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Correlation Risk Modeling and Management

An Applied Guide including the Basel III Correlation Framework—with Interactive Models in Excel®/VBA

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Correlation Risk Modeling and Management

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Preface

Correlation risk is the risk that the correlation between two or more financial variables changes unfavorably. Correlation risk was highlighted in the global financial crisis in 2007 to 2009, when correlations between many financial variables such as the default correlation between debtors or the default correlation between a debtor and an insurer increased dramatically. This led to huge unexpected losses for many financial institutions, which in part triggered the global financial crisis.

This book is the first to address financial correlation risk in detail. In Chapter 1, we introduce the basic properties of correlation risk, before we show in Chapter 2 how correlations behave in the real world. We then discuss whether correlation risk can be quantified using standard statistical correlation measures such as Pearson's ρ, Spearman's rank correlation coefficient, and Kendall's τ in Chapter 3. We address specific financial correlation measures in Chapter 4, and discuss whether the copula correlation model is appropriate to measure financial correlations in Chapter 5. Often, as in the Basel III framework, a shortcut to the Gaussian copula is applied, such as the one-factor Gaussian copula (OFGC) model. This approach, which is applied in the Basel framework to derive credit risk, is discussed in Chapter 6. In Chapter 7 we address a fairly new correlation family, the elegant but somewhat coarse top-down correlation models. Chapter 8 discusses stochastic correlation models, which are a new and promising way to model financial correlations. In Chapters 9 and 10, we introduce new concepts to quantify market and credit correlation risk. In Chapter 11 we address the challenging task of hedging correlation risk. Chapter 12 evaluates the proposed correlation concepts in the Basel III framework, which are designed to mitigate correlated credit and market risk. Chapter 13 deals with the future of correlation modeling, which may include neural networks, fuzzy logic, genetic algorithms, chaos theory, and combinations of these concepts.

Figure P.1 gives an overview of the main correlation models that will be addressed in this book. We will discuss the conceptual, mathematical, and computational properties of the models and evaluate their benefits and limitations for finance.

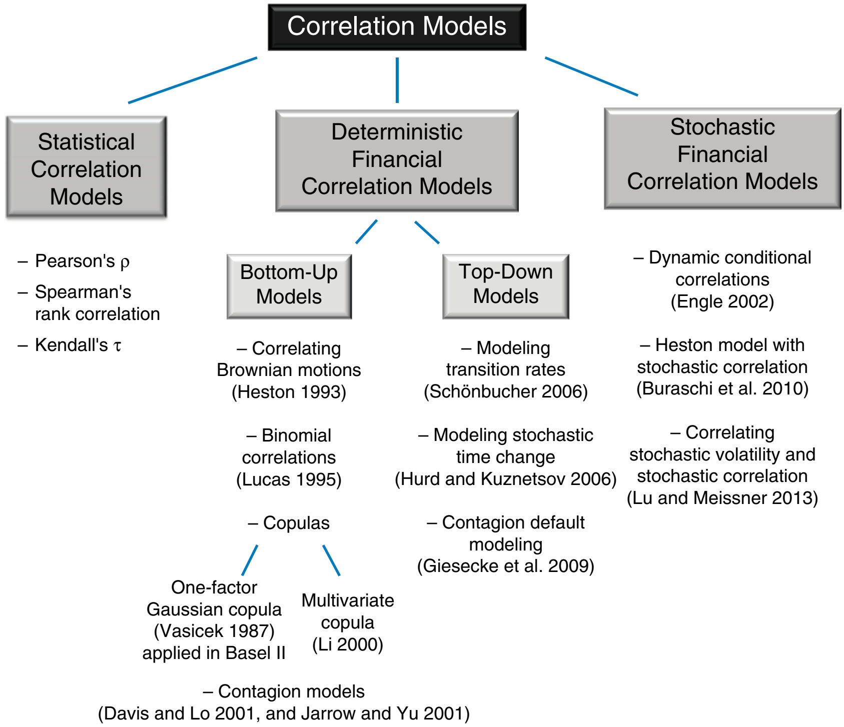

FIGURE P.1 Main Statistical and Financial Correlation Models

TARGET AUDIENCE

This book should be valuable to anyone who is exposed to financial correlations and financial correlation risk. So it should be of interest to upper management, risk managers, analysts, traders, compliance departments, model validation groups, controllers, reporting groups, brokers, and others. The book contains questions and problems at the end of each chapter, which should facilitate using the book in a classroom. The answers to the problems are available to instructors; please e-mail gunter@dersoft.com.

BASEL III

This book addresses new risk measures, especially the new correlation risk measures of the Basel III accord. We discuss the Basel-applied value at risk (VaR) concept, which includes correlated market risk, in the introductory

Chapter 1, section 1.3.3. We address the one-factor Gaussian copula (OFGC) correlation model, which underlies the Basel credit correlation framework, in Chapter 6. We revisit the VaR concept for a multi-asset portfolio in Chapter 9, section 9.4. In Chapter 12, we discuss the Basel III correlation framework in detail, deriving credit value at risk (CVaR) and required capital (RC). In particular, we address credit value adjustment (CVA) with general and specific wrong-way risk (WWR), which includes the correlation between general market factors as well as the correlation between specific entities.

ADDITIONAL MATERIALS

This book comes with 26 supporting spreadsheets, models, and documents. They can be downloaded at www.wiley.com/go/correlationriskmodeling; password: gunter123.

The supporting documents can also be downloaded from the author's website www.dersoft.com/correlationbook/downloads.

Below is a breakdown of the supporting documents by file.

For a general refresher on the basics of mathematical finance:

- Math refresher.docx

Chapter 1

- 2-asset VaR.xlsx
- Matrix primer.xlsx
- Exchange option.xls
- Quanto option.xls
- Dependence and Correlation.xlsm
- Log returns.xlsx

Chapter 2

- Correlation fitting.docx

Chapter 3

- Lookback option.xls

Chapter 4

- GBM path with jumps.xlsm
- 2-asset default time Copula.xlsm

Chapter 5

- CDO Gauss educational.xlsm

Chapter 6

- OFGC educational.xls
- Base correlation generation.xlsm

Chapter 7

- Base correlation generation.xlsm

Chapter 8

- GBM path with jumps.xlsm
- Stochastic correlation.xlsx

Chapter 9

- VaR educational.xlsm
- VaR n asset cora gora.xlsm
- Exchange option cora.docx
- Math refresher.docx

Chapter 10

- CDS with default correlation.xlsm
- CDS three correlated entities pricing code.docx

Chapter 11

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- Option on the better of two.xlsm
- Correlation swap.xls
- Interest rate swap pricing model.xls

Chapter 12

- CVAR.xlsm
- Basel double default.xlsm

I welcome feedback. If you have a suggestion or comment, or if you spot an error, please email me at gunter@dersoft.com. There is an errata page at www.dersoft.com/correlationbook/errata.docx.

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Many of my students, colleagues, and friends have significantly contributed to the writing of this book. I would like to thank the master of financial engineering (MFE) classes of 2012 and 2013 at the Shidler College of Business at the University of Hawaii for detailed discussions and number crunching, especially for the empirical Chapter 2. In particular, Martin Chang, Zhen Chen, Clint Davis, Charles Demarest, Barrett Gady, Susan Globokar, Ziyan Jiang, Thuy Le, Stefan Mayr, Dongfang Nie, Amirarsalan Pakravan, Babak Saadat, Alex Schnurrer, Jun Sheng, Wenjing Tang, Manogaran Thanabalan, Ryuichi Umeda, Eugene Wong, Amir Yousefi, and Elke Zeller contributed strongly.

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I would also like to thank the editors Gemma Diaz, Chris Gage, Emilie Herman, Nick Wallwork, and Jules Yap of John Wiley & Sons, for their encouragement, support, and competent work.

> ***Why don't you make the book more fun?***
>
> **—Jasmine Meissner, 7**

Yeah, well, while I enjoyed writing this book, I can only hope that the reader also enjoys the book and learns from it. I am happy to receive feedback; you can e-mail me at gunter@dersoft.com.

About the Author

After a lectureship in mathematics and statistics at the Economic Academy Kiel, Gunter Meissner, PhD, joined Deutsche Bank in 1990, trading interest rate futures, swaps, and options in Frankfurt and New York. He became Head of Product Development in 1994, responsible for originating algorithms for new derivatives products, which at the time were lookback options, multi-asset options, quanto options, average options, index amortizing swaps, and Bermuda swaptions. In 1995/1996 Gunter was Head of Options at Deutsche Bank Tokyo. From 1997 to 2007, he was Professor of Finance at Hawaii Pacific University and from 2008 to 2013 Director of the master in financial engineering program at the Shidler College of Business at the University of Hawaii. Currently, he is President of Derivatives Software (www.dersoft.com), founder and CEO of Cassandra Capital Management (www.cassandracm.com), and Adjunct Professor of Mathematical Finance at NYU-Courant.

Gunter Meissner has published numerous papers and books on derivatives and risk management, and is a frequent speaker at conferences and seminars. He can be reached at gunter@dersoft.com.

CHAPTER 1

Some Correlation Basics: Properties, Motivation, Terminology

Behold, the fool saith, "Put not all thine eggs in the one basket."
—Mark Twain

In this chapter we introduce the basic concepts of financial correlations and financial correlation risk. We show that correlations are critical in many areas of finance such as investments, trading, and risk management, as well as in financial crises and in financial regulation. We also show how correlation risk relates to other risks in finance such as market risk, credit risk, systemic risk, and concentration risk.

1.1 WHAT ARE FINANCIAL CORRELATIONS?

Heuristically (meaning nonmathematically), we can define two types of financial correlations: static and dynamic.

Static financial correlations measure how two or more financial assets are associated within a certain time period.

Examples are:

- The classic value at risk (VaR) model. It answers the question: What is the maximum loss of correlated assets in a portfolio with a certain probability for a given time period? This time period can be 10 days as Basel III requires, as well as shorter or longer (see Chapter 1, section 1.3.3 and Chapter 9, section 9.4 for more on VaR and correlation).

- The original copula approach for collateralized debt obligations (CDOs). It measures the default correlations between all assets in the CDO for a certain time period, which is typically identical to the maturity date of the CDO (see Chapter 5 for details).
- The binomial default correlation model of Lucas (1995), which is a special case of the Pearson correlation model. It measures the probability of two assets defaulting together within a short time period (see Chapter 3 for details).

Besides the static correlation concept, there are dynamic correlations:

Dynamic financial correlations measure how two or more financial assets move together in time.

Examples are:

- In practice, pairs trading, a type of statistical arbitrage, is performed. Let's assume the movements of assets x and y have been highly correlated in time. If now asset x performs poorly with respect to y, then asset x is bought and asset y is sold with the expectation that the gap will narrow.
- Within the deterministic correlation approaches, the Heston 1993 model correlates the Brownian motions dz_1 and dz_2 of assets 1 and 2. The core equation is $dz_1(t) = \rho\, dz_2(t) + \sqrt{(1-p^2)}\, dz_3(t)$ where dz_1 and dz_2 are correlated in time with correlation parameter p. See Chapter 3 for details.
- Correlations behave randomly and unpredictably. Therefore, it is a good idea to model them as a stochastic process. Stochastic correlation processes are by construction time dependent and can replicate correlation properties well. See Chapter 8 for details.

> ***Suddenly everything was highly correlated.***
>
> —*Financial Times*, April 2009

1.2 WHAT IS FINANCIAL CORRELATION RISK?

Financial correlation risk is the risk of financial loss due to adverse movements in correlation between two or more variables.

These variables can comprise any financial variables. For example, the positive correlation between Mexican bonds and Greek bonds can hurt

Mexican bond investors if Greece bond prices decrease, which happened in 2012 during the Greek crisis. Or the negative correlation between commodity prices and interest rates can hurt commodity investors if interest rates rise. A further example is the correlation between a bond issuer and a bond insurer, which can hurt the bond investor (see the example displayed in Figure 1.1).

Correlation risk is especially critical in risk management. An increase in the correlation of asset returns increases the risk of financial loss, which is often measured by the value at risk (VaR) concept. For details see section 1.3.3 and Chapter 9, sections 9.4 and 9.5. An increase in correlation is typical in a severe systemic crisis. For example, in the Great Recession from 2007 to 2009, financial assets and financial markets worldwide became highly correlated. Risk managers who had in their portfolios assets with negative or low correlations suddenly witnessed many of them decline together; hence asset correlations increased sharply. For more on systemic risk, see section 1.3.4, "The Global Financial Crisis of 2007 to 2009 and Correlation," as well as Chapter 2, which displays empirical findings of correlations.

Correlation risk can also involve variables that are nonfinancial, such as economic or political events. For example, the correlation between the increasing sovereign debt and currency value can hurt an exporter, as occurred in Europe in 2012, where a decreasing euro hurt U.S. exporters. Geopolitical tensions as, for example, in the Middle East can hurt airline companies due to increasing oil prices, or a slowing gross domestic product (GDP) in the United States can hurt Asian and European exporters and investors, since economies and financial markets are correlated worldwide.

Let's look at correlation risk via an example of a credit default swap (CDS). A CDS is a financial product in which the credit risk is transferred from the investor (or CDS buyer) to a counterparty (CDS seller). Let's assume an investor has invested $1 million in a bond from Spain. He is now worried about Spain defaulting and has purchased a credit default swap from a French bank, BNP Paribas. Graphically this is displayed in Figure 1.1.

The investor is protected against a default from Spain, since in case of default, the counterparty BNP Paribas will pay the originally invested $1 million to the investor. For simplicity, let's assume the recovery rate and accrued interest are zero.

The value of the CDS, i.e., the fixed CDS spread s, is mainly determined by the default probability of the reference entity Spain. However, the spread s is also determined by the joint default correlation of BNP Paribas and Spain. If the correlation between Spain and BNP Paribas increases, the present value of the CDS for the investor will decrease and he will suffer a paper loss. The worst-case scenario is the joint default of Spain and BNP Paribas, in

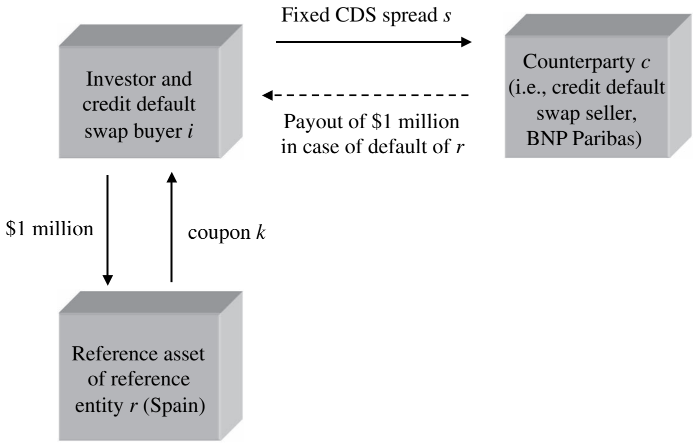

FIGURE 1.1 An Investor Hedging His Spanish Bond Exposure with a CDS

which case the investor will lose his entire investment in the Spanish bond of $1 million.

In other words, the investor is exposed to default correlation risk between the reference asset *r* (Spain) and the counterparty *c* (BNP Paribas). Since both Spain and BNP Paribas are in Europe, let's assume that there is a positive default correlation between the two. In this case, the investor has wrong-way correlation risk or short wrong-way risk (WWR). Let's assume the default probability of Spain and BNP Paribas both increase. This means that the exposure to the reference entity Spain increases (since the CDS has a higher present value for the investor) *and* it is more unlikely that the counterparty BNP Paribas can pay the default insurance. We will discuss wrong-way risk, which is a key term in the Basel II and III accords, in Chapter 12.

The magnitude of the correlation risk is expressed graphically in Figure 1.2.

From Figure 1.2 we observe that for a correlation of −0,3 and higher, the higher the correlation, the lower the CDS spread. This is because an increasing ρ means a higher probability of the reference asset and the counterparty defaulting together. In the extreme case of a perfect correlation of 1, the CDS is worthless. This is because if Spain defaults, so will the insurance seller BNP Paribas.

We also observe from Figure 1.2 that for a correlation from −1 to about −0.3, the CDS spread increases slightly. This seems counterintuitive at first. However, an increase in the negative correlation means a higher probability of either Spain *or* BNP Paribas defaulting. In the case of Spain defaulting, the CDS buyer will get compensated by BNP Paribas. However, if the insurance seller BNP Paribas defaults, the CDS buyer will lose his

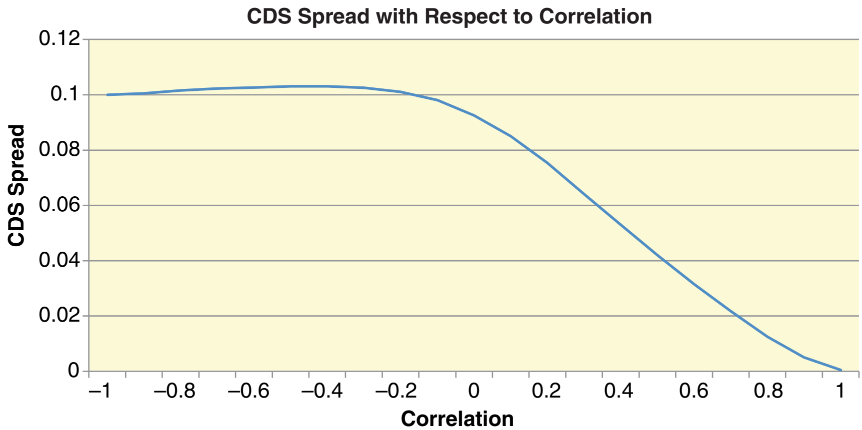

FIGURE 1.2 CDS Spread *s* of a Hedged Bond Purchase (as Displayed in Figure 1.1) with Respect to the Default Correlation between the Reference Entity *r* and the Counterparty *c*

insurance and will have to repurchase it. This may have to be done at a higher cost. The cost will be higher if the credit quality of Spain has decreased since inception of the original CDS. For example, the CDS spread may have been 3% in the original CDS, but may have increased to 6% due to a credit deterioration of Spain. For more details on pricing CDSs with counterparty risk and the reference asset–counterparty correlation, see Chapter 10, section 10.1, as well as Kettunen and Meissner (2006).

We observe from Figure 1.2 that the dependencies between a variable (here the CDS spread) and correlation may be nonmonotonous; that is, the CDS spread sometimes increases and sometimes decreases if correlation increases. We will also encounter this nonmonotony feature of correlation when we discuss the mezzanine tranche of a CDO in Chapter 5.

1.3 MOTIVATION: CORRELATIONS AND CORRELATION RISK ARE EVERYWHERE IN FINANCE

Why study financial correlations? That's an easy one. Financial correlations appear in many areas in finance. We will briefly discuss five areas: (1) investments and correlation, (2) trading and correlation, (3) risk management and correlation, (4) the global financial crisis and correlation, and (5) regulation and correlation. Naturally, if an entity is exposed to correlation, this means that the entity has correlation risk (i.e., the risk of a change in the correlation).

1.3.1 Investments and Correlation

From our studies of the Nobel Prize–winning capital asset pricing model (CAPM) (Markowitz 1952; Sharpe 1964) we remember that an increase in diversification increases the return/risk ratio. Importantly, high diversification is related to low correlation. Let's show this in an example. Let's assume we have a portfolio of two assets, *X* and *Y*. They have performed as in Table 1.1.

Let's define the return of asset *X* at time *t* as x_t, and the return of asset *Y* at time *t* as y_t. A return is calculated as a percentage change, $(S_t - S_{t-1})/S_{t-1}$, where *S* is a price or a rate. The average return of asset *X* for the time frame 2009 to 2013 is $\mu_X = 29.03\%$; for asset *Y* the average return is $\mu_Y = 20.07\%$. If we assign a weight to asset *X*, w_X, and a weight to asset *Y*, w_Y, the portfolio return is

$$\mu_P = w_X \, \mu_X + w_Y \, \mu_Y \tag{1.1}$$

where $w_X + w_Y = 1$.

The standard deviation of returns, called *volatility*, is derived for asset *X* with equation (1.2):

$$\sigma_X = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \mu_X)^2} \tag{1.2}$$

where x_t is the return of asset *X* at time *t* and *n* is the number of observed points in time. The volatility of asset *Y* is derived accordingly. Equation 1.2 can be computed with =stdev in Excel and std in MATLAB. From our example in Table 1.1, we find that $\sigma_X = 44.51\%$ and $\sigma_Y = 47.58\%$.

Let's now look at the covariance. The covariance measures how two variables covary (i.e., move together). More precisely, the covariance

TABLE 1.1 Performance of a Portfolio with Two Assets

Year	Asset *X*	Asset *Y*	Return of Asset *X*	Return of Asset *Y*
2008	100	200		
2009	120	230	20.00%	15.00%
2010	108	460	−10.00%	100.00%
2011	190	410	75.93%	−10.87%
2012	160	480	−15.79%	17.07%
2013	280	380	75.00%	−20.83%
		Average	29.03%	20.07%

measures the strength of the linear relationship between two variables. The covariance of returns for assets X and Y is derived with equation (1.3):

$$\mathrm{Cov}_{XY} = \frac{1}{n-1}\sum_{t=1}^{n}(x_t - \mu_X)(y_t - \mu_Y) \tag{1.3}$$

For our example in Table 1.1 we derive $\mathrm{Cov}_{XY} = -0.1567$. Equation (1.3) is = Covariance.S in Excel and cov in MATLAB. The covariance is not easy to interpret, since it takes values between $-\infty$ and $+\infty$. Therefore, it is more convenient to use the Pearson correlation coefficient ρ_{XY}, which is a standardized covariance; that is, it takes values between -1 and $+1$. The Pearson correlation coefficient is:

$$\rho_{XY} = \frac{\mathrm{Cov}_{XY}}{\sigma_X \sigma_Y} \tag{1.4}$$

For our example in Table 1.1, $\rho_{XY} = -0.7403$, showing that the returns of assets X and Y are highly negatively correlated. Equation (1.4) is 'correl' in Excel and 'corrcoef' in MATLAB. For the derivation of the numerical examples of equations (1.2) to (1.4) and more information on the covariances, see Appendix 1A and the spreadsheet "Matrix primer.xlsx," sheet "Covariance matrix," at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

We can calculate the standard deviation for our two-asset portfolio P as

$$\sigma_P = \sqrt{w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_X w_Y \mathrm{Cov}_{XY}} \tag{1.5}$$

With equal weights, i.e., $w_X = w_Y = 0.5$, the example in Table 1.1 results in $\sigma_P =$ 16.66%.

Importantly, the standard deviation (or its square, the variance) is interpreted in finance as risk. The higher the standard deviation, the higher the risk of an asset or a portfolio. Is standard deviation a good measure of risk? The answer is: It's not great, but it's one of the best we have. A high standard deviation may mean high upside potential, so it penalizes possible profits! But a high standard deviation naturally also means high downside risk. In particular, risk-averse investors will not like a high standard deviation, i.e., high fluctuation of their returns.

An informative performance measure of an asset or a portfolio is the risk-adjusted return, i.e., the return/risk ratio. For a portfolio it is μ_P/σ_P, which we derived in equations (1.1) and (1.5). In Figure 1.3 we observe one of the few free lunches in finance: the lower (preferably negative) the correlation of the assets in a portfolio, the higher the return/risk ratio. For a rigorous proof, see Markowitz (1952) and Sharpe (1964).

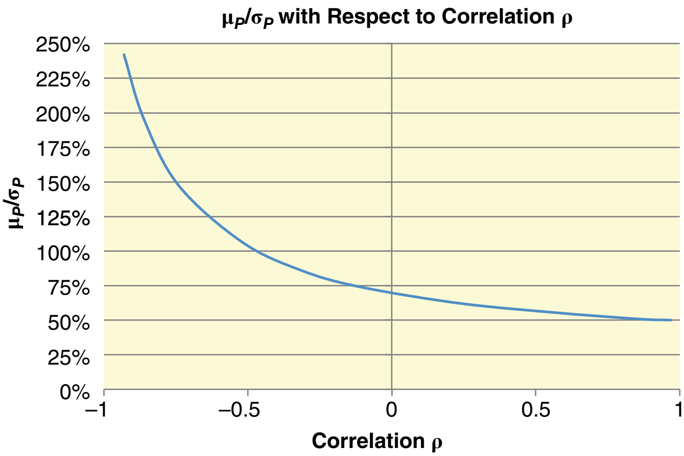

FIGURE 1.3 The Negative Relationship of the Portfolio Return/Risk Ratio μ_P/σ_P with Respect to the Correlation ρ of the Assets in the Portfolio (Input Data are from Table 1.1)

Figure 1.3 shows the high impact of correlation on the portfolio return/risk ratio. A high negative correlation results in a return/risk ratio of close to 250%, whereas a high positive correlation results in a 50% ratio. The equations (1.1) to (1.5) are derived within the framework of the Pearson correlation approach. We will discuss the limitations of this approach in Chapter 3.

> *Only by great risks can great results be achieved.*
>
> —Xeres

1.3.2 Trading and Correlation

In finance, every risk is also an opportunity. Therefore, at every major investment bank and hedge fund *correlation desks* exist. The traders try to forecast changes in correlation and attempt to financially gain from these changes in correlation. We already mentioned the correlation strategy "pairs trading." Generally, *correlation trading* means trading assets whose prices are determined at least in part by the comovement of one or more asset in time. Many types of correlation assets exist.

1.3.2.1 Multi-Asset Options A popular group of correlation options are multi-asset options, also termed rainbow options or mountain range options. Many different types are traded. The most popular ones are listed here. S_1 is the price of asset 1 and S_2 is the price of asset 2 at option maturity. K is the strike price, i.e., the price determined at option start, at which the underlying asset can be bought in the case of a call, and the price at which the underlying asset can be sold in the case of a put.

- Option on the better of two. Payoff = $\max(S_1, S_2)$.
- Option on the worse of two. Payoff = $\min(S_1, S_2)$.
- Call on the maximum of two. Payoff = $\max[0, \max(S_1, S_2) - K]$.
- Exchange option (as a convertible bond). Payoff = $\max(0, S_2 - S_1)$.
- Spread call option. Payoff = $\max[0, (S_2 - S_1) - K]$.
- Option on the better of two or cash. Payoff = $\max(S_1, S_2, \text{cash})$.
- Dual-strike call option. Payoff = $\max(0, S_1 - K_1, S_2 - K_2)$.
- Portfolio of basket options. Payoff = $\left[\sum_{i=1}^{n} n_i S_i - K, 0\right]$, where n_i is the weight of assets i.

Importantly, the prices of these correlation options are highly sensitive to the correlation between the asset prices S_1 and S_2. In the list above, except for the option on the worse of two, the lower the correlation, the higher the option price. This makes sense since a low, preferable negative correlation means that if one asset decreases, on average the other increases. So one of the two assets is likely to result in a high price and a high payoff. Multi-asset options can be conveniently priced using closed form extensions of the Black-Scholes-Merton 1973 option model; see Chapter 9 for details.

Let's look at the evaluation of an exchange option with a payoff of $\max(0, S_2 - S_1)$. The payoff shows that the option buyer has the right to give away asset 1 and receive asset 2 at option maturity. Hence, the option buyer will exercise her right if $S_2 > S_1$. The price of the exchange option can be derived easily. We first rewrite the payoff equation $\max(0, S_2 - S_1) = S_1 \max[0, (S_2/S_1) - 1]$. We then input the covariance between asset S_1 and S_2 into the implied volatility function of the exchange option using a variation of equation (1.5):

$$\sigma_E = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\mathrm{Cov}_{AB}} \tag{1.5a}$$

where σ_E is the implied volatility of S_2/S_1, which is input into the standard Black-Scholes-Merton 1973 option pricing model.

For an exchange option pricing model and further discussion, see Chapter 9, section 9.2.2 and the model "Exchange option.xls" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

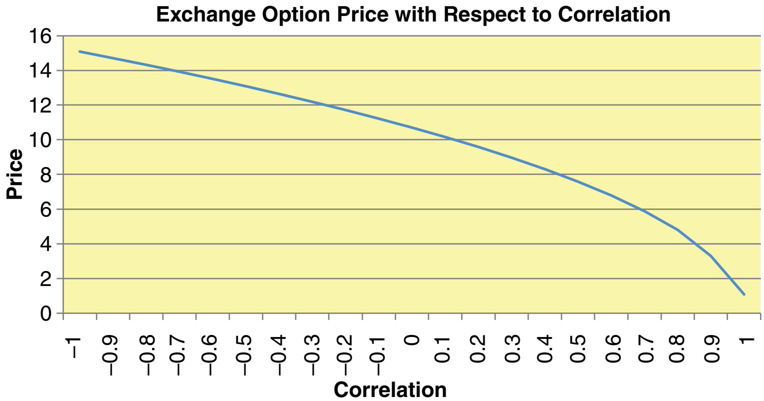

FIGURE 1.4 Exchange Option Price with Respect to Correlation of the Assets in the Portfolio
For details on an exchange option as pricing and correlation risk management, see Chapter 9, section 9.2.2.

Importantly, the exchange option price is highly sensitive to the correlation between the asset prices S_1 and S_2, as seen in Figure 1.4.

From Figure 1.4 we observe the strong impact of the correlation on the exchange option price. The price is close to 0 for high correlation and \$15.08 for a negative correlation of -1. As in Figures 1.2 and 1.3, the correlation approach underlying Figure 1.4 is the Pearson correlation model. We will discuss the limitations of the Pearson correlation model in Chapter 3.

1.3.2.2 Quanto Option Another interesting correlation option is the quanto option. This is an option that allows a domestic investor to exchange his potential option payoff in a foreign currency back into his home currency at a fixed exchange rate. A quanto option therefore protects an investor against currency risk. For example, an American believes the Nikkei will increase, but she is worried about a decreasing yen, which would reduce or eliminate her profits from the Nikkei call option. The investor can buy a quanto call on the Nikkei, with the yen payoff being converted into dollars at a fixed (usually the spot) exchange rate.

Originally, the term *quanto* comes from the word *quantity*, meaning that the amount that is reexchanged to the home currency is unknown, because it depends on the future payoff of the option. Therefore the financial institution that sells a quanto call does not know two things:

1. How deep in the money the call will be, i.e., which yen amount has to be converted into dollars.

2. The exchange rate at option maturity at which the stochastic yen payoff will be converted into dollars.

The correlation between (1) and (2) i.e., the price of the underlying S' and the exchange rate X, significantly influences the quanto call option price. Let's consider a call on the Nikkei S' and an exchange rate X defined as domestic currency per unit of foreign currency (so \$/1 yen for a domestic American) at maturity.

If the correlation is positive, an increasing Nikkei will also mean an increasing yen. That is favorable for the call seller. She has to settle the payoff, but only needs a small yen amount to achieve the dollar payment. Therefore, the more positive the correlation coefficient, the lower the price for the quanto option. If the correlation coefficient is negative, the opposite applies: If the Nikkei increases, the yen decreases in value. Therefore, more yen are needed to meet the dollar payment. As a consequence, the lower the correlation coefficient, the more expensive the quanto option. Hence we have a similar negative relationship between the option price and correlation as in Figure 1.2.

Quanto options can be conveniently priced closed form applying an extension of the Black-Scholes-Merton 1973 model. For a pricing model and a more detailed discussion on a quanto option, see the "Quanto option.xls" model at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

1.3.2.3 Correlation Swap The correlation between assets can also be traded directly with a correlation swap. In a correlation swap a fixed (i.e., known) correlation is exchanged with the correlation that will actually occur, called realized or stochastic (i.e., unknown) correlation, as seen in Figure 1.5.

Paying a fixed rate in a correlation swap is also called *buying correlation*. This is because the present value of the correlation swap will increase for the correlation buyer if the realized correlation increases. Naturally the fixed rate receiver is *selling correlation*.

The realized correlation ρ in Figure 1.5 is the correlation between the assets that actually occurs during the time of the swap. It is calculated as:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j} \tag{1.6}$$

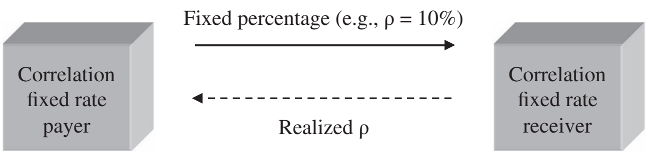

FIGURE 1.5 A Correlation Swap with a Fixed 10% Correlation Rate

where $\rho_{i,j}$ is the Pearson correlation between asset i and j, and n is the number of assets in the portfolio.

The payoff of a correlation swap for the correlation fixed rate payer at maturity is:

$$N\left(\rho_{\text{realized}} - \rho_{\text{fixed}}\right) \tag{1.7}$$

where N is the notional amount. Let's look at an example of a correlation swap.

Correlation swaps can indirectly protect against decreasing stock prices. As we will see in this chapter in section 1.4, as well as in Chapter 2, when stocks decrease, typically the correlation between the stocks increases. Hence a fixed correlation payer protects himself indirectly against a stock market decline.

EXAMPLE 1.1: PAYOFF OF A CORRELATION SWAP

What is the payoff of a correlation swap with three assets, a fixed rate of 10%, a notional amount of \$1,000,000, and a 1-year maturity?

First, the daily log returns $\ln(S_t/S_{t-1})$ of the three assets are calculated for 1 year.[1] Let's assume the realized pairwise correlations of the log returns at maturity are as displayed in Table 1.2.

The average correlation between the three assets is derived by equation (1.6). We apply the correlations only in the shaded area from Table 1.2, since these satisfy $i > j$. Hence we have $\rho_{\text{realized}} = \frac{2}{3^2 - 3}(0.5 + 0.3 + 0.1) = 0.3$. Following equation (1.7), the payoff for the correlation fixed rate payer at swap maturity is $\$1{,}000{,}000 \times (0.3 - 0.1) = \$200{,}000$.

TABLE 1.2 Pairwise Pearson Correlation Coefficient at Swap Maturity

	$S_{j=1}$	$S_{j=2}$	$S_{j=3}$
$S_{i=1}$	1	0.5	0.1
$S_{i=2}$	0.5	1	0.3
$S_{i=3}$	0.1	0.3	1

1. Log returns $\ln(S_1/S_0)$ are an approximation of percentage returns $(S_1 - S_0)/S_0$. We typically use log returns in finance since they are additive in time, whereas percentage returns are not. For details see Appendix 1B.

Currently, year 2013, there is no industry-standard valuation model for correlation swaps. Traders often use historical data to anticipate ρ_{realized}. In order to apply swap valuation techniques, we require a term structure of correlation in time. However, no correlation term structure currently exists. We can also apply stochastic correlation models to value a correlation swap. Stochastic correlation models are currently emerging and will be discussed in Chapter 8.

1.3.2.4 Buying Call Options on an Index and Selling Call Options on Individual Components Another way of buying correlation (i.e., benefiting from an increase in correlation) is to buy call options on an index such as the Dow Jones Industrial Average (the Dow) and sell call options on individual stocks of the Dow. As we will see in Chapter 2, there is a positive relationship between correlation and volatility. Therefore, if correlation between the stocks of the Dow increases, so will the implied volatility[2] of the call on the Dow. This increase is expected to outperform the potential loss from the increase in the short call positions on the individual stocks.

Creating exposure on an index and hedging with exposure on individual components is exactly what the "London whale," JPMorgan's London trader Bruno Iksil, did in 2012. Iksil was called the London whale because of his enormous positions in credit default swaps (CDSs).[3] He had sold CDSs on an index of bonds, the CDX.NA.IG.9, and hedged them by buying CDSs on individual bonds. In a recovering economy this is a promising trade: Volatility and correlation typically decrease in a recovering economy. Therefore, the sold CDSs on the index should outperform (decrease more than) the losses on the CDSs of the individual bonds.

But what can be a good trade in the medium and long term can be disastrous in the short term. The positions of the London whale were so large that hedge funds short-squeezed him: They started to aggressively buy the CDS index CDX.NA.IG.9. This increased the CDS values in the index and created a huge (paper) loss for the whale. JPMorgan was forced to buy back the CDS index positions at a loss of over $2 billion.

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2. Implied volatility is volatility derived (implied) by option prices. The higher the implied volatility, the higher the option price.

3. Simply put, a credit default swap (CDS) is an insurance against default of an underlying (e.g., a bond). However, if the underlying is not owned, a long CDS is a speculative instrument on the default of the underlying (just like a naked put on a stock is a speculative position on the stock going down). See Meissner (2005) for more.

1.3.2.5 Paying Fixed in a Variance Swap on an Index and Receiving Fixed on Individual Components A further way to buy correlation is to pay fixed in a variance swap on an index and to receive fixed in variance swaps on individual components of the index. The idea is the same as the idea with respect to buying a call on an index and selling a call on the individual components: If correlation increases, so will the variance. As a consequence, the present value for the variance swap buyer, the fixed variance swap payer, will increase. This increase is expected to outperform the potential losses from the short variance swap positions on the individual components.

In the preceding trading strategies, the correlation between the assets was assessed with the Pearson correlation approach. As mentioned, we will discuss the limitations of this model in Chapter 3.

1.3.3 Risk Management and Correlation

After the global financial crisis from 2007 to 2009, financial markets have become more risk averse. Commercial banks, investment banks, as well as nonfinancial institutions have increased their risk management efforts. As in the investment and trading environment, correlation plays a vital part in risk management. Let's first clarify what risk management means in finance.

Financial risk management is the process of identifying, quantifying, and, if desired, reducing financial risk.

The three main types of financial risk are:

1. Market risk.
2. Credit risk.
3. Operational risk.

Additional types of risk may include systemic risk, liquidity risk, volatility risk, and the topic of this book, correlation risk. We will concentrate in this introductory chapter on market risk. Market risk consists of four types of risk: (1) equity risk, (2) interest rate risk, (3) currency risk, and (4) commodity risk.

There are several concepts to measure the market risk of a portfolio, such as value at risk (VaR), expected shortfall (ES), enterprise risk management (ERM), and more. VaR is currently (year 2013) the most widely applied risk management measure. Let's show the impact of asset correlation on VaR.[4]

First, what is value at risk (VaR)? VaR measures the maximum loss of a portfolio with respect to a certain probability for a certain time frame. The equation for VaR is:

$$\mathrm{VaR}_P = \sigma_P \, \alpha \sqrt{x} \tag{1.8}$$

4. We use a variance-covariance VaR approach in this book to derive VaR. Another way to derive VaR is the nonparametric VaR. This approach derives VaR from simulated historical data. See Markovich (2007) for details.

where VaR_P is the value at risk for portfolio P, and α is the abscise value of a standard normal distribution corresponding to a certain confidence level. It can be derived as =normsinv(confidence level) in Excel or norminv (confidence level) in MATLAB. α takes the values $-\infty < \alpha < +\infty$. x is the time horizon for the VaR, typically measured in days; σ_P is the volatility of the portfolio P, which includes the correlation between the assets in the portfolio. We calculate σ_P via

$$\sigma_P = \sqrt{\beta_h C \beta_v} \tag{1.9}$$

where β_h is the horizontal β vector of invested amounts (price time quantity), β_v is the vertical β vector of invested amounts (also price time quantity),[5] and C is the covariance matrix of the returns of the assets.

Let's calculate VaR for a two-asset portfolio and then analyze the impact of different correlations between the two assets on VaR.

EXAMPLE 1.2: DERIVING VaR OF A TWO-ASSET PORTFOLIO

What is the 10-day VaR for a two-asset portfolio with a correlation coefficient of 0.7, daily standard deviation of returns of asset 1 of 2%, of asset 2 of 1%, and \$10 million invested in asset 1 and \$5 million invested in asset 2, on a 99% confidence level?

First, we derive the covariances (Cov):

$$\begin{aligned} Cov_{11} &= \rho_{11}\,\sigma_1\,\sigma_1 = 1 \times 0.02 \times 0.02 = 0.0004^6 \\ Cov_{12} &= \rho_{12}\,\sigma_1\,\sigma_2 = 0.7 \times 0.02 \times 0.01 = 0.00014 \\ Cov_{21} &= \rho_{21}\,\sigma_2\,\sigma_1 = 0.7 \times 0.01 \times 0.02 = 0.00014 \\ Cov_{22} &= \rho_{22}\,\sigma_2\,\sigma_2 = 1 \times 0.01 \times 0.01 = 0.0001 \end{aligned} \tag{1.10}$$

(continued)

5. More mathematically, the vector β_h is the transpose of the vector β_v, and vice versa: $\beta_h{}^T = \beta_v$ and $\beta_v{}^T = \beta_h$. Hence we can also write equation (1.9) as $\sigma_P = \sqrt{\beta_h C \beta_h{}^T}$. See the spreadsheet "Matrix primer.xls," sheet "Matrix Transpose," at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

6. The attentive reader realizes that we calculated the covariance differently in equation (1.3). In equation (1.3) we derived the covariance from scratch, inputting the return values and means. In equation (1.10) we are assuming that we already know the correlation coefficient ρ and the standard deviation σ.

(*continued*)

Hence our covariance matrix is $C = \begin{pmatrix} 0.0004 & 0.00014 \\ 0.00014 & 0.0001 \end{pmatrix}$

Let's calculate σ_P following equation (1.9). We first derive $\beta_h C$

$$(10\ 5)\begin{pmatrix} 0.0004 & 0.00014 \\ 0.00014 & 0.0001 \end{pmatrix} = (10 \times 0.0004 + 5 \times 0.00014\ 10 \times 0.00014 + 5 \times 0.0001) = (0.0047\ 0.0019)$$

and then $(\beta_h C)\beta_v = (0.0047 \quad 0.0019)\begin{pmatrix} 10 \\ 5 \end{pmatrix} = 10 \times 0.0047 + 5 \times 0.0019 = 5.65\%$[7]

Hence we have $\sigma_P = \sqrt{\beta_h C \beta_v} = \sqrt{5.65\%} = 23.77\%$.

We find the value for α in equation (1.8) from Excel as = normsinv (0.99) = 2.3264, or from MATLAB as norminv(0.99) = 2.3264.

Following equation (1.8), we now calculate the VaR_P as $0.2377 \times 2.3264 \times \sqrt{10} = 1.7486$.

Interpretation: We are 99% certain that we will not lose more than \$1.75486 million in the next 10 days due to market price changes of asset 1 and 2.

