n-r} p^{r-2}.$$

$$\sum_{r=0}^{n} r^2 f_r = \sum_{r=0}^{n} rf_r + Nn(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!} q^{n-r} p^{r-2}.$$

Since $q + p = 1$, we have

$$\sum_{r=0}^{n} r^2 f_r = Nnp + Nn(n-1)p^2 (q+p)^{n-2}$$

$$= Nnp + Nn(n-1)p^2.$$

The standard deviation of a set $x_1, x_2, x_3, \ldots x_m$ of m observations, with a mean value of $\overline{x}$ is given by the formula

$$\sigma^2 = \frac{1}{m} \sum_{i=1}^{m} x_i^2 - \overline{x}^2.$$

In the present case, this may be written

$$\sigma^2 = \frac{1}{N} \sum_{r=0}^{n} r^2 f_r - \frac{1}{N^2}\left(\sum_{r=0}^{n} r f_r\right)^2.$$

Hence,

$$\sigma^2 = \frac{1}{N}\left(Nnp + Nn(n-1)p^2\right) - \frac{1}{N^2}(Nnp)^2.$$

This gives

$$\sigma^2 = np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq.$$

Therefore, $\sigma = \sqrt{npq}$.

ILLUSTRATION

For direct calculation of the mean and the standard deviation for the data in the previous coin-tossing problem, we may use the following table in which x_i denotes numbers of heads and f_i denotes the corresponding expected frequencies:

x_i	f_i	$f_i x_i$	$f_i x_i^2$
0	2	0	0
1	8	8	8
2	12	24	48
3	8	24	72
4	2	8	32
Totals	32	64	160

The mean is given by

$$\overline{x} = \frac{64}{32} = 2 \quad \text{(obviously)}.$$

This agrees with $np = 4 \times \frac{1}{2}$.

The standard deviation is given by

$$\sigma = \sqrt{\left[\frac{160}{32} - 2^2\right]} = 1.$$

This agrees with $\sqrt{npq} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}}$.

Note:
If the experiment were carried out N times instead of 32 times, all values in the last three columns of the above table would be multiplied by a factor of $\frac{N}{32}$ which would then cancel out in the remaining calculations.

EXAMPLE

Three dice are rolled 216 times. Construct a binomial distribution and show the frequencies of occurrence for 0,1,2 and 3 sixes.

Evaluate the Mean and the standard deviation of the distribution.

Solution

The probability of success in obtaining a six with a single throw of a die is $\frac{1}{6}$ and the corresponding probability of failure is $\frac{5}{6}$.

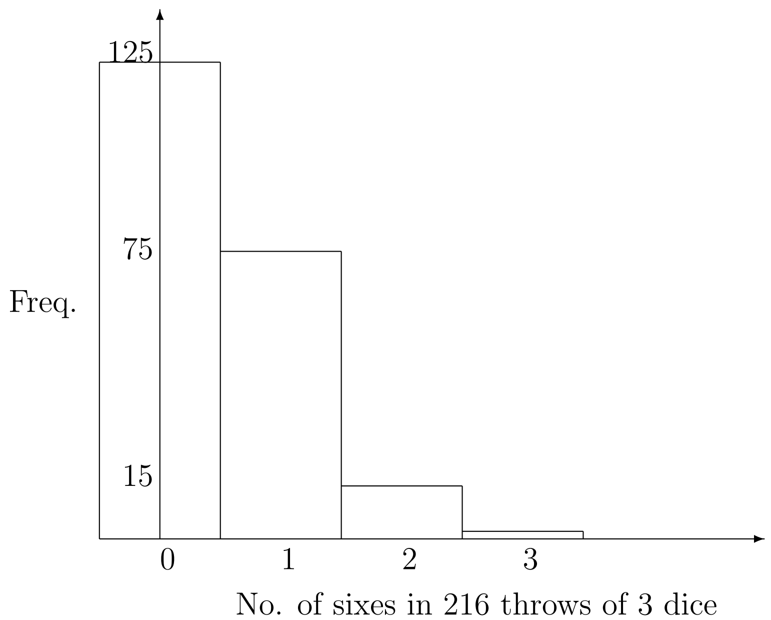
For a single throw of three dice, we require the expansion

$$\left(\frac{1}{6}+\frac{5}{6}\right)^3 \equiv \left(\frac{1}{6}\right)^3 + 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3.$$

This shows that the probabilities of 0,1,2 and 3 sixes are $\frac{125}{216}$, $\frac{75}{216}$, $\frac{15}{216}$ and $\frac{1}{216}$, respectively

Hence, in 216 throws of the three dice we may expect 0 sixes, 125 times; 1 six, 75 times; 2 sixes, 15 times and 3 sixes, once.