The number \$1.7486 million is the 10-day VaR on a 99% confidence level. This means that on average once in a hundred 10-day periods (so once every 1,000 days) this VaR number of \$1.7486 million will be exceeded. If we have roughly 250 trading days in a year, the company is expected to exceed the VaR about once every four years. The Basel Committee for Banking Supervision (BCBS) considers this to be too often. Hence, it requires banks, which are allowed to use their own models (called internal model-based approach), to hold capital for assets in the trading book[8] in the amount of at least 3 times the 10-day VaR (plus a specific risk charge for credit risk). In example 1.2, if a bank is granted the minimum of 3 times the VaR, a VaR

7. The spreadsheet "2-asset VaR.xlsx," which derives the values in example 1.2, can be found at www.wiley.com/go/correlationriskmodeling, section under "Chapter 1."
8. Assets that are marked-to-market, such as stocks, futures, options, and swaps, are in the trading book. Some assets, such as loans and certain bonds, which are not marked-to-market, are in the banking book.

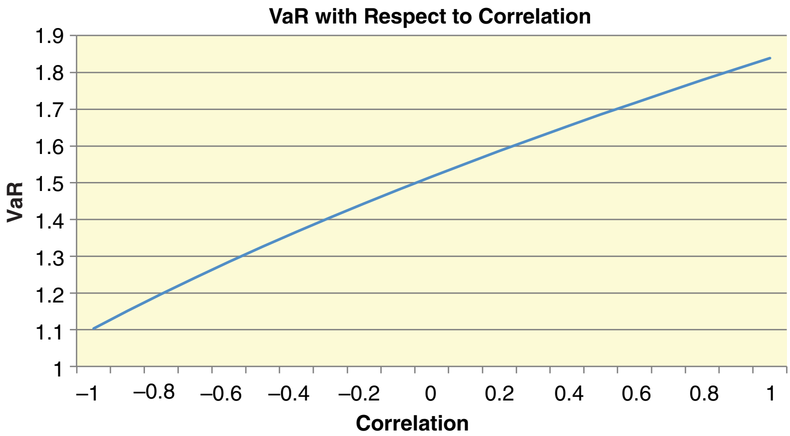

FIGURE 1.6 VaR of the Two-Asset Portfolio of Example 1.2 with Respect to Correlation ρ between Asset 1 and Asset 2

capital charge for assets in the trading book of \$1,7486 million $\times$ 3 = \$5.2539 million is required by the Basel Committee.[9]

Let's now analyze the impact of different correlations between the asset 1 and asset 2 on VaR. Figure 1.6 shows the impact.

As expected, we observe from Figure 1.6 that the lower the correlation, the lower the risk, measured by VaR. Preferably the correlation is negative. In this case, if one asset decreases, the other asset on average increases, hence reducing the overall risk. The impact of correlation on VaR is strong. For a perfect negative correlation of -1, VaR is \$1.1 million; for a perfect positive correlation, VaR is close to \$1.9 million.

9. In a recent Consultative Document (May 2012), the Basel Committee has indicated that it is considering replacing VaR with expected shortfall (ES). Expected shortfall measures tail risk (i.e., the size and probability of losses beyond a certain threshold). See www.bis.org/publ/bcbs219.pdf for details. Loosely speaking, VaR answers the question: "What is the maximum loss in good times?" Expected shortfall answers the question: "What is the loss in bad times?"

There are no toxic assets, just toxic people.

1.3.4 The Global Financial Crisis of 2007 to 2009 and Correlation

Currently, in 2013, the global financial crisis of 2007 to 2009 seems almost like a distant memory. The U.S. stock market has recovered from its low in March 2009 of 6,547 points and has more than doubled to over 15,000. World economic growth is at a moderate 2.5%. However, the U.S. unemployment rate is stubbornly high at around 8% and has not decreased to pre-crisis levels of about 5%. Most important, to fight the crisis, countries engaged in huge stimulus packages to revive their faltering economies. As a result, enormous sovereign deficits are plaguing the world economy. The European debt crisis, with Greece, Cyprus, and other European nations virtually in default, is a major global economic threat. The U.S. debt is also far from benign with a debt-to-GDP ratio of over 80%. One of the few nations that is enjoying these enormous debt levels is China, which is happy buying the debt and taking in the proceeds.

A crisis that brought the financial and economic system worldwide to a standstill is naturally not monocausal, but has many reasons. Here are the main ones:

- An extremely benign economic and risk environment from 2003 to 2006 with record low credit spreads, low volatility, and low interest rates.
- Increasing risk taking and speculation of traders and investors who tried to benefit in these presumably calm times. This led to a bubble in virtually every market segment, such as the housing market, mortgage market (especially the subprime mortgage market), stock market, and commodity market. In 2007, U.S. investors had borrowed 470% of the U.S. national income to invest and speculate in the real estate, financial, and commodity markets.
- A new class of structured investment products, such as collateralized debt obligations (CDOs), CDO-squareds, constant-proportion debt obligations (CPDOs), constant-proportion portfolio insurance (CPPI), as well as new products like options on credit default swaps (CDSs), credit indexes, and the like.
- The new copula correlation model, which was trusted naively by many investors and which could presumably correlate the $n(n-1)/2$ assets in a structured product. Most CDOs contained 125 assets. Hence there

are $125(125 - 1)/2 = 7{,}750$ asset correlation pairs to be quantified and managed. For details see Chapters 5 and 6.

- A moral hazard of rating agencies, which were paid by the same companies whose assets they rated. As a consequence, many structured products received AAA ratings and gave the illusion of little price and default risk.
- Risk managers and regulators who lowered their standards in light of the greed and profit frenzy. We recommend an excellent (anonymous) paper in the *Economist*: "A Personal View of the Crisis, Confessions of a Risk Manager."

The topic of this book is correlation risk, so let's concentrate on the correlation aspect of the crisis. Around 2003, two years after the Internet bubble burst, the risk appetite of the financial markets increased, and investment banks, hedge funds, and private investors began to speculate and invest in the stock markets, commodity markets, and especially the real estate market.

In particular, residential mortgages became an investment object. The mortgages were packaged in collateralized debt obligations (CDOs; see Chapter 5 for a detailed discussion), and then sold off to investors nationally and internationally. The CDOs typically consist of several tranches; that is, the investor can choose a particular degree of default risk. The equity tranche holder is exposed to the first 3% of mortgage defaults, the mezzanine tranche holder is exposed to the 3% to 7% of defaults, and so on. The new copula correlation model derived by Abe Sklar in 1959 and transferred to finance by David Li in 2000 could presumably manage the default correlations in the CDOs (see Chapters 5 and 6 for details).

The first correlation-related crisis, which was a forerunner of the major one to come in 2007 to 2009, occurred in May 2005. General Motors was downgraded to BB and Ford was downgraded to BB+, so both companies were now in junk status. A downgrade to junk status typically leads to a sharp bond price decline, since many mutual funds and pension funds are not allowed to hold junk bonds.

Importantly, the correlation of the bonds in CDOs that referenced investment grade bonds decreased, since bonds of different credit qualities are typically lower correlated. This led to huge losses of hedge funds, which had put on a strategy where they were short the equity tranche of the CDO and long the mezzanine tranche of the CDO. Figure 1.7 shows the dilemma. Hedge funds had shorted the equity tranche[10] (0% to 3% in Figure 1.7) to

10. Shorting the equity tranche means being short credit protection or selling credit protection, which means receiving the (high) equity tranche contract spread.

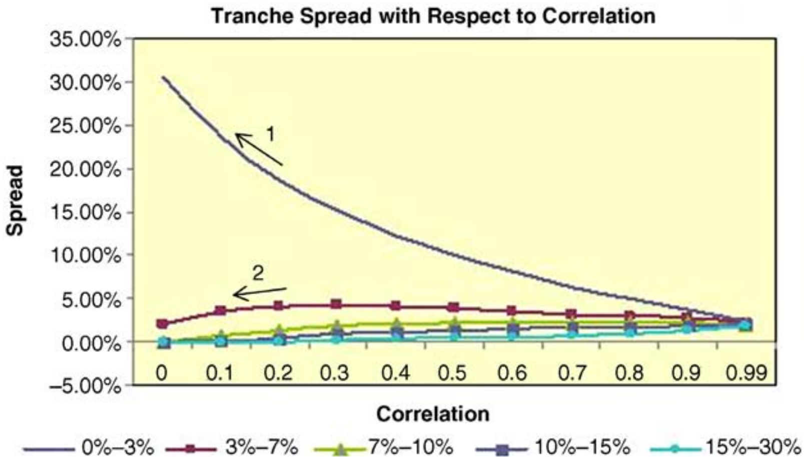

FIGURE 1.7 CDO Tranche Spread with Respect to Correlation

collect the high equity tranche spread. They had then presumably hedged[11] the risk by going long the mezzanine tranche[12] (3% to 7% in Figure 1.7). However, as we can see from Figure 1.7, this hedge is flawed.

When the correlations of the assets in the CDO decreased, the hedge funds lost on both positions.

1. The equity tranche spread increased sharply; see arrow 1. Hence the fixed spread that the hedge funds received in the original transaction was now significantly lower than the current market spread, resulting in a paper loss.
2. In addition, the hedge funds lost on their long mezzanine tranche positions, since a lower correlation lowers the mezzanine tranche spread; see arrow 2. Hence the spread that the hedge funds paid in the original transactions was now higher than the market spread, resulting in another paper loss.

As a result of the huge losses, several hedge funds, such as Marin Capital, Aman Capital, and Baily Coates Cromwell, filed for bankruptcy. It is important to point out that the losses resulted from a lack of understanding of the correlation properties of the tranches in the CDO. The CDOs

11. To hedge means to protect or to reduce risk. See Chapter 11, section 11.1, for details.
12. Going long the mezzanine tranche means being long credit protection or buying credit protection, which means paying the (fairly low) mezzanine tranche contract spread.

themselves can hardly be blamed or be called toxic for their correlation properties.

From 2003 to 2006 the CDO market, mainly referencing residential mortgages, had exploded, increasing from \$64 billion to \$455 billion. To fuel the CDOs, more and more questionable subprime mortgages were given, named NINJA loans, standing for “no income, no job or assets.” When housing prices started leveling off in 2006, the first mortgages started to default. In 2007 more and more mortgages defaulted, finally leading to a real estate market collapse. With it the huge CDO market collapsed, leading to the stock market and commodity market crash and a freeze in the credit markets. The financial crisis spread to the world economies, creating a global severe recession, now called the Great Recession.

In a systemic crash like this, naturally many types of correlations increase (see also Figure 1.8). From 2007 to 2009, default correlations of the mortgages in the CDOs increased. This actually helped equity tranche investors, as we can see from Figure 1.7: If default correlations increase, the equity tranche spread decreases, leading to an increase in the value of the equity tranche. However, this increase was overcompensated by a strong increase in default probability of the mortgages. As a consequence, tranche spreads increased sharply, resulting in huge losses for the equity tranche investors as well as investors in the other tranches.

Correlations between the tranches of the CDOs also increased during the crisis. This had a devastating effect on the super-senior tranches. In normal times, these tranches were considered extremely safe since (1) they were AAA rated and (2) they were protected by the lower tranches. But with the increased tranche correlation and the generally deteriorating credit market, these super-senior tranches were suddenly considered risky and lost up to 20% of their value.

To make things worse, many investors had leveraged the super-senior tranches, termed leveraged super-senior (LSS) tranches, to receive a higher spread. This leverage was typically 10 to 20 times, meaning an investor paid \$10,000,000 but had risk exposure of \$100,000,000 to \$200,000,000. What made things technically even worse was that these LSSs came with an option for the investors to unwind the super-senior tranche if the spread had widened (increased). Many investors started to purchase the LSS spread at very high levels, realizing a loss and increasing the LSS tranche spread even further.

In addition to the overinvestment in CDOs, the credit default swap (CDS) market also exploded from its beginnings in the mid-1990s from about \$8 trillion in 2004 to almost \$60 trillion in 2007. CDSs are typically used as insurance to protect against default of a debtor, as we explained in Figure 1.1. No one will argue that an insurance contract is toxic. On the contrary, it is the principle of an insurance contract to spread the risk to a wider audience and hence reduce individual risk, as we can see from health insurance or life insurance contracts.

CDSs, though, can also be used as speculative instruments. For example, the CDS seller (i.e., the insurance seller) hopes that the insured event (e.g., default of a company or credit deterioration of the company) will not occur. In this case the CDS seller keeps the CDS spread (i.e., the insurance premium) as income, as American International Group (AIG) tried to do in the crisis. A CDS buyer who does not own the underlying asset is speculating on the credit deterioration of the underlying asset, just like a naked put option holder speculates on the decline of the underlying asset.

So who is to blame for the 2007–2009 global financial crisis? The quants, who created the new products such as CDSs and CDOs and the models to value them? The upper management and the traders, who authorized and conducted the overinvesting and extreme risk taking? The rating agencies, who gave an AAA rating to many of the CDOs? The regulators, who approved the overinvestments? The risk managers, who allowed the excessive risk taking?

The entire global financial crisis can be summed up in one word: Greed! It was the upper management, the traders, and the investors who engaged in excessive trading and irresponsible risk taking to receive high returns, huge salaries, and generous bonuses. For example, the London unit of AIG had sold close to $500 billion in CDSs without much reinsurance! Their main hedging strategy seemed to have been: Pray that the insured contracts don't deteriorate. The investment banks of the small Northern European country of Iceland had borrowed 10 times Iceland's national GDP and invested it. With this leverage, Iceland naturally went de facto into bankruptcy in 2008, when the credit markets deteriorated. Lehman Brothers, before filing for bankruptcy in September 2008, reported a leverage of 30.7 (i.e., $691 billion in assets and only $22 billion in stockholders' equity). The true leverage was even higher, since Lehman tried to hide the leverage with materially misleading repo transactions.[13] In addition, Lehman had 1.5 million derivatives transactions with 8,000 different counterparties on its books.

Did the upper management and traders of hedge funds and investment banks admit to their irresponsible leverage, excessive trading, and risk taking? No. Instead they created the myth of the toxic asset, which is absurd.

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13. *Repo* stands for repurchase transaction. It can be viewed as a short-term collateralized loan.

It is like a murderer saying, "I did not shoot that person. It was my gun!" Toxic are not the financial products, but humans and their greed.

Most traders were well aware of the risks that they were taking. In the few cases where traders did not understand the risks, the asset itself cannot be blamed; rather, the incompetence of the trader is the reason for the loss. While it is ethically disappointing that the investors and traders did not admit to their wrongdoing, at the same time it is understandable. If they admitted to irresponsible trading and risk taking, they would immediately be prosecuted.

Naturally, risk managers and regulators have to take part of the blame for allowing the irresponsible risk taking. The moral hazard of the rating agencies, being paid by the same companies whose assets they rate, also needs to be addressed.

We will discuss the role of financial models, their benefits, and their limitations at the beginning of Chapter 3.

1.3.5 Regulation and Correlation

Correlations are critical inputs in regulatory frameworks such as the Basel accords, especially in regulations for market risk and credit risk. We will discuss the correlation approaches of the Basel accords in this book. First, let's clarify.

1.3.5.1 What Are Basel I, II, and III? Basel I, implemented in 1988; Basel II, implemented in 2006; and Basel III, which is currently being developed and implemented until 2018, are regulatory guidelines to ensure the stability of the banking system.

The term *Basel* comes from the beautiful city of Basel in Switzerland, where the honorable regulators meet. None of the Basel accords has legal authority. However, most countries (about 100 for Basel II) have created legislation to enforce the Basel accords for their banks.

1.3.5.2 Why Basel I, II, and III? The objective of the Basel accords is to "provide incentives for banks to enhance their risk measurement and management systems" and "to contribute to a higher level of safety and soundness in the banking system." In particular, Basel III is being developed to address the deficiencies of the banking system during the financial crisis of 2007 to 2009. Basel III introduces many new ratios to ensure liquidity and adequate leverage of banks. In addition, new correlation models will be implemented that deal with double defaults in insured risk transactions as displayed in Figure 1.1. Correlated defaults in a multi-asset portfolio quantified with the Gaussian copula, correlations in derivatives transactions termed credit value adjustment (CVA), and correlations in what is called wrong-way risk (WWR) are currently being discussed. We devote the entire Chapter 12 to addressing the benefits and limitations of these correlation approaches in Basel III.

1.4 HOW DOES CORRELATION RISK FIT INTO THE BROADER PICTURE OF RISKS IN FINANCE?

As already mentioned in section 1.3.3, we differentiate three main types of risks in finance:

1. Market risk
2. Credit risk
3. Operational risk

Additional types of risk may include systemic risk, concentration risk, liquidity risk, volatility risk, legal risk, reputational risk, and more. Correlation risk plays an important part in market risk and credit risk, and is closely related to systemic risk and concentration risk. Let's discuss it.

1.4.1 Correlation Risk and Market Risk

Correlation risk is an integral part of market risk. Market risk, comprised of equity risk, interest rate risk, currency risk, and commodity risk. Market risk is typically measured with the value at risk (VaR) concept. VaR has a covariance matrix of the assets in the portfolio as an input. So market risk implicitly incorporates correlation risk, i.e., the risk that the correlations in the covariance matrix change. We have already studied the impact of different correlations on VaR in section 1.3.3, "Risk Management and Correlation."

Market risk is also quantified with expected shortfall (ES), also termed conditional VaR or tail risk. Expected shortfall measures market risk for extreme events, typically for the worst 0.1%, 1%, or 5% of possible future scenarios. A rigorous valuation of expected shortfall naturally includes the correlation between the asset returns in the portfolio, as VaR does.[14]

14. Unfortunately, different authors use different definitions (and notation) for ES. To study ES, we recommend the original ES paper by Artzner et al. (1997), an educational paper by Yamai and Yoshiba (2002), as well as Acerbi and Tasche (2001) and McNeil, Frey, and Embrechts (2005).

1.4.2 Correlation Risk and Credit Risk

Correlation risk is also a critical part of credit risk. Credit risk is comprised of (1) migration risk and (2) default risk. Migration risk is the risk that the credit quality of a debtor decreases, i.e., migrates to a lower credit state. A lower credit state typically results in a lower asset price, so a paper loss for the creditor. We already studied in section 1.2 the effect of correlation risk of an investor who has hedged his bond exposure with a CDS. We derived that the investor is exposed to the correlation between the reference asset and the counterparty, the CDS seller. The higher the correlation, the higher the CDS paper loss for the investor and, importantly, the higher the probability of a total loss of the investment.

The degree to which defaults occur together (i.e., default correlation) is critical for financial lenders such as commercial banks, credit unions, mortgage lenders, and trusts, which give many types of loans to companies and individuals. Default correlations are also critical for insurance companies, which are exposed to credit risk of numerous debtors. Naturally, a low default correlation of debtors is desired to diversify the credit risk. Table 1.3 shows the default correlation from 1981 to 2001 of 6,907 companies, of which 674 defaulted.

The default correlations in Table 1.3 are one-year default correlations averaged over the time period 1981 to 2001. We will see how to calculate default correlations in Chapter 4, especially in section 4.2, "The Binomial Correlation Measure" (Lucas 1995).

From Table 1.3, we observe that default correlations between industries are mostly positive with the exception of the energy sector. This sector is typically viewed as a recession-resistant, stable sector with little or no correlation to other sectors. We also observe that the default correlation within sectors is higher than between sectors. This suggests that systematic factors (e.g., a recession or structural weakness such as the general decline of a sector) have a greater impact on defaults than do idiosyncratic factors. Hence if General Motors defaults, it is more likely that Ford will default, rather than Ford benefiting from the default of its rival.

Since the intrasector default correlations are higher than intersector default correlations, a lender is advised to have a sector-diversified loan portfolio to reduce default correlation risk.

Defaults are binomial events, either default or no default. So principally we can use a simple correlation model such as the binomial model of Lucas (1995) to analyze them, which we will do in Chapter 4, section 4.2. However, we can also analyze defaults in more detail and look at term structure of defaults. Let's assume a creditor has given loans to two debtors. One debtor is

TABLE 1.3 Default Correlation of 674 Defaulted Companies by Industry

One-Year U.S. Default Correlations—Non-Investment-Grade Bonds 1981–2001												
	Auto	**Cons**	**Ener**	**Fin**	**Build**	**Chem**	**HiTech**	**Insur**	**Leis**	**Tele**	**Trans**	**Util**
Auto	3.80%	1.30%	1.20%	0.40%	1.10%	1.60%	2.80%	−0.50%	1.00%	3.90%	1.30%	0.50%
Cons	1.30%	2.80%	−1.40%	1.20%	2.80%	1.60%	1.80%	1.10%	1.30%	3.20%	1.30%	1.90%
Ener	1.20%	−1.40%	6.40%	−2.50%	−0.50%	0.40%	−0.10%	−1.60%	−1.00%	−1.40%	−0.10%	0.70%
Fin	0.40%	1.20%	−2.50%	**5.20%**	2.60%	0.10%	2.30%	3.00%	1.60%	3.70%	1.50%	4.50%
Build	1.10%	2.80%	−0.50%	2.60%	**6.10%**	1.20%	2.30%	1.80%	2.30%	**6.50%**	4.20%	1.30%
Chem	1.60%	1.60%	0.40%	0.10%	1.20%	3.20%	1.40%	−1.10%	1.10%	2.80%	1.10%	1.00%
HiTech	2.80%	1.80%	−0.10%	0.40%	2.30%	1.40%	3.30%	0.00%	1.10%	2.80%	1.10%	1.00%
Insur	−0.50%	1.10%	−1.60%	3.00%	1.80%	−1.10%	0.00%	**5.60%**	1.20%	−2.60%	2.30%	1.40%
Leis	1.00%	1.30%	−1.00%	1.60%	2.30%	1.10%	1.40%	1.20%	2.30%	4.00%	2.30%	0.60%
Tele	3.90%	3.20%	−1.40%	3.70%	**6.50%**	2.80%	4.70%	−2.60%	4.00%	**10.70%**	3.20%	−0.80%
Trans	1.30%	2.70%	−0.10%	1.50%	4.20%	1.10%	1.90%	2.30%	2.30%	3.20%	4.30%	−0.20%
Util	0.50%	1.90%	0.70%	4.50%	1.30%	1.00%	1.00%	1.40%	0.60%	−0.80%	−0.20%	**9.40%**

Correlations above 5% are bold.

Source: Standard & Poor's (S&P) 500.

TABLE 1.4 Term Structure of Default Probabilities for an A-Rated Bond and a CC-Rated Bond in 2002

	Year									
	1	2	3	4	5	6	7	8	9	10
A	0.02%	0.07%	0.13%	0.14%	0.15%	0.17%	0.18%	0.21%	0.24%	0.25%
CC	23.83%	13.29%	10.31%	7.62%	5.04%	5.13%	4.04%	4.62%	2.62%	2.04%

Source: Moody's.

A rated, and one is CC rated. A historical default term structure these bonds is displayed in Table 1.4.

For most investment grade bonds, the term structure of default probabilities increases in time, as we see from Table 1.4 for the A-rated bond. This is because the longer the time horizon, the higher the probability of adverse internal events such as mismanagement, or adverse external events such as increased competition or a recession. For bonds in distress, however, the default term structure is typically inverse, as seen for the CC-rated bond in Table 1.4. This is because for a distressed company the immediate future is critical. If the company survives the coming problematic years, the probability of default decreases.

For a creditor, the default correlation of her debtors is critical. As mentioned, a creditor will benefit from a low default correlation of her debtors, which spreads the default correlation risk. We can correlate the default term structures in Table 1.4 with the famous (now infamous) copula model, which will be discussed in Chapter 4. This will allow us to answer such questions as: "What is the joint probability of debtor 1 defaulting in year 3 and debtor 2 defaulting in year 5?"

> *Correlations always increase in stressed markets.*
>
> —John Hull

1.4.3 Correlation Risk and Systemic Risk

So far, we have analyzed correlation risk with respect to market risk and credit risk and have concluded that correlations are a critical input when quantifying market risk and credit risk. Correlations are also closely related to systemic risk, which we define here.

SYSTEMIC RISK

The risk of a financial market or an entire financial system collapsing.

An example of systemic risk is the collapse of the entire credit market in 2008. At the height of the crisis in September 2008, when Lehman Brothers filed for bankruptcy, the credit markets were virtually frozen with essentially no lending activities. Even as the Federal Reserve guaranteed interbank loans, lending resumed only very gradually and slowly.

The stock market crash starting in October 2007 with the Dow Jones Industrial Average at 14,093 points and then falling by 53.54% to 6,547 points by March 2009 is also a systemic market collapse. All but one of the Dow 30 stocks had declined. Walmart was the lone Dow stock that was up during the crisis. Of the S&P 500 stocks, 489 declined during this time frame. The 11 stocks that were up were:

1. Apollo Group (APOL), educational sector; provides educational programs for working adults and is a subsidiary of the University of Phoenix.
2. AutoZone (AZO), auto industry; provides auto replacement parts.
3. CF Industries (CF), agricultural industry; provides fertilizer.
4. DeVry Inc. (DV), educational sector; holding company of several universities.
5. Edward Lifesciences (EW), pharmaceutical industry; provides products to treat cardiovascular diseases.
6. Family Dollar (FDO), consumer staples.
7. Gilead Pharmaceuticals (GILD), pharmaceutical industry; provides HIV, hepatitis medications.
8. Netflix (NFLX), entertainment industry; provides Internet subscription service.
9. Ross Stores (ROST), consumer staples.
10. Southwestern Energy (SWN), energy sector.
11. Walmart (WMT), consumer staples.

From this list we can see that the consumer staples sector (which provides such basic necessities as food and household items) fared well during the crisis. The educational sector also typically thrives in a crisis, since many unemployed seek to further their education.

Importantly, systemic financial failures such as the one from 2007 to 2009 typically spread to the economy, with a decreasing GDP, increasing unemployment, and therefore a decrease in the standard of living.

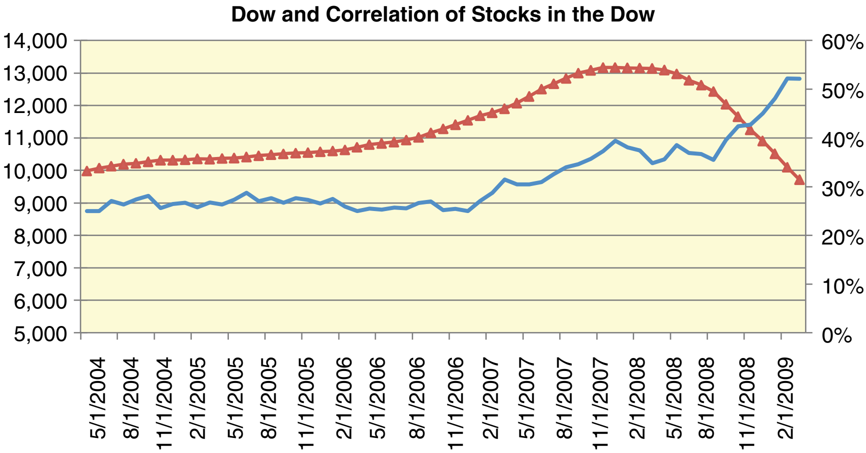

FIGURE 1.8 Relationship between the Dow (graph with triangles, numerical values on left axis) and Correlation between the Stocks in the Dow (numerical values on right axis)

Systemic risk and correlation risk are highly dependent. Since a systemic decline in stocks involves almost the entire stock market, correlations between the stocks increase sharply. Figure 1.8 shows the relationship between the percentage change of the Dow Jones Industrial Average, short "Dow," and the correlation between the stocks in the Dow before the crisis from May 2004 to October 2007 and during the crisis from October 2007 to March 2009.

In Figure 1.8 we downloaded daily closing prices of all 30 stocks in the Dow and put them into monthly bins. We then derived monthly 30×30 correlation matrices using the Pearson correlation measure and averaged the matrices. We then smoothed the graph by taking the one-year moving average.

From Figure 1.8 we can observe a somewhat stable correlation from 2004 to 2006, when the Dow increased moderately. In the time period from January 2007 to February 2008 we observe that the correlation in the Dow increases when the Dow increases more strongly. Importantly, in the time of the severe decline of the Dow from August 2008 to March 2009, we observe a sharp increase in the correlation from noncrisis levels of on average 27% to over 50%. In Chapter 2, we will observe empirical correlations in detail, and we will find that at the height of the crisis in February 2009 the correlation of the stocks in the Dow reached a high of 96.97%. Hence, portfolios that were considered well diversified in benign times experienced a sharp increase in correlation and hence unexpected losses due to the combined, highly correlated decline of many stocks during the crisis. We will quantify this correlation risk and its associated potential losses in detail in Chapters 9 and 10.

1.4.4 Correlation Risk and Concentration Risk

Concentration risk is a fairly new risk category and therefore not yet uniquely defined. We provide a sensible definition.

CONCENTRATION RISK

The risk of financial loss due to a concentrated exposure to a particular group of counterparties.

Concentration risk can be quantified with the concentration ratio. For example, if a creditor has 10 loans of equal size, the concentration ratio would be $1/10 = 0.1$. If a creditor has only one loan to one counterparty, the concentration ratio would be 1. Naturally, the lower the concentration ratio, the more diversified is the default risk of the creditor, assuming the default correlation between the counterparties is smaller than 1.

We can also categorize counterparties into groups, for example sectors. We can then analyze sector concentration risk. The higher the number of different sectors a creditor has lent to, the higher is the sector diversification. High sector diversification reduces default risk, since intrasector defaults are more highly correlated than counterparties in different sectors, as seen in Table 1.3.

Naturally, concentration risk and correlation risk are closely related. Let's verify this in an example.

EXAMPLE 1.3: CONCENTRATION RATIO AND CORRELATION

CASE A

The commercial bank C has lent \$10,000,000 to a single company, W. So C's concentration ratio is 1. Let's assume company W has a default probability (P_W) of 10%. Hence the expected loss (EL) for bank C is $\$10,000,000 \times 0.1 = \$1,000,000$ (see Figure 1.9).

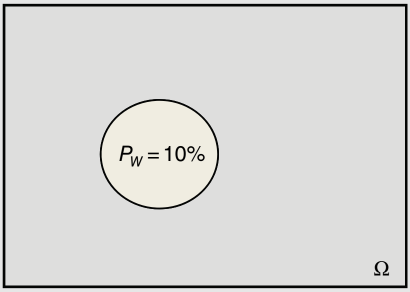

FIGURE 1.9 Probability Space for the Default Probability of a Single Loan to *W*

CASE B

The commercial bank *C* has lent \$5,000,000 to company *X* and \$5,000,000 to company *Y*. Let's assume both *X* and *Y* have a 10% default probability. So *C*'s concentration ratio is reduced to ½.

If the default correlation between *X* and *Y* is bigger than 0 and smaller than 1, we derive that the worst-case scenario [i.e., the default of *X* and *Y*, $P(X \cap Y)$, with a loss of \$1,000,000] is reduced, as seen in Figure 1.10.

The exact joint default probability $P(X \cap Y)$ depends on the correlation model and correlation parameter values, which will be discussed in Chapters 3 to 8. For any model, though, if default correlation between *X* and *Y* is 1, then there is no benefit from the lower concentration ratio. The probability space would have the form as in Figure 1.9.

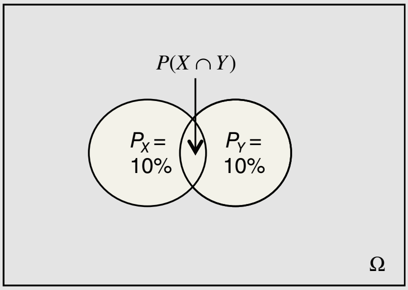

FIGURE 1.10 Probability Space for Loans to Companies *X* and *Y*

(continued)

(*continued*)

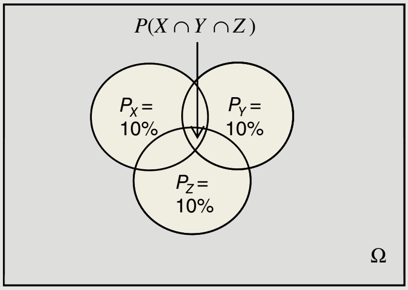

FIGURE 1.11 Probability Space for Loans to Companies X, Y, and Z

CASE C

If we further decrease the concentration ratio, the worst-case scenario (i.e., the expected loss of 10%) decreases further. Let's assume the lender C gives loans to three companies, X, Y, and Z, of \$3.33 million each. Let's assume that the default probabilities of X, Y, and Z are 10% each. Therefore the concentration ratio decreases to ⅓. The probabilities are displayed in Figure 1.11.

Hence from Figures 1.9 to 1.11 we observe the benefits of a lower concentration ratio. The worst-case scenario, an expected loss of \$1,000,000, reduces with a decreasing concentration ratio.

A decreasing concentration ratio is closely related to a decreasing correlation coefficient. Let's show this. The defaults of companies X and Y are expressed as two binomial variables, which take the value 1 if in default, and 0 otherwise. Equation (1.11) gives the joint probability of default for the two binomial events:

$$P(X \cap Y) = \rho_{XY}\sqrt{P_X(1-P_X)P_Y(1-P_Y)} + P_X P_Y \tag{1.11}$$

where ρ_{XY} is the correlation coefficient and

$$\sqrt{P_X(1-P_X)} \tag{1.12}$$

is the standard deviation of the binomially distributed variable X.

Let's assume again that the lender C has given loans to X and Y of \$5,000,000 each. Both X and Y have a default probability of 10%. Following equation (1.12), this means that the standard deviation for X and Y is $\sqrt{0.1 \times (1-0.1)} = 0.3$.

Let's first look at the case where the default correlation is $\rho_{XY} = 1$. This means that X and Y cannot default individually. They can only default together or survive together. The probability that they default together is 10%. Hence the expected loss is the same as in case A: EL = (\$5,000,000 + \$5,000,000) $\times$ 0.1 = \$1,000,000. We can verify this with equation (1.11) for the joint probability of two binomial events,
$P(X \cap Y) = 1 \times \sqrt{0.1\,(1-0.1) \times 0.1\,(1-0.1)} + 0.1 \times 0.1 = 10\%$. The probability space is graphically the same as Figure 1.9 with $P_X = P_Y =$ 10% as the probability event.

If we now decrease the correlation coefficient, we can see from equation (1.11) that the worst-case scenario, the joint default probability of X and Y, $P(X \cap Y)$, will decrease. For example, $\rho_{XY} = 0.5$ results in $P(X \cap Y) = 5.5\%$, and $\rho_{XY} = 0$ results in $P(X \cap Y) = 1\%$. Interestingly, even a slightly negative correlation coefficient can result in a positive joint default probability if the standard deviation of the binomial events is fairly low and the default probabilities are high. In our example, the standard deviation of both entities is 30% and a default probability of both entities is 10%. Together with a negative correlation coefficient of -0.1, following equation (1.11), this leads to a joint default probability of 0.1%. We will discuss the binomial correlation model in more detail in Chapter 4, section 4.2.

In conclusion, we have shown the beneficial aspect of a lower concentration ratio, which is closely related to a lower correlation coefficient. In particular, both a lower concentration ratio and a lower correlation coefficient reduce the worst-case scenario for a creditor, the joint probability of default of his debtors.

In Chapter 12, section 12.2, we will verify this result and find that a higher (copula) correlation between assets results in a higher credit value at risk (CVaR). CVaR measures the maximum loss of a portfolio of correlated debt with a certain probability for a certain time frame. Hence CVaR measures correlated default risk and is analogous to the VaR concept for correlated market risk, which we discussed in section 1.3.

1.5 A WORD ON TERMINOLOGY

As mentioned in section 1.3.2, "Trading and Correlation," we find the terms *correlation desks* or *correlation trading* in trading practice. Correlation trading means that traders trade assets or execute trading strategies whose value is at least in part determined by the comovement of two or more assets in time. We already mentioned pairs trading, the exchange option, and the quanto option as examples of correlation trading. In trading practice, the term *correlation* is typically applied quite broadly, referring to any comovement of asset prices in time.

However, in financial theory, especially in recent publications, the term *correlation* is often defined more narrowly, referring only to the linear Pearson correlation model, as in Cherubini, Luciano, and Vecchiato (2004), Nelsen (2006), or Gregory (2010). These authors refer to other than Pearson correlation coefficients as dependence measures or measures of association. However, in financial theory the term *correlation* is also often applied generally to describe dependencies, as in the terms *credit correlation*, *default correlation*, or *volatility-asset return correlation*, which are quantified by non-Pearson models as in Heston (1993), Lucas (1995), or Li (2000).

In this book, we will refer to the Pearson coefficient, discussed in Chapter 3, section 3.1, as correlation coefficient and the coefficients derived by non-Pearson models as dependency coefficients. In accordance with most literature, we will refer to all methodologies that measure some form of dependency as correlation models or dependency models. Ordinal dependence measures, discussed in sections 3.2 and 3.3, which are related to the Pearson correlation approach, will be termed rank correlation measures.

1.6 SUMMARY

There are two types of financial correlations: (1) *Static* correlations measure how two or more financial assets are associated within a certain time period, for example a year. (2) *Dynamic* financial correlations measure how two or more financial assets move together in time.

Correlation risk can be defined as the risk of financial loss due to adverse movements in correlation between two or more variables. These variables can be financial variables such as correlated defaults of two debtors or non-financial such as the correlation between political tensions and an exchange rate. Correlation risk can be nonmonotonous, meaning that the dependent variable, for example the CDS spread, can sometimes increase and sometimes decrease when the correlation parameter value increases.

Correlations and correlation risk are critical in many areas in finance, such as investments, trading, and especially in risk management, where different correlations result in very different degrees of risk. Correlations also play a key role in a systemic crisis, where correlations typically increase and can lead to high unexpected losses. As a result, the Basel III accord has introduced several correlation concepts and measures to reduce correlation risk (see Chapter 12 for details).

Correlation risk can be categorized as its own type of risk. However, correlation parameters and correlation matrices are critical inputs and hence a part of market risk and credit risk. Market risk and credit risk are highly sensitive to changing correlations. Correlation risk is also closely related to concentration risk, as well as systemic risk, since correlations typically increase in a systemic crisis.

The term *correlation* is not uniquely defined. In trading practice, *correlation* is applied quite broadly and refers to the comovements of assets in time, which may be measured by different correlation concepts. In financial theory, the term *correlation* is often defined more narrowly, referring only to the linear Pearson correlation coefficient. Non-Pearson correlation measures are termed *dependence measures* or *measures of association*.

APPENDIX 1A: DEPENDENCE AND CORRELATION

Dependence

In statistics, two events are considered dependent if the occurrence of one affects the probability of another. Conversely, two events are considered independent if the occurrence of one does not affect the probability of another. Formally, two events, A and B, are independent if and only if the joint probability equals the product of the individual probabilities:

$$P(A \cap B) = P(A)P(B) \tag{1A.1}$$

Solving equation (1A.1) for $P(A)$, we get

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

Following the Kolmogorov definition $\frac{P(A \cap B)}{P(B)} \equiv P(A|B)$, we derive

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B) \tag{1A.2}$$

where $P(A|B)$ is the conditional probability of A with respect to B. $P(A|B)$ reads "probability of A given B." In equation (1A.2), the probability of A, $P(A)$, is not affected by B, since $P(A) = P(A|B)$; hence the event A is independent from B.

From equation (1A.2) we also derive

$$P(B) = \frac{P(A \cap B)}{P(A)} = P(B|A) \tag{1A.3}$$

Hence from equation (1A.1) it follows that A is independent from B and B is independent from A.

Example of Statistical Independence The historical default probability of company A is $P(A) = 3\%$, the historical default probability of company B is $P(B) = 4\%$, and the historical joint probability of default is $3\% \times 4\% = 0.12\%$. In this case $P(A)$ and $P(B)$ are independent. This is because from equation (1A.2), we have

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B) = 3\% = \frac{3\% \times 4\%}{4\%} = 3\%$$

Since $P(A) = P(A|B)$, the default probability of company A is independent from the default probability of company B. Using equation (1A.3), we can do the same exercise for the default probability of company B, which is independent from the default probability of company A.

Correlation

As mentioned in section 1.5, the term *correlation* is not uniquely defined. In trading practice, the term *correlation* is used quite broadly, referring to any comovement of asset prices in time. In statistics, correlation is typically defined more narrowly and typically refers to the linear dependency derived in the Pearson correlation model. Let's look at the Pearson covariance and relate it to the dependence discussed earlier.

A covariance measures how strong the linear relationship between two variables is. These variables can be deterministic (meaning their outcome is known), as the historical default probabilities in equation 1A.1. For random variables (variables with an unknown outcome such as flipping a coin), the Pearson covariance is derived with expectation values:

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y) \tag{1A.4}$$

where $E(X)$ and $E(Y)$ are the expected values of (X) and (Y) respectively, also known as the mean, and $E(XY)$ is the expected value of the product of the random variables X and Y.

The covariance in equation (1A.4) is not easy to interpret. Therefore, often a normalized covariance, the correlation coefficient, is used. The Pearson correlation coefficient $\rho(XY)$ is defined as

$$\rho(X,Y)=\frac{\text{Cov}(X,Y)}{\sigma(X)\sigma(Y)} \tag{1A.5}$$

where $\sigma(X)$ and $\sigma(Y)$ are the standard deviations of X and Y, respectively. While the covariance takes values between $-\infty$ and $+\infty$, the correlation coefficient conveniently takes values between -1 and $+1$.

Independence and Uncorrelatedness

From equation (1A.1) above we find that the condition for independence of two random variables is $E(XY) = E(X)\ E(Y)$. From equation (1A.4) we see that $E(XY) = E(X)\ E(Y)$ is equal to a covariance of zero. Therefore, if two variables are independent, their covariance is zero.

Is the reverse also true? Does a zero covariance mean independence? The answer is no. Two variables can have a zero covariance even when they are dependent! Let's show this with an example. For the parabola $Y = X^2$, Y is clearly dependent on X, since Y changes when X changes. However, the correlation of the function $Y = X^2$ derived by equations (1A.4) or (1A.5) is zero! This can be shown numerically and algebraically. For a numerical derivation, see the simple spreadsheet "Dependence and Correlation.xlsm," sheet 1, at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

Algebraically, we have from equation (1A.4):

$$\text{Cov}(X,Y)=E(XY)-E(X)E(Y)$$

Inputting $Y = X^2$, we derive

$$\begin{aligned}\text{Cov}(X,Y) &= E(X\,X^2)-E(X)E(X^2)\\ &= E(X^3)-E(X)E(X^2)\end{aligned}$$

Let X be a uniform variable bounded in $[-1,+1]$. Then the mean $E(X)$ is zero and we have

$$\begin{aligned}\text{Cov}(X,Y) &= 0-0\,E(X^2)\\ &= 0\end{aligned}$$

For a numerical example, see the spreadsheet "Dependence and Correlation.xlsm," sheet 2, at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

In conclusion, the Pearson covariance or correlation coefficient can give values of zero; that is, it tells us that the variables are uncorrelated, even if the variables are dependent! This is because the Pearson correlation concept measures only linear dependence. It fails to capture nonlinear relationships. This shows the limitation of the Pearson correlation concept for finance, since most financial relationships are nonlinear. See Chapter 3 for a more detailed discussion on the Pearson correlation model.

APPENDIX 1B: ON PERCENTAGE AND LOGARITHMIC CHANGES

In finance, growth rates are expressed as relative changes, $(S_t - S_{t-1})/S_{t-1}$, where S_t and S_{t-1} are the prices of an asset at time t and $t - 1$, respectively. For example, if $S_t = 110$, and $S_{t-1} = 100$, the relative change is $(110 - 100)/100 = 0.1 = 10\%$.

We often approximate relative changes with the help of the natural logarithm:

$$(S_t - S_{t-1})/S_{t-1} \approx \ln (S_t/S_{t-1}) \tag{1B.1}$$

This is a good approximation for small differences between S_t and S_{t-1}. $\mathrm{Ln}(S_t/S_{t-1})$ is called a log return. The advantage of using log returns is that they can be added over time. Relative changes are not additive over time. Let's show this in two examples.

Example 1: A stock price at t_0 is \$100. From t_0 to t_1, the stock increases by 10%. Hence the stock increases to \$110. From t_1 to t_2 the stock increases again by 10%. So the stock price increases to $\$110 \times 0.1 = \$121$. This increase of 21% is higher than adding the percentage increases of $10\% + 10\% = 20\%$. Hence percentage changes are not additive over time.

Let's look at the log returns. The log return from t_0 to t_1 is $\ln(110/100) = 9.531\%$. From t_1 to t_2 the log return is $\ln(121/110) = 9.531\%$. When adding these returns, we get $9.531\% + 9.531\% = 19.062\%$. This is the same as the log return from t_0 to t_2; that is, $\ln(121/100) = 19.062\%$. Hence log returns are additive in time.[15]

Let's now look at another, more extreme example.

Example 2: A stock price in t_0 is \$100. It moves to \$200 in t_1 and back to \$100 in t_2. The percentage change from t_0 to t_1 is $(\$200 - \$100)/\$100 = 100\%$.

15. We could also have solved for the absolute value 121, which matches a logarithmic growth rate of 9.531%: $\ln(x/110) = 9.531\%$, or $\ln(x) - \ln(110) = 9.531\%$, or $\ln(x) = \ln(110) + 9.531\%$. Taking the power of e, we get $e^{(\ln(x))} = x = e^{(\ln(110)+0.09531)} = 121$.

The percentage change from t_1 to t_2 is (\$100 − \$200)/(200) = −50%. Adding the percentage changes, we derive +100% − 50% = +50%, although the stock has not increased from t_0 to t_2! Naturally, this type of performance measure is incorrect and not allowed in accounting.

Log returns give the correct answer: The log return from t_0 to t_1 is ln(200/100) = 69.31%. The log return from t_1 to t_2 is ln(100/200) = −69.31%. Adding these log returns in time, we get the correct return of the stock price from t_0 to t_2 of 69.31% − 69.31% = 0%.

These examples are displayed in the simple spreadsheet "Log returns.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

PRACTICE QUESTIONS AND PROBLEMS

1. What two types of financial correlations exist?
2. What is wrong-way correlation risk or short wrong-way risk?
3. Correlations can be nonmonotonous. What does this mean?
4. Correlations are critical in many areas in finance. Name five.
5. High diversification is related to low correlation. Why is this considered one of the few free lunches in finance?
6. Create a numerical example and show why a lower correlation results in a higher return/risk ratio.
7. What is correlation trading?
8. What is pairs trading?
9. Name three correlation options in which a lower correlation results in a higher option price.
10. Name one correlation option where a lower correlation results in a higher option price.
11. Create a numerical example of a two-asset portfolio and show that a lower correlation coefficient leads to a lower VaR number.
12. Why do correlations typically increase in a systemic market crash?
13. In 2005, a correlation crisis with respect to CDOs occurred that led to huge losses for several hedge funds. What happened?
14. In the global financial crisis of 2007 to 2009, many investors in the presumably safe super-senior tranches got hurt. What exactly happened?
15. What is the main objective of the Basel III accord?
16. The Basel accords have no legal authority. Why do most developed countries implement them?
17. How is correlation risk related to market risk and to credit risk?
18. How is correlation risk related to systemic risk and to concentration risk?

19. How can we measure the joint probability of occurrence of a binomial event as default or no default?
20. Can it be that two binomial events are negatively correlated but they have a positive probability of joint default?
21. What is value at risk (VaR) and credit value at risk (CVaR)? How are they related?
22. Correlation risk is quite broadly defined in trading practice, referring to any comovement of assets in time. How is the term *correlation* defined in statistics?
23. What do the terms *measure of association* and *measure of dependence* refer to in statistics?

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CHAPTER 2

Empirical Properties of Correlation: How Do Correlations Behave in the Real World?

Anything that relies on correlation is charlatanism.
—Nassim Taleb

In this chapter we show that, contrary to common beliefs, financial correlations display statistically significant and expected properties. We show that correlation levels as well as correlation volatility are generally higher in economic crises, which should be taken into consideration by traders and risk managers. We also find strong mean reversion in correlations as well as expected behavior of autocorrelation. The distribution of correlations is typically not normal or lognormal.

2.1 HOW DO EQUITY CORRELATIONS BEHAVE IN A RECESSION, NORMAL ECONOMIC PERIOD, OR STRONG EXPANSION?

In our study, we observed daily closing prices of the 30 stocks in the Dow Jones Industrial Average (Dow) from January 1972 to October 2012. This resulted in 10,303 daily observations of the Dow stocks and hence $10{,}303 \times 30 = 309{,}090$ closing prices. We built monthly bins and derived 900 correlation values (30×30) for each month, applying the Pearson correlation approach (see Chapter 3 for details). Since we had 490 months in the study, all together we derived $490 \times 900 = 441{,}000$ correlation values. We eliminated the unity correlation values on the diagonal of each correlation

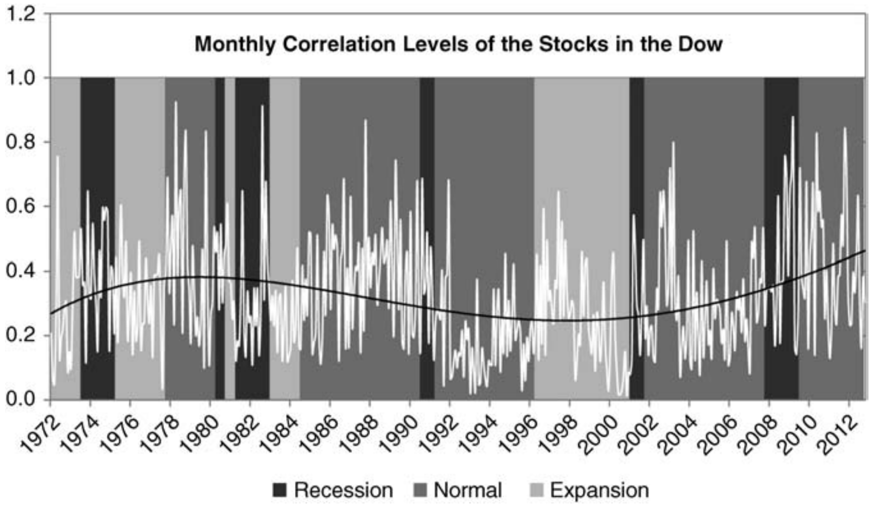

FIGURE 2.1 Average Correlation of Monthly 30 × 30 Dow Stock Bins
The light gray background represents an expansionary economic period, the medium gray background a normal economic period, and the dark gray background a recession. The horizontal line shows the polynomial trend line of order 4.

matrix and derived 441,000 – (30 × 490) = 426,300 correlation values as inputs.

The composition of the Dow is changing in time, with successful stocks being put into the Dow and unsuccessful stocks being removed. Our study is comprised of the Dow stocks that represent the Dow at each particular point in time.

Figure 2.1 shows the 490 monthly averaged correlation levels from 1972 to 2012 with respect to the state of the economy. We differentiate three states: an *expansionary period* with gross domestic product (GDP) growth rates of 3.5% or higher, a *normal economic period* with growth rates between 0% and 3.49%, and a *recession* with two consecutive quarters of negative growth rates.

Figure 2.2 shows the volatility of the averaged monthly correlations. For the calculation of volatility, see Chapter 1, section 1.3.1.

From Figures 2.1 and Figures 2.2 we observe the somewhat erratic behavior of Dow correlation levels and volatility. However, Table 2.1 reveals some expected results.

From Table 2.1 we observe that correlation levels are lowest in strong economic growth times. The reason may be that in strong growth periods equity prices react primarily to idiosyncratic, not macroeconomic factors. In recessions, correlation levels typically increase, as shown in Table 2.1.

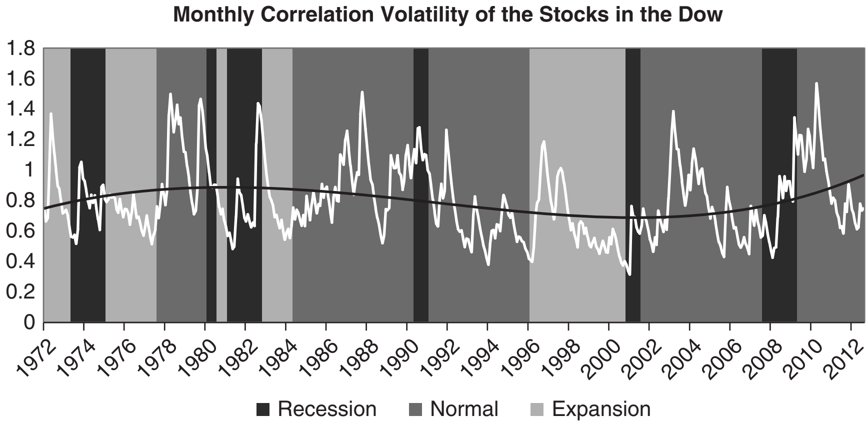

FIGURE 2.2 Correlation Volatility of the Average Correlation of Monthly 30 × 30 Dow Stock Bins with Respect to the State of the Economy. The horizontal line shows the polynomial trend line of order 4.

In addition, we have already displayed in Chapter 1, section 1.4, Figure 1.8, that correlation levels increased sharply in the Great Recession from 2007 to 2009. In a recession, macroeconomic factors seem to dominate idiosyncratic factors, leading to a downturn of multiple stocks.

A further expected result in Table 2.1 is that correlation volatility is lowest in an economic expansion and highest in worse economic states. We did expect a higher correlation volatility in a recession compared to a normal economic state. However, it seems that high correlation levels in a recession remain high without much additional volatility. Generally, correlation volatility is high, as we can also observe from the somewhat erratic correlation function in Figure 2.1. We will analyze whether the correlation volatility is an indicator for future recessions in section 2.5. Altogether, Table 2.1 displays the higher correlation risk in bad economic times, which traders and risk managers should consider in their trading and risk management.

TABLE 2.1 Correlation Level and Correlation Volatility with Respect to the State of the Economy

	Correlation Level	Correlation Volatility
Expansionary period	27.46%	71.17%
Normal economic period	32.73%	83.40%
Recession	36.96%	80.48%

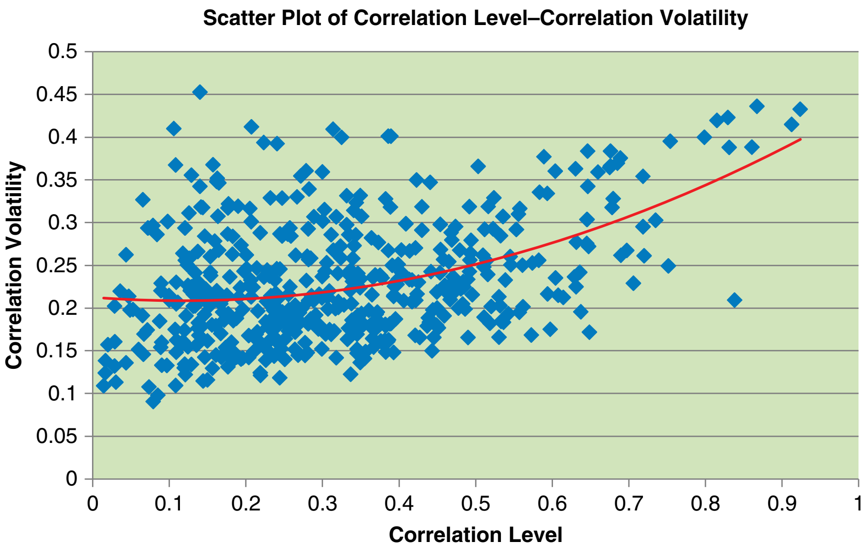

FIGURE 2.3 Positive Relationship between Correlation Level and Correlation Volatility with a Polynomial Trend Line of Order 2

From Table 2.1 we observe a generally positive relationship between correlation level and correlation volatility. This is verified in more detail in Figure 2.3.

2.2 DO EQUITY CORRELATIONS EXHIBIT MEAN REVERSION?

Mean reversion is the tendency of a variable to be pulled back to its long-term mean. In finance, many variables, such as bonds, interest rates, volatilities, credit spreads, and more, are assumed to exhibit mean reversion. Fixed coupon bonds, which do not default, exhibit strong mean reversion: A bond is typically issued at par, for example at \$100. If the bond does not default, at maturity it will revert to exactly that price of \$100, which is typically close to its long-term mean.

Interest rates are also assumed to be mean reverting: In an economic expansion, typically demand for capital is high and interest rates rise. These high interest rates will eventually lead to a cooling off of the economy, possibly leading to a recession. In this process, capital demand decreases and interest rate decrease from their high levels towards their long-term mean, eventually falling below their long-term mean. Being in a recession, eventually economic activity increases again, often supported by monetary and fiscal policy. In this reviving economy, demand for capital increases, in turn increasing interest rates to their long-term means.

2.2.1 How Can We Quantify Mean Reversion?

Mean reversion is present if there is a negative relationship between the change of a variable, $S_t - S_{t-1}$, and the variable at $t-1$, S_{t-1}. Formally, mean reversion exists if

$$\frac{\partial(S_t - S_{t-1})}{\partial S_{t-1}} < 0 \tag{2.1}$$

where

S_t: price at time t
S_{t-1}: price at the previous point in time $t - 1$
∂: partial derivative coefficient

Equation (2.1) tells us: If S_{t-1} increases by a very small amount, $S_t - S_{t-1}$ will decrease by a certain amount, and vice versa. This is intuitive: If S_{t-1} has decreased and is low at $t - 1$ (compared to the mean of S, μ_S), then at the next point in time t, mean reversion will pull up S_{t-1} to μ_S and therefore increase $S_t - S_{t-1}$. If S_{t-1} has increased and is high in $t - 1$ (compared to the mean of S, μ_S), then at the next point in time t, mean reversion will pull down S_{t-1} to μ_S and therefore decrease $S_t - S_{t-1}$. The degree of the pull is the degree of the mean reversion, also called mean reversion rate, mean reversion speed, or gravity.

Let's quantify the degree of mean reversion. Let's start with the discrete Vasicek 1977 process, which goes back to Ornstein-Uhlenbeck 1930:

$$S_t - S_{t-1} = a\,(\mu_S - S_{t-1})\Delta t + \sigma_S\,\varepsilon\sqrt{\Delta t} \tag{2.2}$$

where

S_t: price at time t
S_{t-1}: price at the previous point in time $t - 1$
a: degree of mean reversion, also called mean reversion rate or gravity, $0 \le a \le 1$
μ_S: long-term mean of S
σ_S: volatility of S
ε: random drawing from a standardized normal distribution at time t, $\varepsilon(t)$: $n \sim (0,1)$

We can compute ε as =normsinv(rand()) in Excel/VBA and norminv (rand) in MATLAB.

We are currently interested only in mean reversion, so for now we will ignore the stochasticity part in equation (2.2), $\sigma_S\,\varepsilon\sqrt{\Delta t}$.

For ease of explanation, let's assume $\Delta t = 1$. Then, from equation (2.2) we see that a mean reversion parameter of $a = 1$ will pull S_{t-1} to the long-term mean μ_S completely at every time step. For example, if S_{t-1} is 80 and μ_S is 100, then $a\,(\mu_S - S_{t-1}) = 1 \times (100 - 80) = 20$, so the S_{t-1} of 80 is mean reverted up to its long-term mean of 100. Naturally, a mean reversion parameter a of 0.5 will lead to a mean reversion of 50% at each time step, and a mean reversion parameter a of 0 will result in no mean reversion.

Let's now quantify mean reversion. Setting Δt to 1, equation (2.2) without stochasticity reduces to

$$S_t - S_{t-1} = a\,(\mu_S - S_{t-1}) \tag{2.3}$$

or

$$S_t - S_{t-1} = a\,\mu_S - a\,S_{t-1} \tag{2.4}$$

To find the mean reversion rate a, we can run a standard regression analysis of the form

$$Y = \alpha + \beta\,X$$

Following equation (2.4), we are regressing $S_t - S_{t-1}$ with respect to S_{t-1}:

$$\underbrace{S_t - S_{t-1}}_{Y} = \underbrace{a\,\mu_S}_{\alpha} \underbrace{-\,a\,S_{t-1}}_{\beta X} \tag{2.5}$$

Importantly, from equation (2.5), we observe that the regression coefficient β is equal to the negative mean reversion parameter a.

We now run a regression of equation (2.5) to find the empirical mean reversion of our correlation data. Hence S represents the 30×30 Dow stock monthly average correlations from 1972 to 2012. The regression analysis is displayed in Figure 2.4.

The regression function in Figure 2.4 displays a strong mean reversion of 77.51%. This means that on average in every month, a deviation from the long-term correlation mean (34.83% in our study) is pulled back to that long-term mean by 77.51%. We can observe this strong mean reversion also by looking at Figure 2.1. An upward spike in correlation is typically followed by a sharp decline in the next time period, and vice versa.

Let's look at an example of modeling correlation with mean reversion.

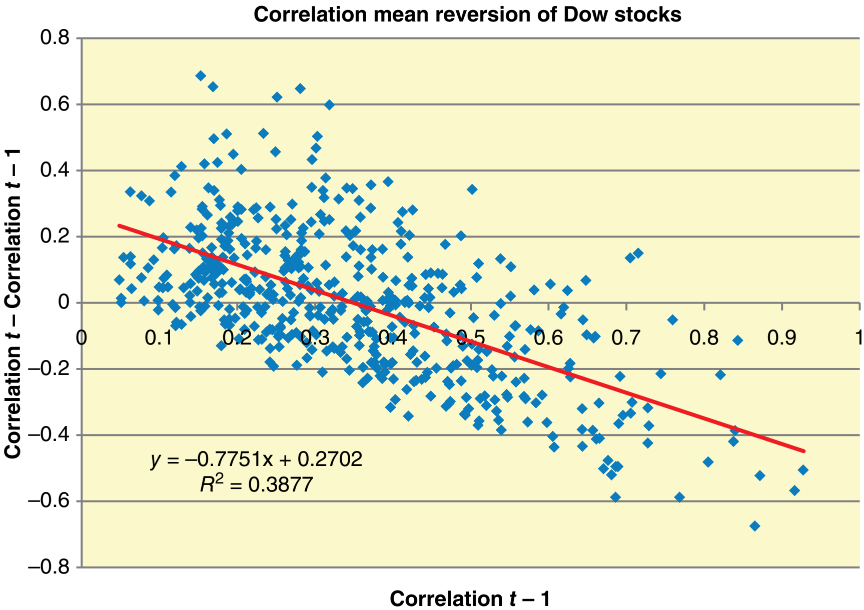

FIGURE 2.4 Regression Function (2.5) for 490 Monthly Average Dow Correlations from 1972 to 2012

EXAMPLE 2.1: EXPECTED CORRELATION

The long-term mean of the correlation data is 34.83%. In February 2012, the averaged correlation of the 30 × 30 Dow correlation matrices was 26.15%. From the regression function from 1972 to 2012, we find that the average mean reversion is 77.51%. What is the expected correlation for March 2012 following equation (2.3) or (2.4)?

Solving equation (2.3) for S_t, we have $S_t = a\,(\mu_S - S_{t-1}) + S_{t-1}$. Hence the expected correlation in March is

$$S_t = 0.7751 \times (0.3483 - 0.2615) + 0.2615 = 0.3288$$

As a result, we find that the mean reversion rate of 77.51% increases the correlation in February 2012 of 26.15% to an expected correlation in March 2012 of 32.88%.[1]

1. Note that we have omitted any stochasticity, which is typically included when modeling financial variables, as shown in equation (2.2).

2.3 DO EQUITY CORRELATIONS EXHIBIT AUTOCORRELATION?

Autocorrelation is the degree to which a variable is correlated to its past values. Autocorrelation can be quantified with the Nobel Prize–winning autoregressive conditional heteroscedasticity (ARCH) model of Robert Engle (1982) or its extension, generalized autoregressive conditional heteroscedasticity (GARCH) by Tim Bollerslev (1986), see Chapter 8, section 8.3, for more details. However, we can also regress the time series of a variable to its past time series values to derive autocorrelation. This is the approach we will take here.

In finance, positive autocorrelation is also termed *persistence*. In mutual fund or hedge fund performance analysis, an investor typically wants to know if an above-market performance of a fund has persisted for some time (i.e., is positively correlated to its past strong performance).

The question whether autocorrelation exists is an easy one. Autocorrelation is the "reverse property" to mean reversion: The stronger the mean reversion (i.e., the more strongly a variable is pulled back to its mean), the lower the autocorrelation (i.e., the less it is correlated to its past values), and vice versa.

For our empirical correlation analysis, we derive the autocorrelation (AC) for a time lag of one period with equation (2.6):

$$AC(\rho_t, \rho_{t-1}) = \frac{Cov(\rho_t, \rho_{t-1})}{\sigma(\rho_t)\sigma(\rho_{t-1})} \tag{2.6}$$

where

AC: autocorrelation

ρ_t: correlation values for time period t (in our study the monthly average of the 30×30 Dow stock correlation matrices from 2/1/1972 to 12/13/2012, after eliminating the unity correlation on the diagonal)

ρ_{t-1}: correlation values for time period $t-1$ (i.e., the monthly correlation values starting and ending one month prior than period t)

Cov: covariance; see equation (1.3) for details.

Equation (2.6) is algebraically identical with the Pearson correlation coefficient equation (1.4) in Chapter 1. The autocorrelation just uses the correlation values of time period t and time period $t-1$ as inputs.

Following equation (2.6), we find the one-period lag autocorrelation of the correlation values from 1972 to 2012 to be 22.49%. As mentioned earlier, autocorrelation is the opposite property of mean reversion. Therefore, not surprisingly, the autocorrelation of 22.49% and the mean reversion in our study of 77.51% (see previous section 2.2) add up to 1.

Figure 2.5 shows the autocorrelation with respect to different time lags.

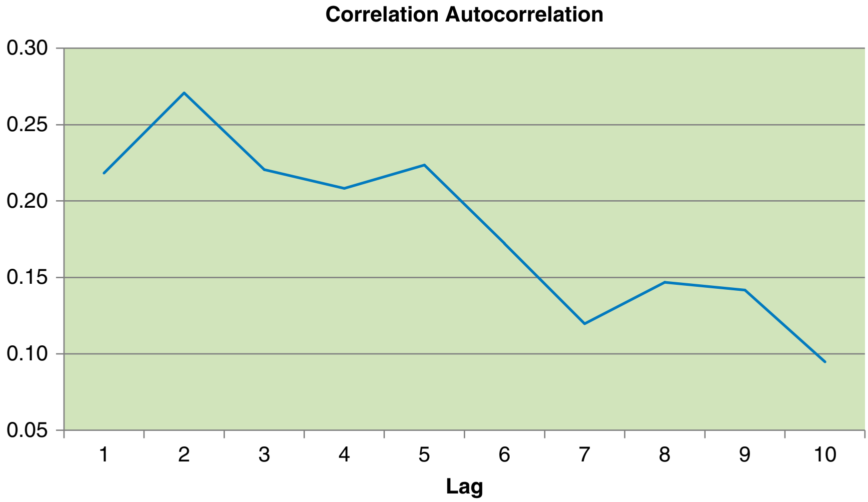

FIGURE 2.5 Autocorrelation of Monthly Average 30 × 30 Dow Stock Correlations from 1972 to 2012. The time period of the lags is months.

From Figure 2.5 we observe that time lag 2 autocorrelation is highest, so autocorrelation with respect to two months prior produces the highest autocorrelation. Altogether we observe the expected decay in autocorrelation with respect to time lags of earlier periods.

2.4 HOW ARE EQUITY CORRELATIONS DISTRIBUTED?

The input data of our distribution tests are daily correlation values between all 30 Dow stocks from 1972 to 2012. This resulted in 426,300 correlation values. The distribution is shown in Figure 2.6.

From Figure 2.6, we observe the mostly positive correlations between the stocks in the Dow. In fact, 77.23% of all 426,300 correlation values were positive.

We tested 61 distributions for fitting the histogram in Figure 2.6, applying three standard fitting tests: (1) Kolmogorov-Smirnov, (2) Anderson-Darling, and (3) chi-squared. Not surprisingly, the versatile Johnson SB distribution with four parameters, γ and δ for the shape, μ for location, and σ for scale, provided the best fit. Standard distributions such as normal distribution, lognormal distribution, or beta distribution provided a poor fit.

We also tested the correlation distribution between the Dow stocks for different states of the economy. The results were slightly but not significantly

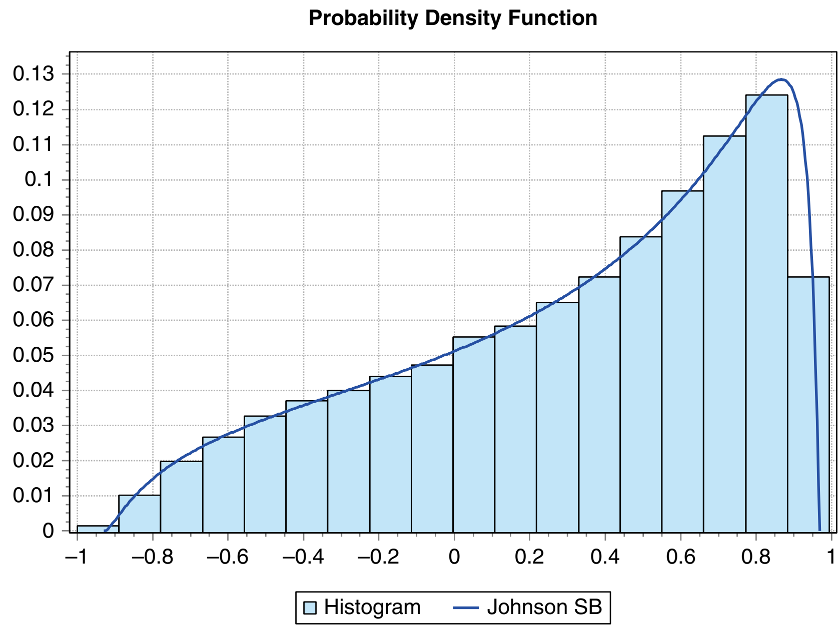

FIGURE 2.6 Histogram of 426,300 Correlations between the Dow 30 Stocks from 1972 to 2012
The continuous line shows the Johnson SB distribution, which provided the best fit.

different; see the file "Correlation Fitting.docx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 2."

2.5 IS EQUITY CORRELATION VOLATILITY AN INDICATOR FOR FUTURE RECESSIONS?

In our study from 1972 to 2012, six recessions occurred:

1. A severe recession in 1973–1974 following the first oil price shock.
2. A short recession in 1980.
3. A severe recession in 1981–1982 following the second oil price shock.
4. A mild recession in 1990–1991.
5. A mild recession in 2001 after the Internet bubble burst.
6. The Great Recession from 2007 to 2009 following the global financial crisis.

TABLE 2.2 Decrease in Correlation Volatility Preceding a Recession

	% Change in Correlation Volatility before Recession	Severity of Recession (% Change of GDP)
1973–1974	−7.22%	−11.93%
1980	−10.12%	−6.53%
1981–1982	−4.65%	−12.00%
1990–1991	0.06%	−4.05%
2001	−5.55%	−1.80%
2007–2009	−2.64%	−14.75%

The decrease in correlation volatility is measured as a six months change of six-month moving average correlation volatility. The severity of the recession is measured as the total GDP decline during the recession.

Table 2.2 displays the relationship of a change in the correlation volatility preceding the start of a recession.

From Table 2.2 we observe the severity of the 2007–2009 Great Recession, which exceeded the severity of the oil price shock–induced recessions in 1973–1974 and 1981–1982.

From Table 2.2 we also notice that, except for the mild recession in 1990–1991, before every recession a downturn in correlation volatility occurred. This coincides with the fact that correlation volatility is low in an expansionary period (see Table 2.1), which often precedes a recession. However, the relationship between a decline in volatility and the severity of the recession is statistically nonsignificant. The regression function is almost horizontal and the R^2 is close to zero. Studies with more data, going back to 1920, are currently being conducted.

2.6 PROPERTIES OF BOND CORRELATIONS AND DEFAULT PROBABILITY CORRELATIONS

Our preliminary studies of 7,645 bond correlations and 4,655 default probability correlations display properties similar to those of equity correlations. Correlation levels were higher for bonds (41.67%) and slightly lower for default probabilities (30.43%) compared to equity correlation levels (34.83%). Correlation volatility was lower for bonds (63.74%) and slightly higher for default probabilities (87.74%) compared to equity correlation volatility (79.73%).

Mean reversion was present in bond correlations (25.79%) and in default probability correlations (29.97%). These levels were lower than the very high equity correlation mean reversion of 77.51%.

The default probability correlation distribution is similar to the equity correlation distribution (see Figure 2.4) and can be replicated best with the Johnson SB distribution. However, the bond correlation distribution shows a more normal shape and can be best fitted with the generalized extreme value distribution and quite well with the normal distribution. Some fitting results can be found in the file "Correlation Fitting.docx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 2." The bond correlation and default probabilities results are currently being verified with a larger sample database.

2.7 SUMMARY

The following are the main findings of the empirical correlation analysis.

- Our study confirmed that the worse the state of the economy, the higher are equity correlations. Equity correlations were extremely high in the Great Recession of 2007 to 2009 and reached 96.97% in February 2009.
- Equity correlation volatility is lowest in an expansionary period and higher in normal and recessionary economic periods. Traders and risk managers should take these higher correlation levels and higher correlation volatility that markets exhibit during economic distress into consideration.
- Equity correlation levels and equity correlation volatility are positively related.
- Equity correlations show very strong mean reversion. The Dow correlations from 1972 to 2012 showed a monthly mean reversion of 77.51%. Hence, when modeling correlation, mean reversion should be included in the model.
- Since equity correlations display strong mean reversion, they display low autocorrelation. Autocorrelations show the typical decrease with respect to time lags.
- The equity correlation distribution showed a distribution that can be replicated well with the Johnson SB distribution. Other distributions such as normal, lognormal, and beta distributions did not provide a good fit.
- First results show that bond correlations display properties similar to those of equity correlations. Bond correlation levels and bond correlation volatilities are generally higher in bad economic times. In addition, bond correlations exhibit mean reversion, although lower mean reversion than equity correlations exhibit.
- First results show that default correlations also exhibit properties seen in equity correlations. Default probability correlation levels are slightly

lower than equity correlations levels, and default probability correlation volatilities are slightly higher than equity correlations. Studies with more data are currently being conducted.

PRACTICE QUESTIONS AND PROBLEMS

1. In which state of the economy are equity correlations the highest?
2. In which state of the economy is equity correlation volatility high?
3. What follows from questions 1 and 2 for risk management?
4. What is mean reversion?
5. How can we quantify mean reversion? Name two approaches.
6. What is autocorrelation?
7. For equity correlations, we see the typical decrease of autocorrelation with respect to time lags. What does that mean?
8. How are mean reversion and autocorrelation related?
9. What is the distribution of equity correlations?
10. When modeling stocks, bonds, commodities, exchange rates, volatilities, and other financial variables, we typically assume a normal or lognormal distribution. Can we do this for equity correlations?

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CHAPTER 3

Statistical Correlation Models— Can We Apply Them to Finance?

Great achievements involve great risk.
—Dalai Lama

Correlation models measure the degree of association between two or more variables. In this chapter we discuss three popular statistical correlation measures:

1. The Pearson correlation measure.
2. The Spearman rank correlation.
3. The Kendall τ.

We will analyze the properties of these correlation measures and evaluate whether it is appropriate to apply them to financial variables.

Let's first generally assess the role of models in finance.

> ***Models are not perfect. That doesn't mean they are not useful.***
> **—Robert Merton**

3.1 A WORD ON FINANCIAL MODELS

The financial reality is extremely complex, with thousands of investors, who may behave irrationally, and numerous markets such as equity, fixed income, commodities, foreign exchange, real estate, and more, which are correlated. In addition, numerous financial institutions and a great number of financial products such as stocks, bonds, indexes, exchange-traded funds (ETFs), structured products, and derivatives exist. Naturally, there is no financial model that can replicate the immense complexity of these financial systems and their products. In the 1980s and 1990s, econometricians actually tried to replicate this complexity by models with hundreds of equations and variables. However, these models have failed to produce convincing results.

Does this mean financial modeling is senseless? No. Financial models are useful tools to help us understand the financial system. The value at risk (VaR) model, for example (discussed in Chapters 1 and 9), can give us a good estimate about our market risk. The copula model (which will be discussed in this chapter and Chapter 6 and which is applied in the Basel framework) can give us a good estimate about the credit value at risk (CVaR) of a portfolio. The Black-Scholes-Merton (BSM) model can give us a good idea about the value of an option.

Importantly, however, we have to be constantly aware of the limitations of any financial model. In this respect, there are three main aspects to consider.

3.1.1 The Financial Model Itself

In physics we have models and relationships that are accurate and constant in time. For example, the relationship $E = mc^2$ is true and will be true in the future in normal physical environments. However, financial models such as VaR, CVaR, and BSM are models that depend on market prices as inputs. These market prices are determined by human beings and can therefore behave randomly and unexpectedly. (That's why we often use random models in financial modeling, since we believe they can better replicate random human behavior; see Chapter 8.) Therefore, we always have to be aware that any financial model is at best an approximation of reality and should never be trusted uncritically.

We also have to assess whether the model actually has problems with respect to approximating reality. The VaR model, for example, assumes a normal distribution of asset returns. However, in reality we find that asset returns have fat tails, so it would be better to use a model with higher kurtosis. The Black-Scholes-Merton (BSM) option pricing model assumes a constant volatility for all strikes. However, it is well known that traders apply a volatility smile in currencies markets (i.e., higher volatility for out-of-the-money calls and puts) and a volatility skew in equity markets (i.e., higher volatility for out-of-the-money puts). Risk managers and traders have to critically observe whether a model should be applied to price and hedge, or the model risk is too high; that is, the application of the model to replicate reality is not feasible.

In rare cases, a financial model has mathematical inconsistencies. For example, this is the case when pricing up-and-out calls and puts and down-and-out calls and puts on the BSM model. If the knock-out strike KO is equal to the strike K and the interest rate r equals the underlying asset return q, the model is insensitive to changes in implied volatility. In the case of $KO = K$ and $r = q = 0$, the model is insensitive to changes in volatility and option maturity. Similarly, lookback options cannot be valued on a standard extension of the BSM model if the interest rate is equal to the return (i.e., $r = q$). In this case, a new algorithm must and can be found. For details, see the model, "Lookback option.xls," at www.wiley.com/go/correlationriskmodeling under "Chapter 3." Naturally, traders and risk managers have to be aware of mathematical inconsistencies of their models to avoid incorrect pricing and hedging.

3.1.2 The Calibration of the Model

Calibrating a model means finding the values for the parameters of the model, so that the model can produce the prices that are found in the market. Once we find those parameter values, the model can then be applied to value products for which few or no market prices are available. A critical issue is what time frame should be observed when calibrating the parameter values. This was a significant problem in the 2007–2009 global financial crisis. Risk managers fed their VaR, CVaR, and collateralized debt obligation (CDO, discussed in Chapters 5 and 6) models the benign volatility and correlation data from noncrisis years, especially from 2003 to 2006. Hence the risk numbers that came out of the models significantly underestimated the catastrophic events from 2007 to 2009. Naturally, no model can produce realistic outputs when it is fed unrealistic inputs. In programming terminology: *Garbage in, garbage out!*

Models also need to be stress-tested. This means that extreme scenarios such as economic recessions and systemic market crashes are simulated. This can give risk managers and traders a good estimate of the risks of their portfolios in distressed times. Not surprisingly, Basel III and the U.S. Federal Reserve are requiring financial institutions to perform stress tests. In 2012, 15 of 19 financial institutions passed the Fed's required stress tests, i.e., "had enough capital to withstand a severe recession." See www.nytimes.com/2012/03/14/business/jpmorgan-passes-stress-test-raises-dividend.html for more details.

3.1.3 Mindfulness about Models

As mentioned earlier, no financial model is or will ever be able to replicate exactly the complexity of the financial system. Therefore we have to constantly be aware of the limitations of financial models. These limitations were ignored in the crisis of 2007 to 2009, when many traders and risk managers blindly trusted the new copula correlation model. When real estate prices declined sharply in 2007 to 2009, and structured products such as CDOs, which referenced mortgages, declined by 50% or more, the losses were not anticipated by the copula model for two reasons:

1. The correlation assumptions of the copula model were violated in the systemic crash. The copula model assumes a negative correlation between the equity tranche and senior tranches, as we already saw in Figure 1.3. However, correlation increased sharply during the crisis, and equity tranche values *and* senior tranche values both declined.
2. In addition, the copula models were calibrated with the benign data from low-risk periods, as mentioned previously.

In conclusion, there needs to be human judgment when the outputs of models are assessed. The outputs have to be viewed in consideration of the limitations of any financial model. As David Li, who transferred the copula model to finance, put it: "The most dangerous thing is when people believe everything that comes out of it [the copula model]."

We will now address correlation models used in statistics, which measure associations between two or more variables, and discuss their usefulness in finance.

3.2 STATISTICAL CORRELATION MEASURES

In the following section, we analyze the most widely applied correlation concept in science, the Pearson correlation model. We find that the Pearson correlation model, despite its popularity, has severe limitations when applied in financial analysis.

3.2.1 The Pearson Correlation Approach and Its Limitations for Finance

From our Statistics 101 class we all remember the Pearson product moment correlation coefficient or Pearson correlation coefficient ρ. The Pearson approach measures the strength of the linear association between two variables. In fact, we have already applied the Pearson approach numerous times in this book, for example in Chapter 1 when discussing correlations in investments, trading, and market risk management. Let's now look at the Pearson correlation model in detail. The Pearson correlation coefficient ρ is defined:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)} \tag{3.1}$$

where X and Y are sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ with the elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n \in R$.

In section 1.3, we already used this equation with x_t being the asset returns of asset X at time t and y_t being the asset returns of asset Y at time t. $\sigma(X)$ and $\sigma(Y)$ in equation (3.1) are the standard deviation of X and Y, respectively. The covariance in equation (3.1) is defined as in equation (3.2):

$$\mathrm{Cov}(X,Y) = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y) \tag{3.2}$$

If we deal with random sets (whose outcome is unknown, such as rolling a die), the covariance is quantified with expectation values. Hence the covariance is $E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$, where $E(X)$ is the expected value of X and $E(Y)$ is the expected value of Y. $E(XY)$ is the expected value of the product of the random variables X and Y. Also, the variances of X and Y are $\sigma_X^2 = E(X^2) - E(X)^2$ and $\sigma_Y^2 = E(Y^2) - E(Y)^2$, respectively. Hence for random sets equation (3.1) assumes the equivalent form

$$\rho_1(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}} \tag{3.3}$$

The application of the Pearson correlation coefficient and the related least squares linear regression analysis is a standard statistical tool in finance; see for example Fitch (2006), who regresses correlations between asset returns with sector specific regional factor loadings. Das et al. 2006 linearly regress the mean probability of default with market volatility and debt to asset ratios. Altman et al. (2005) apply the Pearson correlation approach and its extensions to verify the negative correlation between default rates and recovery rates.

However, the limitations of the Pearson correlation approach in finance are evident for five reasons:

1. Linear dependencies, which are evaluated in equations (3.1) and (3.3), do not appear often in finance. We have already seen in Figures 1.2 to 1.4 and 1.6 to 1.8 that financial relationships are typically nonlinear.
2. Zero correlation derived in equations (3.1) and (3.3) does not necessarily mean independence. This is because only the first two moments, mean and standard deviation, are considered in equations (3.1) and (3.3). For example, the parabola $Y = X^2$ will lead to $\rho = 0$, which is arguably misleading. See Appendix 1A of Chapter 1 for details.
3. Linear correlation measures are natural dependence measures only if the joint distribution of the variables is elliptical.[1] However, only a few distributions such as the multivariate normal distribution and the multivariate Student's t distribution are special cases of elliptical distributions, for which linear correlation measure can be meaningfully interpreted.[2]
4. The variances of the sets X and Y have to be finite. However, for distributions with strong kurtosis, for example the Student's t distribution with $v \leq 2$, the variance is infinite.
5. In contrast to the copula approach discussed in Chapters 5 and 6, which is invariant to strictly increasing transformations, the Pearson correlation approach is typically not invariant to transformations. For example, the Pearson correlation between pairs X and Y is in general different from the Pearson correlation between the pairs $\ln(X)$ and $\ln(Y)$. Hence the information value of the Pearson correlation coefficient after data transformation is limited.

For these reasons, the application of the Pearson correlation concept in finance is questionable. The linear Pearson correlation coefficient can at best serve as an approximation for the typically nonlinear relationship between financial variables.

Let's now discuss two ordinal correlation measures and evaluate their usefulness for financial applications.

3.2.2 Spearman's Rank Correlation

Ordinal correlation measures such as Spearman's rank correlation and Kendall's τ have gained popularity in finance in the recent past. Let's discuss them both and then assess whether they are applicable to finance.

1. An elliptical distribution is a generalization of multivariate normal distributions.
2. See Embrechts, McNeil, and Straumann (1999) and Bingham and Kiesel (2002) for details.

TABLE 3.1 Performance of a Portfolio with Two Assets

	Asset X	Asset Y	Return of Asset X	Return of Asset Y
2008	100	200		
2009	120	230	20.00%	15.00%
2010	108	460	−10.00%	100.00%
2011	190	410	75.93%	−10.87%
2012	160	480	−15.79%	17.07%
2013	280	380	75.00%	−20.83%
		Average	29.03%	20.07%

The Spearman's rank correlation concept is an ordinal correlation measure. This means that the numerical values of the elements in a set are not relevant for deriving the correlation, just the order of the elements. The Spearman's correlation coefficient is sometimes referred to as the Pearson correlation coefficient for ranked variables. It will result in a perfect correlation coefficient of 1 if an increase in the elements x_i is always accompanied by an increase in y_i, regardless of the numerical increase, and vice versa. The Spearman correlation approach is nonparametric in the sense that it can be applied without requiring knowledge of the joint distribution of the variables.