The corresponding histogram is as follows:



From the previous Theorem, the mean value is

$$3 \times \frac{1}{6} = \frac{1}{2}$$

and the standard deviation is

$$\sqrt{3 \times \frac{1}{6} \times \frac{5}{6}} = \frac{\sqrt{15}}{6}.$$

"JUST THE MATHS"

SLIDES NUMBER

19.7

PROBABILITY 7
(The Poisson distribution)

by

A.J.Hobson

19.7.1 The theory

UNIT 19.7 - PROBABILITY 7

THE POISSON DISTRIBUTION

19.7.1 THE THEORY

We recall that, in a binomial distribution of n trials, the probability, P_r, that an event occurs exactly r times out of a possible n is given by

$$P_r = \frac{n!}{(n-r)!r!} p^r q^{n-r},$$

where p is the probability of success in a single trial and $q = 1 - p$ is the probability of failure.

Now suppose that n is very large compared with r and that p is very small compared with 1.

Then,

(a)

$$\frac{n!}{(n-r)!} = n(n-1)(n-2).....(n-r+1) \simeq n^r.$$

ILLUSTRATION

If $n = 120$ and $r = 3$, then

$$\frac{n!}{(n-r)!} = \frac{120!}{117!} = 120 \times 119 \times 118 \simeq 120^3.$$

(b)

$$q^r = (1-p)^r \simeq 1,$$

so that

$$q^{n-r} \simeq q^n = (1-p)^n.$$

We may deduce that

$$P_r \simeq \frac{n^r p^r (1-p)^n}{r!} = \frac{(np)^r (1-p)^n}{r!}$$

or

$$P_r \simeq \frac{(np)^r}{r!}\left[1 - np + \frac{n(n-1)}{2!}p^2 - \frac{n(n-1)(n-2)}{3!}p^3 + \ldots\right]$$

$$\simeq \frac{(np)^r}{r!}\left[1 - np + \frac{(np)^2}{2!} - \frac{(np)^3}{3!} + \ldots\right].$$

Hence,

$$P_r \simeq \frac{(np)^r}{r!} e^{-np}.$$

The number, np, in this formula is of special significance, being the average number of successes to be expected a single set of n trials. If we denote np by μ, we obtain the **"Poisson distribution"** formula,

$$P_r \simeq \frac{\mu^r e^{-\mu}}{r!}$$

Notes:

(i) Although the formula has been derived from the binomial distribution, as an approximation, it may also be used in its own right, in which case we drop the approximation sign.

(ii) The Poisson distribution is more use than the binomial distribution when n is a very large number, the binomial distribution requiring the tedious evaluation of its various coefficients.

(iii) The Poisson distribution is of particular use when the average frequency of occurrence of an event is known, but not the number of trials.

EXAMPLES

1. The number of cars passing over a toll-bridge during the time interval from 10a.m. until 11a.m. is 1,200.

   (a) Determine the probability that not more than 4 cars will pass during the time interval from 10.45a.m. until 10.46a.m.

   (b) Determine the probability that 5 or more cars pass during the same interval.

   Solution

   The number of cars which pass in 60 minutes is 1200 so that the average number of cars passing, per minute, is $20 = \mu$.

   (a) The probability that not more than 4 cars pass in a one-minute interval is the sum of the probabilities for 0,1,2,3,4 cars.

   That is,

$$\sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$\left[\frac{(20)^0}{0!} + \frac{(20)^1}{1!} + \frac{(20)^2}{2!} + \frac{(20)^3}{3!} + \frac{(20)^4}{4!}\right] e^{-20} =$$

$$8221 e^{-20} \simeq 1.69 \times 10^{-5}.$$

(b) The probability that 5 or more cars will pass in a one-minute interval is the probability of failure in (a). In other words,

$$1 - \sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$1 - 8221e^{-20} \simeq 0.99998$$

2. A company finds that, on average, there is a claim for damages which it must pay 7 times in every 10 years. It has expensive insurance to cover this situation.