Let's look at the example in section 1.3.1 in Chapter 1. We have two assets, which have performed as in Table 3.1.

We had derived the Pearson correlation coefficient for the assets' returns in Table 1.1 as −0.7403. Let's now derive the Spearman rank correlation coefficient.

1. We first have to order the return set pairs of X and Y with respect to the set X. This is done in columns 2 and 3 of Table 3.2.
2. We then derive the ranks of X_i and Y_i. This is done in columns 4 and 5.

TABLE 3.2 Ranked Asset Returns to Derive the Spearman Correlation Coefficient

	Ranked Return of X_i	Assigned (same year) Return of Y_i	Rank of X_i	Rank of Y_i	d_i	d_i^2
2012	−15.79%	17.07%	1	4	−3	9
2010	−10.00%	100.00%	2	5	−3	9
2009	20.00%	15.00%	3	3	0	0
2013	75.00%	−20.83%	4	1	3	9
2011	75.93%	−10.87%	5	2	3	9
						Sum = 36

3. We now derive the difference of the ranks in column 6 and square the difference in column 7.

The Spearman rank correlation coefficient ρ_S is defined as

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} \tag{3.4}$$

For our example in Table 3.2, we derive $\rho_S = 1 - \frac{6X36}{5(5^2-1)} = -0.8$. Since the Spearman correlation coefficient is defined between -1 and $+1$, we find that the returns of assets X and Y are highly negatively correlated according to the Spearman rank correlation concept. The -0.8 Spearman correlation is similar to the Pearson correlation coefficient of -0.7403, which we had derived in Chapter 1. Before we evaluate the usefulness of the Spearman correlation coefficient for finance, let's discuss another rank correlation measure.

3.2.3 Kendall's τ

Kendall's τ is a further, fairly popular ordinal correlation measure applied in finance. As with the Spearman's correlation coefficient, the Kendall τ is nonparametric and will result in a perfect correlation coefficient of 1 if an increase in the variable x is always accompanied by an increase in y, regardless of the numerical increase, and vice versa. In most other cases, the two rank correlation measures are not equal.

The Kendall τ is defined as

$$\tau = \frac{n_c - n_d}{n(n-1)/2} \tag{3.5}$$

where n_c is the number of concordant data pairs and n_d is the number of discordant pairs.

A concordant pair is defined as any pair of observations where $x_t > y_t$ and $x_{t^*} > y_{t^*}$ or $x_t < y_t$ and $x_{t^*} < y_{t^*}$, where $t \neq t^*$.

A discordant data pair is where $x_t > y_t$ and $x_{t^*} < y_{t^*}$ or $x_t > y_t$ and $x_{t^*} < y_{t^*}$, where $t \neq t^*$.

A pair is neither concordant nor discordant if $x_t = y_t$ or $x_{t^*} = y_{t^*}$.

Let's calculate the Kendall τ for our example in Table 3.2. We have five observation pairs and therefore $5 \times (5 - 1)/2 = 10$ combinations of pairs to evaluate. We have the two concordant pairs {(1,4),(2,5)}, {(4,1),(5,2)} and the four discordant pairs {(1,4),(4,1)}, {(1,4),(5,2)}, {(2,5),(4,1)}, {(2,5),(5,2)}. The pairs {(1,4),(3,3)}, {(2,5),(3,3)}, {(3,3),(4,1)}, and {(3,3),(5,2)} are neither concordant nor discordant. From equation (3.5), we derived the Kendall $\tau = \frac{2-4}{5(5-1)/2} = -0.2$. Since Kendall's τ is defined between -1 and $+1$, we can interpret the -0.2 as: The association between the returns of assets X and Y is slightly negative when calculated by the Kendall τ concept.

3.3 SHOULD WE APPLY SPEARMAN'S RANK CORRELATION AND KENDALL'S τ IN FINANCE?

Rank correlation measures have been popular in analyzing rating categories (i.e., the categories AAA, AA, A, … , to D), since these are ordinal. Cherubini and Luciano (2002) apply Spearman's rank correlation and Kendall's τ to analyze the dependence of market prices and counterparty risk measured by rating categories in a copula setting. Burtschell, Gregory and Laurent (2008) compare Kendall's τ to various copulas and find significant difference in the correlation approaches when inferring CDO tranche spreads. Anderson (2010) analyzes CDS correlations and finds that Spearman's rank correlations for CDS spreads more than doubled during the financial crisis from July 2007 to March 2009.

Ordinal rank correlation measures are an appropriate tool if the observations are ordinal. The problem with applying ordinal rank correlations to cardinal observations is that ordinal correlation are less sensitive to outliers. To show this, let's double the outliers of the returns of asset X in Table 3.2. We derive Table 3.3.

The values in Table 3.3 result in an increase of the Pearson correlation coefficient from -0.7402 to -0.6108 in Table 3.2, which will increase risk when input into VaR. However, since the numerical value of outliers in the rank correlations Spearman and Kendall are irrelevant, the correlations in the rank correlation measures do not change. This is an unwelcome property, especially in risk management. For example, a severe loss that may have

TABLE 3.3 Table 3.2 but with Increased Outliers for Asset X

	Ranked Return of X_i	Assigned (same year) Return of Y_i	Rank of X_i	Rank of Y_i	d_i	d_i^2
2012	−31.58%	17.07%	1	4	−3	9
2010	−10.00%	100.00%	2	5	−3	9
2009	20.00%	15.00%	3	3	0	0
2013	75.00%	−20.83%	4	1	3	9
2011	151.86%	−10.87%	5	2	3	9
						Sum = 36

occurred in the past is not numerically assessed. This can lead to the illusion of less risk than is actually present!

A special problem with the Kendall τ is when many nonconcordant and many nondiscordant pairs occur, which are omitted in the calculation. This may lead to only a few concordant and discordant pairs, which can distort the Kendall τ coefficient. To a certain degree this is the case in our example of Table 3.2. Of the 10 observation pairs, four are neither concordant nor discordant, leaving just six pairs to be evaluated.

We can conclude that the application of statistical correlation measures to assess financial correlations is limited. The main concern with the Pearson correlation coefficient is that it evaluates linear relationships. However, financial variables are mostly nonlinear. In addition, the limited interpretation for nonelliptical data is problematic; see point 3, section 3.2.1. Statistical rank correlation measures should not be applied to cardinal financial variables, especially since the sensitivity to outliers is low. These outliers, for example high losses, are critical when evaluating correlations and risk. Statistical rank correlation measures are appropriate only if the financial variables are ordinal as, for example, rating categories.

Since the application of the statistical correlation concepts is limited in finance, quants have developed specific financial correlation measures, which we will discuss in Chapter 4.

3.4 SUMMARY

In this chapter, we first generally assessed the value of financial modeling. The financial reality is extremely complex, with numerous markets, complex products, and—most critically—investors who can behave irrationally. No financial model will ever be able to replicate this complex financial reality perfectly. However, this does not mean financial models are useless. Financial models can give a good approximation of the reality and help us better understand the behavior of financial processes. They can further help us forecast future crises and help us understand and manage financial risk.

In this chapter we also discussed statistical correlation approaches and investigated whether they are appropriate for financial modeling. By far the most widely applied correlation concept in statistics is the Pearson correlation model. The reason for the popularity of the Pearson model is its mathematical simplicity and high intuition. The Pearson correlation model is widely applied in finance. But should we actually apply it to financial modeling? The answer is "not really," especially not for complex financial correlations, as, for example, correlations in a CDO; see Chapter 4.

The Pearson approach suffers from a variety of problems: most importantly, it measures only *linear* relationships. However, most financial correlations are nonlinear. As a result, zero correlation derived by the Pearson approach does not necessarily mean independence (see also Appendix 1A of Chapter 1), so the Pearson correlation outcome can be quite misleading. The Pearson correlation approach can at best serve as a good approximation of the mostly nonlinear financial correlations found in practice. When applying the simple, linear Pearson correlation model to financial correlations, we should constantly be aware of its severe limitations.

Ordinal or rank correlations measures such as Spearman's rank correlation and Kendall's τ do not consider numerical values but just the order of the elements (i.e., higher or lower) when deriving correlations. For financial variables that are ordinal, such as rating categories, ordinal correlation measures are appropriate. However, the application of ordinal correlation measures to cardinal data is not appropriate, since ordinal correlation measures ignore the extreme values of outliers. This can give the illusion of less risk than is present.

PRACTICE QUESTIONS AND PROBLEMS

1. Discuss briefly why financial modeling is useful.
2. What are the general limitations of financial modeling?
3. How do models in physics and models in finance differ?
4. Name three critical aspects that have to be considered when applying financial models in reality.
5. What problems with financial modeling occurred in particular in the Great Recession of 2007 to 2009?
6. What is the main limitation of the Pearson correlation approach?
7. Name three other limitations of the Pearson correlation approach.
8. Does a Pearson correlation coefficient of zero mean independence?
9. In the Pearson correlation model, what values do covariances take, and what values does the correlation coefficient take?
10. Should we apply the Pearson correlation model to finance?
11. What is the main difference between cardinal correlation measures such as the Pearson model and ordinal correlation measures such as Spearman's rank correlation and Kendall's τ?
12. What is a severe limitation when applying Spearman's rank correlation and Kendall's τ to finance?
13. When should we apply Spearman's rank correlation and Kendall's τ in financial modeling?

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CHAPTER 4

Financial Correlation Modeling—Bottom-Up Approaches

Fortune sides with him who dares.

—Virgil

In this chapter we address correlation models, which were specifically designed to measure the association of financial variables. We will concentrate on bottom-up correlation models, which collect information, quantify it, and then aggregate the information to derive an overall correlation result.

4.1 CORRELATING BROWNIAN MOTIONS (HESTON 1993)

One of the most widely applied correlation approaches used in finance was generated by Steven Heston in 1993. Heston applied the approach to negatively correlate stochastic stock returns $dS(t)/S(t)$ and stochastic volatility $\sigma(t)$. The core equations of the original Heston model are the two stochastic differential equations (SDEs):

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma(t) dz_1(t) \tag{4.1}$$

and

$$d\sigma^2(t) = a[m_\sigma^2 - \sigma^2(t)]\, dt + \xi\, \sigma(t)\, dz_2(t) \tag{4.2}$$

where

S: variable of interest, e.g. a stock price
μ: growth rate of S
σ: volatility of S; hence σ^2 is the variance rate of S
dz: standard Brownian motion, i.e. $dz(t) = \varepsilon(t)\sqrt{dt}$, $\varepsilon(t)$ is *i.i.d.* (independently and identically distributed). In particular $\varepsilon(t)$ is a random drawing from a standardized normal distribution at time t, $\varepsilon(t) = n \sim (0, 1)$. We can compute ε as =normsinv(rand()) in Excel/VBA and norminv(rand) in MATLAB
a: mean reversion rate (gravity), i.e. degree with which σ^2 at time t, $\sigma^2(t)$, is pulled back to its long term mean m_σ^2. a can take the values $0 \leq a \leq 1$ (see Chapter 2, section 2.2 for details)
m_σ^2: long-term mean of σ^2
ξ: volatility of the volatility σ.

In equation (4.1), the variable S follows the standard geometric Brownian motion (GBM), which is also applied in the Black-Scholes-Merton option pricing model (which, however, assumes a constant volatility σ). For a model that generates the GBM in equation (4.1), and equation (4.1) with random jumps, see the model "GBM path with jumps.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 4." Equation (4.2) models the stochastic variance rate with the mean-reverting Cox-Ingersoll-Ross (CIR) process; see Cox, Ingersoll, and Ross (1985).

Importantly, the correlation between the stochastic processes (4.1) and (4.2) is introduced by correlating the two Brownian motions dz_1 and dz_2. The instantaneous correlation between the Brownian motions is

$$\mathrm{Corr}[dz_1(t), dz_2(t)] = \rho\, dt \tag{4.3}$$

The definition (4.3) can be conveniently modeled with the identity

$$dz_1(t) = \sqrt{\rho}\, dz_2(t) + \sqrt{1-\rho}\, dz_3(t) \tag{4.4}$$

where dz_2(t) and dz_3(t) are independent, and $dz(t)$ and $dz(t')$ are independent, $t \neq t'$.

Equation (4.4) only allows a positive correlation between dz_1 and dz_2 (since the correlation parameter ρ is input as a square root). We can rewrite equation (4.4) to allow negative correlation by applying $\sqrt{\rho_1} = \alpha$. Equation (4.4) then changes to

$$dz_1(t) = \alpha\, dz_2(t) + \sqrt{1-\alpha^2} dz_3(t) \tag{4.5}$$

From equation (4.5) we observe that for a dependence coefficient of $\alpha = 1$, the critical Brownian motions $dz_1(t)$ and $dz_2(t)$ are equal at every time t. For $\alpha = 0$, the Brownian motions $dz_1(t)$ and $dz_2(t)$ are not correlated since $dz_1(t) = dz_3(t)$. For $\alpha = -1$, $dz_1(t)$ and $dz_2(t)$ have an inverse correlation.

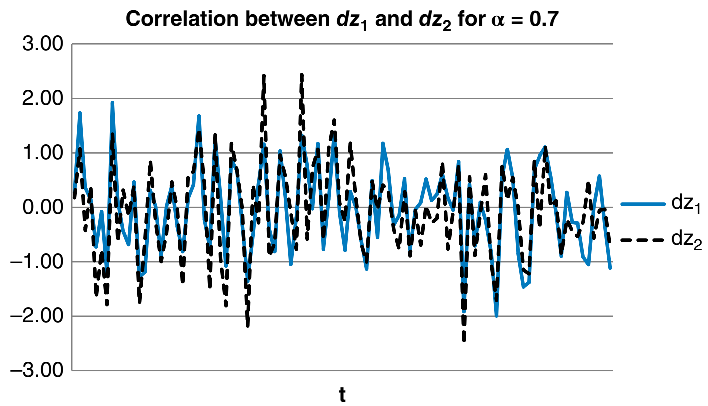

FIGURE 4.1A Positive Correlation between the Brownian Motions dz_1 and dz_2 Derived by Equation (4.5) with $\alpha = 0.7$

Equations (4.4) and (4.5) are mathematically and computationally convenient. If dz_2 and dz_3 are standard normal, it follows by construction that dz_1 will also be standard normal for any value of $-1 \leq \sqrt{\rho} = \alpha \leq 1$.

Figure 4.1a and Figure 4.1b show the correlation between dz_1 and dz_2 for different dependence parameters α.

The Heston correlation approach is a dynamic, versatile, and mathematically rigorous correlation model. It allows us to positively or negatively correlate stochastic processes and permits dynamic correlation modeling since $dz(t)$ is a function of t. Hence it is not surprising that the approach is an integral part of correlation modeling in finance.

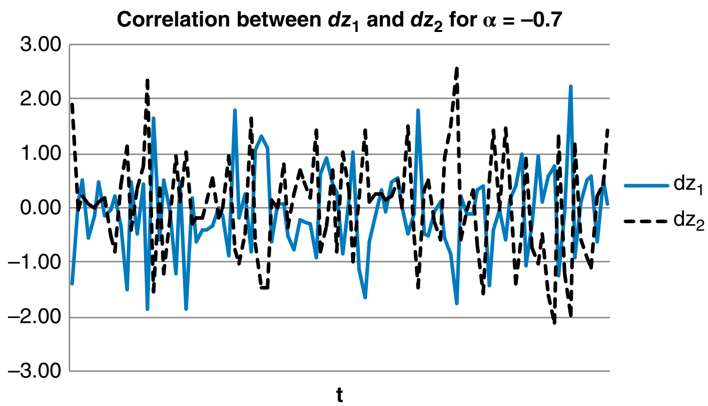

FIGURE 4.1B Negative Correlation between the Brownian Motions dz_1 and dz_2 Derived by Equation (4.5) with $\alpha = -0.7$

4.1.1 Applications of the Heston Model

One prominent application of the Heston model is in the stochastic alpha beta rho (SABR) model of Hagan et al. (2002), where stochastic interest rates and stochastic volatility are correlated to derive realistic volatility smiles and skews. For extensions of the SABR model, see West (2005), Henry-Labordere (2007), Kahl and Jaeckel (2009), and Benhamou, Gobet, and Mohammed (2009) as well as Chapter 9, section 9.2.1. Huang and Yildirim (2008) use the Heston approach to correlate the volatility of the inflation process and the volatility of the nominal discount bond process to value Treasury inflation-protected security (TIPS) futures. Langnau (2009) combines the Heston approach with the local volatility model of Dupire (1994). The result is a dynamic local correlation model (LCM), which matches the implied volatility skew of equity index options well.

In credit risk modeling, Zhou (2001) derives analytical equations for joint default probabilities in a Black-Cox first passage time framework applying Heston correlations. Zhou's equations help to explain empirical default properties, such as (1) default correlations and asset price correlations are positively related, and (2) default correlations are small over short time horizons. They typically first increase in time, then plateau out, and then gradually decline, as found by Lucas (1995). Brigo and Pallavicini (2008) apply two Heston correlations. The first correlates two factors that drive the interest rate process, while the second correlates the interest rate process with the default intensity process. Meissner, Rooder and Fan (2013) apply the Heston approach in a reduced form framework. They correlate the Brownian motion of a LIBOR market model (LMM) modeled reference asset and an LMM modeled counterparty, and investigate the impact on the CDS spread. They find that just correlating the LMM processes results in a rather low impact on the CDS spread; that is, it leads to higher CDS spreads than correlating the default processes directly. See Chapter 10, section 10.1 for details.

A variation of the Heston approach will be discussed in Chapter 8, section 8.5.

4.2 THE BINOMIAL CORRELATION MEASURE

A further popular correlation measure, mainly applied to default correlation, is the binomial correlation approach of Douglas Lucas (1995). Let's assume we have two entities (individuals, companies, or sovereigns) X and Y. We define the binomial events

$$1_X = 1_{\{\tau_X \leq T\}} \tag{4.6}$$

and

$$1_Y = 1_{\{\tau_Y \le T\}} \tag{4.7}$$

where τ_X is the default time of entity X and τ_Y is the default time of entity Y. 1_X is the indicator variable of entity X.

We read the equation (4.6) as: If entity X defaults before or at time T (i.e., $\tau_X \le T$), then 1_X takes the value 1 and the value 0 otherwise. The same applies to entity Y.

Furthermore, let $P(X)$ and $P(Y)$ be the default probability of X and Y respectively, and $P(XY)$ is the joint probability of default. The standard deviation of a one-trial binomial event is $\sqrt{P(X) - (P(X))^2}$, where P is the probability of outcome X. Hence, modifying the Pearson correlation equation (3.3), we derive the joint default dependence coefficient of the binomial events $1_{\{\tau_X \le T\}}$ and $1_{\{\tau_Y \le T\}}$ as

$$\rho(1_{\{\tau_X \le T\}}, 1_{\{\tau_Y \le T\}}) = \frac{P(XY) - P(X)P(Y)}{\sqrt{(P(X) - (P(X))^2}\sqrt{(P(Y) - (P(Y))^2}} \tag{4.8}$$

By construction, equation (4.8) can only model binomial events, for example default and no default. With respect to equation (3.3), we observe that in equation (3.3) X and Y are sets of $i = 1,\ldots, n$ variates, with $i \in \Re$. $P(X)$ and $P(Y)$ in equation (4.8), however, are scalars, for example the default probabilities of entities X and Y for a certain time period T, respectively, $0 \le P(X) \le 1$, and $0 \le P(Y) \le 1$. Hence the binomial correlation approach of equation (4.8) is a limiting case of the Pearson correlation approach of equation (3.3). As a consequence, the significant shortcomings of the Pearson correlation approach for financial modeling apply also to the binomial correlation model.

4.2.1 Application of the Binomial Correlation Measure

The binomial correlation approach [equation (4.8)] had been applied by rating agencies to value collateralized debt obligations (CDOs); for a discussion see Bank for International Settlements (2004) and Schönbucher (2004). However, the rating agencies have replaced the binomial correlation approach with a structural Merton-based model in combination with Monte Carlo (see Meissner, Garnier, and Laute 2008). Hull and White (2001) apply the binomial correlation measure to price CDSs with counterparty risk. They find that the impact of the counterparty risk on the CDS is small if the binomial correlation between the reference asset and the counterparty is small. The impact increases if the binomial correlation increases and the creditworthiness of the counterparty declines.

Numerous studies have applied the binomial correlation measure to analyze historical default correlations. Most of the studies show little statistical evidence of default correlation. Erturk (2000) finds no statistically significant evidence of default correlation for less than one-year intervals for 1,500 investment grade entities in the United States. Similarly, Nagpal and Bahar (2001) find low binomial correlation coefficients within 11 sectors in the United States from 1981 and 1999. Li and Meissner (2006) study intrasector and intersector default correlations of 10,348 U.S. companies from 1981 to 2003. Intersector default correlations show 80.76% positive default dependencies. However, only 8.97% of these were statistically significant at a 5% level. Intersector default correlations increased to 100% positive in recessionary periods. Of these, again 8.97% were statistically significant at the 5% level.

4.3 COPULA CORRELATIONS

A fairly recent and famous as well as infamous correlation approach applied in finance is the copula approach. Copulas go back to Abe Sklar (1959). Extensions are provided by Schweizer and Wolff (1981) and Schweizer and Sklar (1983). One-factor copulas were introduced to finance by Oldrich Vasicek in 1987. More versatile, multivariate copulas were applied to finance by David Li in 2000.

When flexible copula functions were introduced to finance in 2000, they were enthusiastically embraced but then fell into disgrace when the global financial crisis hit in 2007. Copulas became popular because they could presumably solve a complex problem in an easy way: It was assumed that copulas could correlate multiple assets, for example the 125 assets in a CDO, with a single (although multidimensional) function. We will devote the entire Chapter 5 to discussing the benefits and limitations of the Gaussian copula for valuing CDOs. Let's first look at the math of the copula correlation concept.

Copula functions are designed to simplify statistical problems. They allow the joining of multiple univariate distributions to a single multivariate distribution. Formally, a copula function C transforms an n-dimensional function on the interval [0, 1] into a unit-dimensional one:

$$C : [0,1]^n \rightarrow [0,1] \tag{4.9}$$

More explicitly, let $G_i(u_i) \in [0, 1]$ be a univariate, uniform distribution with $u_i = u_1, \ldots, u_n$, and $i \in N$. Then there exists a copula function C such that

$$C[G_1(u_1), \ldots, G_n(u_n)] = F_n\,[F_1^{-1}(G_1(u_1)), \ldots, F_n^{-1}(G_n(u_n)); \rho_F] \qquad (4.10)$$

where $G_i(u_i)$ are called marginal distributions, F_n is the joint cumulative distribution function, F_i^{-1} is the inverse of F_i, and ρ_F is the correlation structure of F_n.

Equation (4.10) reads: Given are the marginal distributions $G_1(u_1)$ to $G_n(u_n)$. There exists a copula function that allows the mapping of the marginal distributions $G_1(u_1)$ to $G_n(u_n)$ via F^{-1} and the joining of the (abscise values) $F^{-1}(G_i(u_i))$ to a single, n-variate function $F_n\,[F_1^{-1}(G_1(u_1)), \ldots, F_n^{-1}(G_n(u_n))]$ with correlation structure of ρ_F.

If the mapped values $F_i^{-1}(G_i(u_i))$ are continuous, it follows that C is unique. For detailed properties and proofs of equation (4.10), see Sklar (1959) and Nelsen (2006). A short proof is given in the Appendix 4B.

Numerous types of copula functions exist. They can be broadly categorized in one-parameter copulas as the Gaussian copula[1] and the Archimedean copula family, the most popular being Gumbel, Clayton, and Frank copulas. Often cited two-parameter copulas are Student's t, Fréchet, and Marshall-Olkin. Figure 4.2 shows an overview of popular copula functions.

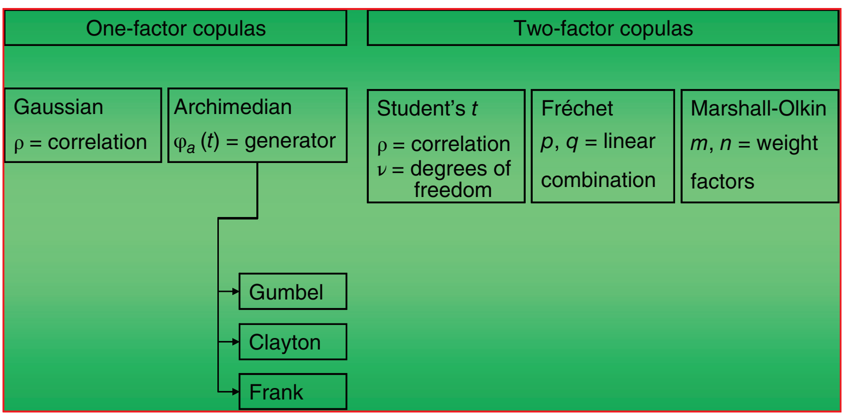

FIGURE 4.2 Popular Copula Functions in Finance

1. Strictly speaking, only the *bivariate* Gaussian copula is a one-parameter copula, the parameter being the copula correlation coefficient. A multivariate Gaussian copula may incorporate a correlation *matrix*, containing various correlation coefficients.

4.3.1 The Gaussian Copula

Due to its convenient properties, the Gaussian copula C_G is among the most applied copulas in finance. In the n-variate case, it is defined

$$C_G[G_1(u_1), \ldots, G_n(u_n)] = M_n[N^{-1}(G_1(u_1)), \ldots, N^{-1}(G_n(u_n)); \rho_M] \quad (4.11)$$

where M_n is the joint, n-variate cumulative standard normal distribution with ρ_M, the $n \times n$ symmetric, positive-definite correlation matrix of the n-variate normal distribution M_n. N^{-1} is the inverse of a univariate standard normal distribution.

If the $G_x(u_x)$ are uniform, then the $N^{-1}(G_x(u_x))$ are standard normal and M_n is standard multivariate normal. For a proof, see Cherubini et al. 2004.

It was David Li (2000) who transferred the copula approach of equation (4.11) to finance. He defined the cumulative default probabilities Q for entity i at a fixed time t, $Q_i(t)$ as marginal distributions. Hence we derive the Gaussian default time copula C_{GD},

$$C_{GD}[Q_i(t), \ldots, Q_n(t)] = M_n\ [N^{-1}(Q_1(t)), \ldots, N^{-1}(Q_n(t)); \rho_M] \quad (4.12)$$

Equation (4.12) reads: Given are the marginal distributions, that is, the cumulative default probabilities Q of entities $i = 1$ to n at times t, $Q_i(t)$. There exists a Gaussian copula function C_{GD}, which allows the mapping of the marginal distributions $Q_i(t)$ via N^{-1} to standard normal and the joining of the (abscise values) $N^{-1}Q_i(t)$ to a single n-variate standard normal distribution M_n with the correlation structure p_M.

More precisely, in equation (4.12) the term N^{-1} maps the cumulative default probabilities Q of asset i for time t, $Q_i(t)$, percentile to percentile a univariate standard normal distribution. So the 5th percentile of $Q_i(t)$ is mapped to the 5th percentile of the standard normal distribution, the 10th percentile of $Q_i(t)$ is mapped to the 10th percentile of the standard normal distribution, and so forth. As a result, the $N^{-1}(Q_i(t))$ in equation (4.12) are abscise (x-axis) values of the standard normal distribution. For a numerical example, see example 4.1 and Figure 4.3. The $N_i^{-1}(Q_i(t))$ are then joined to a single n-variate distribution M_n by applying the correlation structure of the multivariate normal distribution with correlation matrix ρ_M. The probability of n correlated defaults at time t is given by M_n.

We will now look at the Gaussian copula in an example.

EXAMPLE 4.1: DERIVING THE JOINT PROBABILITY OF DEFAULT OF TWO ENTITIES WITH THE GAUSSIAN COPULA

Let's assume we have two companies, B and Caa, with their estimated default probabilities for year 1 to 10 as displayed in Table 4.1.

TABLE 4.1 Default Probability and Cumulative Default Probability of Companies B and Caa

Default Time t	Company B Default Probability	Company B Cumulative Default Probability $Q_B(t)$	Company Caa Default Probability	Company Caa Cumulative Default Probability $Q_{Caa}(t)$
1	6.51%	6.51%	23.83%	23.83%
2	7.65%	14.16%	13.29%	37.12%
3	6.87%	21.03%	10.31%	47.43%
4	6.01%	27.04%	7.62%	55.05%
5	5.27%	32.31%	5.04%	60.09%
6	4.42%	36.73%	5.13%	65.22%
7	4.24%	40.97%	4.04%	69.26%
8	3.36%	44.33%	4.62%	73.88%
9	2.84%	47.17%	2.62%	76.50%
10	2.84%	50.01%	2.04%	78.54%

Default probabilities for investment grade companies typically increase in time, since uncertainty increases with time. However, in Table 4.1 we have two companies currently in distress. For these companies the next few years will be the most difficult. If they survive these next years, their default probability will decrease.

Let's now find the joint default probabilities of the companies B and Caa for any time t with the Gaussian copula function (4.12). First we map the cumulative default probabilities $Q(t)$, which are in columns 3 and 5 in Table 4.1, to the standard normal distribution via $N^{-1}(Q(t))$. Computationally this can be done with = normsinv(Q(t)) in Excel or norminv(Q(t)) in MATLAB. Graphically the mapping $N^{-1}(Q(t))$ can be represented in two steps, which are displayed in Figure 4.3. In the

(*continued*)

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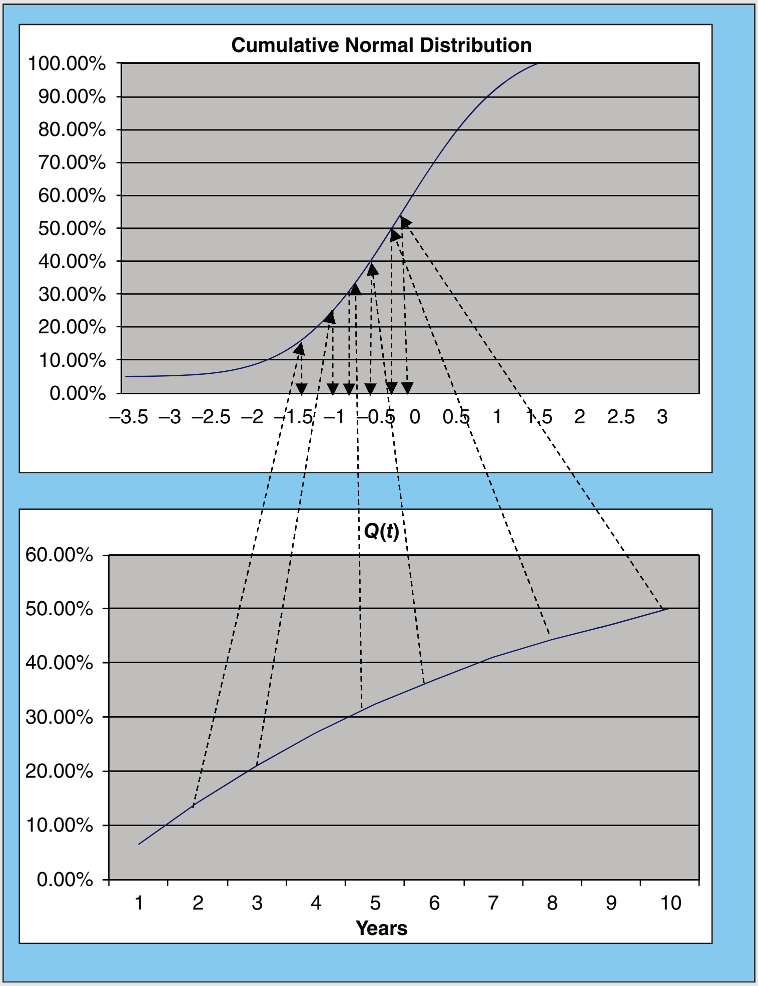

FIGURE 4.3 Graphical Representation of the Copula Mapping $N^{-1}(Q(t))$

lower graph of Figure 4.3, the cumulative default probability of asset B, $Q_B(t)$, is displayed. We first map these cumulative probabilities percentile to percentile to a cumulative standard normal distribution in the upper graph of Figure 4.3 (up arrows). In a second step the

TABLE 4.2 Cumulative Default Probabilities and Corresponding Standard Normal Percentiles of Companies *B* and *Caa*

Default Time t	Company B Cumulative Default Probability $Q_B(t)$	Company B Cumulative Standard Normal Percentiles $N^{-1}(Q_B(t))$	Company Caa Cumulative Default Probability $Q_{Caa}(t)$	Company Caa Cumulative Standard Normal Percentiles $N^{-1}(Q_{Caa}(t))$
1	6.51%	−1.5133	23.83%	−0.7118
2	14.16%	−1.0732	37.12%	−0.3287
3	21.03%	−0.8054	47.43%	−0.0645
4	27.04%	−0.6116	55.05%	0.1269
5	32.31%	−0.4590	60.09%	0.2557
6	36.73%	−0.3390	65.22%	0.3913
7	40.97%	−0.2283	69.26%	0.5032
8	44.33%	−0.1426	73.88%	0.6397
9	47.17%	−0.0710	76.50%	0.7225
10	50.01%	0.0003	78.54%	0.7906

abscise (x-axis) values of the cumulative normal distribution are found (down arrows).

The same mapping procedure is done for company *Caa*; the cumulative default probabilities of company *Caa*, which are displayed in Table 4.1 in column 5, are mapped percentile to percentile to a cumulative standard normal distribution via $N^{-1}(Q_{Caa}(t))$.

We have now derived the percentile to percentile mapped cumulative default probability values of our companies to a cumulative standard normal distribution. These values are displayed in Table 4.2, columns 3 and 5.

We can now use the derived $N^{-1}(Q_B(t))$ and $N^{-1}(Q_{Caa}(t))$ and apply them to equation (4.12). Since we have only $n = 2$ companies B and *Caa* in our example, equation (4.12) reduces to

$$M_2[N^{-1}(Q_B(t)), N^{-1}(Q_{Caa}(t)); \rho] \tag{4.13}$$

From equation (4.13) we see that since we have only two assets in our example, we have only one correlation coefficient ρ, not a correlation matrix ρ_M.

(*continued*)

(*continued*)

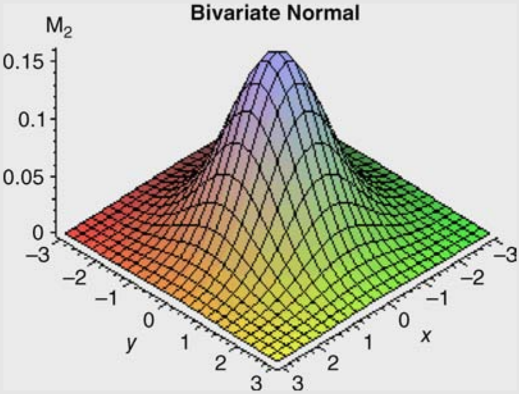

FIGURE 4.4 Bivariate (Noncumulative) Normal Distribution M_2

Importantly, the copula model now assumes that we can apply the correlation structure ρ_M or a single ρ of the multivariate distribution (in our case the Gaussian multivariate distribution M) to the transformed marginal distributions $N^{-1}(Q_B(t))$ and $N^{-1}(Q_{Caa}(t))$. This is done for mathematical and computational convenience.

The bivariate normal distribution M_2 is displayed in Figure 4.4.

The code for the bivariate cumulative normal distribution M can be found on the Internet. It is also displayed at "2-asset default time Copula.xlsm," at www.wiley.com/go/correlationriskmodeling, under "Chapter 4" in Module 1.

We now have all necessary ingredients to find the joint default probabilities of our companies B and Caa. For example, we can answer the question: What is the joint default probability Q of companies B and Caa in the next year assuming a one-year Gaussian default correlation of 0.4? The solution is:

$$Q(t_B \leq 1 \cap t_{Caa} \leq 1) \equiv M(x_B \leq -1.5133 \cap x_{Caa} \leq -0.7118, \rho = 0.4) = 3.44\% \qquad (4.14)$$

where t_B is the default time of company B and t_{Caa} is the default time of company Caa. x_B and x_{Caa} are the mapped abscise values of the bivariate normal distribution, which are derived from Table 4.2.

In another example, we can answer the question: What is the joint probability of company B defaulting in year 3 and company Caa defaulting in year 5? It is

$$Q(t_B \leq 3 \cap t_{Caa} \leq 5) \equiv M(x_B \leq -0.8054 \cap x_{Caa} \leq 0.2557, \rho = 0.4) = 16.93\% \tag{4.15}$$

Equations (4.14) and (4.15) show why this type of copula is also called default time copula. We are correlating the default times of two or more assets t_i. A spreadsheet that correlates the default times of two assets can be found at "2-asset default time Copula.xlsm," at www.wiley.com/go/correlationriskmodeling, under "Chapter 4." The numerical value of 3.44% of equation (4.14) is in cell Q17.

4.3.2 Simulating the Correlated Default Time for Multiple Assets

The preceding example considers only two assets. We will now find the default time for an asset that is correlated to the default times of all other assets in a portfolio using the Gaussian copula.

To derive the default time τ of asset i, τ_i, which is correlated to the default times of all other assets $i = 1,..., n$, we first derive a sample $M_n(\cdot)$ from a multivariate copula [r.h.s. of equation (4.13) in the Gaussian case], $M_n(\cdot) \in [0, 1]$. This is done via Cholesky decomposition, which is explained in Appendix 4A. The sample includes the default correlation via the default correlation matrix ρ_M of the n-variate standard normal distribution M_n. An example of a default correlation matrix was displayed in Chapter 1 in Table 1.3. We equate the sample $(\cdot)$ from M_n, $M_n(\cdot)$ with the cumulative individual default probability Q of asset i at time τ, $Q_i(\tau_i)$. Therefore,

$$M_n(\cdot) = Q_i(\tau_i) \tag{4.16}$$

or

$$\tau_i = Q_i^{-1}(M_n(\cdot)) \tag{4.17}$$

There is no closed-form solution for equations (4.16) or (4.17). To find the solution, we first take the sample $M_n(\cdot)$ and use equation (4.16) to equate

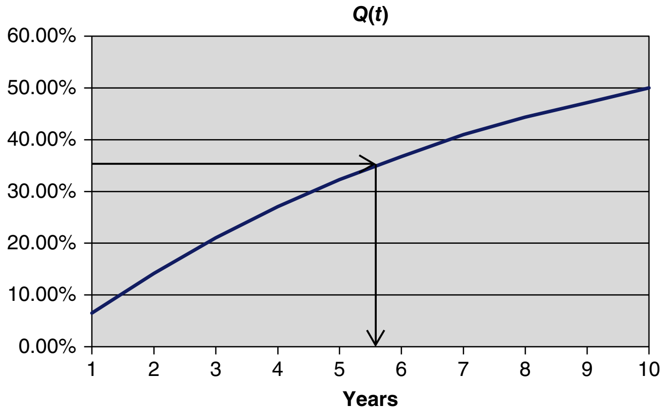

FIGURE 4.5 Finding the Default Time τ of 5.5 Years from Equation (4.16) for a Random Sample of the n-Variate Normal Distribution $M_n(\cdot)$ of 35%
Source: CDO mapping explained.xls.

it to $Q_i(\tau_i)$. This can be done with a search procedure such as Newton-Raphson. We can also use a simple lookup function in Excel.

Let's assume the random drawing from $M_n(\cdot)$ was 35%. We now equate 35% with the market given function $Q_i(\tau_i)$ and find the expected default time of asset i, τ_i. This is displayed in Figure 4.5, where $\tau_i = 5.5$ years. We repeat this procedure numerous times, for example 100,000 times, and average each τ_i of every simulation to find our estimate for τ_i. Importantly, the estimated default time of asset i, τ_i, includes the default correlation with the other assets in the portfolio, since the correlation matrix is an input of the n-variate standard normal distribution M_n.

4.3.3 Finding the Correlated Default Time in a Continuous Time Framework Using Survival Probabilities

In the idealized intensity model framework, we admit a continuous exponential default intensity function $\lambda_i(t)$. The default intensity for entity i is the default probability of entity i for a future time period, assuming the default of the entity i has not occurred until the beginning of the future period. For example, the default intensity from the end of year 6 to the end of year 7 (the seventh year) is the default probability for that time period, conditional on no default until the end of year 6. The default intensity from the end of year 6 to the end of year 7 is higher than the forward default probability for that time period, since when standing at the end of year 6, defaulting in year 7 is higher. Let's look at a numerical example.

EXAMPLE 4.2: FORWARD DEFAULT PROBABILITY AND DEFAULT INTENSITY

Let's assume that the 6-year default probability Q_6 of entity i is 36.73% and the 7-year default probability Q_7 of entity i is 40.97%. What is forward default probability in year 7 and what is the default intensity in year 7?

The forward default probability, which is viewed today at time 0, is

$$q(0)_{6,7} = Q(7) - Q(6) = 40.97\% - 36.73\% = 4.24\%$$

The forward default intensity, viewed at the end of time 6, is

$$\begin{aligned}\lambda(6)_{6,7} &= (Q_7 - Q_6)/(1 - Q_6) \\ &= 40.97\% - 36.73\%/(1 - 36.73\%) \\ &= 6.70\%\end{aligned}$$

$(1 - Q_6)$ represents the survival probability until the end of year 6.

We can find the probability of survival of entity i until t, $\Pr[\tau_i > t]$ as the area under the given default intensity function for which the default time τ_i is bigger than t. This is displayed in Figure 4.6.

The default intensity function in Figure 4.6 is similar to the default probability curve of our bond B in Table 4.1, column 2. However, default intensity functions can have different shapes. For investment grade bonds,

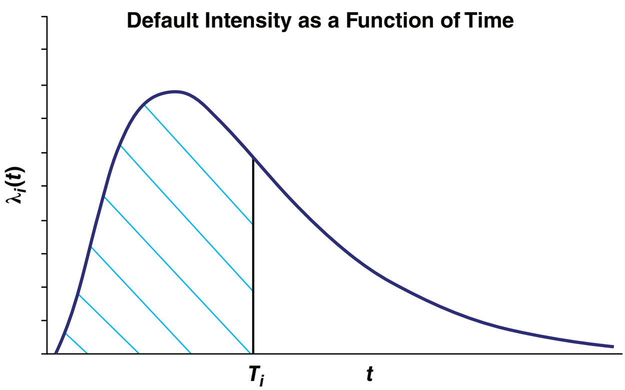

FIGURE 4.6 Survival Probability of Entity i, $\Pr[\tau_i > t]$, which is the Striped Area

they typically increase in time, since uncertainty and therefore default probabilities increase in time.

Formally, we can approximate the survival probability $\Pr[\tau_i > t]$ in Figure 4.6 as

$$\Pr[\tau_i > t] = \exp\left\{ -\int_0^{\tau_i} \lambda_i(t)dt \right\} \tag{4.18}$$

where τ_i is the default time of asset i, which we are looking for. In equation (4.18) we are discounting with the default intensity λ_i to find the survival probability. This methodology was derived by Lando (1998), and Duffie and Singleton (1999). They found that the present value of a risky claim (as a risky bond) can be derived by discounting with the default-adjusted rate. For example, if a Treasury bond is discounted with the risk-free rate r, a risky bond can be discounted with $r + \lambda$, where λ is the default intensity, see Lando (1998) and Duffie and Singleton (1999) for details.

To find τ_i, we first draw a random sample $(\cdot)$ from the n-variate standard normal distribution (M_n), $(M_n(\cdot)$. We then equate the survival probability with the barrier $M_n(\cdot)$; that is,

$$\exp\left\{ -\int_0^{\tau_i} \lambda_i(t)dt \right\} = M_n(\cdot) \tag{4.19}$$

or

$$\int_0^{\tau_i} \lambda_i(t)dt = -\ln[M_n(\cdot)] \tag{4.20}$$

and solve numerically for τ_i. In the case of a constant default intensity λ_i, equation (4.20) simplifies and we find the correlated default time closed form as

$$\tau_i = \frac{-\ln[M_n(\cdot)]}{\lambda_i} \tag{4.21}$$

Importantly, the derived default time of entity i, τ_i, is correlated to the default times of the other assets in the portfolio, since the barrier $M_n(\cdot)$ includes the default correlation via the default correlation matrix of M_n, ρ_M; see equation (4.11).

4.3.4 Copula Applications

There are numerous applications of copula functions in finance.

- One prominent copula application is the valuing of structured products such as CDOs. We will devote a whole chapter, Chapter 5, to the topic of valuing CDOs with copulas.
- A further prominent application of the multivariate Gaussian copula is the modeling of credit rating changes by CreditMetrics. First, a copula dependence coefficient is derived for all asset pairs. This is often derived from equity correlation. A correlated sample from the bivariate copula equation (4.13) is then derived. The sample is then compared to the historical rating percentile to determine whether a rating change occurs. Monte Carlo simulation is conducted to derive the entire rating distribution. This approach has to be applied to all company pairs in question. Hence it is computationally quite intensive. For details see Finger (2009).
- Copulas are also popular tools to model CDSs with counterparty risk. Typically the bivariate Gaussian copula is applied to model the default correlation between the CDS seller and the reference asset; see Kim and Kim (2003), Hamp, Kettunen, and Meissner (2007) and Brigo and Chourdakis (2009).
- Recently copula functions have also been applied outside the credit risk framework. Copulas have been applied to constant maturity spread options, foreign exchange cross options, and basket options; see Qu (2005). Outside of finance, copulas are applied in civil engineering, meteorology, and medicine.

4.3.5 Limitations of the Gaussian Copula

As with any model, the Gaussian copula has limitations with respect to its application to financial reality. The main limitations are discussed next.

4.3.5.1 Tail Dependence In a crisis, correlations typically increase, as studies by Das et al. (2007) and Duffie et al. (2009) show and as we derived in Figure 1.3 in Chapter 1 and in the empirical Chapter 2. Hence it would be desirable to apply a correlation model with high comovements in the lower tail of the joint distribution. Following the tail dependence definition of Joe (1999), a bivariate copula has lower tail dependence if

$$\lim_{y_1 \downarrow 0, y_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(y_1)|(\tau_2 < N_2^{-1}(y_2)] > 0 \tag{4.22}$$

where τ_i is the default time of asset i, y_i is the marginal distribution of asset i, and N^{-1} is the inverse of the standard normal distribution. Equation (4.22) reads: If the functions y_1 and y_2 both approach 0 from above, tail dependence exists if the following holds: The probability of τ_1 being smaller than $N_1^{-1}(y_1)$, given that τ_2 is smaller than $N_2^{-1}(y_2)$, is bigger than 0. However, it can be easily shown that the Gaussian copula has no tail dependence for any correlation parameter ρ: $\lim_{y_1 \downarrow 0, y_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(y_1)|(\tau_2 < N_2^{-1}(y_2)] = 0$, $\rho \in \{-1, 1\}$. In contrast, the Student's t copula, equation (4.10) with F_n being the n-variate Student's t distribution and F^{-1} being the inverse of F, satisfies equation (4.22) for any $\rho \in \{-1, 1\}$. Hence it may be more desirable to apply the Student's t copula in financial crisis modeling. Figure 4.7 (a) to (d) shows several copula scatter plots.

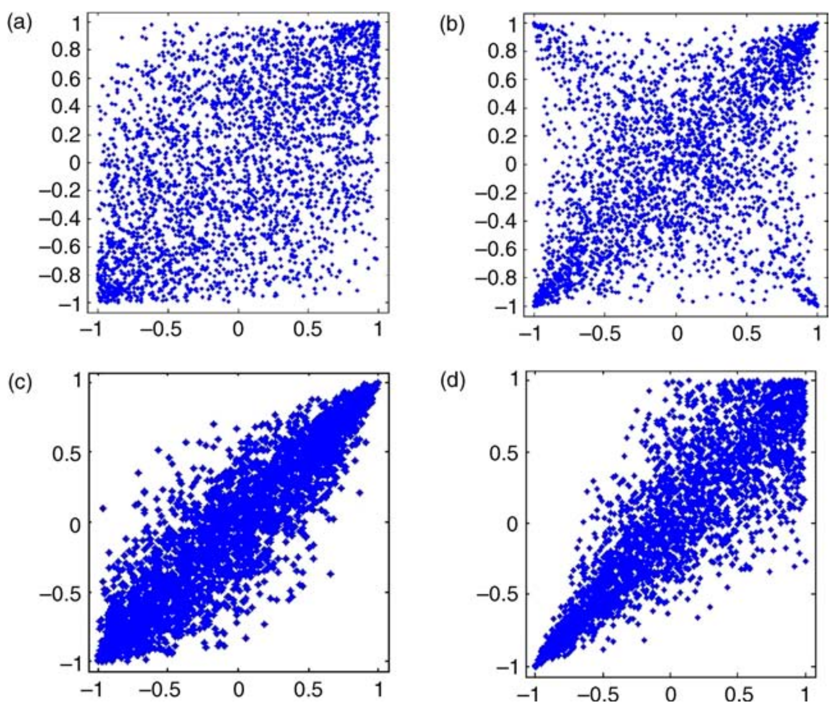

FIGURE 4.7 Scatter Plots of Different Copula Models: (a) Bivariate Gaussian Copula with $\rho = 0.5$; (b) Bivariate Student's t Copula with $\rho = 0.5$, dof Degrees of Freedom $= 1$; (c) Bivariate Gumbel Copula with $\alpha = 4$; (d) Bivariate Clayton Copula with $\alpha = 5$

As seen in Figure 4.7 (c), the Gumbel copula exhibits high tail dependence, especially for negative comovements. Since correlations typically increase when asset prices decrease, as we verified in Chapter 2, the Gumbel copula might also be a good correlation approach for financial modeling.

4.3.5.2 Calibration A further criticism of the Gaussian copula is the difficulty to calibrate it to market prices. In practice, typically a single correlation parameter (not a correlation matrix) is used to model the default correlation between any two entities in a CDO; see Chapter 6 for details. Conceptually this correlation parameter should be the same for the entire CDO portfolio. However, traders randomly alter the correlation parameters for different tranches in order to derive desired tranche spreads. Traders increase the correlation for extreme tranches such as the equity tranche or senior tranches, referred to as the correlation smile. This is similar to the often cited implied volatility smile in the Black-Scholes-Merton model. Here traders increase the implied volatility, especially for out-of-the-money puts but also for out-of-the-money calls, to increase the option price. We will discuss this limitation further in Chapter 6, especially section 6.6.

Another criticism of the Gaussian copula is that for certain parameter constellations it may not be possible to imply a market CDO tranche spread for a correlation parameter between 0 and 1. Kherraz (2006) tests the large homogeneous portfolio (LHP) version of the Gaussian copula (see Chapter 6, section 6.1, and Chapter 7, section 7.1) and finds that the lowest 40% and highest 20% of losses of the equity tranche cannot be explained by the model. However, Kherraz uses a fairly high default probability of 40% in his study and does not mention the frequency or timing of the occurrences. Finger (2009) tests the calibration of the LHP model with base correlation, a correlation with zero attachment point, which is bootstrapped from the implied correlation (see JPMorgan 2004 for details on base correlation). Finger finds calibration failures for just 20 days for the iTraxx and 21 days for the CDX indes before July 2007. He finds no calibration failures after July 2007.

Several other studies, such as Hull and White (2004), Andersen and Sidenius (2004), and Burtschell, Gregory, and Laurent (2008), test the one-factor Gaussian copula as well as other copulas such as Marshall-Olkin, Clayton, or double-t. None of the studies finds any calibration failures for these copulas.

4.3.5.3 Risk Management A further criticism of the copula approach is that the copula model is static and consequently allows only limited risk management; see Finger (2009) or Donnelly and Embrechts (2010). The original copula models of Vasicek (1987) and Li (2000) and several extensions of the models such as Hull and White (2004) or Gregory and Laurent (2004) do have a one-period time horizon (i.e., are static). In particular, there is no stochastic process for the critical underlying variables default intensity and default correlation. However, even in these early copula formulations, back-testing and stress-testing the variables for different time horizons can give valuable sensitivities; see Whetten and Adelson (2004) and Meissner, Hector, and Rasmussen (2008).

In addition, the copula variables can be made a function of time as in Hull et al. (2005). However, this still does not create a fully dynamic stochastic process with drift and noise, which allows flexible hedging and risk management.

In the following section we discuss further bottom-up financial correlation models, the contagion correlation approach.

4.4 CONTAGION CORRELATION MODELS

The basic idea in contagion correlation modeling is that the default intensity of an entity is a function of the default of another entity. Hence contagion default modeling incorporates counterparty risk (i.e., the direct impact of a defaulting entity on the default intensity of another entity).

Contagion default modeling was pioneered by Davis and Lo (1999, 2001) and Jarrow and Yu (2001). Davis and Lo model the latent variable Z of entity i, Z_i with equation

$$Z_i = X_i + (1 - X_i)\left(1 - \prod_{\substack{i=1 \\ i \neq j}}^{n} (1 - X_j K_{ij})\right) \tag{4.23}$$

where

Z_i: binomial default indicator variable of entity i
X_i and X_j: Bernoulli random variable[2] of entity i and j, respectively
K_{ij}: contagion variable (i.e., the degree of with the default of j impacts the default intensity of entity i)

Let's understand equation (4.23). Z_i is a binomial default indicator variable of entity i. This means if $Z_i = 1$, entity i defaults, and if $Z_i = 0$,

2. A Bernoulli random variable can take values of 0 and 1 with certain probabilities for each value. See http://mathworld.wolfram.com/BernoulliDistribution.html for details.

entity i survives. Entity i can default directly, when it is not being affected by entity j. In this case the Bernoulli random variable $X_i = 1$. Entity i can also default indirectly (i.e., when it is affected by the default of entity j). In this case, the Bernoulli random variable $X_j = 1$. The degree of infection is modeled with the Bernoulli random contagion variable K_{ij}. Formally,

$$\begin{aligned} \Pr(X_i = 1) &= p \\ \Pr(X_j = 1) &= q \\ \Pr(K_{ij} = 1) &= r \end{aligned} \tag{4.24}$$

where p, q, and r and input parameters that are $\in [0, 1]$. In a dynamic setting, the persistence of the contagion variable K_{ij} may be modeled as an exponentially decreasing function of time t. A parameter $g(t)$ (gravity) determines the degree of decreasing contagion in t; that is, $K_{ij}(t) = e^{-g(t)t}$ where $g(t) > 0$ and $\partial g/\partial t < 0$.

Jarrow and Yu (2001) introduce default intensity contagion with a set of linear equations:

$$\lambda_A(t) = a_1 + a_2\, 1_{\{\tau_B \le t\}} \tag{4.25}$$

$$\lambda_B(t) = b_1 + b_2\, 1_{\{\tau_A \le t\}} \tag{4.26}$$

where λ_X is the default probability of X, and a_1, a_2, b_1, and b_2 are parameters that are bigger than zero and have to be calibrated. τ_X is the default time of entity X. In equation (4.25) 1 is an indicator variable. 1 takes the value 1 if the default time of entity B, τ_B, is smaller than a certain time t. We can simulate τ_B randomly, for example with a copula model, which we have done in equations (4.14) or (4.17). Multiple sampling will result in many outcomes of the experiment $\tau_B \le t$. For example, if 10% of the outcomes are that $\tau_B \le t$, then the probability of $\tau_B \le t$ (i.e., the probability of the entity B defaulting before t) is 10%. From equation (4.25) we see that the higher the probability of $1_{\{\tau_B \le t\}}$, the higher the default probability of A, λ_A. The same logic applies to equation (4.26).

Introducing symmetric contagion among all entities creates the problem of circularity, which Jarrow and Yu (2001) call "looping defaults." In this case, the construction of a joint distribution is rather complex. Jarrow and Yu solve the problem by introducing the concept of asymmetric dependence; that is, the default of primary entities impacts the default intensity of secondary entities, but not vice versa. In this case, the joint default distribution conveniently becomes the product of the individual primary default times.

Contagion correlation modeling can be combined with conditionally independent default (CID) correlation modeling. These combinations are discussed in Chapter 6, section 6.5.

4.5 SUMMARY

In this chapter, we evaluated correlation approaches that were especially designed to model financial correlations. We concentrated on bottom-up approaches, which collect information, quantify it, and then aggregate the information to derive an overall correlation result.

One of the most widely applied correlation concepts is **correlating Brownian motions,** introduced by Steven Heston (1993). In the Heston model, the Brownian motions of two variables are correlated with a simple equation. The model was originally designed to replicate the negative correlation between stock returns and volatility. However, the model has been applied to other financial relationships such as stochastic interest rates and stochastic volatility, as in the popular stochastic alpha beta rho (SABR) model; to stochastic interest rates and stochastic default intensities; and to many more relationships. Altogether, the Heston approach is mathematically rigorous, dynamic, and flexible. Therefore it is one of the most valuable and applied correlation models in finance.

The **binomial correlation model** of Douglas Lucas (1995) models by design binomial events, for example default or no default. The binomial model is a special limiting case of the Pearson correlation model. Whereas in the Pearson model the inputs are sets of variables, in the binomial model the inputs are scalars. Since the binomial correlation model is a special case of the Pearson correlation model, the significant shortcomings of the Pearson correlation approach for financial modeling also apply to the binomial correlation model.

Copula correlations were first enthusiastically embraced, but then fell into disgrace when the global financial crisis hit in 2007. Copulas go back to Abe Sklar in 1959 and were introduced to finance by Oldrich Vasicek (1987) and David Li (2000). Copula functions simplify statistical problems. They allow the joining of multiple univariate distributions to a single multivariate distribution. In this way copulas can evaluate n correlation functions with a single (although n-dimensional) function. Many different types of copulas and extensions exist. The Gaussian copula is the most popular one due to its simplicity and convenient programming.

The bottom line is that copulas are rigorous statistical approaches that can have value in finance. However, severe limitations of copulas for finance exist: (1) Most copulas, especially the Gaussian copula, have low tail dependence. (2) Calibration to market prices is problematic, especially for one-factor copulas. (3) Copulas are principally static; however, they can be extended to be dynamic such as in Hull and White (2005) and Albanese et al. (2011).

Contagion correlation modeling, pioneered by Davis and Lo (1999, 2001) and Jarrow and Yu (2001), is based on the idea that the default of one entity impacts the default intensity of another entity. The degree of the impact can be modeled with an exponentially decreasing function of time. However, introducing symmetric contagion among all entities creates the problem of circularity. In this case, the construction of a joint distribution is rather complex. One solution is to model asymmetric dependence; that is, the default of primary entities impacts the default intensity of secondary entities, but not vice versa.

APPENDIX 4A: CHOLESKY DECOMPOSITION

The Gaussian copula model creates a multidimensional normal distribution from standard normal marginal distributions. Monte Carlo simulations derive samples from the distribution, which are compared with the default threshold. The standard procedure to derive correlated samples from a multivariate normal distribution is Cholesky decomposition. We will outline the method here.

Given is the n-dimensional correlation matrix Σ[3].

$$\Sigma = \begin{bmatrix} c_{11} & c_{12} \ldots c_{1n} \\ c_{21} & c_{22} \ldots c_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ c_{n1} & c_{n2} \ldots c_{nn} \end{bmatrix}$$

We decompose Σ into $\Sigma = M\ M^T$, where M is a special symmetric, positive definite, lower triangular matrix, and M^T is the transpose of M[4]:

3. The reader may study some basic matrix algebra at the spreadsheet "Matrix primer.xlsx," at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."
4. The matrix transpose A^T is the matrix obtained by exchanging A's rows and columns. Hence if we have a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, it follows that $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$. See the spreadsheet "Matrix primer.xlsx," sheet "Matrix Transpose," at www.wiley .com/go/correlationriskmodeling, under "Chapter 1."

$$\begin{bmatrix} c_{11} & c_{12}\ldots c_{1n} \\ c_{21} & c_{22}\ldots c_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ c_{n1} & c_{n2}\ldots c_{nn} \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & \ldots 0 \\ m_{21} & m_{22} & \ldots 0 \\ \cdot & \cdot & \cdot \\ m_{n1} & m_{n2} & \ldots m_{nn} \end{bmatrix} \times \begin{bmatrix} m_{11} & m_{12}\ldots m_{nn} \\ 0 & m_{22}\ldots m_{2n} \\ \cdot & \cdot \quad\quad \cdot \\ 0 & 0 \quad \ldots m_{nn} \end{bmatrix}$$

From the decomposed matrix, we can find equations for m_{ij} (see the following example). We then generate uncorrelated random samples from a standard normal distribution ε, [ε = normsinv(rand()) in Excel and norminv(rand) in MATLAB] and find correlated random values x_i from $x_i = M\,\varepsilon_i$.

Let's look at an example of Cholesky decomposition for three assets.

Example: Cholesky Decomposition for Three Assets

Given is the correlation matrix Σ with elements c_{11} to c_{33}, which we decompose into $\Sigma = M\ M^T$,

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \times \begin{bmatrix} m_{11} & m_{21} & m_{31} \\ 0 & m_{22} & m_{32} \\ 0 & 0 & m_{33} \end{bmatrix}$$

We can find the equations for m_{ij} from matrix multiplication:

$$c_{11} = m_{11} \times m_{11} \ \rightarrow m_{11} = \sqrt{c_{11}}$$
$$c_{21} = m_{21} \times m_{11} \ \rightarrow m_{21} = c_{21}/m_{11}$$
$$c_{22} = m_{21} \times m_{21} + m_{22} \times m_{22} \ \rightarrow m_{22} = \sqrt{c_{22} - (m_{21})^2}$$
$$c_{31} = m_{31} \times m_{11} \ \rightarrow m_{31} = c_{31}/m_{11}$$
$$c_{32} = m_{31} \times m_{21} + m_{32} \times m_{22} \ \rightarrow m_{32} = (c_{32} - m_{31} \times m_{21})/m_{22}$$
$$c_{33} = m_{31} \times m_{31} + m_{32} \times m_{32} + m_{33} \times m_{33} = \ \rightarrow m_{33} = \sqrt{c_{33} - (m_{31})^2 - (m_{32})^2}$$

We now generate uncorrelated random samples from a standard normal distribution ε (ε = normsinv(rand()) in Excel) and find correlated random values x_i from $x_i = M\,\varepsilon_i$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \times \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

Hence the values for the three correlated random samples x_i are

$$x_1 = m_{11} \times \varepsilon_1$$
$$x_2 = m_{21} \times \varepsilon_1 + m_{22} \times \varepsilon_2$$
$$x_3 = m_{31} \times \varepsilon_1 + m_{32} \times \varepsilon_2 + m_{33} \times \varepsilon_3$$

We can now apply Monte Carlo simulation (i.e., simulate the equations for x_i multiple times to derive robust values for the x_i).

Try to solve the numerical Cholesky decomposition end-of-chapter problem, number 16.

APPENDIX 4B: A SHORT PROOF OF THE GAUSSIAN DEFAULT TIME COPULA

Given are the cumulative default probabilities Q of entities $i = A, B,...,n$ at various times t, $Q_i(t_i)$. There exists a copula function C:

$$C[Q_A(t_A), Q_B(t_B), \ldots, Q_n(t_n)] = M_n(t_A, t_B, \ldots, t_n) \tag{4B.1}$$

where M_n is an n-dimensional Gaussian distribution function.

Proof:

Let R_i, $i = A, B,..., n$ be a uniform random variable. We define

$$\Pr[R_A \leq Q_A(t_A), R_B \leq Q_B(t_B), \ldots, R_n \leq Q_n(t_n)] \tag{4B.2}$$

Applying $R_i \leq Q_i(t_i) = Q_i^{-1}(R_i) \leq t_i$ to equation (4B.2), we derive

$$\Pr[Q_A^{-1}(R_A) \leq t_A, Q_B^{-1}(R_B) \leq t_B, \ldots, Q_n^{-1}(R_n) \leq t_n] \tag{4B.3}$$

Let T_i be the abscise value of the default distribution $Q_i^{-1}(R_i)$. Hence

$$\Pr[T_A \leq t_A, T_A \leq t_B, \ldots, T_n \leq t_n] \tag{4B.4}$$

For the n-dimensional Gaussian distribution M_n, equation (4B.4) is

$$M_n(t_A, t_B, \ldots, t_n)$$

PRACTICE QUESTIONS AND PROBLEMS

1. The original Heston (1993) model correlates the Brownian motion of which two financial variables? What is the most significant result of the original Heston model?

2. To create negative correlation between asset 1 and asset 2 in the Heston (1993) model, what value does the correlation coefficient α take in equation $dz_1(t) = \alpha\, dz_2(t) + \sqrt{1-\alpha^2} dz_3(t)$?
3. The Heston model is one of the most widely applied correlation models in finance. Why?
4. What is the difference between the Pearson correlation model and the binomial correlation model of Lucas (1995)?
5. What are the limitations of the binomial correlation model of Lucas (1995)?
6. What is the basic principle of the copula correlation model?
7. Why is the Gaussian copula model the most popular copula model in finance?
8. What does "In the copula mapping process, the marginal distributions are preserved" mean?
9. Given are the marginal default probabilities 5% for asset 1 and 7% for asset 2. If the Gaussian correlation coefficient is 0.3, what is the joint probability of default, assuming asset 1 and asset 2 are jointly bivariately distributed?
10. Given are the 5-year default probability of entity *i* of 40% and the 6-year default probability of entity *i* of 45%. What is the forward default probability in year 6 and what is the default intensity in year 6?
11. What are the limitations of the Gaussian copula for financial applications?
12. Since the Gaussian copula has low tail dependence, which other copulas seem more suitable to model financial correlations?
13. Can the copula model be blamed for the great recession of 2007 to 2009?
14. What is the basic idea in contagion correlation models?
15. Name the limitations of contagion models.
16. Derive correlated samples x_1, x_2, and x_3 from the correlation matrix $\Sigma = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.4 & 1 \end{bmatrix}$ applying Cholesky decomposition (see Appendix 4A for details).

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CHAPTER 5

Valuing CDOs with the Gaussian Copula—What Went Wrong?

Take risks: if you win, you will be happy; if you lose, you will be wise.

—Author Unknown

When the global financial crisis hit in 2007 to 2009, the Gaussian copula was widely blamed for the crisis, especially when applied to valuing collateralized debt obligations (CDOs); see "Recipe for Disaster: The Formula That Killed Wall Street" (*Wired*, 2009), "Wall Street Wizards Forgot a Few Variables" (*New York Times*, 2009), or "The Formula That Felled Wall Street" (*Financial Times*, 2009). In this chapter we analyze the pricing methodology of CDOs and evaluate whether the Gaussian copula is to blame. Let's first look at some CDO basics.

5.1 CDO BASICS—WHAT IS A CDO? WHY CDOs? TYPES OF CDOs

Before we evaluate whether the Gaussian copula is to be blamed for the global financial crisis of 2007 to 2009, let's first discuss some basic properties of CDOs.

5.1.1 What Is a CDO?

A collateralized debt obligation (CDO) is a financial structure in which the credit risk from a pool of securities is transferred from one counterparty, the originating bank, to another, the investor. The investor can invest in different CDO tranches. Each tranche has a different degree of credit risk. The credit risk is distributed with a waterfall principle: If losses accumulate and the detachment level of a tranche is breached, additional credit losses flow into the adjacent higher tranche. A CDO is typically arranged by a special purpose vehicle (SPV), which is AAA rated to minimize counterparty risk.

5.1.2 Why CDOs?

There are three main parties in a CDO:

1. The originator (or protection buyer), who transfers the credit risk.
2. The investor, who assumes the credit risk.
3. The special purpose vehicle (SPV), which manages the CDO.

The motivation for the originator is naturally to transfer the credit risk, which improves his credit rating, frees credit lines, reduces regulatory capital, and lowers funding cost. The motivation for the investor is to receive high yields. The motivation for the SPV is fee income.

CDOs include several sound financial properties:

- *Diversification.* Since typically 125 assets are in a CDO, a skilled originator will choose assets with a low correlation to achieve high diversification benefits (see Chapter 1, section 1.3.1, "Investments and Correlation").
- *Subordination.* This means that mezzanine and higher tranches are protected by lower tranches, since lower tranches absorb default losses from the underlying basket of credits first.
- *Overcollateralization.* Typically the assets in a CDO have a higher value than the liabilities that the SPV owes to the investors. This overcollateralization adds an additional element of protection for investors.

The drawback of CDOs lies in their relative pricing complexity. We have to find the default probability function with respect to time of 125 assets for the duration of the CDO, which can be up to 10 years. This alone is difficult to estimate. Furthermore, we have to correlate the default functions of the 125 assets! This is where the copula function comes in.

First let's have a look at where the CDO market is today.

From Table 5.1 we observe that the CDO market is recovering nicely since 2009; however, the CDO issuance is far below the record 2006 levels.

TABLE 5.1 Global CDO Issuance in USD Millions

Year	Total CDO Issuance (in USD millions)
2003	86,629.8
2004	157,820.7
2005	251,265.3
2006	520,644.6
2007	481,600.7
2008	61,886.8
2009	4,336.0
2010	8,665.1
2011	31,131.3
2012	45,399.8
until 7/15/2013	44,403.0

Source: www.Sifma.com.

5.1.3 Types of CDOs

There are three main types of CDOs, which are displayed in Figure 5.1.

In a cash CDO, the originating bank sells assets to the SPV, which then creates tranches. Each tranche is exposed to a certain degree of default risk. The first losses from asset defaults flow into the equity tranche. Further losses flow into the next higher mezzanine tranche, and so on. Figure 5.2 shows the cash flows of a typical cash CDO.

In a synthetic CDO, assets are not sold from the originating bank to the SPV, but the SPV assumes the credit risk via selling credit default swaps (CDSs). The SPV receives the CDS spreads from the originating bank and the cash from the investor, and invests these cash flows into risk-free assets. A synthetic CDO is displayed in Figure 5.3.

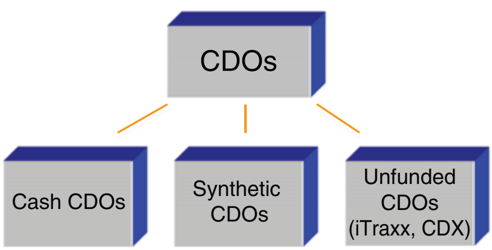

FIGURE 5.1 Main Types of CDOs

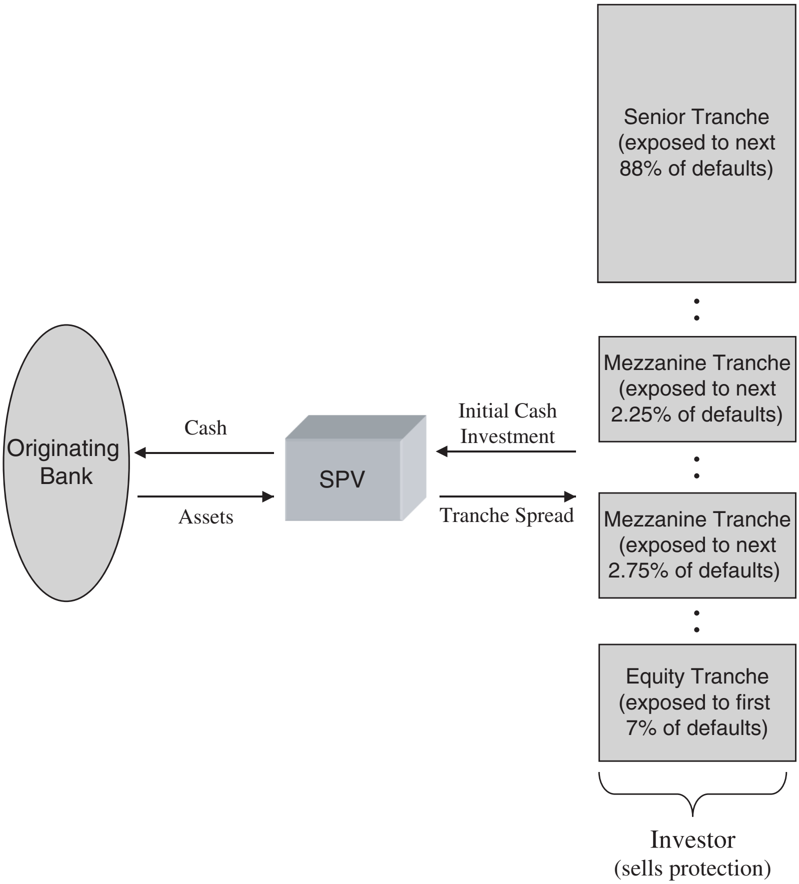

FIGURE 5.2 A Cash CDO

A third type of CDOs are unfunded CDOs such as the family of CDX indexes or the iTraxx indexes, also called credit default swap indexes. The most popular CDX index is the CDX.NA.IG, which references 125 investment grade CDSs in North America. The most popular iTraxx index is the iTraxx Europe, which references 125 investment grade CDSs in Europe. Importantly, the CDX and iTraxx indexes are unfunded; therefore no initial principal amount is exchanged between the buyer (investor) and the seller. Hence the trading of the CDX and iTraxx indexes is similar to buying and selling futures contracts. The cash flows of an unfunded, tranched CDO are displayed in Figure 5.4.

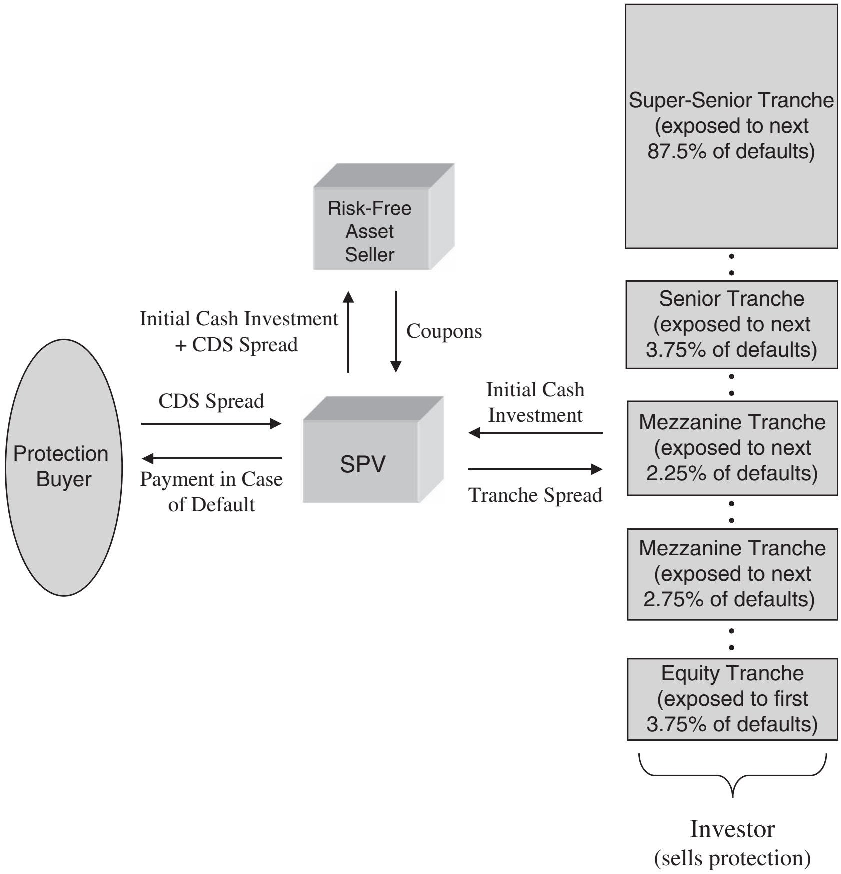

FIGURE 5.3 A Synthetic CDO

5.2 VALUING CDOs

There are three main input factors when valuing a CDO:

1. The default probability of each of the 125 assets.
2. The default correlation between the 125 assets in the portfolio.
3. The recovery rate in case of default.

Let's discuss briefly how to derive the default probability function before we concentrate on the most significant element, the default correlation.

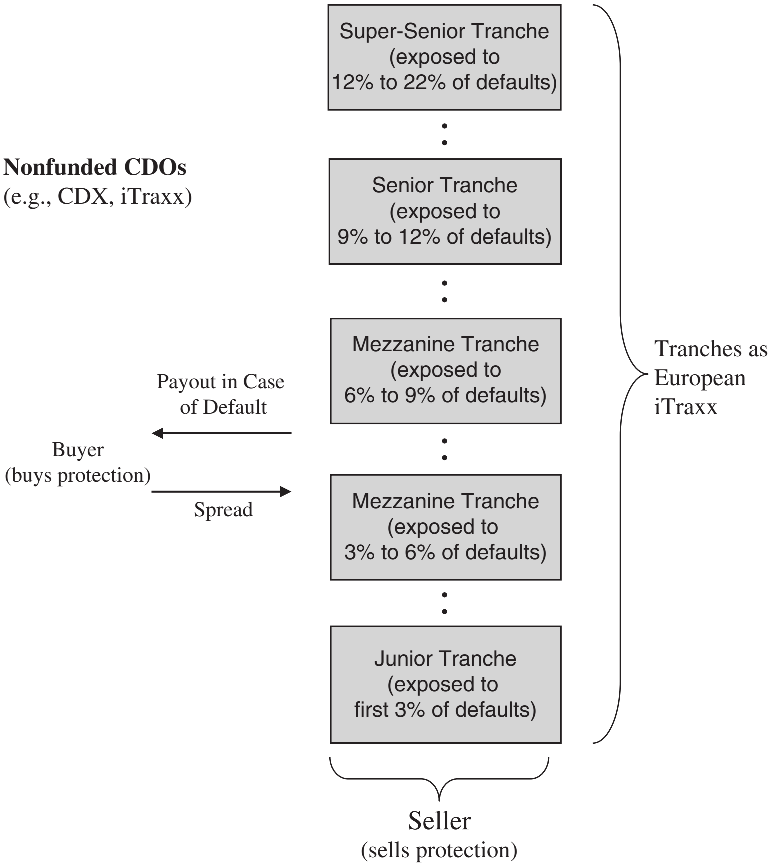

FIGURE 5.4 A Tranched, Nonfunded CDO Such as the iTraxx

5.2.1 Deriving the Default Probability for Each Asset in a CDO

Most investment banks, hedge funds, and SPVs use an extension of the seminal Merton 1974 model to derive the default probability for each asset in a CDO. Let's calculate this default probability.

In 1973, Fischer Black and Myron Scholes, and separately Robert Merton, created their famous Black-Scholes-Merton (BSM) option pricing model. The well-known equation for a call is

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{5.1}$$

where $d_1 = \dfrac{\ln\left(\dfrac{S_0}{Ke^{-rT}}\right) + \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

C: call price
S_0: current stock price
N: cumulative standard normal distribution
K: strike price
r: continuously compounded risk-free interest rate
T: option maturity, measured in years
σ: implied volatility of S

One year later, in 1974, Robert Merton transferred the option framework of equation (5.1) to corporate finance. He applied the equation Equity = Assets – Liabilities, and argued that the equity value of a company has similar properties as a call: If the asset value of a company increases, equity increases with unlimited upside potential. In addition, the value of equity is asymmetric, since it can only go to zero. This is the case when the asset value drops below the debt value, which is the case of default. With this rationale, Merton derived

$$E = V_0\,N(d_1) - D\,e^{-rT}N(d_2) \tag{5.2}$$

with $d_1 = \dfrac{\ln\left(\dfrac{V_0}{\mathrm{D}e^{-rT}}\right) + \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

where

V_0: current asset value of the company
D: debt of the company
σ: implied volatility of V
T: time to maturity of debt D

Other variables are defined as in equation (5.1).

Note that equations (5.1) and (5.2) are mathematically identical. Just the variables are redefined.

The asymmetric payoff of equity implies, as is the case with a call, that there is time value of equity, as seen in Figure 5.5.

Figure 5.5 outlines the relationship between a company's equity value and its asset value at a certain point in time before debt maturity. If we assume that the asset value grows with a certain rate r, we derive the probability of

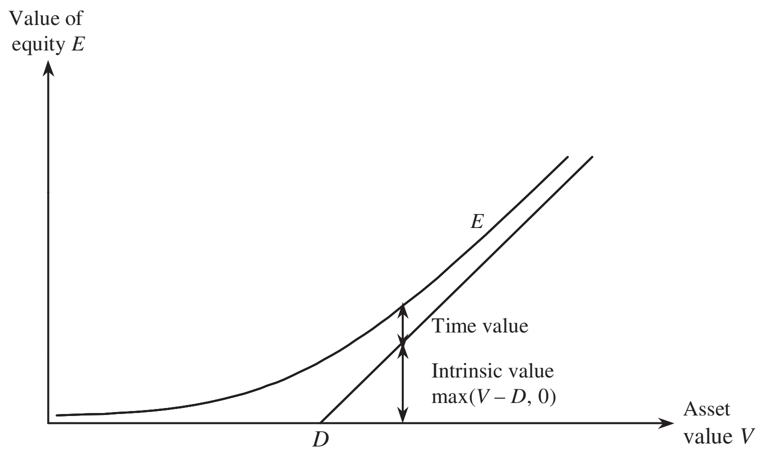

FIGURE 5.5 Equity Value with Respect to Asset Value in the Merton 1974 Model

default as the probability of the asset value being smaller than the value of debt at debt maturity T, as seen in Figure 5.6.

In Figure 5.6, using the terminology of Moody's KMV, EDF is the expected default frequency (i.e., the default probability), and DD is the distance to default, which is a representation of the risk-neutral d_2 of equation (5.2). DD is the difference between the expected asset value and debt value at debt maturity T. There is an inverse relationship between EDF and DD.

Importantly, in equation (5.1) the probability of exercising a call option at option maturity T is $\text{Prob}(S_T > K) = N(d_2)$. The probability of not exercising the call option is $1 - N(d_2) = N(-d_2)$. In analogy, the probability of the asset value V being smaller than the debt value D at time T, which means default at T, follows from the Merton 1974 model of equation (5.2) as $N(-d_2)$. Hence, the default probability in the Merton model is derived conveniently with a closed form solution as $N(-d_2)$.[1]

The ingenious Merton 1974 model outlines the principles of a company's default using structural properties such as asset and debt. The main limitations of the model are that only one form of debt D is modeled and that default can occur only at debt maturity T. Naturally, numerous extensions of the model have been created to bring the model in line with the complexities of reality. In particular:

- The first passage time models of Black and Cox (1976); Kim, Ramaswamy, and Sundaresan (1993); Longstaff and Schwartz (1995);

1. For an analysis of this property, see Meissner (2007).

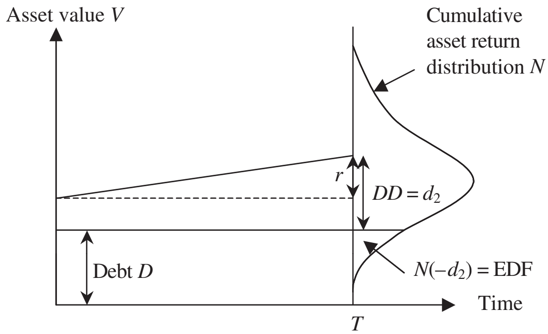

FIGURE 5.6 Default Probability EDF $= N(-d_2)$ in the Merton Model for Asset Value $V <$ Debt D

and Briys and de Varenne (1997) evaluate the default probability before debt maturity T by introducing an exogenous, continuous default barrier. Once the asset value falls below the barrier, default occurs. Hence the first time passage models effectively turn the European-style model of equation (5.2) into an American-style model.

- The asset return distribution at debt maturity T does not grow with the risk-free rate r and is not assumed normally distributed (see Figure 5.6). Instead a real-world historical asset growth rate and asset distribution is applied. For example, Moody's KMV database contains 30 years of information on over 6,000 public and 150,000 private company default events.
- The debt value is not considered constant as in Figure 5.6. Instead, empirical data is used to project a realistic increase or decrease in debt.
- Other default criteria besides asset and debt value are taken into consideration, such as liquidity risk and systemic risk, as well as company-specific data (product line, competition, quality of management, etc.).

The Merton model, which we just discussed, is called a *structural approach*, since it uses the capital structure of the entity as inputs to derive the default probability. A different way to determine the default probability of an entity is the *reduced form approach*. Here market prices such as bond prices or credit default swap prices are the inputs to derive the default probabilities; see Jarrow and Turnbull (1995); Jarrow, Lando, and Turnbull (1997); and Duffie and Singleton (1999). The approach is called reduced form since it does not apply the capital structure of an entity as inputs.

Let's now discuss the critical aspect of the Gaussian copula with respect to valuing CDOs.