   The premium has just been increased, and the firm is considering letting the insurance lapse for 12 months as it can afford to meet a single claim.

   Assuming a Poisson distribution, what is the probability that there will be at least two claims during the year ?

   Solution

   Using $\mu = \frac{7}{10} = 0.7$, the probability that there will be at <u>most</u> one claim during the year is given by

$$P_0 + P_1 = e^{-0.7} + e^{-0.7}0.7 = e^{0.7}(1 + 0.7).$$

The probability that there will be at <u>least</u> two claims during the year is given by

$$1 - e^{-0.7}(1 + 0.7) \simeq 0.1558$$

3. There is a probability of 0.005 that a welding machine will produce a faulty joint when it is operated. The machine is used to weld 1000 rivets. Determine the probability that at least three of these are faulty.

Solution

First, we have $\mu = 0.005 \times 1000 = 5$.

Hence the probability that at <u>most</u> two will be faulty is given by

$$P_0 + P_1 + P_2 = e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right].$$

That is,

$$e^{-5}[1 + 5 + 12.5] \simeq 0.125$$

Hence, the probability that at least three will be faulty is approximately

$$1 - 0.125 = 0.875$$

"JUST THE MATHS"

SLIDES NUMBER

19.8

PROBABILITY 8
(The normal distribution)

by

A.J.Hobson

UNIT 19.8 - PROBABILITY 8

THE NORMAL DISTRIBUTION

19.8.1 LIMITING POSITION OF A FREQUENCY POLYGON

The distribution considered here is appropriate to examples where the number of trials is large and hence the calculation of frequencies and probabilities, using the binomial distribution, would be inconvenient

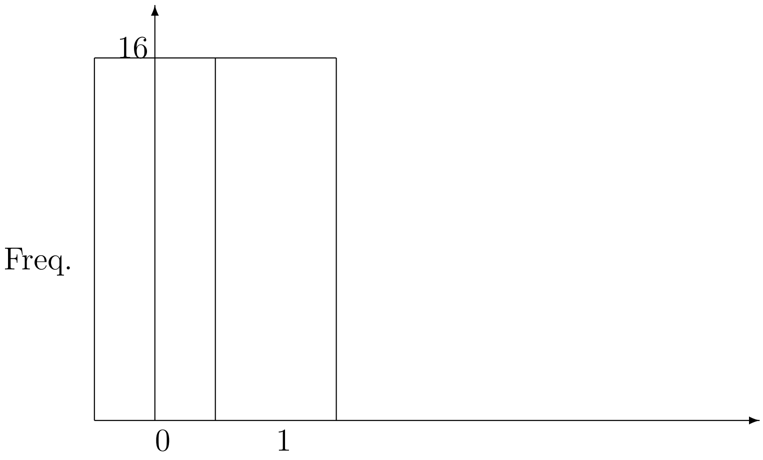
We introduce the **"normal distribution"** by considering the histograms of the binomial distribution for a toss of 32 coins as the number of coins increases.

The probability of obtaining a head is $\frac{1}{2}$ and the probability of obtaining a tail is also $\frac{1}{2}$.

(i) One Coin

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^1 =$$

$$32\left(\frac{1}{2}+\frac{1}{2}\right) = 16 + 16.$$

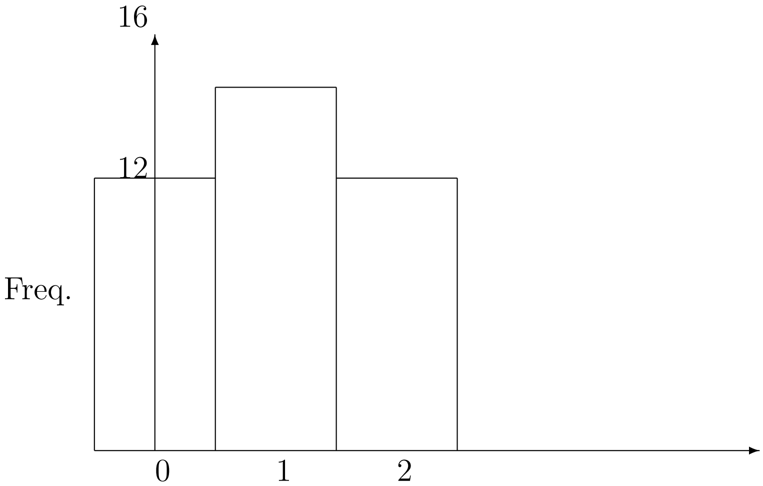


No. of heads in 32 tosses of 1 coin

(ii) Two Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^2 =$$

$$32\left(\left[\frac{1}{2}\right]^2 + 2\left[\frac{1}{2}\right]\left[\frac{1}{2}\right] + \left[\frac{1}{2}\right]^2\right) = 8 + 16 + 8.$$

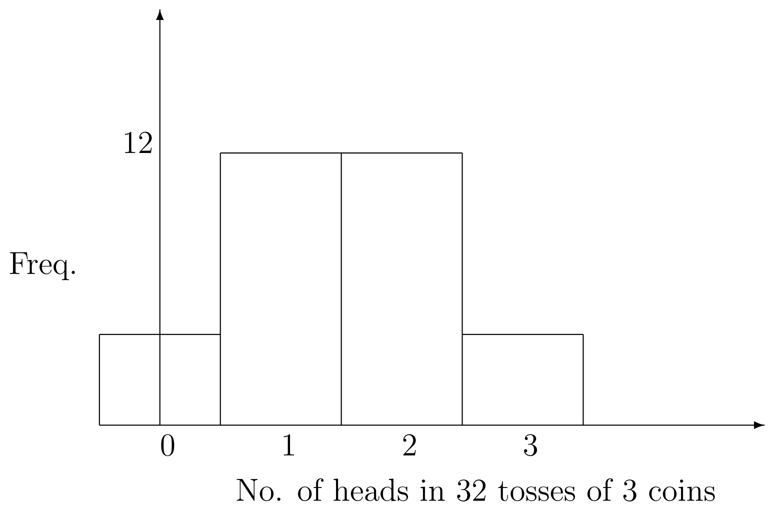


No. of heads in 32 tosses of 2 coins

(iii) Three Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^3 =$$

$$32\left(\left[\frac{1}{2}\right]^3+3\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]+3\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^2+\left[\frac{1}{2}\right]^3\right)=4+12+12+4.$$

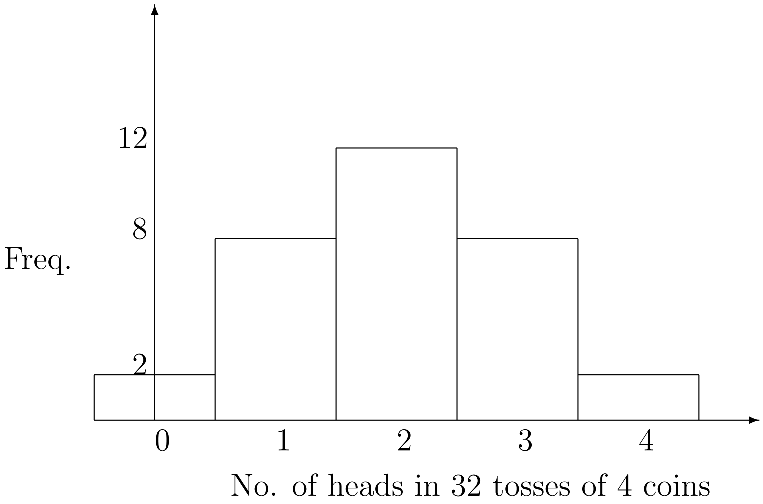


(iv) Four Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^4 =$$

$$32\left(\left[\frac{1}{2}\right]^4+4\left[\frac{1}{2}\right]^3\left[\frac{1}{2}\right]+6\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]^2+4\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^3+\left[\frac{1}{2}\right]^4\right)$$

$$= 2+8+12+8+2.$$



As the number of coins increases, the frequency polygon approaches a symmetrical bell-shaped curve.

This is true only when the histogram itself is either symmetrical or nearly symmetrical.

DEFINITION

As the number of trials increases indefinitely, the limiting position of the frequency polygon is called the **"normal frequency curve"**.