5.2.2 Deriving the Default Correlation of the Assets in a CDO

In the previous section, we derived the individual default probability λ of each asset i, λ_i, in the CDO. The probability of default of an asset λ_i is now mapped via

$$N^{-1}(\lambda_i) \tag{5.3}$$

where N^{-1} is the inverse of a standard normal distribution (=normsinv(λ_i) in Excel, norminv(λ_i) in MATLAB). Equation (5.3) maps the default probabilities to a standard normal distribution. For example, if $\lambda_i = 5\%$, then $N^{-1}(0.05) = -1.645$, which is the x-axis value of the 5th percentile of a standard normal distribution.[2] This procedure allows a comparison of the default probabilities with samples from an n-variate normal asset distribution M_n.

We will now determine the *default threshold*. This is the value that, when breached, will constitute default of the entity or asset in question. To derive the threshold, typically the popular Gaussian copula model is applied. We slightly rewrite the right side of equation (4.12) and derive the default threshold as

$$M_n[N^{-1}(u_1), \ldots, N^{-1}(u_n); \rho_M] \tag{5.4}$$

M_n is the n-variate Gaussian distribution, N^{-1} is again the inverse of a standard normal distribution, and u_x is a uniform random vector $u_x \varepsilon$ [0, 1]; =rand() in Excel/VBA or randn() in MATLAB. ρ_M is the asset correlation matrix. An example of an asset correlation matrix is shown in Table 5.2.

We now look at a certain time frame t and derive the mapped default probability of asset i at time t, $N^{-1}(\lambda_{i,t})$, following equation (5.3). We also derive M_n in equation (5.4) for a certain time t, $M_{n,t}$, and then derive a sample $M_{n,t}(\cdot)$ using Cholesky decomposition, which was explained in Appendix 4A of Chapter 4. If the mapped individual default probability $N^{-1}(\lambda_{i,t})$ is bigger than the threshold sample $M_{n,t}(\cdot)$, default of asset i occurs and vice versa. Formally:

$$\tau_{i,t} = 1_{\left\{N^{-1}(\lambda_{i,t}) > M_{n,t}(\cdot)\right\}} \tag{5.5}$$

In equation (5.5), 1 is an indicator variable. That is, 1 assumes the value 1 if $N^{-1}(\lambda_{i,t}) > M_{n,t}(\cdot)$ and zero otherwise. We now perform Monte Carlo simulations; that is, we derive multiple results (e.g., 100,000) of equation (5.5) and average those results. This gives us a certain probability of default of

2. See Chapter 4, section 4.3 for details of copula mapping.

TABLE 5.2 (Fictitious) Asset Correlation Matrix Underlying Figure 5.7

Asset Correlation Matrix

1	0.15	0.15	0.15	0.15	0.15	0.05	0.05	0.05	0.05
0.15	1	0.15	0.15	0.15	0.05	0.05	0.05	0.05	0.05
0.15	0.15	1	0.15	0.15	0.15	0.15	0.05	0.05	0.05
0.15	0.15	0.15	1	0.15	0.15	0.05	0.05	0.05	0.05
0.15	0.15	0.15	0.15	1	0.15	0.15	0.05	0.05	0.05
0.15	0.05	0.15	0.15	0.15	1	0.15	0.05	0.05	0.05
0.05	0.05	0.15	0.05	0.15	0.15	1	0.15	0.05	0.05
0.05	0.05	0.05	0.05	0.05	0.05	0.15	1	0.15	0.05
0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.15	1	0.05
0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	1

asset i at time t. For example, if the average result of equation (5.5) for a certain asset i in the CDO is 0.1, then the default probability of this asset is 10% at time t. We apply equation (5.5) for all n assets in the CDO. This gives us the *correlated* default distribution of all assets in the CDO. The defaults in the distribution are correlated since the threshold $M_{n,t}(\cdot)$ includes the correlation of the defaults via the correlation matrix ρ_M. Figure 5.7 shows a possible default distribution generated by the Gaussian copula model.

In Figure 5.7, the defaults are put into 10% bins. We observe that there is approximately a 19% probability that 10% of the assets default,

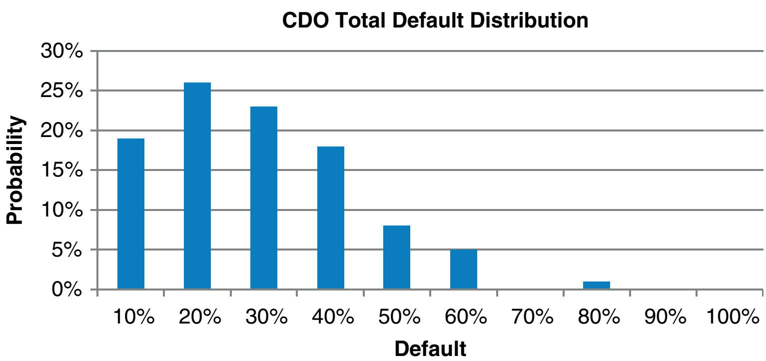

FIGURE 5.7 A Loss Distribution of the Gaussian Copula Model
Inputs are: 10 assets, default probability of every asset 5%, recovery rate 5%, correlations as in Table 5.2. See the model "CDO Gauss educational.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 5," for the generation of the loss distribution.

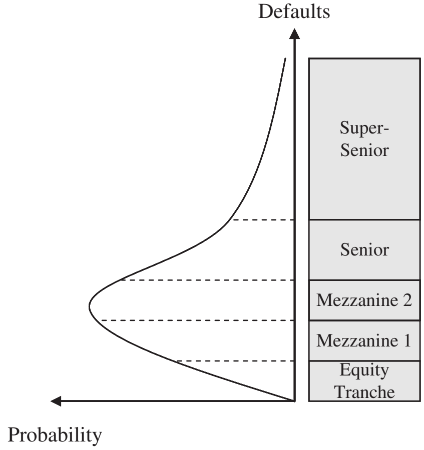

FIGURE 5.8 Mapping of the Default Distribution to Tranches

approximately a 26% probability that 20% of the assets default, and so forth. We see that the loss distribution is somewhat lognormal; however, other simulations display other shapes.

We now map the default distribution to the tranches of the CDO. Assuming a continuous default distribution, the mapping is shown in Figure 5.8.

Figure 5.8 gives us the correlated default probability of each tranche. The tranche spread s, which is effectively a coupon that the tranche investor receives (see Figures 5.2 to 5.4) is directly related to the default probability λ via equation (5.6):

$$s \approx \lambda(1 - R) \tag{5.6}$$

where R is the recovery rate.

Equation (5.6) is also called the "credit triangle," since three parameters are involved and two parameters are necessary to derive the remaining third. If the recovery rate is already included in the loss distribution, we have $s \approx \lambda$. This relationship is intuitive since the default probability λ is the risk that the investors take, and they should be compensated for this risk by receiving a similar amount, the spread s. The relationship $s \approx \lambda\ (1 - R)$ was formally derived by Lando (1998) with $R = 0$ and by Duffie and Singleton (1999) with $R \neq 0$.

Once we have derived the correlated default probability distribution λ, we can derive the loss distribution L via

$$L = \text{EAD}\,\lambda\,(1 - R) \tag{5.7}$$

where EAD is the exposure at default, which for a CDO is the invested amount in the tranche. Equation (5.7) assumes that the default probability distribution of λ does not include the recovery rate R.

The model "CDO Gauss educational.xlsm" that derives the default distribution and loss distribution in a Gaussian copula framework can be found at www.wiley.com/go/correlationriskmodeling, under "Chapter 5."

5.2.3 Recovery Rate

The default probability of the assets in the CDO and the default correlation of the assets are the critical inputs when valuing a CDO. A third input is the recovery rate in case the asset defaults. However, the recovery rate is not as critical an input as the default probability and the default correlation.

Generally, recovery rates depend on the type of security, seniority, country, and state of the economy. The United States enjoys one of the highest recovery rates due to its lenient Chapter 11 bankruptcy law, whereas recovery rates in Japan are among the lowest. Several studies find that recovery rates are higher in an economic expansion than in a recession (Altman 2002; Doshi 2011).

Recovery rates are often approximated using historical recovery rates of defaulted companies. Interestingly, the rating agency Fitch assigns a lower recovery rate to higher-rated entities. The logic is that higher-rated entities will only default in a recession, in which recovery rates are lower. Lower-rated entities are assigned a higher recovery rate, since they can also default in an economic expansion. Fitch refers to this concept as "tiered recovery rates." This is in line with S&P's model to forecast recovery rates with respect to ratings, as seen in Figure 5.9.

In addition, the thinner the tranche, the higher the loss severity, since thinner tranches can be wiped out more quickly. Equity and junior mezzanine tranches are typically thinner than senior tranches. Hence, some rating agencies typically assign lower ratings and in some cases lower recovery rates to equity and junior tranches.

In the standardized iTraxx and CDX indexes, the assumed recovery rates range from 20% to 35%, depending on the credit quality of the index; see www.markit.com/en/products/data/indices/credit-and-loan-indices/itraxx/news.page for details.

5.3 CONCLUSION: THE GAUSSIAN COPULA AND CDOs—WHAT WENT WRONG?

As mentioned in the beginning of this chapter, the Gaussian copula is occasionally blamed for the global financial crisis of 2007 to 2009. Let's get the facts straight.

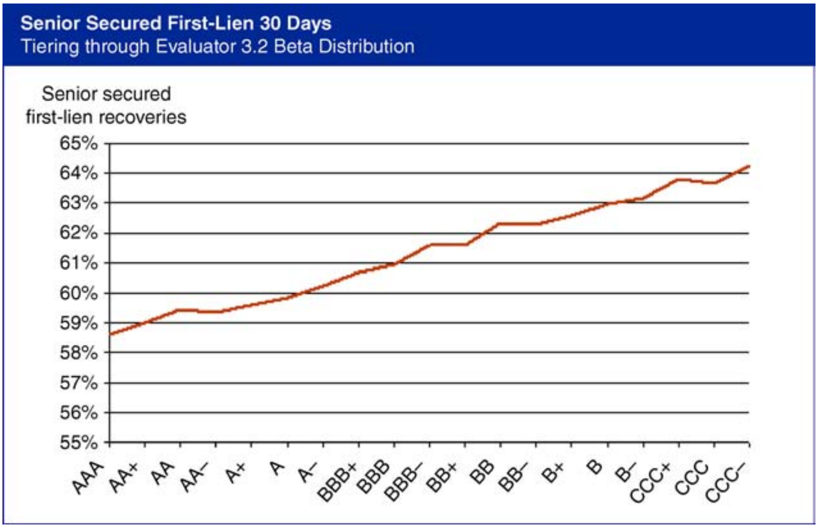

FIGURE 5.9 Recovery Rates from S&P's 3.2 Beta Model
Source: Ghetti and Cheng (2006).

5.3.1 Complexity of CDOs

As mentioned in section 5.1, CDOs are useful financial products since they include sound financial principles such as diversification, subordination, and overcollateralization. However, from a valuation perspective CDOs are complex instruments. Generating the default probabilities of 125 assets for the maturity of a CDO, typically 5 to 10 years, is not an easy task. This is because many variables such as future economic environment, sector-specific developments, products, changing competition, and changing company management are difficult to predict. In addition, typically the assets in the CDO have never defaulted before, so empirical data of analogous companies has to be analyzed.

Even more difficult is finding the default correlation between all the 125 assets in a CDO, which principally requires us to generate a 125-by-125 default correlation matrix. In addition, these correlations are typically quite unstable, as we have seen in Chapter 2.

5.3.2 The Gaussian Copula Model to Value CDOs

The Gaussian copula model is a mathematically rigorous and adequate model to value a CDO. The copula model allows the joining of n default probability functions to a single n-variate distribution. The correlation structure of the newly created n-variate distribution is then applied.

Naturally the Gaussian copula has its limitations. We discussed some of those limitations in Chapter 4, section 4.3. They include low tail dependence, difficulties in calibration, and traders violating correlation assumptions by using their own tranche-specific correlation inputs. In addition, the original copula function is static; that is, it has a one-period time horizon. However, default probability functions and correlation matrices can be derived for different time horizons. This does not create a truly stochastic process with drift and noise, but it gives valuable information for different times t. One-factor copulas can be made dynamic such as in Hull, Presdescu, and White (2005). However, the one-factor copulas, which assume a single correlation value for all assets in a CDO, are simplistic and should not be applied when valuing complex CDOs.

The main problem in 2007 and 2008 when valuing CDOs with the Gaussian copula was inadequate calibration. Benign default probability functions were applied and low default correlations between the assets in the CDO were input in correlation matrices. When data from noncrisis periods are input into a model, it cannot be expected that the model will produce correct outputs in a crisis! In programming terminology: *Garbage in, garbage out*. In the future, crisis scenarios have to be tested; that is, default probabilities and default correlations from crisis periods have to be applied. Basel III and the U.S. Federal Reserve have adopted this approach by requiring financial institutions to stress-test their models. In conclusion, “Don’t blame the models; blame the people who misuse them.”

5.4 SUMMARY

In this chapter we discussed the Gaussian copula correlation model and its application for valuing collateralized debt obligations (CDOs). Several non-quantitative articles have blamed the copula model for the global financial crisis of 2007 and 2009.

There are three main types of CDOs: (1) cash CDOs, (2) synthetic CDOs, and (3) unfunded CDOs such as the iTraxx or CDX indexes. However, many variations of these three basic types exist. The three main players in a CDO are (1) the originator (or protection buyer), who tranfers the credit risk; (2) the investor, who assumes the credit risk; and (3) the special purpose vehicle (SPV), which manages the CDO. CDOs have been misleadingly deemed toxic, especially by those who do not want to take responsibiliy for their incompetence and trading losses. CDOs include sound financial principles such as diversification, subordination, and overcollateralization.

The drawback of CDOs lies in their relative pricing complexity, especially with respect to correlation. If the CDO has 125 assets, we have to evaulate a 125×125 asset correlation matrix. Here is where the copula function comes in. It allows the joining of n (for example $n = 125$) univariate distributions to *one*, however, n-dimensional distribution. Cholesky decomposition lets us easily sample from this distribution (see Appendix 4A of Chapter 4). This sample serves as a default threshold: If the individual default probability of an asset is equal to or exceeds the threshold, default of the asset occurs. Monte Carlo simulations are then conducted, and the average of the outcomes constitutes the default probability of the asset. Conveniently, the default correlation of the assets is included in the default probability, since they are incorporated in the threshold.

Naturally, as with every model, the Gaussian copula has its limitations, such as low tail dependence, problems in calibration, and its principally static nature. However, the main problem in the 2007 to 2009 crisis was the overinvestment in CDOs, the lack of hedging, and, importantly, the data feed. Benign default probabilities and default correlation data from noncrisis periods were input into the model. Of course, it cannot be expected that this non-crisis data can realistically value the behavior of a financial structure like a CDO in a severe crisis. This is why the central banks and the Basel III Committee for Banking Supervision have required all financial institutions to perform stress tests to evaluate the risks under extreme crisis scenarios.

PRACTICE QUESTIONS AND PROBLEMS

1. What is the basic idea of a CDO?
2. Name the three main types of CDOs.
3. Which are the three main players in a CDO? Why is the SPV typically AAA rated?
4. Name the motives of these three players to enter into a CDO.
5. Name the three financial principles that are incorporated in a CDO, and explain them briefly.
6. What is the default probability of an entity based on the Merton 1974 model, if the current asset value $V_0 = \$4{,}000{,}000$, the debt value $D = \$3{,}000{,}000$, the maturity T of the debt is in 1 year, the risk-free interest rate r is 2%, and the volatility of the assets σ is 20%? (A simple model that derives the answer is available upon request.)
7. In the Merton 1974 model, there is a closed form solution for the default probability. What is it?

8. The elegant Merton 1974 model principally serves as a basis for more realistic extensions. What are the limitations of the Merton 1974 model?
9. The Merton 1974 model is the basis for all structural models. Why is the Merton model called structural? Why are reduced form models called reduced form?
10. When valuing the default probability in a CDO, why do we map the default probability of asset i, λ_i to standard normal via $N^{-1}(\lambda_i)$?
11. The multivariate copula function M_n serves as the default threshold. How is the default of asset i derived in the copula model?
12. The credit triangle is $s \approx \lambda\,(1 - R)$, where s is the credit spread, λ is the default intensity, and R is the recovery rate. When $R = 0$, we have $s \approx \lambda$. Explain the intuition of $s \approx \lambda$.
13. The recovery rate is often modeled as being higher, the lower the credit rating of an asset. This seems counterintuitive. But why is it rational?
14. Can the Gaussian copula be blamed for the global financial crisis of 2007 to 2009?
15. What were the main reasons for the misevaluation of CDOs before and during the crisis?

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CHAPTER 6

The One-Factor Gaussian Copula (OFGC) Model—Too Simplistic?

Make everything as simple as possible, but not simpler.
—Albert Einstein

In Chapter 5 we discussed the standard copula model. It joins n marginal distributions to a single n-variate distribution. The n marginal distributions are correlated in a correlation matrix. This matrix is $n \times n$ dimensional, so if the collateralized debt obligation (CDO) has 125 assets, the matrix is 125×125 dimensional. This is mathematically and computationally quite challenging. Often financial institutions take a shortcut, putting the assets into sectors and correlating the different sectors. This reduces the dimension of the correlation matrix.

A further shortcut is to assume that all assets in the portfolio have the same pairwise correlation. This seems simplistic, and it is. However, if the assets in the portfolio are homogeneous, i.e., they are very similar, e.g., they have the same or similar credit ratings and/or they belong to the same sector, this assumption may be tolerable.

If we simplify further, we can also assume that the default probability of all assets in the portfolio is the same. This again seems simplistic, and again it is. However, if the assets in the portfolio are homogeneous, for example they have the same or similar credit rating and/or they belong to the same sector, this simplification may be adequate.

A model in which the correlations and the default probabilities are assumed to be the same for all assets is called homogeneous, or a *large homogeneous portfolio (LHP)*. In 1987 Oldrich Vasicek developed a methodology to price the credit risk for such an LHP, called the *one-factor Gaussian copula (OFGC)* model. The OFGC is a special type of the *conditionally independent default (CID) correlation approach*, which we will explain in this chapter.

Let's just look at some basics that are necessary to evaluate the credit risk in a portfolio. We need three main inputs.

1. *Default intensity*. As explained in Chapter 4, example 4.2, default intensity for period t to $t+1$ is the default probability from t to $t+1$ conditional on no default until period t. If $t=0$ (i.e., we look at a time period starting today), default intensity and default probability are identical.
2. *Default correlation*, which measures the likelihood that two or more assets will default together. The standard copula model discussed in Chapter 4 includes a default correlation matrix of the assets in the portfolio (displayed in Table 1.3). The OFGC applies a conditionally independent default (CID) correlation approach, which includes a single correlation coefficient, i.e., it assumes the same pairwise correlation between all assets in the portfolio.
3. *Recovery rate*. This can be modeled as explained in the previous chapter (section 5.2.3) or derived by historical data.

Let's look at the evolution of the one-factor Gaussian copula, which is displayed in Table 6.1.

TABLE 6.1 Large Homogeneous Portfolio (LHP) Valued by the One-Factor Gaussian Copula (OFGC) Model and Extensions

	Default Intensity λ	**Correlation Coefficient ρ**	**Recovery Rate R**
Vasicek's 1987 LHP, valued on OFGC (used in Basel III to calculate credit value at risk CVaR, see Chapter 12)	Same for all assets i	Same for all assets i	Same for all assets i
Extension of LHP: OFGC with different λ_i (used to value homogeneous CDOs)	Different λ_i for each asset i, λ_i can be a function of t, $\lambda_i(t)$	Same for all assets i	Same for all assets i
Multivariate Gaussian Copula David Li (2000) (is typically applied to value CDOs, see Chapter 5)	λ is a function of i and t, $\lambda_i(t)$	Different for each asset pair since a correlation matrix ρ_M is applied	Different for each asset class

6.1 THE ORIGINAL ONE-FACTOR GAUSSIAN COPULA (OFGC) MODEL

We first define a variable i, $i = 1,\ldots, n$. The variable i represents a certain company i, whose asset is part of a portfolio, for example a CDO. We then derive an auxiliary default indicator variable x_i for every company i. x_i can be thought of as the overall strength of company i. The x_i are derived by

$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i \tag{6.1}$$

where

ρ: default correlation parameter between the assets in the portfolio, $0 \leq \rho \leq 1$; ρ is assumed identical and constant for all asset pairs in the portfolio.

M: systematic market factor, which impacts all companies. M can be thought of as the general economic environment, for example the return of the S&P 500. M is a random drawing from a standard normal distribution, formally $M = n \sim (0, 1)$. M is the same as ε in Chapter 4, section 4.1.

Z_i: the idiosyncratic factor of company i. Z_i expresses ith company's individual strength, possibly measured by company i's stock price return. Just like M, Z_i is a random drawing from a standard normal distribution.

x_i: the value for x_i results from equation (6.1) and is interpreted as a default indicator variable for company i. The lower x_i is, the earlier is the default time T for company i. x_i is by construction standard normally distributed.

The variables M, Z_i, and the resulting x_i in equation (6.1) are sometimes referred to as *latent variables* or *frailty variables*, because the lower M or Z_i is, the lower is x_i and hence the earlier is the default time of company i. Equation (6.1) is the key equation in the OFGC. It can be graphically represented with Figure 6.1.

Although equation (6.1) is rather simple, it includes three important properties:

1. The key property of equation (6.1) is that we do not model the default correlation between the assets i in the portfolio directly, but instead we *condition defaults on* M. We assume that ρ is identical for all asset pairs. Therefore, we have the same relationship between every asset i and M: If ρ is one, every asset i has a perfect correlation with M; hence all assets are

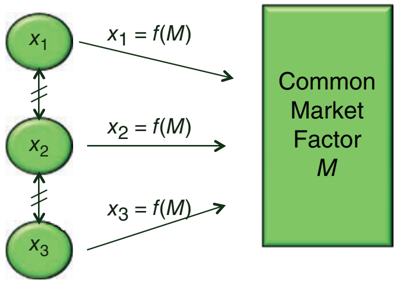

FIGURE 6.1 Graphical Representation of the Correlation Concept of Equation (6.1) for $n = 3$ Entities
The x_i are not directly correlated, but indirectly correlated by conditioning on the common factor M.

perfectly correlated. If ρ is zero, all assets i depend only on their idiosyncratic factor Z_i; hence the assets are independent. For a ρ of 0.7071 (and therefore $\sqrt{\rho} = 0.5$), all x_i are determined equally by M and Z_i. Importantly, once we have determined M (by a random drawing from a standard normal distribution), the assets i are conditionally (on M) independent. Therefore we name this approach *conditionally independent* (*CID*) correlation modeling.

2. Since M and Z_i are random drawings from a standard normal distribution, it conveniently follows that x_i is standard normal.
3. The higher x_i, the higher the default time t_i; hence the later the default of asset i.

We will now derive how the OFGC can evaluate the spreads of the tranches in a CDO.

6.2 VALUING TRANCHES OF A CDO WITH THE OFGC

The road map for deriving the fair spread of each tranche consists of the following six steps. They are displayed in Table 6.2.

The six steps to derive the fair spread of a CDO tranche are:

1. Drawing random samples for M and Z_i and deriving x_i:

   We start with drawing random samples from a standard normal distribution [=normsinv(rand()) in Excel or norminv(rand) in MATLAB]. For every simulation we draw one sample M and one sample for each asset i, Z_i, (Table 6.2, columns 2 and 3). Together with a (market given) ρ we derive x_i from equation (6.1) for every asset i (Table 6.2, column 4).

TABLE 6.2 A Sample-Simulation for Deriving the Correlated Default Time of Five Assets (see last column)

	M	Z_i	Correlated x_i	$N(x_i)=P_i$	$1 - P_i$	Years to Default	Default in Year
Asset 1		−0.5560	−1.6454	0.0499	0.9501	1.00	1
Asset 2		0.3338	−1.0162	0.1548	0.8452	3.28	4
Asset 3	−1.7710	1.3042	−0.3300	0.3707	0.6293	9.03	No default
Asset 4		0.7198	−0.7433	0.2287	0.7713	5.06	No default
Asset 5		−0.5345	−1.6302	0.0515	0.9485	1.03	2

Default intensity λ for all assets 5%, copula correlation coefficient $\rho = 0.5$, maturity 4 years. See the spreadsheet at "Ofgc educational.xls" at www.wiley.com/go/correlationriskmodeling under "Chapter 6."

2. Converting the x_i into probabilities P_i:

   Next we convert the x_i, which are $-\infty \leq x_i \leq \infty$, to cumulative probabilities P_i using the cumulative standard normal distribution N [=normsdist(x_i) in Excel, normcdf(x_i) in MATLAB], hence $N(x_i) = P_i$, $0 \leq P_i \leq 1$ (Table 6.2, column 5). The usage of standard normal distributions for M and Z_i and the resulting cumulative normal distribution via $N(x_i)$ is why the approach is called the Gaussian copula. Our simulated default probabilities P_i are uniform [=rand() in Excel, rand in MATLAB], since we just reversed =normsinv(rand()) from step 1. Graphically this is displayed in Figure 6.2.
3. Deriving market survival thresholds $1 - P_i$:

   We calculate market thresholds $1 - P_i$ (Table 6.2, column 6). From step 2 we see that the $1 - P_i$ are calculated in the same way for every asset i. However, the numerical values of each $1 - P_i$ differ, since they depend on the random drawing for every asset i, Z_i (the $1 - P_i$ in each simulation will be identical only if the random drawings Z_i are identical by coincidence and/or in the case of $\rho = 1$).
4. Deriving individual survival probabilities of asset i, s_i:
   - Constant (flat) default intensity curve λ in time:

     In this case we take the market given default intensity λ for asset i, λ_i, at time t and derive the survival probability s for asset i at time t via $(1-h_i)^t = s_i^t$.[1]
   - Default intensity is a function of time $\lambda(t)$:

1. We do this because it is easy to work with survival probabilities. For example, if the survival probability for year 1 is 90% and the survival probability for the second year is also 90%, the survival probability from time 0 to the end of year 2 is 90% × 90% = 81%, assuming the survival probabilities are independent.

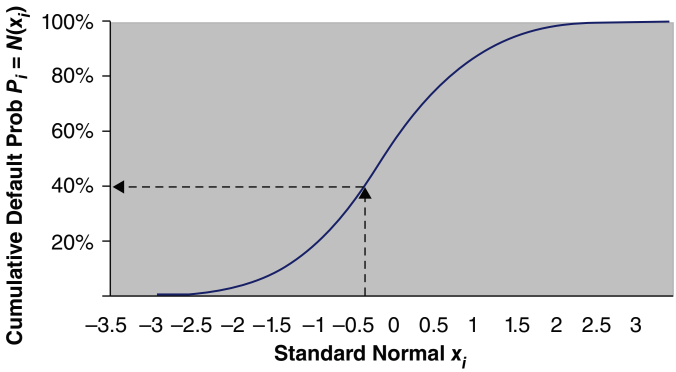

FIGURE 6.2 A Sample Simulation, Which Transforms x_i to a Cumulative Default Probability $N(x_i) = P_i$
Source: Meissner (2008).

If we have a market given nonconstant default intensity curve for each company i with respect to time, we have to derive the idiosyncratic survival curve for every asset i. We take the annual, market given default probability curve $p_i(t)$ for every asset i. We derive the default intensity (also called hazard rate) at a specific time T, $\lambda_i(T)$, from $h_i(T) = \frac{p_i(T)}{1 - \sum_{t=1}^{T-1} p_i(t)}$, $t = 1,..., T$. $\lambda_i(T)$ is the default probability from time T to time $T + 1$, assuming no default until time T. We find the annual survival probabilities $s_i(t) = 1 - \lambda_i(t)$. We derive the cumulative survival time $S_i(T) = \prod_{t=1}^{T} s_i(t)$. A curve of $S_i(t)$ is shown in Figure 6.3. It is generated from default probabilities of 5% in year 1 to 14% in year 10, linearly increasing.

5. Deriving the correlated default time t (Table 6.2, column 7):
   - Constant (flat) default intensity curve λ:

     We derive the default time t of asset i by equating asset i's (market given) survival probability at t, s_i^t with the market survival threshold $1 - P_i$, which we derived in steps 2 and 3

$$s_i^t = 1 - P_i \tag{6.2}$$

We solve equation (6.2) for the default time t of asset i, t_i.[2] We then use Monte Carlo simulation to derive many default times t for an asset i. We average the default times t for each i. Note that the default time t

2. We solve equation (6.2) for t by taking the natural logarithm of both sides: $\ln s_i^t = \ln(1 - P_i)$ or $t \ln(s_i) = \ln(1 - P_i)$ or $t = \ln(1 - P_i)/\ln(s_i)$.

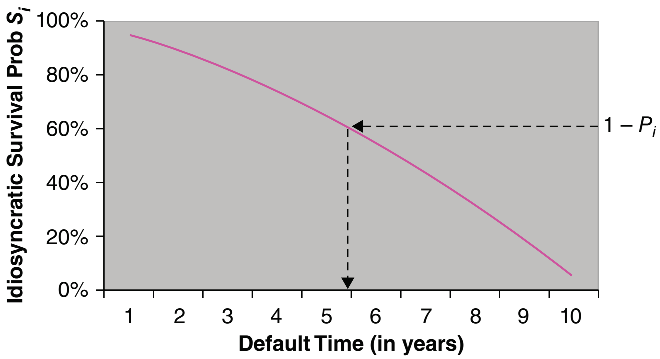

FIGURE 6.3 Derivation of the Correlated Default Time t When the Default Intensity Curve Is Nonconstant
Source: Meissner (2008).

includes the default time correlation of all assets in the portfolio, since $1 - P_i$ includes the default correlation.

- Default intensity as a function of time $\lambda(t)$:

  We relate the simulated survival probabilities from step 3, $1 - P_i$, to asset i's idiosyncratic cumulative survival probability curve S_i, generated in step 4. We find the default time of asset i with a lookup function; see Figure 6.3.

  We then use Monte Carlo simulation to derive many default times t for an asset i and average the default times t for each i.

  The fact that we find the default time t by equating the idiosyncratic survival probability S_i with the simulated survival threshold $1 - P_i$, relates to the standard Gaussian copula, which we discussed in Chapter 5. In the standard Gaussian copula the default of an asset at time τ was determined by equating the inverse of the default intensity with a market threshold; see equation (5.5).

  In Table 6.2, $M = -1.7710$. M can be interpreted as the economic environment. M is a random drawing from a standard normal distribution and takes values $-\infty \leq M \leq \infty$. So in the displayed simulation in Table 6.2, $M = -1.7710$ means that we have a somewhat negative economic environment. This is why several of the names in Table 6.2 default. In particular, assets 1, 2, and 5 default, since their Z_i are also relatively low. Asset 1's Z_i is the lowest; therefore it defaults earliest (i.e., in year 1).

6. Deriving the tranche spread:

   Once we have derived the average default time for each asset i, it is easy to find the fair tranche spread. Each tranche of a CDO consists of a portfolio of credit default swaps (CDSs), as displayed in Figure 6.4. The number of

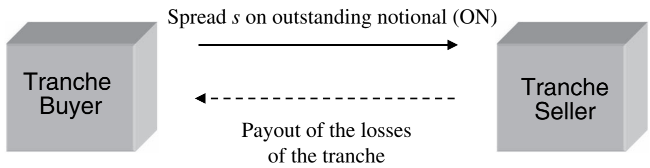

FIGURE 6.4 Cash Flows of a CDO Tranche

CDSs in a certain tranche is determined by the attachment and detachment point. For example, the equity tranche of the iTraxx and CDX indexes contains 0% to 3% of defaults of the 125 assets. Hence $3\% \times 125 = 3.75$ of all defaults fall into the equity tranche (so the fourth default falls to 75% into the equity tranche and to 25% into the next higher tranche).

Since we have derived the expected default time of every asset i (step 5), we also know the losses at any time t. From the losses of every asset i, we can find the outstanding notional (ON) of the CDO at any time t. With this outstanding notional in hand, we can price the tranche.

Each CDO tranche is evaluated with simple swap valuation techniques. The present value (PV) of the spread leg of tranche j is

$$PV(\text{Spread tranche}_j) = \text{Spread}_j\, E\left(\sum_{t=1}^{n} e^{-rt}\text{ON}_j(t)\right) \tag{6.3}$$

where E stands for expected value, ON$j(t)$ is the outstanding notional of tranche j at time t, and r is the continuously compounded risk-free interest rate. The present value of the payout leg of tranche j is

$$PV(\text{Payout leg}_j) = E\left(\sum_{t=1}^{n} e^{-rt}(\text{ON}_j(t-1) - \text{ON}_j(t))(1-R)\right) \tag{6.4}$$

where R is the recovery rate.

Equating (6.3) and (6.4), setting to zero and solving for Spread$_j$ gives the fair market spread of tranche j:

$$\text{Spread}_j = \frac{E\left(\sum_{t=1}^{n} e^{-rt}(\text{ON}_j(t-1) - \text{ON}_j(t))(1-R)\right)}{E\left(\sum_{t=1}^{n} e^{-rt}\text{ON}_j(t)\right)} \tag{6.5}$$

For an educational model showing steps 1 to 6, see "Ofgc educational.xls" at www.wiley.com/go/correlationriskmodeling, under "Chapter 6."

Let's derive the fair tranche spread in a numerical example.

EXAMPLE 6.1: DERIVING THE FAIR TRANCHE SPREAD OF A CDO WITH THE ONE-FACTOR GAUSSIAN COPULA (OFGC) MODEL

Let's look at a CDO with a 3-year maturity. The starting notional is \$1,000,000,000, with 125 equally weighted companies. Hence each asset has a notional of \$8,000,000.

Next assume that the spread payments and payouts are annually in arrears. The recovery rate for every asset is 40%. Interest rates are constant at 10%. We consider an equity tranche with a detachment point of 3%. Hence the equity tranche has a starting notional of \$30,000,000.

Let's also assume that from our analysis of steps 1 to 5, we derive that one asset defaults after 1.5 years and one asset defaults in 2.5 years. Hence the starting notional of \$30,000,000 reduces to \$22,000,000 for t_2 (end of year 2) and to \$14,000,000 for t_3 (end of year 3).

From equation (6.5), the numerator is

$$e^{-0.1\times 1}\times 0\times 0.6+e^{-0.1\times 2}\times(\$30{,}000{,}000-\$22{,}000{,}000)\times 0.6$$
$$+e^{-0.1\times 3}\times(\$22{,}000{,}000-\$14{,}000{,}000)\times 0.6=\$8{,}782{,}153$$

From equation (6.5), the denominator is

$$e^{-0.1\times 1}\times\$30{,}000{,}000+e^{-0.1\times 2}\times\$22{,}000{,}000$$
$$+e^{-0.1\times 3}\times\$14{,}000{,}000=\$55{,}528{,}654$$

Hence the fair equity tranche spread, paid annually in arrears, is

$$\$8{,}782{,}153/\$55{,}528{,}654=15.82\%$$

Note that we abstracted from any accrued interest on the bond of the asset. Furthermore, we have abstracted from any accrued interest on the spread premium.

6.2.1 Randomness in the OFGC Model

The reader might be surprised by all the randomness in generating the default time t. Since both M and Z_i are random drawings from a normal distribution, does it matter what value ρ has in equation (6.1)? A similar question is: Since all P_i are random, uniform probabilities [=rand() in Excel], why don't we just start the process at step 2 and derive uniform drawings; hence P_i = rand()?

The answer is that ρ in equation (6.1) represents the default correlation between the assets in the CDO. We generate *one* M for every asset i and a unique Z for every asset i in each Monte Carlo simulation. Hence ρ serves as a weighting factor between M and Z_i. The higher ρ, the more the x_i depend on the common factor M, hence the higher the default correlation of the assets in the CDO, and vice versa. This is discussed in detail in the following section.

6.3 THE CORRELATION CONCEPT IN THE OFGC MODEL

As mentioned previously, a key property of the OFGC model is that the default correlation between the assets in the CDO is not modeled directly, but instead indirectly by conditioning the defaults on a common market factor M.

The higher ρ in equation (6.1), the higher is the dependence of each asset i on the factor M, hence the higher the correlation between the assets. This is expressed in the standard deviation of the simulated survival probabilities $1 - P_i$. The higher ρ is, the lower is the standard deviation of the $1 - P_i$'s in a simulation, hence the higher is the probability that the assets default together.

Figure 6.5 shows the idiosyncratic survival curves of two assets $i = 1$ and $i = 2$. If ρ is high (close to 1), the $1 - P_i$'s of each asset i will have a low standard deviation in each simulation (i.e., will have similar values). Since the simulated $1 - P_i$'s are quite similar, the probability of the assets defaulting together is high, as seen in Figure 6.5.

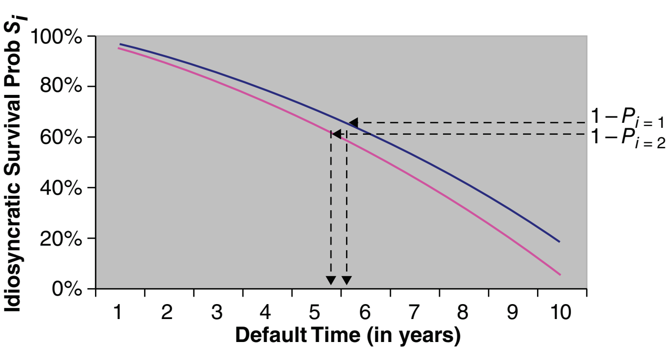

FIGURE 6.5 A High Correlation, Resulting in Similar Default Times of Assets $i = 1$ and $i = 2$
Source: Meissner (2008).

In the extreme case of $\rho = 1$, each simulation will generate the same $1 - P_i$ for every asset i (since the x_i and the resulting $1 - P_i$ depend only on the common factor M). If additionally the hazard rates of every asset i are identical, the OFGC will generate the same default time for every asset i.

For a low correlation, e.g., $\rho = 0.05$, it follows from equation (6.1), that the x_i, P_i, and $1 - P_i$ are mainly determined by the idiosyncratic Z_i. In this case the survival times $1 - P_i$ in each simulation have a high standard deviation; that is, they are quite different (unless by coincidence the random drawings Z_i are similar). Therefore, the OFGC model typically generates quite different default times, as seen in Figure 6.6.

6.3.1 The Loss Distribution of the OFGC Model

As seen in section 6.2, deriving the fair tranche spread with the OFGC model, there are two main input factors that determine the price of a CDO. One is the default probability of the assets in the CDO. Naturally, the higher the default probability, the higher the spread of the tranches in the CDO. The second crucial input factor is the default correlation of the assets in the CDO.

Figure 6.7 shows the loss distribution with respect to correlation of a 10-asset CDO with a 2-year maturity and a 10% hazard rate.

As seen in Figure 6.7, for zero correlation, the OFGC model displays a somewhat lognormal distribution of losses. For medium correlation, the losses are more evenly distributed and descending. For very high correlation, the probability of extreme events increases. Hence, there is a high

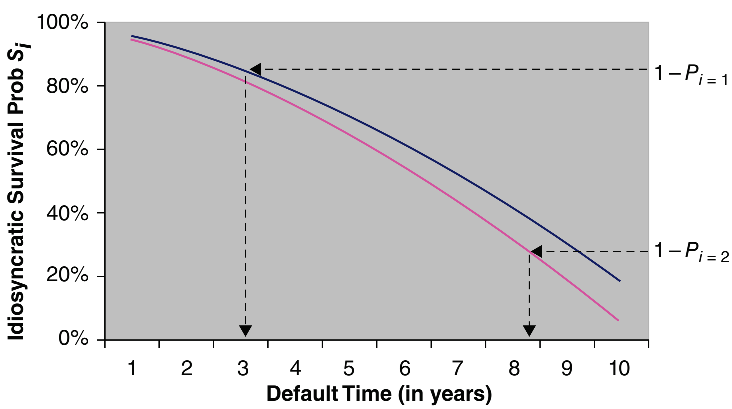

FIGURE 6.6 A Low Correlation, Resulting in Quite Different Default Times for Assets $i = 1$ and $i = 2$
Source: Meissner (2008).

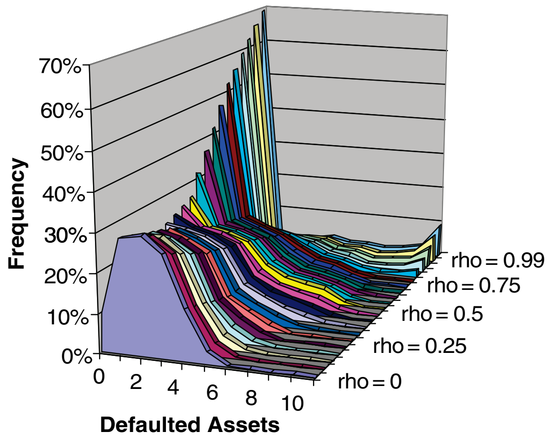

FIGURE 6.7 Loss Distribution of a 10-Asset CDO with Respect to Default Correlation of the Assets in the CDO
Source: Meissner (2008).

probability of zero losses and an increased probability that all assets default. For a model deriving the loss distribution of the OFGC, see "Base correlation generation.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 6."

6.3.2 The Tranche Spread–Correlation Relationship

From Figure 6.8 we can deduct the tranche spread–correlation relationship in the OFGC model. Interestingly, the equity tranche spread is negatively related to default correlation, whereas the senior tranche is positively related.

The negative relationship between the equity tranche spread and default correlation is intuitive: The higher the default correlation of the companies in the CDO, the higher the probability of extreme events; that is, the probability of many or no defaults is high. The high probability of no defaults reduces the riskiness and hence reduces the equity tranche spread. The high probability of many defaults at the same time increases the riskiness of the equity tranche, which increases the equity tranche spread. However, this effect does not impact the equity tranche significantly, since the losses are capped at the detachment level.

The opposite logic applies to the senior tranche: If default correlation is high, many defaults may occur at the same time. Therefore, the senior tranche may be impacted; hence the riskiness and the spread are high.

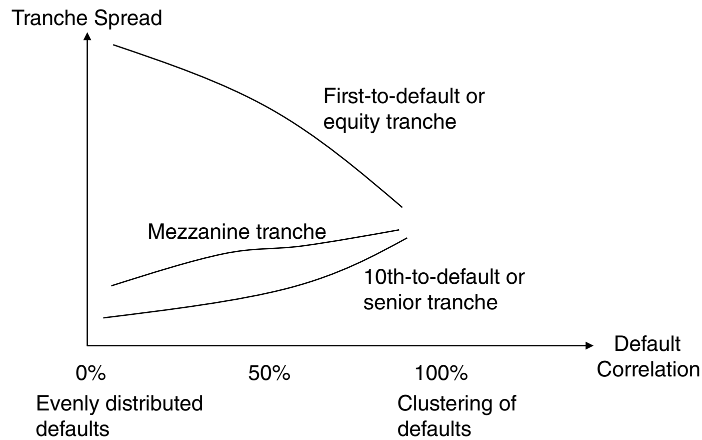

FIGURE 6.8 Relationship between Tranche Spread and Default Correlation in the OFGC Model

It should be mentioned that the relationship between tranche spread and correlation in Figure 6.8 is a specific result of the OFGC model. Other models with different distribution assumptions and correlation approaches derive a different spread-correlation relationship, e.g., a positive relationship between equity spread and correlation.[3]

6.4 THE RELATIONSHIP BETWEEN THE OFGC AND THE STANDARD COPULA

The core equation of the OFGC was displayed in equation 6.1.