THEOREM

In a binomial distribution for N samples of n trials each, where the probability of success in a single trial is p, it may be shown that, as n increases indefinitely, the frequency polygon approaches a smooth curve, called the **"normal curve"**, whose equation is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}},$$

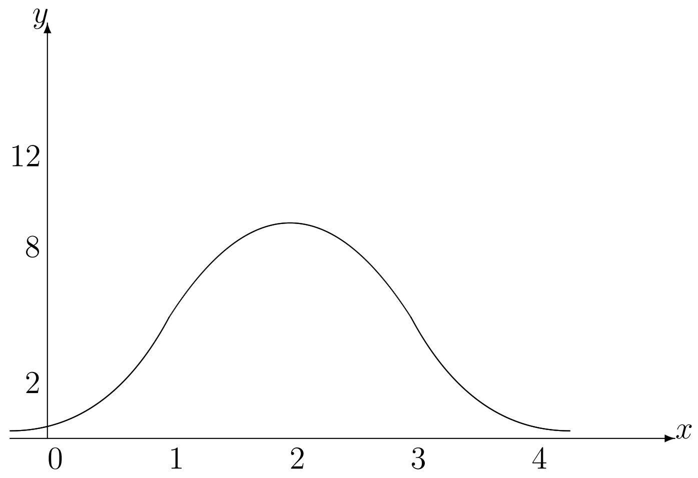
where

$\overline{x}$ is the mean of the binomial distribution $= np$;

σ is the standard deviation of the binomial distribution $= \sqrt{np(1-p)}$;

y is the frequency of occurence of the value, x.

For example, the histogram for 32 tosses of 4 coins approximates to the following normal curve:



Notes:

(i) We omit the proof of the Theorem.

(ii) The larger the value of n, the better is the level of approximation.

(iii) The normal curve is symmetrical about the straight line $x = \overline{x}$, since the value of y is the same at $x = \overline{x} \pm h$ for any number, h.

(iv) If the relative frequency (or probability) with which the value, x, occurs is denoted by P, then $P = y/N$ and the relationship can be written

$$P = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

The graph of this equation is called the **"normal probability curve"**.

(v) Symmetrical curves are easier to deal with if the vertical axes of co-ordinates is the line of symmetry.

The normal probability curve can be simplified if we move the origin to the point $(\overline{x}, 0)$ and plot $P\sigma$ on the vertical axis instead of P.

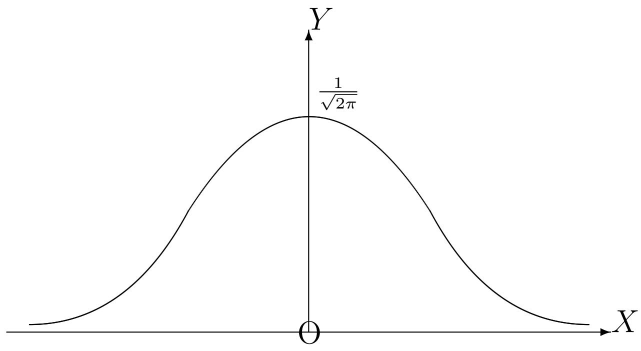
Letting $P\sigma = Y$ and $\frac{x-\overline{x}}{\sigma} = X$ the equation of the normal probability curve becomes

$$Y = \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}}.$$

This equation represents the **"standard normal probability curve"**.

From any point on the standard normal probability curve, we may obtain the values of the original P and x values by using the formulae

$$x = \sigma X + \overline{x} \quad \text{and} \quad P = \frac{Y}{\sigma}.$$



(vi) If the probability of success, p, in a single trial is **not** equal to or approximately equal to $\frac{1}{2}$, then the distribution given by the normal frequency curve and the two subsequent curves will be a poor approximation and is seldom used for such cases.

19.8.2 AREA UNDER THE NORMAL CURVE

For the histogram of a binomial distribution corresponding to values of x, suppose that $x = a$ and $x = b$ are the values of x at the base-centres of two particular rectangles, where $b > a$ and all rectangles have width 1.

The area of the histogram from $x = a - \frac{1}{2}$ to $x = b + \frac{1}{2}$ represents the number of times which we can expect values of x, between $x = a$ and $x = b$ inclusive, to occur.