$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i \tag{6.1}$$

By construction, x_i in equation (6.1) is standard normal. Therefore, we can easily create the cumulative standard normal distribution of the x_i:

$$N(x_i) \tag{6.1a}$$

where N is the one-dimensional cumulative standard normal distribution. Importantly, $N(x_i)$ includes the default correlation between the $i = 1,\ldots, n$ assets via the correlated x_i, derived in equation (6.1).

3. See R. Jarrow and D. van Deventer, "Synthetic CDO Equity: Short or Long Correlation Risk?" *The Journal of Fixed Income*, 17 (2008): 4.

In comparison, the n-dimensional cumulative standard normal distribution M_n was generated in the standard Gaussian copula framework by equation (4.12):

$$M_n[N^{-1}(Q_1(t)), ..., N^{-1}(Q_n(t)); \rho_M] \tag{4.12}$$

where $Q_i(t)$ is the cumulative default distribution of asset i with respect to t and ρ_M is the correlation matrix of the assets in the portfolio.

Let's look at four differences between the OFGC of equations (6.1) and (6.1a) and the standard copula of equation (4.12).

1. The correlations between the assets i in equation (6.1) are modeled indirectly by conditioning the auxiliary variable of asset i, x_i, on a common factor M. In contrast, equation (4.12) applies the typical correlation matrix ρ_M (for an example, see Chapter 1, Table 1.3).
2. As a consequence, in the OFGC all asset pairs in the portfolio have the same correlation. The standard Gaussian copula is richer, as it can model asset pair correlation individually in the correlation matrix.
3. The cumulative normal distribution in equation (6.1a), which includes the correlation between the assets i via x_i, is conveniently one-dimensional. The cumulative normal distribution in equation (4.12) is n-dimensional.
4. The bivariate case of the standard Gaussian copula is equivalent with the OFGC: Sampling from equation (4.12) is achieved by Cholesky decomposition (as explained in Appendix 4A of Chapter 4). In the bivariate case, Cholesky sampling of two correlated variables x_1 and x_2 from equation (4.12) reduces to

$$x_1 = \varepsilon_1$$

$$x_2 = \sqrt{\rho}\,\varepsilon_1 + \sqrt{1-\rho}\,\varepsilon_2$$

where ε_1 and ε_2 are independent samples from $n \sim (0, 1)$. This is equivalent to samples $i = 1, 2$ from equation (6.1).

6.5 EXTENSIONS OF THE OFGC

The OFGC is principally static, i.e. has a one period time horizon. However, the static property of the OFGC can be relaxed, as in Hull, Presdescu, and White (2005), who apply a dynamic OFGC model. Hence they modify equation (6.1) and model

$$dz_i(t) = \sqrt{\rho(t)}\,dM(t) + \sqrt{1-\rho(t)}dZ_i(t) \tag{6.6}$$

where $dM(t)$ and $dZ_i(t)$ are $n \sim (0,1)$ and independent. It follows from equation (6.6) that $dz_i(t)$ is also $n \sim (0,1)$. The dependence on $M(t)$ again determines indirectly the correlation between assets i. For example, if $\rho(t) = 1$, $dz_i(t)$ depends only on $dM(t)$; hence all assets i have the same Brownian motion at time t. If $\rho(t) = 0$, $dz_i(t) = dZ_i(t)$; hence the Brownian motions of assets i are uncorrelated at time t.

Furthermore, more common factors M can be modeled. In this case equation (6.1) generalizes to

$$x_i = \sum_{k=1}^{m} \sqrt{p_{i,k}} M_k + Z_i \sum_{k=1}^{m} \sqrt{p_{i,k}}$$

and the correlation between x_i and x_j is $\sum_{k=1}^{m} \sqrt{p_{i,k} p_{j,k}}$.

Numerous further extensions of the OFGC approach exist. One of the most popular is the one-factor Student's t copula.

$$\overline{x}_i = \sqrt{\rho} M + \sqrt{1-\rho} Z_i \tag{6.7}$$

where M and Z_i are independent and $n \sim (0,1)$. $x_i = \overline{x}_i \sqrt{W}$ where W follows an inverse gamma distribution. It follows that the latent variable x_i is Student's t distributed.

Another popular extension of the OFGC in equation (6.1) is the double-t copula. It is defined as

$$x_i = \sqrt{\rho} M_S + \sqrt{1-\rho} Z_{S,i} \tag{6.8}$$

where M_S and $Z_{S,i}$ are independent and follow a Student's t distribution. Since the Student's t distribution is not stable under convolution, the latent variable x_i in equation (6.8) is not Student's t distributed.

Another extension of the OFGC is integrating a binomial representation of stochastic correlation. Burtschell et al. (2008) model the latent variable x_i as

$$x_i = B_i(\sqrt{\rho_1} M + \sqrt{1-\rho_1} Z_i) + (1 - B_i)(\sqrt{\rho_2} M + \sqrt{1-\rho_2} Z_i)$$

where M and Z_i are defined as in equation (6.1). B_i is a Bernoulli random variable. We define a Bernoulli cutoff $B^* \in [0,1]$ and model

$$B_i = \begin{cases} 0 & \text{if} \quad r < B^* \\ 1 & \text{if} \quad r \geq B^* \end{cases}$$

where r is a random drawing from a uniform distribution $n \in [0, 1]$. If we set $p_1 > p_2$, the cutoff level B^* can be set low to model high correlation in distressed times.

A further extension of the OFGC is creating a local correlation model (LCM) [see Turc et al. (2005)], where the correlation is state-dependent. In particular, Turc et al. assume that the correlation ρ is dependent on the state of the economy M. Hence the OFGC changes to

$$x_i = -\sqrt{\rho(M)}M + \sqrt{1-\rho(M)}Z_i$$

The approach is similar in nature to the local volatility model of Dupire (1994), where volatility at time t, σ_t, is a function of the state of the underlying S at time t and t, $\sigma_t(S_t, t)$. Whereas Dupire is able to model the implied volatility skew and smile in the equity option market well, the local correlation model is able to reproduce the implied correlation smile of CDO tranches spreads quite accurately. As a result, the marked-to-market and hedge ratios of the local correlation model outperform those of the original OFGC.

A further extension of the OFGC is by Schönbucher and Schubert (2001), who integrate stochastic dynamics into the Gaussian copula model. Andersen and Sidenius (2004/2005) introduce randomness to the factor M with their random factor loading (RFL) model, allowing default correlation to be higher in a recession. Andersen (2006) adds jumps in both factors and residuals to the RFL model. Willeman (2005) applies lognormal jumps, and Baxter (2006) uses Brownian variance-gamma jumps to model the credit process.

6.5.1 Further Extensions of the OFGC Model: Hybrid CID–Contagion Modeling

As derived in section 6.1, the OFGC applies a conditionally independent default (CID) correlation approach. CID models offer realistic correlation features such as default clustering. For example, if the correlation coefficient ρ is high and the market environment M is negative [see equation (6.1)], many entities will default together.

As mentioned in Chapter 4, section 4.4, contagion correlation approaches can model counterparty default contagion. Hence it is not surprising that several models incorporate the CID common factor feature as well as contagion properties. In these models, typically a contagion term is simply added to the CID process. Schönbucher and Schubert (2001), Frey and Backhaus (2003), Giesecke and Weber (2004), and Yu (2007) propose

$$\lambda_i = \alpha_i M + Z_i + \sum_{j \neq i} \beta_{i,j} N_j \tag{6.9}$$

where M and Z_i are defined as in equation (6.1). $\beta_{i,j}$ is a function that models the contagion of firm i to a default of firm j, and N_j is a default counting process $N_j = \sum_{j \geq 1} 1_{\{Tj \leq t\}}$ where T_j is the stopping time (i.e., default time) of firm j. A special case of equation (6.9) is derived by Giesecke and Weber (2006) with $Z_i = 0$ and $\alpha_i M = c_i$. Hence the deterministic function c_i, which models the base intensity, may not incorporate a systematic factor M.

Alternatively, Schönbucher (2004), Giesecke and Goldberg (2004), and Duffie, Eckner, Horel, and Saita (2009) suggest

$$\lambda_i = \alpha_i M + Z_i + \beta_i \overline{U}$$

where β_i is a deterministic function and $\overline{U}$ is a common factor, which is unobservable. However, the factor $\overline{U}$ is transformed into an observable process $U = E(\overline{U}_t | \mathcal{F}_t)$, where $\mathcal{F}_t$ is the filtration, which contains all events. The filtered process U is updated with observable information, in particular information about default events, which constitutes the contagion of firm i on defaults of other firms j.

6.6 CONCLUSION—IS THE OFGC TOO SIMPLISTIC TO EVALUATE CREDIT RISK IN PORTFOLIOS?

Let's answer this question by first looking at the benefits and limitations of the OFGC model.

6.6.1 Benefits of the OFGC Model

The OFGC model is simple. Similar to the Black-Scholes-Merton model, the OFGC model has high intuition and is easy to implement. More complex approaches such as nonfactor copulas have to use multivariate methods such as Gaussian quadrature or recursion techniques (Andersen, Sidenius, and Basu 2003; Hull and White 2004); inverse fast Fourier transforms (Laurent and Gregory 2003); or saddle point approximations (Martin et al. 2005) to generate the cumulative loss distribution. However, the OFGC model uses a simple univariate function $N(x_i)$ to generate the simulated loss distribution, since the x_i already include the correlation. The default time of an asset can be derived easily by equating the market survival threshold with individual survival probability; see equation (6.2).

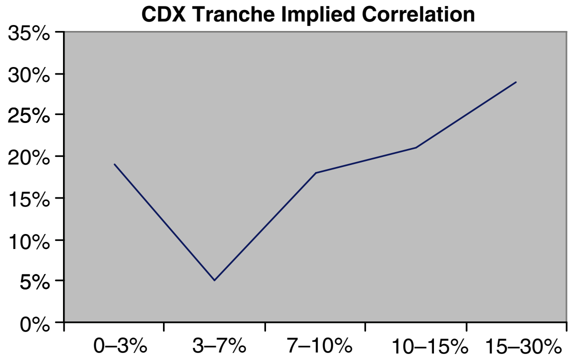

FIGURE 6.9 CDX Implied Correlation, Also Called Compound Correlation, Backed Out of the OFGC Model, November 2004

6.6.2 Limitations of the OFGC Model

The OFGC model is simple. It is essentially static, with no underlying stochastic process.[4] Hence dynamic delta and gamma hedging are difficult to implement.

The most significant drawback is that traders do not seem to agree with the model. Just as option traders increase the implied volatility to derive higher prices for out-of-the-money puts and calls in the Black-Scholes-Merton model, CDO traders alter the crucial input factor correlation in the OFGC. The often cited *correlation smile* is shown in Figure 6.9.

However, there is a crucial difference between the volatility smile of options and the correlation smile of CDOs. Whereas an increase in the implied correlation increases the senior tranche spread, an increase in the implied correlation decreases the equity tranche spread. This is because the equity tranche spread has a negative dependence on implied correlation; see Figure 6.8. Hence CDO traders arbitrarily decrease the equity tranche spread and arbitrarily increase the senior tranche spread.

In practice traders do not like to work with implied correlation. It does not allow easy interpolation (e.g., the pricing of an off-the-run tranche as for example the 2% to 8% tranche). Hence traders typically derive a base correlation curve, which has an attachment point of zero and the detachment points of the standard tranches, hence 0%–3%, 0%–7%, 0%–10%, 0%–15%, and 0%–30% in the case of the CDX. The derived base correlation

4. See Schönbucher and Schubert 2001 for integrating stochastic dynamics into the Gaussian copula model.

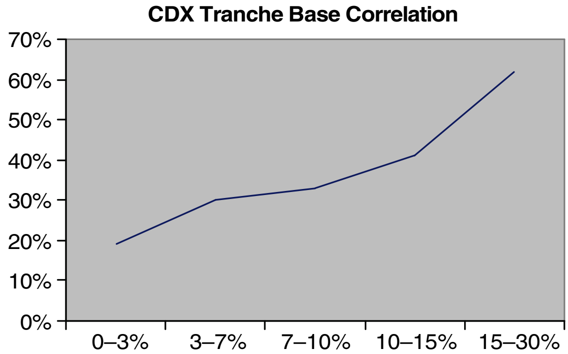

FIGURE 6.10 CDX Base Correlation, Backed Out of the OFGC Model, November 2004

curve is typically upward sloping and allows easier interpolation of off-the-run tranches, as seen in Figure 6.10.

The transformation of forward implied correlation to spot base correlation reminds us of the calculation of spot rates from Eurodollar future rates in the interest rate market. For a model that bootstraps the base correlation curve from CDO tranche spreads, see "Base correlation generation.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 6."

In conclusion, the OFGC is an elegant, simple, and intuitive model that traders like. It bears benefits and limitations similar to those of the Black-Scholes-Merton model. Similar to the Black-Scholes-Merton model, the benefits are simplicity and intuition. One limitation of the OFGC with respect to application is that traders violate the assumptions of the model. They randomly alter the crucial input factor correlation to derive desired tranche spreads.

While traders like simplicity, simplicity comes at a cost. The critical question is whether the assumptions of the OFGC, i.e., the same default probability of all assets in the portfolio and the same correlation between all asset pairs in the portfolio are too simplistic to derive the credit risk of that portfolio. The answer is: Only in rare cases, if the assets in the portfolio are very homogeneous, i.e., they have similar default probabilities and similar default correlations, is the OFGC an adequate model. Most portfolios of investment banks, however, are highly diversified with assets from different sectors and different geographical regions and hence have different default probabilities and default correlations. In this case the OFGC is an inappropriately simplistic model. It is a bit surprising that the Basel III accord applies the OGFC to evaluate credit risk for the portfolios of financial institutions. For more details, see Chapter 12, "Correlation and Basel II and III," especially section 12.2, "Basel II and III's Credit Value at Risk (CVaR) Approach."

6.7 SUMMARY

In this chapter we discussed a shortcut of the Gaussian copula function: the one-factor Gaussian copula (OFGC). We evaluated whether it is too simplistic, especially with respect to valuing CDOs.

The OFGC was created by Oldrich Vasicek in 1987. The OFGC makes the following strong simplistic assumptions: (1) All assets in a portfolio have the same default probability, (2) all assets in a portfolio have the same pairwise default correlation, and, less critically, (3) all assets in the portfolio have the same recovery rate. These assumptions constitute a *large homogeneous portfolio*. In fact, the simplistic assumptions of the OFGC are justifiable only if the portfolio in question is very homogeneous; for example, it contains assets of the same sector with the same or similar credit ratings.

The OFGC applies the *conditionally independent default (CID)* correlation approach. In this approach, the assets are not correlated directly, but indirectly by conditioning on a common factor that is shared by all assets. For example, all assets depend on the current state of the economy. The higher the dependence of the assets on the state of the economy, the higher is also the correlation between the assets. For example in the extreme case, if all assets' dependence on the common factor is 1, all assets are perfectly correlated. If the dependence on the common factor is 0, the assets are uncorrelated. For dependence values on the common factor between 0 and 1, naturally there is a partial correlation between the assets.

The correlated default time of an asset is derived in a similar fashion as in the standard Gaussian copula: A threshold is created that contains the default correlations of the assets in the portfolio. This threshold is equated with the survival probability of the asset, and in the case of a constant default probability function, this equation can be solved for the default time t. In case of a nonconstant default probability function, a search procedure finds the default time. Monte Carlo simulations are applied to derive numerous default times, and the result is averaged to determine the final default time. The default times of the different tranches are then mapped to the tranches of the CDO to find the tranche spread.

Since the OFGC is simplistic, many extensions exist that attempt to bring the OFGC closer to reality. A dynamic OFGC model can be created, multiple common factors can be introduced, or different distributions for the latent variables can be applied.

In conclusion, the OFGC is a simple, intuitive model that traders like. However, simplicity comes at a cost. The assumptions of identical default probability of all assets and identical default correlation between all assets in the portfolio are justifiable only for a portfolio with highly homogeneous assets, possibly in the same sector and with similar credit ratings. For the heterogeneous portfolios of most investment banks the OFGC is too simplistic. In addition, as with the Black-Scholes-Merton option pricing model, traders seem to disagree with the OFGC: They randomly alter the tranche correlations to derive desired tranche spreads; this violates the basic principle of the OFGC, which assumes a constant CDO-wide, tranche-nonspecific default correlation.

PRACTICE QUESTIONS AND PROBLEMS

1. Name the three strongly simplistic assumptions of the one-factor Gaussian copula (OFGC) model.
2. For which portfolios are those assumptions justifiable?
3. The correlation concept of the OFGC is incorporated in the simple equation (6.1) $x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$. Explain the correlation concept with this equation.
4. Equation (6.1) applies the conditionally independent default (CID) correlation approach. Explain the term *conditionally independent*.
5. Why are the variables M, Z_i, and the resulting x_i in equation (6.1) called "latent" and "frailty" variables?
6. In equation (6.1), the x_i are standard normally distributed. How are the x_i transformed into probabilities?
7. In equation (6.2), $s_i^t = 1 - P_i$, s_i^t is the survival probability of asset i at time t, and $1 - P_i$ is the default threshold, which includes the correlation. Solve equation (6.2) for the default time t of asset i. What is the default time of asset i if $s_i^t = 80\%$ and $P_i = 50\%$?
8. Calculate the fair equity tranche spread of a CDO for the following CDO with a three-year maturity: The starting notional is $2,000,000,000, with 125 equally weighted companies. Hence each asset has a notional value of $16,000,000.

   Let's assume spread payments and payouts are annually in arrears. The recovery rate for every asset is 30%. Interest rates are constant at 5%. We consider an equity tranche with a detachment point of 3%. Hence the equity tranche has a starting notional value of $60,000,000.

   Let's assume that we have derived that one asset defaults after 1.5 years and one asset defaults at 2.5 years. Hence the starting notional of \$60,000,000 reduces to \$44,000,000 for t_2 (end of year 2) and to \$28,000,000 for t_3 (end of year 3).

What is the equity tranche spread derived by the OFGC?

9. The tranche spread of the equity tranche and the senior tranche behave very differently with respect to changes in the correlation of the assets in the CDO. Draw a graph showing the tranche spread–correlation dependence for the equity tranche and a senior tranche.
10. Explain the graph that you created in problem 9.
11. Name the main differences between the standard Gaussian copula and the OFGC.
12. The OFGC is a first, simplistic approach to derive the tranche spread in a CDO and the credit risk in portfolios. Name three extensions of the OFGC.
13. Should we apply the OFGC to value CDOs? Should we apply the OFGC to value credit risk in portfolios?
14. Why do "traders seem to disagree with the OFGC"?
15. Explain the correlation smile that traders apply to derive tranche spreads. How is the correlation smile related to the volatility smile when pricing options?

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CHAPTER 7

Financial Correlation Models—Top-Down Approaches

Imagination is more important than knowledge.

—Albert Einstein

Financial credit models, which derive correlated default risk, can be characterized by the way the portfolio default intensity distribution is derived. In the bottom-up models of Chapters 4, 5, and 6, the distribution of the portfolio intensity is an aggregate of the individual entities' default intensity. In a top-down model the evolution of the portfolio intensity distribution is derived directly, i.e., abstracting from the individual entities' default intensities.

Top-down models are typically applied in practice if:

- The default intensities of the individual entities are unavailable or unreliable.
- The default intensities of the individual entities are unnecessary. This may be the case when evaluating a homogeneous portfolio such as an index of homogeneous entities.
- The sheer size of a portfolio makes the modeling of individual default intensities problematic.

Top-down models are typically more parsimonious and computationally efficient, and can often be calibrated better to market prices than bottom-up models. Although seemingly important information such as the individual entities' default intensities is disregarded, a top-down model can typically capture properties of the portfolio such as volatility or correlation smiles better than a bottom-up model. In addition, the individual entities' default information can often be inferred by random thinning techniques.

In this chapter we analyze the correlation modeling of several top-down approaches. In particular we revisit Vasicek's 1987 one-factor Gaussian copula (OFGC) model, and we discuss the Markov chain models of Hurd and Kuznetsov (2006a, 2006b) and Schönbucher (2006), as well as the top-down contagion model of Giesecke, Goldberg, and Ding (2011). Top-down models are mathematically somewhat more complex than bottom-up models are. So readers who are not mathematically well equipped may take this chapter with caution. We hope mathematicians will like it.

7.1 VASICEK'S 1987 ONE-FACTOR GAUSSIAN COPULA (OFGC) MODEL REVISITED

The one-factor Gaussian copula model can be considered a top-down correlation model, since it abstracts from the individual default intensities of each asset i. Rather, *one* default intensity is assumed for all assets in the portfolio.

We devoted the entire Chapter 6 to the one-factor Gaussian copula model (OFGC), where we discussed properties and practical applications such as valuing CDOs. In this more theoretical chapter, we briefly show that a realistic default distribution can be derived with the OFGC.

Vasicek 1987 assumes (1) a constant and identical default intensity of all entities in a portfolio and (2) the same default correlation between the entities. These two conditions constitute a large homogeneous portfolio (LHP), which is evaluated with the one-factor Gaussian copula (OFGC) framework.

The OFGC model allows creating a loss distribution to find $k = 1,\ldots, n$ defaults of a basket of n entities at time T. We start with the core equation:

$$x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i \tag{6.1}$$

where variables are defined as in Chapter 6, section 6.1.

We now map the cumulative default probabilities $Q(T)$, which are identical for all entities in the portfolio, to standard normal via $N^{-1}(Q(T))$, where N^{-1} is the inverse of the cumulative standard normal distribution. We equate the $N^{-1}(Q(T))$ with the correlated market frailty variable x_i of equation (6.1); hence $x_i = N^{-1}(Q(T))$. This equation satisfies the OFGC property that $\text{Prob}(x_i < x) = \text{Prob}(T_i < T)$; that is, the frailty variable x_i (which includes the default correlation) is mapped percentile to percentile to default times T_i.[1]

1. For more on the copula mapping, see Chapter 4, section 4.3.

Inputting $x_i = N^{-1}(Q(T))$ into equation (6.1) and solving for Z_i, we derive

$$Z_i = \frac{N^{-1}(Q(T)) - \sqrt{p}\,M}{\sqrt{1-p}} \tag{7.1}$$

The correlation between the $i = 1,\ldots, n$ entities is modeled indirectly by conditioning on M. Once we determine the value of M (by a random drawing from a standard normal distribution), it follows that defaults of the entities are mutually independent. In particular, the cumulative default probability of the idiosyncratic factor Z_i, $N(Z_i)$ can be expressed as the cumulative default probability dependent on M, $Q(T|M)$. Hence we have

$$Q(T|M) = N\left(\frac{N^{-1}(Q(T)) - \sqrt{p}M}{\sqrt{1-p}}\right) \tag{7.2}$$

Equation (7.2) gives the cumulative default probability conditional on the market factor M. We now have to find the unconditional default probabilities. We do this by first discretely integrating over M. Since M is standard normal, this is computationally easy; we can use the discrete Gaussian quadrature (Norm (x) – Norm $(x - 1)$) in MATLAB. We now have to derive all possible $k = 0,\ldots, n$ default combinations. We do this by applying the binomial distribution B, hence $B(k; n, Q(T|M))$ and weighing it with the piecewise integrated units of M. The result is a distribution of the number of defaults until T, as shown in Figure 7.1.

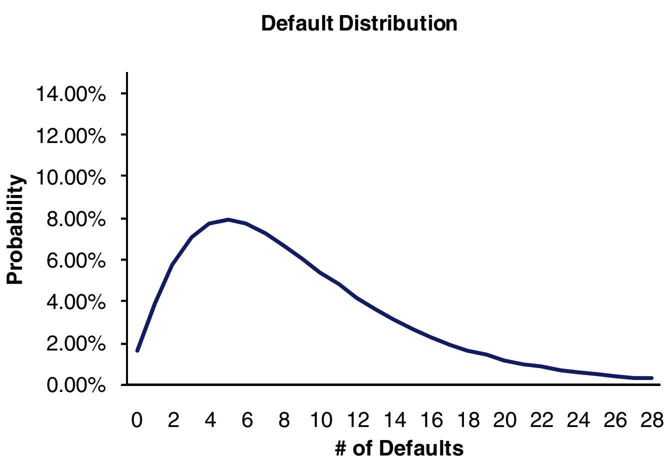

FIGURE 7.1 Unconditional Default Distribution Derived from the OFGC Model Parameters $Q(T) = 7.3\%$, $\rho = 10\%$, portfolio size 125 entities, recovery rate 40%.

A spreadsheet that derives the default distribution in the OFGC framework can be found at "Base correlation generation.xlsm" at www.wiley.com/go/correlationriskmodeling under "Chapter 7."

7.2 MARKOV CHAIN MODELS

In this section, we discuss two models that generate correlation in the Markov chain framework.

7.2.1 Inducing Correlation via Transition Rate Volatilities

Philipp Schönbucher (2006) generates different transition and default correlation properties via different transition rate[2] volatilities in a time-inhomogeneous, finite-state Markov chain[3] framework. The model is inspired by the Heath-Jarrow-Morton (HJM) (1992) interest rate model. Whereas the HJM model generates an interest rate term structure at future times t, Schönbucher creates a stochastic evolution of transition rates to derive the loss distribution at future times t. In analogy to the HJM model, Schönbucher applies the current (time 0) term structure of transition rates as inputs. Hence the model does not require any calibration.

Specifically, the model consists of a time-inhomogeneous, hypothetical Markov chain of cumulative losses $L(t), t \geq 0$ with discrete states $\{0, 1, 2,..., I\}$ of the I entities of the portfolio. The generator matrix[4] $A(t,T)$, $t \leq T$, of transition probabilities satisfies the usual conditions; see Jarrow et al. (1997) on deriving the risk-neutral generator matrix for continuous and discrete time. Integrating the Kolmogorov differential equations $\frac{dP(t,T)}{dT}\frac{1}{P(t,T)} = A(T)$, we find the transition probability matrix $\Lambda(t,T)$, which reproduces the loss distribution $\mathbf{p}(t,T) = (p_0(t,T),..., p_I(t,T))$. The components of the loss distribution, the probabilities $p_n(t,T)$, are set so that

$$p_n(t, T) = P[L(T) = n|F_t] \tag{7.3}$$

2. Transition rates are probabilities to move from one credit state to another.

3. A Markov chain is a stochastic "memoryless" process. This means that only the present information, not the past, is relevant. A discrete Markov process is referred to as a Markov chain, although occasionally authors (such as Jarrow et al. 1997) use continuous time when referring to a Markov chain.

4. A generator matrix is a "starting matrix," which serves as a basis to derive matrices at future times. In our context the matrices are transition matrices.

That is, $p_n(t,T)$ represents the probability of exactly n losses in the portfolio, viewed at time t for maturity T. F_t is the filtration, which contains all events.

In order to create a no-arbitrage framework and a unique correspondence of transition probabilities to the loss distribution, Schönbucher initially allows only one-step transitions (i.e., only to the next lower rating class). Therefore, the transition probability matrix at time t for maturity T, $\Lambda(t,T)$, contains only zero entries, except on the diagonal and directly adjacent higher nodes:

$$\Lambda(t,T) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & .. & 0 & 0 \\ 0 & a_{2,2} & a_{2,3} & .. & 0 & 0 \\ : & : & : & .. & : & : \\ 0 & 0 & 0 & .. & a_{I-1,I-1} & a_{I-1,I} \\ 0 & 0 & 0 & .. & 0 & 0 \end{pmatrix}$$

where $a_{i,i}$, $0 \le i \le I$ is the probability of staying in the same state and $a_{i,j}$, $0 \le j \le I$ is the transition probability of moving from state i to state j. The transition probabilities evolve stochastically in time to reproduce the arbitrage-free term structure of loss distributions $\mathbf{p}(t,T)$ at future times t with maturity T. In particular, the transition probability of entity n, seen at time t for maturity T, $a_n(t,T)$, $0 < n <$ I, follows a standard generalized Wiener process; that is,

$$da_n(t,T) = \mu_{a_n}(t,T)dt + \sigma_{a_n}(t,T)dz \tag{7.4}$$

where μ_{a_n} is the drift rate of a_n, and σ_{a_n} is the volatility of a_n. Equation (7.4) brings us to the correlation properties of the model. Default correlation is induced by the dynamics of the transition volatility $\sigma_{a_n}(t,T)$. Schönbucher specifies a parameter constellation in which an increase in the factor loading of the transition rates a_n increases the volatility of a_n, and vice versa. Importantly, in this framework, a higher volatility of a_n means a higher transition rate of all entities n to a lower state, hence a higher default correlation; conversely, a lower volatility of a_n means a lower transition rate of all entities n to a lower state, hence lower default correlation. The model can also replicate local correlation by specifying a higher volatility, hence higher correlation only for a short period of time (i.e., $\frac{\partial\sigma_{a_n}(t,T)}{\partial t} = x$ for the current time t) and a lower correlation for a future time $t + dt$ $\frac{\partial\sigma_{a_n}(t+dt,T)}{\partial t} = y$, where $x > y$.

7.2.2 Inducing Correlation via Stochastic Time Change

To the best of our knowledge, it was Peter Clark (1973) who first applied stochastic time processes to financial modeling. Clark proposed a stochastic time process $T(t)$ with independent increments drawn from a lognormal distribution. $T(t)$ is a *directing process*, a stochastic clock that determines the speed of the evolution of the stock price process $S(t)$, forming the new process $S(T(t))$. This new process $S(T(t))$ serves as a *subordinator process* for the stock price process $S(t)$. Clark finds that the subordinated distributions can explain future cotton prices better than alternative standard distributions.

The variance-gamma model of Dilip Madan, Carr, and Chang (1998) applies stochastic time change to option pricing, generalizing previous work by Madan and Seneta (1990) and Madan and Milne (1991). The model consists of a standard Brownian motion, whose drift μ, however, is evaluated at random time changes t, which are modeled by a gamma process. The model has the same subordinated structure as Clark (1973):

$$S(t;\mu,\sigma,\upsilon) = \mu\,\Gamma(t;1,\upsilon) + \sigma\,dz\,(\Gamma(t;1,\upsilon)) \tag{7.5}$$

with variables as defined in equation (7.4), and $\Gamma(t;1,\upsilon)$ is a gamma distribution with unit mean and variance υ. By controlling the skew via μ and the kurtosis via υ, the model is able to match volatility smiles in the market well. Further models that apply stochastic time change to option pricing are Geman, Madan, and Yor (2001), Carr et al. (2003), and Cont and Tankov (2004).

The stochastic time models just discussed help to explain certain phenomena in financial practice. In the following, we discuss Hurd and Kuznetsov (2006a, 2006b), who were the first to induce correlation via stochastic time change. Their time-homogeneous Markov chain model of K discrete rating classes $Y_t \in \{1, 2,..., K)$ assumes that transition and default intensities are identical for entities in the same rating category. Hence the model does not directly reference individual transition and default intensities, and therefore it qualifies primarily as top-down.

At the core of the model is a continuous-time, time-homogeneous Markov chain with time-constant generator matrix L_Y:

$$L_Y = \begin{pmatrix} l_{1,1} & l_{1,2} & l_{1,3} & .. & l_{1,K} \\ l_{2,1} & l_{2,2} & l_{2,3} & .. & l_{2,K} \\ : & : & : & .. & : \\ l_{K-1,1} & l_{K-1,2} & l_{K-1,3} & .. & l_{K-1,K} \\ 0 & 0 & 0 & .. & 0 \end{pmatrix}$$

where K is the absorbing bankruptcy state. This means that once an entity has defaulted, it stays in default. This is not necessarily the case in the United States, where many companies emerge from bankruptcy. Hence the model could be extended to include a nonabsorbing bankruptcy state. The $l_{i,j}$, $i \in \{1, 2,\ldots, K-1)$, $j = \in \{1, 2,\ldots, K)$, are the instantaneous transition intensities of migrating from rating class i to j under the historical (real-world or reference) measure P.

Hurd and Kuznetsov further introduce a vector-valued process:

$$X_t = \{r_t, u_t, \lambda_t\} \tag{7.6}$$

where r_t is the risk-free interest rate, the recovery rate is $R_t = e^{-u_t}$ and, importantly, λ_t is the stochastic migration intensity process. The vector $\mathbf{X}_t$ captures macroeconomic data and represents a common factor that affects all entities. The credit migration process of the rating classes $Y_t \in \{1, 2,\ldots, K\}$ is conditioned on the vector $\mathbf{X}_t$. Hence the Y_t are conditionally independent, applying the conditionally independent default (CID) approach, discussed in Chapter 6. Hence $\mathbf{X}_t$ has an interpretation similar to that of the scalar M in equation (6.1), $x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$. The main motivation for this approach is again to reduce complexity.

More specifically, the correlation dynamics of the model can be derived by a probability measure change. From the generator matrix L_Y we have

$$E^P(Y_{t+dt} = j | Y_t = i) = l_{ij}\, dt \tag{7.7}$$

where P is the historical probability measure. Hurd and Kuznetsov now introduce a time-changed process, a stochastic clock τ_t, which may have continuous components and jump components. τ_t is a function of λ_t,

$$\tau_t = \int_0^t \lambda_s\, ds \tag{7.8}$$

Under the Girsanov theorem (see Neftci 1996 for an intuitive discussion), we can define a new stochastic process under the risk-neutral measure Q with the changed drift (and jump) but constant volatility:

$$E^Q(Y_{t+dt} = j | Y_t = i) = l_{ij}\, \lambda_t\, dt \tag{7.9}$$

Since λ_t is an element of the conditioning market factor X_t [see equation (7.6)], the migration processes Y_t in the new process under Q are now dependent. Importantly, from equations (7.8) and (7.9) we observe that default correlation is induced by the speed of the stochastic clock τ_t. An increase in the speed of the clock increases the speed of migration of all entities and hence increases the probability of simultaneous defaults. If the stochastic clock jumps, the probability of simultaneous defaults is even higher.

We find that the induction of correlation via volatility changes (Schönbucher 2006) and the induction of correlation via stochastic time change have a similar interpretation. An increase in transition volatility as well as an increase in the stochastic clock both increase the migration within the transition matrix and hence increase the probability of simultaneous defaults, and vice versa.

7.3 CONTAGION DEFAULT MODELING IN TOP-DOWN MODELS

In a popular credit risk model applied in financial practice, Giesecke, Goldberg, and Ding (2011) derive a random thinning process, which allocates the portfolio intensity to the sum of the individual entities' intensities. Giesecke et al. show that this process uniquely exists and can be realized analytically. More formally, the thinning process Z^k under the reference measure μ is predictable and has the form

$$Z^k = \frac{\lambda^k}{\lambda} \tag{7.10}$$

where λ is the portfolio default intensity and λ^k is the default intensity of entity k, $k = 1,..., m$ and $\lambda = \lambda^1 +...+ \lambda^m$. The model has the property that the thinning processes add up to one, $\sum_{k=1}^{m} Z^k = 1$ and that immediately after entity k defaults, the thinning process of this entity k drops to zero, $Z^k_{\tau^{k+}} = 0$.

The thinning process and a resulting basic default dependence can be explained with an $m = 2$ entity portfolio with an assumed loss distribution N of

$$P(N_T = n) = \begin{cases} (.) & \text{for } n = 0 \\ 1 - e^{-T} & \text{for } n = 1 \\ 1 - e^{-T} - Te^{-T} & \text{for } n = 2 \end{cases} \tag{7.11}$$

where P is an integrable probability measure, n is the number of entities defaulting with the associated probability in (7.11), and N is a Poisson process with stopping time T^2. The thinning process can be parameterized with a nonnegative constant q^{k1}.

$$Z_t^k = \frac{\lambda_t^k}{\lambda_t} = \begin{cases} q^{k1} & \text{for } t \leq T^1 \\ 1_{\tau^{k1}=T^1} & \text{for } T^1 < t \leq T^2 \\ 0 & \text{for } T^2 < t \end{cases} \tag{7.12}$$

From equation (7.12) we observe that the thinning process Z_t^k equals q^{k1} before or at time T^1 and equals 1 if the first entity defaults before or at T^1 since $\sum_{k=1}^{m} Z^k = 1$ and $Z_{\tau^{k+}}^k = 0$, as can be seen above. The parameter q^{k1} governs the joint default dependence structure via $P(\tau^1 \leq T \cap \tau^2 \leq T) = 1 - e^{-T} - (1 - q^{k1})Te^{-T}$. From this equation and equation (7.11), we see that the extreme values of $q^{k1} = 1$ and $q^{k1} = 0$ generate the probability of exactly one default or two defaults, respectively. The name of the entity that defaults is revealed at the default time, highlighting the fact that random thinning allocates the portfolio intensity to the individual entities.

To incorporate a more rigorous joint default dependency, Giesecke et al. (2009) suggest that the joint default distribution is governed by the portfolio intensity λ. In particular, Giesecke et al. suggest that the process of the portfolio default intensity λ has an exponentially mean-reverting drift with a stochastic jump component, which models default contagion:

$$d\lambda_t = g(\lambda_L - \lambda_t) + \delta \, dJ_t \tag{7.13}$$

where J is the jump with magnitude $\delta \geq 0$ at default of an entity. The jump elevates the level of the portfolio default intensity λ (i.e., the default intensity of all entities). After the jump, the contagion reverts exponentially at rate $g \geq 0$ (gravity) to its long-term noncontagious mean λ_L.

In Chapter 4, section 4.4, we discussed contagion default modeling in a bottom-up framework. In this framework, the contagion is modeled at an individual entity level; that is, the default of entity i directly impacts the default intensity of entity j. This had led to problems of circularity, which significantly complicates the derivation of a joint default distribution. In the top-down environment, the default contagion is modeled conveniently at a portfolio level, circumventing the problem of circularity.

Calibrating the parameters in equation (7.13) and those of the thinning process to the CDX high yield index during the crisis in September 2008, Giesecke et al. (2009) find that their model outperforms copula-based hedges. In addition, the mean profit is higher than when using the copula approach.

In an extension to an early version of the Giesecke et al. (2009) model, Giesecke and Tomecek (2005)[5] incorporate a stochastic time change. However, in contrast to Hurd and Kuznetsov (2006a, 2006b), where stochastic time change is applied to induce correlation, Giesecke and Tomecek 2005 utilize the stochastic time change to transform a standard Poisson into a counting process N of default arrival times T_n. The counting process is represented by a standard Poisson process of the form

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_n \leq t\}} \tag{7.14}$$

where

$$S_n = \sum_{i=1}^{n} V_i \tag{7.15}$$

and the V_i are independent and identically distributed (i.i.d.); in particular the V_i are exponential random variables.

The continuous process $G(t) = \int_0^t \lambda_s ds$ defines the time change. G is adapted to the filtration $G = (G_t)t \geq 0$, where G_t represents all information available at time t. Hence the process $G(t)$ is predictable.

The Poisson process (7.14) is mapped to arrival times T_n by the inverse of the time change process G. Hence,

$$T_n = G^{-1}(S_n) \tag{7.16}$$

For a rigorous proof, see Giesecke et al. (2009). Equation (7.16) implies that the Poisson arrivals S_n serve as a Merton-style barrier to derive the arrival times T_n:

$$T_n = G^{-1}(S_n) = \inf\{t : G(t) \geq S_n\} = \inf\left\{t\text{:}\ \int_0^t \lambda_s ds \geq S_n\right\} \tag{7.17}$$

Since $G(t)$ and S_n are generated independently, the model has the form of a doubly stochastic process.

We observe that generating the default time in the copula approach, which we derived in Chapter 5, equation (5.5) $\tau_{i,t} = 1_{\{N^{-1}(\lambda_{i,t}) > M_{n,t}(\cdot)\}}$, is conceptually similar to equation (7.17). However, in equation (5.5), the default time is modeled individually for each entity i with respect to the

5. The first version of Giesecke et al. (2009) model was published in 2004.

entities' default intensity function λ_i. In the top-down approach (7.17), the intensity λ is modeled at a portfolio level. A further difference is that in equation (5.5) the default correlation is elegantly incorporated in the barrier $M_{n,R}(\cdot)$. In the approach (7.17), the default correlation is modeled separately in the core equation (7.17). One benefit of the model (7.17) is that by construction the default times T_n are ordered; that is, $T_1 = \min(T_k)$ and $T_m = \max(T_k)$, $k = 1,\ldots, m$. In the copula model the default distribution is built by numerical integration over unordered default times; refer back to section 7.1.

7.4 SUMMARY

A fairly new, mathematically quite intensive class of correlation models are top-down approaches. In this framework, the evolution of the portfolio intensity distribution is derived directly (i.e., abstracting from the individual entities' default intensities). Top-down models are typically applied if:

- The default intensities of the individual entities are unavailable or unreliable.
- The default intensities of the individual entities are unnecessary. This may be the case when evaluating a homogeneous portfolio such as an index of homogeneous entities.
- The sheer size of a portfolio makes the modeling of individual default intensities problematic.

Vasicek's large homogeneous portfolio (LHP) can be considered a top-down model, since it assumes (1) a constant and identical default intensity of all entities in a portfolio and (2) the same default correlation between the entities. The model is a one-factor version of the Gaussian copula. The model is currently (year 2013) the basis for credit risk management in the Basel II and III accords. The benefits of the model are simplicity and intuition. One of the main shortcomings of the model is that traders randomly alter the correlation parameter for different tranches to achieve desired tranche spreads. Conceptually, however, the correlation parameter should be identical for the entire portfolio.

Within the top-down framework, Philipp Schönbucher (2006) creates a time-inhomogeneous Markov chain of transition rates. Default correlation is introduced by changes in the volatility of transition rates. For certain parameter constellations, higher volatility means faster transition to lower states such as default, and hence implies higher default correlation (and vice versa). Similarly, Hurd and Kuznetsov (2006a; 2006b) induce correlation by a random change in the speed of time. A faster speed of time means faster transition to a lower state, possibly default, and hence increases the default correlation (and vice versa).

In conclusion, top-down models are attractive, elegant, and mathematically rigorous correlation models. They can be applied if a portfolio is highly homogeneous with respect to default probabilities and default correlation. The models do depend on reliable transition data as inputs and come at the cost of relatively high mathematical and computational complexity.