For a large number of trials, we may use the area under the normal curve between $x = a - \frac{1}{2}$ and
$x = b + \frac{1}{2}$.

The **probability** that x will lie between $x = a$ and $x = b$ is represented by the area under the normal probability curve from $x = a - \frac{1}{2}$ and $x = b + \frac{1}{2}$.

We note that the total area under this curve must be 1, since it represents the probability that **any** value of x will occur (a certainty).

To make use of a <u>standard</u> normal probabilty curve for the same purpose, the conversion formulae from x to X and P to Y must be used.

Note:
Tables are commercially available for the area under a standard normal probability curve.

In using such tables, the conversion formulae will usually be necessary.

EXAMPLE

If 12 dice are thrown, determine the probability, using the normal probability curve approximation, that 7 or more dice will show a 5.

Solution

For this example, we use $p = \frac{1}{6}$, $q = \frac{5}{6}$, $n = 12$.

We need the area under the normal probability curve from $x = 6.5$ to $x = 12.5$

The mean of the binomial distribution, in this case, is $\overline{x} = 12 \times \frac{1}{6} = 2$.

The standard deviation is $\sigma = \sqrt{2 \times \frac{1}{6} \times \frac{5}{6}} \simeq \sqrt{1.67} \simeq 1.29$

The required area under the standard normal probability curve will be that lying between

$$X = \frac{6.5 - 2}{1.29} \simeq 3.49 \text{ and } X = \frac{12.5 - 2}{1.29} \simeq 8.14$$

In practice, we take the whole area to the right of $X = 3.49$, since the area beyond $X = 8.14$ is negligible.

Also, the total area to the right of $X = 0$ is 0.5; and, hence, the required area is 0.5 minus the area from $X = 0$ to $X = 3.49$

From tables, the required area is $0.5 - 0.4998 = 0.0002$ and this is the probability that, when 12 dice are thrown, 7 or more will show a 5.

Note:
If we had required the probability that 7 or fewer dice show a 5, we would have needed the area under the normal probability curve from $x = -0.5$ to $x = 7.5$

This is equivalent to taking the whole of the area under the standard normal probability curve which lies to the left of

$$X = \frac{7.5 - 2}{1.29} \simeq 4.26$$

19.8.3 NORMAL DISTRIBUTION FOR CONTINUOUS VARIABLES

So far, the variable, x, has been able to take only the specific values 0,1,2,3.....etc.

Here, we consider the situation when x is a continuous variable.

That is, it may take any value within a certain range appropriate to the problem under consideration.

For a large number of observations of a continuous variable, the corresponding histogram need not have rectangles of class-width 1, but of some other number, say c.

In this case, it may be shown that the normal curve approximation to the histogram has equation

$$y = \frac{Nc}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

The smaller is the value of c, the larger is the number of rectangles and the better is the approximation supplied by the curve.

If we wished to calculate the number of x-values lying between $x = a$ and $x = b$ (where $b > a$), we would need to calculate the area of the histogram from $x = a$ to $x = b$ inclusive, then **divide by** c, since the base-width is no longer 1.

We conclude that the number of these x-values approximates to the area under the normal curve from $x = a$ to $x = b$.

Similarly, the area under the normal probability curve, from $x = a$ to $x = b$ gives an estimate for the probability that values of x between $x = a$ and $x = b$ will occur.

EXAMPLE

A normal distribution of a continuous variable, x, has $N = 2000$, $\overline{x} = 20$ and $\sigma = 5$.

Determine

(a) the number of x-values lying between 12 and 22;

(b) the number of x-values larger than 30.

Solution

(a) The area under the normal probability curve between $x = 12$ and $x = 22$ is the area under the standard normal probability curve from

$$X = \frac{12 - 20}{5} = -1.6 \quad \text{to} \quad X = \frac{22 - 20}{5} = 0.4$$

From tables, this is $0.4452 + 0.1554 = 0.6006$

Hence, the required number of values is approximately $0.6006 \times 2000 \simeq 1201$.

(b) The total area under the normal probability curve to the right of $x = 30$ is the area under the standard normal probability curve to the right of

$$X = \frac{30 - 20}{5} = 2;$$

and, from tables, this is 0.0227

Hence, the required number of values is approximately $0.0227 \times 2000 \simeq 45$.