PRACTICE QUESTIONS AND PROBLEMS

1. What is the difference between bottom-up and top-down correlation models?
2. For which types of portfolios are top-down correlation models appropriate?
3. Why can the one-factor Gaussian copula (OFGC) be considered a top-down model?
4. Markov processes are "memoryless." What does this mean? Give an example.
5. What is a transition rate?
6. Why does a higher transition rate volatility mean higher default correlation in the Schönbucher 2006 model?
7. Why does an increase in stochastic time change mean a higher default correlation in the Hurd-Kuznetsov 2006 model?
8. What is the random thinning process in top-down models, and what does it accomplish?

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CHAPTER 8

Stochastic Correlation Models

I think correlation modeling is basically at the stage volatility modeling was about 15 years ago.

—Vladimir Piterbarg

In finance, many variables such as equities, bonds, commodities, exchange rates, interest rates, volatility, and more are often modeled with a stochastic process. In addition, from our empirical Chapter 2 we derived that financial correlations behave somewhat erratically and randomly. Therefore it seems like a good idea to model financial correlations with a stochastic process.

The modeling of financial correlation with a stochastic process is fairly new, but several promising approaches exist. Let's discuss them. But before we do, let's look at some basics.

8.1 WHAT IS A STOCHASTIC PROCESS?

The reader, who made it all the way to this Chapter 8, hopefully has a good idea what a stochastic process is. But let's have a closer look. Let's start with a deterministic process. A deterministic process is a process with a known outcome. For example, counting numbers by one, or the movement of the sun are deterministic processes. The opposite of a deterministic process is a stochastic process, also called a *random process*. Hence, heuristically (which means nonmathematically), we can define a stochastic process.

STOCHASTIC PROCESS

A process with an unknown outcome.

Examples of a stochastic process are the flipping of a coin or the rolling of a die. Most stochastic processes display the *Markov property*, meaning they are "memoryless." For example, even if the last three rolls of a die all resulted in a 6, the probability of rolling a 6 again is still 1/6.

The formal definition of a stochastic process is not overly enlightening. A stochastic process is simply defined as a collection of ordered random variables X at time t:

$$\{X_t, t \in T\} \tag{8.1}$$

where

X_t: state of the random variable at time t (for example heads or tails when flipping a coin)
T: points in time; for a discrete stochastic process, $T \in N = \{0, 1, 2, 3...\}$.

For a nice rigorous paper on stochastic processes, see Nualart (2008) or Lamberton and Lapeyre (1996).

The terms *stochastic process* and *stochastic model* are often used synonymously, or a stochastic process can be part of a stochastic model.

In finance, one popular stochastic model is the geometric Brownian motion (GBM), which we discussed in Chapter 4, equation (4.1). The stock price in the famous Black-Scholes-Merton 1973 option pricing model follows a GBM. Slightly rewriting equation (4.1), we have

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz \tag{8.2}$$

where

S: variable that follows the GBM, for example a stock price
μ: expected growth rate of S
σ: expected volatility of S
dz: Wiener process or Brownian motion $dz = \varepsilon\sqrt{dt}$, where ε is a random drawing from a standard normal distribution with a mean of 0 and a standard deviation of 1; formally, $\varepsilon = n \sim (0,1)$

Therefore, stochasticity enters equation (8.2) via ε. For a model that generates the GBM in equation (8.2), and equation (8.2) with random jumps, see "GBM path with jumps.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 8."

In the recent past, several stochastic correlation models have been suggested. Let's discuss them.

8.2 SAMPLING CORRELATION FROM A DISTRIBUTION (HULL AND WHITE 2010)

A simple way to model correlation as a stochastic variable is to sample it from a statistical distribution. In finance, we often sample from a standard normal distribution. For example, the ε in the dz term of equation (8.1) is such a sample. However, in some research, samples from other distributions are taken. In the variance-gamma model, introduced by Madan, Carr, and Chang (1998) and briefly discussed in Chapter 7, section 7.2.2, a value from the gamma distribution is sampled at random times t to create a stochastic drift rate of the underlying variable.

Hull and White (2010) extend their dynamic OFGC model, which we discussed in Chapter 6, section 6.5. The core equation is

$$dz_i(t) = \sqrt{\rho(t)}dM(t) + \sqrt{1-\rho(t)}dZ_i(t) \tag{8.3}$$

where

ρ: Asset correlation parameter for the assets of the companies in the portfolio, $0 \leq \rho \leq 1$. ρ is assumed identical and constant for all company pairs in the portfolio.

M: Systematic market factor, which impacts all companies in the portfolio. M can be thought of as the general economic environment, for example, the return of the S&P 500. M is a random drawing from a standard normal distribution, formally $M = n \sim (0, 1)$. M is the same as ε in Chapter 4, section 4.1.

Z_i: Idiosyncratic factor of asset i. Z_i expresses ith company's individual strength, possibly measured by company i's stock price return. Z_i is a random drawing from a standard normal distribution.

z_i: Results from equation (8.3) and is interpreted as a "Default indicator variable" for company i. The higher z_i, the less likely is the default of company i at a certain time T. Hence, z_i is also interpreted as the *asset value* of company i. z_i is by construction standard normal.

Replacing $\sqrt{\rho(t)} = \alpha(t)$ in equation (8.3) to allow for negative correlation between $M(t)$ and $z(t)$, we derive

$$dz_i(t) = \alpha_i(t)dM(t) + \sqrt{1-\alpha^2(t)}dZ_i(t) \tag{8.4}$$

Hull and White now introduce stochastic correlation by sampling $\alpha(t)$ from a beta distribution. This sample serves to create a dependency between $\alpha(t)$ and $dM(t)$. To match empirical credit default index (CDX) prices, they choose the dependency to be $-\sqrt{0.5}$. This creates a positive relationship between α and default probability: If α increases, the market environment dM decreases. This increases the default probability of company i.

The asset correlation between companies i and j is $\alpha_i\ \alpha_j$ (see Chapter 12, section 12.6 for details). A higher α_i or α_j means a higher asset correlation between i and j, and, as derived above, it means a higher joint default probability. This relationship of higher asset correlation and higher default probability was empirically verified by Servigny and Renault (2002) and Das et al. (2006).

Therefore it is not surprising that the approach with the stochastic correlation sample $\alpha(t)$ in the model of equation (8.4) is able to match empirical CDX prices in most cases significantly better than is the case without stochastic correlation; see Hull and White (2010) for details.

8.3 DYNAMIC CONDITIONAL CORRELATIONS (DCCs) (ENGLE 2002)

In equation (3.3) we had defined the Pearson correlation coefficient for a random variable as

$$\rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}} \tag{3.3}$$

Assuming the variables X and Y have a mean of zero [i.e., $E(X) = E(Y) = 0$], equation (3.3) reduces to

$$\rho(X,Y) = \frac{E(XY)}{\sqrt{E(X^2)E(Y^2)}} \tag{8.5}$$

In 2002 Robert Engle introduced dynamic conditional correlations (DCCs) in a model developed by Tim Bollerslev in 1990. The correlation at time t, ρ_t, is conditioned on the information given in the previous period $t-1$. Hence equation (8.5) changes to

$$\rho_t(r_1, r_2) = \frac{E_{t-1}(r_{1,t}r_{2,t})}{\sqrt{E_{t-1}(r_{1,t}^2)E_{t-1}(r_{2,t}^2)}} \tag{8.6}$$

where r is the variable of interest. $r_{1,t}$ may be the return of asset 1, and $r_{2,t}$ may be the return of asset 2 at time t. See Chapter 1, section 1.3.1 on returns.

Conditional correlation is a concept within the Nobel Prize-rewarded autoregressive conditional heteroscedasticity (ARCH) framework (Engle 1982), which was extended to generalized autoregressive conditional heteroscedasticity (GARCH) by Bollerslev (1986).

In the ARCH framework, the variable of interest, the return r, is defined as the product of its standard deviation and an error term.

$$r_{i,t} = \sigma_{i,t}\varepsilon_{i,t} \tag{8.7}$$

where

$r_{i,t}$: return of asset i at time t

$\sigma_{i,t}$: standard deviation of the return of asset i at time t (also called volatility)

$\varepsilon_{i,t}$: random drawing of a standard normal distribution for asset i and time t, $\varepsilon = n \sim (0,1)$

The variance σ^2 or the standard deviation σ in equation (8.7) is modeled with an ARCH process (or one of many extensions such as GARCH, NGARCH, EGARCH, TGARCH,[1] and more) of the form

$$\sigma_t^2 = a_0 + a_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \ldots + a_q\sigma_{t-q}^2\varepsilon_{t-q}^2 \tag{8.8}$$

where $a_0 > 0$, $a_1 \geq 0$, so that σ^2 is positive and $q \in N$.

From equation (8.8) we can observe that the variance is a function of past error terms ε. The error term ε is typically derived from a linear regression of the underlying variable of interest, which in equation (8.7) is the return of asset i. The critical idea in equation (8.8) is that if the past error terms ε_{t-x} are high, so will be the future variance at time t, σ_t^2. The model of equation (8.8) and extensions of the model have been successfully tested; see for example Enders (1995) and Hacker and Hatemi-J (2005). The main contribution of the ARCH and GARCH approach is that the empirical persistence or clustering of volatility can be modeled: In reality, high volatility often persists for a certain period of time, and low volatility often persists for a certain period of time.

1. The N in NGARCH stands for nonlinear, the E in EGARCH stands for exponential, and the T in TGARCH stands for truncated. See Bollerslev (2008) for a nice overview of all ARCH extensions.

The correlation at time t in equation $\rho_t(r_1, r_2) = \frac{E_{t-1}(r_{1,t}r_{2,t})}{\sqrt{E_{t-1}(r_{1,t}^2)E_{t-1}(r_{2,t}^2)}}$ depends on information given at time $t-1$. Given certain assumptions, the correlation can be expressed purely in error terms ε. Let's show this. We express the variance of returns as a function of the return, given the information in the previous period.

$$\sigma_{i,t}^2 = E_{t-1}(r_{i,t}^2) \tag{8.9}$$

Assuming the return r is standard normal, we have $E_{t-1}(r_{i,t}^2) = 1$, and from equation (8.9) $\sigma_{i,t}^2 = 1$. Hence from equation (8.7) we derive that the return r is just the error term ε, and together with $E_{t-1}(r_{i,t}^2) = 1$, equation (8.6) reduces to

$$\rho_t(r_1, r_2) = E_{t-1}(\varepsilon_{1,t}, \varepsilon_{2,t}) \tag{8.10}$$

The conditional correlation expressed in equations (8.6) and (8.10) correlates just two assets, 1 and 2. The model can be generalized to include multiple assets. In this case, we derive the conditional correlation matrix R, which contains the pairwise conditional correlations $\rho_t(r_{ij})$ between the n asset returns. Formally, from equation (8.10) we have

$$E_{t-1}(\varepsilon_{i,t}, \varepsilon_{j,t}) = R_{ij} \tag{8.11}$$

where R: conditional correlation matrix containing the pairwise conditional correlations of the returns of the assets $i = 1,\ldots, n$.

In equation (8.11), the correlation matrix R_{ij} is constant. The approach can be made dynamic; that is, R_{ij} can be time varying, $R_{ij}(t)$. This constitutes *dynamic* conditional correlations (DCCs), as suggested by Engle (2002). Parameterization of the dynamic conditional correlation matrix $R_{ij}(t)$ in a GARCH framework can be achieved by exponential smoothing, with certain parameter constellations allowing mean reversion of the matrix process. See Engle (2002) for details.

8.4 STOCHASTIC CORRELATION—STANDARD MODELS

In this section, we introduce three approaches that model stochastic correlation. The three models are quite closely related.

8.4.1 The Geometric Brownian Motion (GBM)

The geometric Brownian motion, whose basic idea was derived by the biologist Robert Brown in 1827, has already been mentioned several times in this book; see Chapter 4, equation (4.1), and in this Chapter 8, equation (8.2).[2] In finance, variables such as stocks, bonds, commodities, interest rates, and volatility are often modeled with the GBM. When modeling correlation with the GBM, we derive

$$\frac{d\rho}{\rho} = \mu\, dt + \sigma\varepsilon\sqrt{dt} \tag{8.2}$$

where

ρ: correlation between two or more variables
μ: expected growth rate of ρ
σ: expected volatility of ρ
ε: random drawing from a standard normal distribution; formally, $\varepsilon = n \sim (0,1)$

We can compute ε as =normsinv(rand()) in Excel/VBA and norminv(rand) in MATLAB.

Düllmann, Küll, and Kunisch (2008) model correlation with equation (8.2). They study whether stock prices or default rates can better estimate asset correlations. Applying stochastic asset correlation in equation (8.2) rather than constant asset correlation, they find that the stochastic correlation model weakens but does not reject the result that stock prices are superior for estimating asset correlations compared to default rates.

Is the GBM in equation (8.2) a good approach to model correlation? It actually has two limitations:

1. Equation (8.2) is not bounded, meaning correlation ρ can take values bigger than 1 and smaller than –1. From equation (8.2) we see that a value of $\rho > 1$ is more likely to happen when the growth rate μ is high, if the volatility σ is high, and if we have a high value of ε in a simulation.

2. It was actually the Dutch biologist Jan Ingenhousz who first published papers in German and French in 1784 and 1785 on the dispersion of charcoal particles in alcohol. Therefore, he should be credited for what is known today as the Brownian motion.

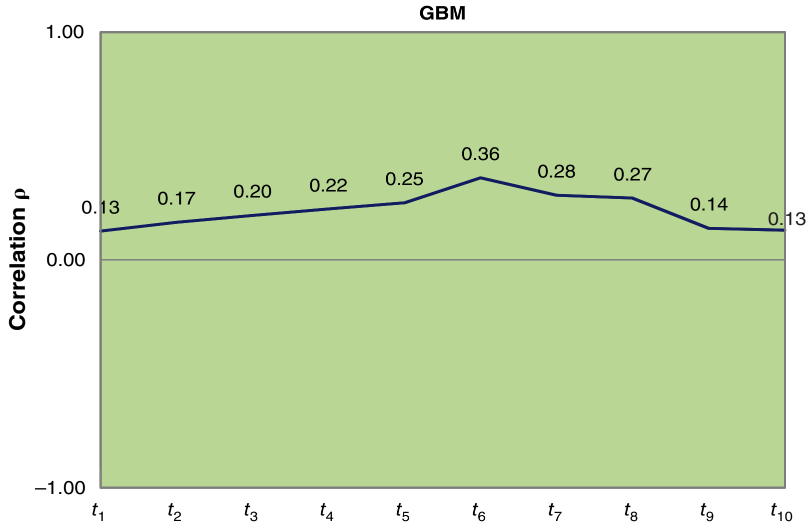

FIGURE 8.1 Sample Correlation Path for 10 Time Steps for Equation (8.12)
The parameter values are $\mu = 1\%$, $\sigma = 30\%$, and $\Delta t = 1$. Starting value in t_0 is 0.1.

Conversely, values of $\rho < -1$ are more likely to occur for low values of μ and high values of σ and ε.

2. Mean reversion (i.e., the tendency of correlation to revert back to its mean) is not modeled with equation (8.2). In the empirical Chapter 2, we derived that financial correlations exhibit strong mean reversion.

For computational purposes, we discretize equation (8.11). With $d\rho = \rho_{t+1} - \rho_t$, we derive

$$\rho_{t+1} = \rho_t + \rho_t\,\mu\Delta t + \rho_t\,\sigma\varepsilon_t\sqrt{\Delta t} \tag{8.12}$$

Figure 8.1 shows a sample path of the GBM.

In Figure 8.1, at each time step, equation (8.12) is applied. The different values for correlation at each time step occur since the random drawing ε is different at each t.[3] For details, see the spreadsheet "Stochastic correlation.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 8."

3. Although the correlation values at t_1 and t_{10} are both 0.13, ε in t_1 and t_{10} are different since ρ increases from t_0 to t_1 from 0.1 to 0.13 and decreases from t_9 to t_{10} from 0.14 to 0.13.

8.4.2 The Vasicek 1977 Model

Another approach to model stochastic correlation is what is known as the Vasicek 1977 model, which, however, should be credited to Uhlenbeck and Ornstein (1930). The model is

$$d\rho = a(m_\rho - \rho_t)dt + \sigma_\rho\, \varepsilon_t \sqrt{dt} \tag{8.13}$$

where a is the mean reversion speed (gravity) (i.e., degree with which the correlation at time t, ρ_t, is pulled back to its long-term mean m_ρ); a can take the values $0 \leq a \leq 1$. m_ρ is the long-term mean of the correlation ρ. Other variables are defined as in equation (8.2).

Equation (8.13) is an improvement to the GBM in equation (8.2) since it includes mean reversion, the tendency of a variable to be pulled back to its long-term mean. We derived in Chapter 2 that financial correlations exhibit strong mean reversion.

The limitation of the Vasicek 1977 model with respect to modeling correlation is that the model is not bounded; correlation values bigger than 1 and smaller than –1 can occur. These values are more likely to occur when mean reversion a is low and volatility σ_P is high.

For computational reasons, we again discretize. With $d\rho = \rho_{t+1} - \rho_t$, equation (8.13) then becomes:

$$\rho_{t+1} = \rho_t + a(m_\rho - \rho_t)\Delta t + \sigma_\rho\, \varepsilon_t \sqrt{\Delta t} \tag{8.14}$$

Figure 8.2 shows a sample path of the Vasicek model.

Comparing Figures 8.1 and 8.2, we observe the higher volatility in Figure 8.2. This is mainly because the relative change $d\rho/\rho$ is modeled in Figure 8.1, whereas the absolute change $d\rho$ is modeled in Figure 8.2; compare equations (8.2) and (8.13).

8.4.3 The Bounded Jacobi Process

The two approaches that we have introduced so far, the geometric Brownian motion and the Vasicek model, both have the limitation that correlation values can become bigger than 1 and smaller than –1. This is an undesired property if the correlation is modeled in the Pearson correlation framework, where the correlation coefficient is bounded between –1 and +1.

A model that can comply with correlation bounds is the bounded Jacobi process.[4] Applying the bounded Jacobi process to correlation, we derive

$$d\rho = a(m_\rho - \rho_t)dt + \sigma_\rho \sqrt{(h - \rho_t)(\rho_t - f)}\varepsilon_t \sqrt{dt} \tag{8.15}$$

4. For a nice paper on the Jacobi process, see Gourieroux and Valery (2004).

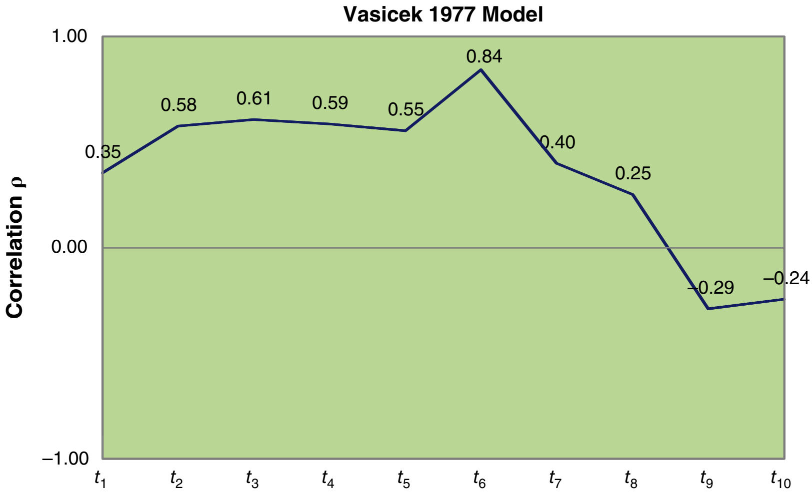

FIGURE 8.2 Sample Correlation Path for 10 Time Steps for Equation (8.14)
The mean reversion parameter $a = 30\%$, and the long-term mean $m_\rho = 10\%$. The other parameter values are the same as in Figure 8.1: Volatility $\sigma = 30\%$ and $\Delta t = 1$. Starting value in t_0 is 0.1.

where h is the upper boundary level, and f is the lower boundary level (i.e., $h \geq \rho \geq f$). Other variables are defined as in equations (8.2) and (8.13).

With equation (8.15) the user can choose specific upper and lower boundaries. For correlation modeling in the Pearson framework, these boundaries are $h = +1$ and $f = -1$. In this case equation (8.15) reduces to

$$d\rho = a(m_\rho - \rho_t)dt + \sigma_\rho \sqrt{(1-\rho_t^2)}\varepsilon_t \sqrt{dt} \tag{8.16}$$

Equation (8.15) requires correlation values within a lower bound f and an upper bound h (otherwise the term $\sqrt{(h-\rho_t)(\rho_t - f)}$ cannot be evaluated). Equation (8.16) requires correlation values within the bounds −1 to +1 (otherwise the term $\sqrt{(1-\rho_t^2)}$ cannot be evaluated). However, for low mean reversion levels a and high volatility σ_P, it can happen that the model generates correlation levels smaller than −1 and higher than +1. Therefore we have to introduce boundary conditions. These boundary conditions for equation (8.15) are

$$\alpha \geq \frac{\sigma^2(h-f)/2}{(m_\rho - f)} \tag{8.17}$$

for the lower bound f and

$$\alpha \geq \frac{\sigma^2 (h - f)/2}{(h - m_\rho)} \tag{8.18}$$

for the higher bound h.

Applying the boundary levels $f = -1$ and $h = +1$, we derive the boundary levels for equation (8.16) as

$$\alpha \geq \frac{\sigma^2}{(m_\rho + 1)} \tag{8.19}$$

for the lower bound f and

$$\alpha \geq \frac{\sigma^2}{(1 - m_\rho)} \tag{8.20}$$

for the higher bound h.

From equations (8.17) to (8.20) we observe the intuitive feature that the bounds are more likely to be satisfied for high values of mean reversion a and low values of volatility σ. See Emmerich (2006) and Wilmott (1998) for the derivation of the boundaries.

Ma (2009) applies the bounded Jacobi process to value correlation dependent quanto options and multi-asset options. This inclusion of stochastic correlation in the Black-Scholes-Merton model improves the valuation of these options compared to the standard Black-Scholes-Merton model.

Discretizing equation (8.16), again applying $d\rho = \rho_{t+1} - \rho_t$, we derive

$$\rho_{t+1} = \rho_t + a(m_\rho - \rho_t)\Delta t + \sigma_\rho \sqrt{(1 - \rho_t^2)}\varepsilon_t \sqrt{\Delta t} \tag{8.21}$$

Figure 8.3 shows a sample path of the equation (8.21).

Comparing Figures 8.2 and 8.3, we observe somewhat minor differences between the correlation modeling with Vasicek in equation (8.14) and the bounded Jacobi process in equation (8.21). The correlation models introduced so far in this chapter can be found in the spreadsheet "Stochastic correlation.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 8."

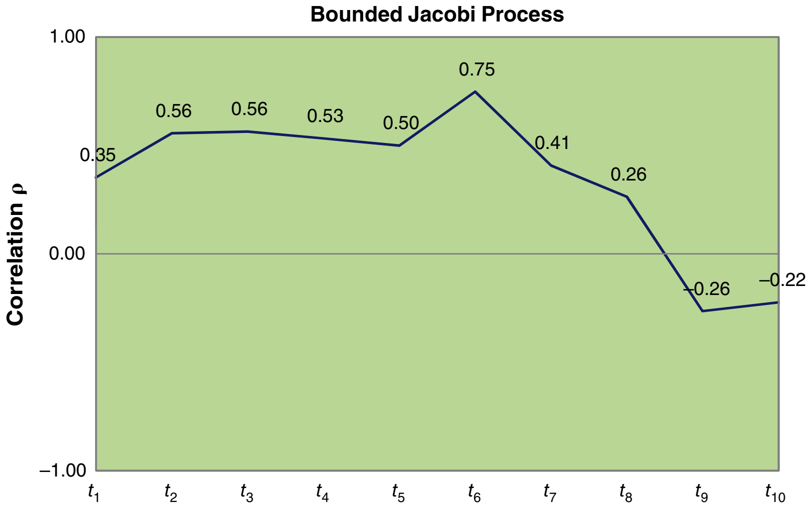

FIGURE 8.3 Sample Correlation Path for 10 Time Steps for Equation (8.21)
As in Figures 8.1 and 8.2, the mean reversion parameter $a = 30\%$, volatility $\sigma = 30\%$, the long-term mean $m_\rho = 10\%$, and $\Delta t = 1$. Starting value in t_0 is 0.1.

8.5 EXTENDING THE HESTON MODEL WITH STOCHASTIC CORRELATION (BURASCHI ET AL. 2010; DA FONSECA ET AL. 2008)

In Chapter 4, section 4.1, we had analyzed and praised the Heston 1993 correlation model. It is a mathematically rigorous, dynamic correlation model, which is widely applied in finance.

Slightly rewriting the equations in Chapter 4.1, the Heston model consists of three main equations.

$$\frac{dS}{S} = \mu\, dt + \sigma_t\, dz_1 \tag{8.22}$$

$$d\sigma_t^2 = a(m_\sigma^2 - \sigma_t^2)dt + \xi\sigma_t\, dz_2 \tag{8.23}$$

where

S: variable of interest e.g., a stock price
μ: growth rate of S
σ: volatility of S
dz: Brownian motion or Wiener process with ε: random drawing from of standard normal distribution with a mean of 0 and a standard deviation of 1. Formally, $\varepsilon = n \sim (0,1)$

a: mean reversion rate (gravity), i.e., degree with which σ^2 at time t, σ_t^2, is pulled back to its long-term mean m_ρ^2. a can take the values $0 \leq a \leq 1$
m_σ^2: long-term mean of the variance rate σ^2
ξ: volatility of the volatility σ.

The stochastic process of S in equation (8.22) and the stochastic variance rate of S, σ^2 in equation (8.23) are correlated with the identity

$$dz_1(t) = \sqrt{\rho_1} dz_2(t) + \sqrt{1 - \rho_1} dz_3(t) \tag{8.24}$$

where $dz_2(t)$ and $dz_3(t)$ are independent, and $dz(t)$ and $dz(t')$ are independent, $t \neq t'$.

Buraschi, Porchia, and Trojani (2010) (the first version of the paper appeared in 2006) and Da Fonseca, Grasselli, and Ielpo (2008) extend the Heston 1993 model with a more rigorous correlation structure. The model is based on the Wishart affine stochastic correlation (WASC) model, introduced by Bru (1991) and extended by Gourieroux and Sufana (2010).

The model is presented as an n-dimensional stochastic process of covariance matrices.[5] For ease of exposition, we will concentrate on n = 2 assets. In this case, S in equation (8.22) expands to a price vector of two assets, S_1 and S_2, formally $S = (S_1, S_2)^T$, where T stands for transpose. The stochastic process for S is

$$dS_t = I_s[\mu\, dt + \sqrt{\Sigma_t} dZ_t] \tag{8.25}$$

where

$I_S = \text{Diag}[S_1, S_2]$, i.e. a diagonal 2×2 matrix, with entries of equation (8.25) on the diagonal and zero entries otherwise
μ: growth rate of the two-dimensional vector S
dZ_t: 2-dimensional Brownian motion
Σ_t: covariance matrix of the returns of asset S_1 and S_2

In our two-asset case, the covariance matrix Σ_t takes the form

$$\Sigma_t = \begin{bmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{21} & \Sigma_t^{22} \end{bmatrix} \tag{8.26}$$

5. We will use some matrix algebra in the following. See "Matrix primer.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1," for some basic matrix operations.

where $\Sigma_t^{11}$and Σ_t^{22} are the variances of the returns of asset 1 and asset 2, respectively, and $\Sigma_t^{12}$and Σ_t^{21} are the covariances of the returns of assets 1 and 2. Note that the covariance of a single asset is equal to its variance; that is, Covariance(*i*,*i*) = Variance(*i*). Therefore, a covariance matrix has variances on its main diagonal and is therefore also called variance-covariance matrix. Also note that the covariance of two assets is commutative; that is, Covariance(*ij*) = Covariance(*ji*); hence in the matrix (8.26) $\Sigma_t^{12} = \Sigma_t^{21}$.

At the core of the model, the covariance matrix (8.26) follows a stochastic process of the form

$$d\Sigma_t = (\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}dW_t Q + Q^T(dW_t)^T\sqrt{\Sigma_t} \tag{8.27}$$

where

Q: volatility of co-volatility matrix $\sqrt{\Sigma_t}$, corresponding to ξ in equation (8.23)
M: negative semidefinite matrix,[6] which controls the degree of mean reversion of Σ_t, corresponding to a in equation (8.23)
Ω: related to the long-term mean of the covariance matrix Σ_t, corresponding to m_σ^2 in equation (8.23)
W: two-dimensional Brownian motion

In the original Heston model, the stochastic process for the underlying asset S and the stochastic process of the variance rate σ^2 are correlated by correlating the Brownian motions of these processes; see equation (8.24). Accordingly, the Brownian motions of equations (8.25) and (8.27) are correlated:

$$dZ(t) = \rho dW(t) + \sqrt{1-\rho^T\rho}\, dB(t) \tag{8.28}$$

where $dW(t)$ and $dB(t)$ are independent, and $dZ(t)$ and $dZ(t')$ are independent, $t \neq t'$.

Equation (8.28) correlates the Brownian motions of equations (8.25) and (8.27). Conveniently, the model admits a closed form solution for the correlation between the underlying return assets S and their variance Σ. For asset 1 we have

$$\mathrm{Corr}(d\ln S_1, d\Sigma^{11}) = \frac{\rho_1 Q_{11} + \rho_2 Q_{21}}{\sqrt{Q_{11}^2 + Q_{21}^2}} \tag{8.29}$$

6. See "Matrix primer.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1," for details.

For asset 2,

$$\mathrm{Corr}(d\ln S_2, d\Sigma^{22}) = \frac{\rho_1 Q_{12} + \rho_2 Q_{22}}{\sqrt{Q_{12}^2 + Q_{22}^2}} \tag{8.30}$$

If we assume that there is no correlation between the volatility of asset 1 and the volatility of asset 2, then $Q_{12} = Q_{21} = 0$. In this case the matrix Q is diagonal and equation (8.29) reduces to

$$\mathrm{Corr}(d\ln S_1, d\Sigma^{11}) = \rho_1 \tag{8.31}$$

Equation (8.30) reduces to

$$\mathrm{Corr}(d\ln S_2, d\Sigma^{22}) = \rho_2 \tag{8.32}$$

In reality we observe a negative relationship between returns S and their variance Σ, sometimes called leverage.[7] This can be modeled for asset 1 with $\rho_1 < 0$ and for asset 2 with $\rho_2 < 0$.

Buraschi et al. (2010) assume a rather simple correlation structure, setting $\rho_1 = 1$ and $\rho_2 = 0$. The negative relationship between the returns S and their variance Σ is then generated with negative values of $Q_{12} = Q_{21}$; see equations (8.29) and (8.30) (Q_{11} and Q_{22} are by definition positive). Da Fonseca (2008) applies a more flexible correlation structure; that is, it allows $\rho \in (-1,1)$ for all ρ.

8.5.1 Critical Appraisal of the Buraschi et al. (2010) and Da Fonseca et al. (2008) Model

The model presented in section 8.5 has numerous parameters and can therefore replicate financial reality well. Especially the negative relationship between returns and volatility (sometimes called leverage) and the higher correlation in a recession (sometimes referred to as asymmetric correlation), which we also found in the empirical Chapter 2, can be modeled. In addition, volatility skews (i.e., higher volatility when returns are negative) and the right balance between correlation persistence and correlation mean reversion can be modeled. Note

7. Unfortunately, the term *leverage* has many meanings in finance. In corporate finance, leverage typically refers to the debt/equity ratio. In derivatives, leverage is the relative change of the value of a derivative with respect to the relative change of underlying. If the derivative is a call C and the underlying is a stock S, the leverage is $\partial C/C/\partial S/S$.

that the higher the persistence, the lower the mean reversion, and vice versa (for details see Chapter 2). Buraschi et al. (2010) also find evidence that part of the hedge fund industries' alpha (i.e., achieving higher returns than the market) can be attributed to hedge funds' exposure to correlation risk.

The drawback of the model lies in its relative mathematical and computational complexity. This may limit its application in reality.

8.6 STOCHASTIC CORRELATION, STOCHASTIC VOLATILITY, AND ASSET MODELING (LU AND MEISSNER 2012)

In a recent paper, Lu and Meissner build a stochastic volatility–stochastic correlation model. Whereas in the Heston 1993 model, extended by Buraschi et al. (2010) and Da Fonseca (2008) (see section 8.5), stochastic returns and stochastic volatility are correlated, Lu and Meissner correlate stochastic volatility and stochastic correlation. Therefore the core equations of the model are

$$d\sigma = a_\sigma(m_\sigma - \sigma_t)dt + \upsilon_\sigma\sqrt{\sigma}dw_1 \tag{8.33}$$

$$d\rho = a_\rho(m_\rho - \sigma_t)dt + \upsilon_\rho\sqrt{1-\rho^2}dw_2 \tag{8.34}$$

$$dw_1 = \rho_w dw_2 + \sqrt{1-\rho_w^2}dw_3 \tag{8.35}$$

where

σ : volatility of the variable of interest, the S&P 500, modeled by the VIX[8]
m_σ : long-term mean of the S&P 500
a_σ : mean reversion of the S&P 500
ν_σ : volatility of volatility σ of the S&P 500
ρ : correlation between an individual stock and the S&P 500
a_ρ, m_ρ, ν_ρ : mean reversion, long-term mean, and volatility of the correlation ρ, respectively
ρ_w : correlation coefficient, which correlates the Brownian motions dw_1 and dw_2

8. The VIX (the Market Volatility Index of the Chicago Board Options Exchange) measures the implied volatility of the S&P 500 (i.e., the volatility implied by S&P 500 index options prices).

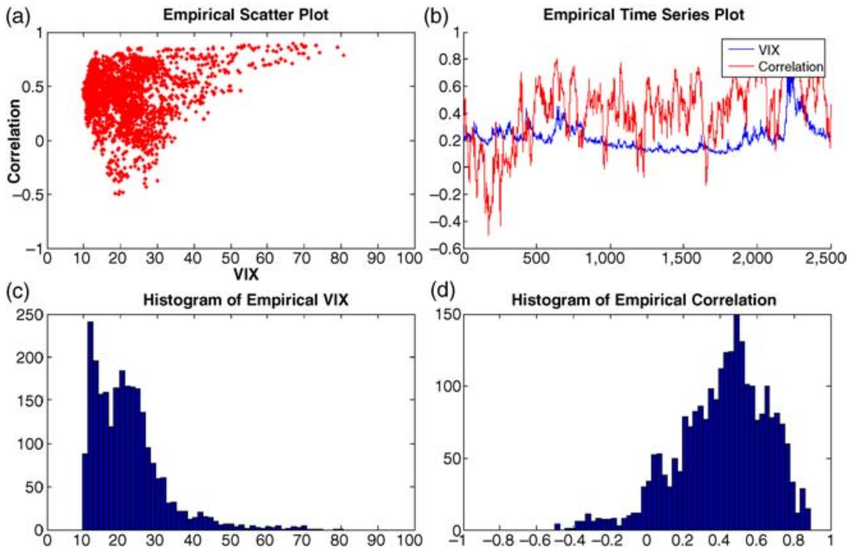

FIGURE 8.4 (a) Empirical Relationship between Implied Volatility (VIX) of the S&P 500 and the Correlation between Chevron Corporation and the S&P 500; (b) Time Series Plot of the Empirical VIX and Empirical Correlation between CVX and the S&P 500; (c) Histogram of VIX; (d) Histogram of the Correlation Coefficient
Data: 01/03/2000 to 07/27/2011.

Equation (8.33) models stochastic volatility with the Cox-Ingersoll-Ross (CIR) (1981) model. Equation (8.34) models stochastic correlation with a modified Jacobi process. The term $\sqrt{1-\rho^2}$ bounds correlation between –1 and +1. However, as discussed in section 8.4, for low mean reversion levels a_ρ and high volatility ν_ρ, it can happen that the model generates correlation levels smaller than –1 and higher than +1. Therefore we have to introduce boundary conditions. These boundary conditions are as displayed in equations (8.19) and (8.20).

The model of equations (8.33) to (8.35) intends to replicate real-world volatility-correlation properties. Figure 8.4 shows some real-world correlation-volatility relationships.

The real-world relationships displayed in Figure 8.4 can be replicated well by the model of equations (8.33) to (8.35), as displayed in Figure 8.5.

Of special interest is the relationship between volatility σ (of the S&P 500 in the example, modeled by the VIX) and the correlation ρ (between a particular stock, Chevron, and the S&P 500 in the example), displayed in the top left chart of Figures 8.4 and 8.5. We observe that the relationship is

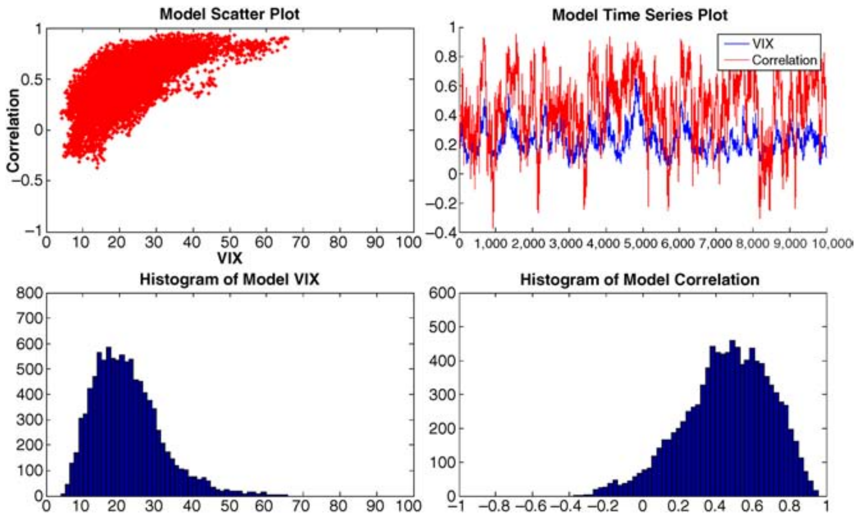

FIGURE 8.5 Simulation Results of the Model of Equations (8.33) to (8.35)

somewhat triangular; that is, (1) it is positive, and (2) the correlation volatility ν_ρ decreases if the volatility σ (represented by the VIX) increases.

The positive relationship between correlation ρ and σ in Figure 8.4 is replicated if ρ_w in equation (8.35) is positive. In addition, the decreasing correlation volatility ν_ρ as a function of the increase in σ (the VIX) is incorporated in the model: If σ increases, ρ increases (if ρ_w is positive). From the term $\upsilon_\rho\sqrt{1-\rho^2}$ it follows that if ρ increases, the volatility of ρ, ν_ρ, decreases. Hence it follows that if σ (represented by the VIX) increases, ν_ρ decreases as displayed in Figure 8.5, top left.

8.6.1 Asset Modeling

The model of equations (8.33) and (8.35) can be applied to model assets. Asset modeling is often done with geometric Brownian motion (GBM), which we discussed in equations (4.1), (8.2), and (8.22). Here is the GBM once again:

$$\frac{dS_i}{S_i} = \mu_i\, dt + \sigma_i\, dz \tag{8.36}$$

where

S_i: asset price of a particular asset i
μ_i: drift of S_i,
σ_i: volatility of S_i

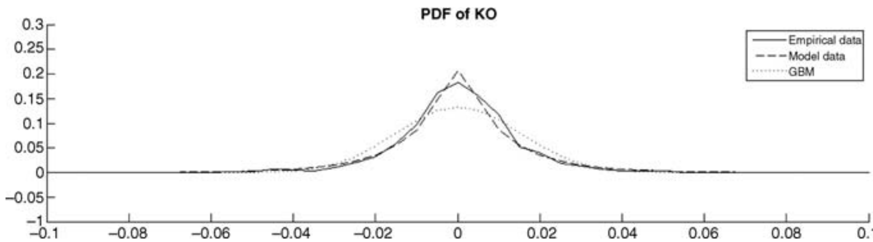

FIGURE 8.6 Probability Density Function (PDF) for Coca-Cola Corporation (KO) Model data are derived with equations (8.33) to (8.35) and equation (8.37).

Lu and Meissner now expand the GBM and model

$$\frac{dS_i}{S_i} = \mu_i\, dt + \sigma_i\, dz + \beta_i\, \rho\sigma dw \tag{8.37}$$

where β_i is constant with $0 \leq \beta_i \leq 1$, and ρ is the correlation between an individual stock and the market, represented by the S&P 500; ρ is modeled as a stochastic process by equation (8.34). σ is the volatility of the market, represented by the VIX of the S&P 500, which is modeled by equation (8.33). The Brownian motions of ρ and σ are correlated via equation (8.35). dw is the Brownian motion of the market component.

Equation (8.37) has a capital asset pricing model (CAPM) interpretation. The first two terms on the right side of equation (8.37) represent the idiosyncratic stock component. The term $\sigma\, dw$ represents the systematic market risk factor, which is shared by all stocks. The impact magnitude of the systematic component on the stock is $\beta_i\, \rho$.

In Figure 8.6, the performance of the model of equations (8.33) to (8.35) and equation (8.37) is compared to the standard GBM of equation (8.35).

From Figure 8.6 we observe that the model of equations (8.33) to (8.35) and equation (8.37) outperforms the standard GBM in equation (8.36). This is verified by standard statistics. The chi-square goodness-of-fit test shows a p-value of 0.8164 ($\text{chi}^2 = 5.986$) between the model distribution and the empirical distribution, while the p-value is 0.054 ($\text{chi}^2 = 19.411$) between the GBM-normal distribution and the empirical distribution. The model gives similar results for other stocks that were tested.

Lu and Meissner (2012) extend the model to include correlation between individual stocks in a portfolio approach, applying the conditionally independent default (CID) correlation concept, which we discussed in Chapter 6. This improves the performance of the model further.

8.7 CONCLUSION: SHOULD WE MODEL FINANCIAL CORRELATIONS WITH A STOCHASTIC PROCESS?

Many assets in finance are modeled with a stochastic process. Assets that are assumed to have little or no mean reversion, such as stocks, exchange rates, or real estate values, are modeled with a non-mean-reverting stochastic process such as the GBM, displayed in equations (4.1), (8.2), and (8.22), or they can be modeled with no-arbitrage, non-mean-reverting models such as the Ho-Lee (1986) model.

Assets that display a certain degree of mean reversion such as bonds, interest rates, or default probabilities are typically modeled with a stochastic process, which includes mean reversion such as the Vasicek 1977 model displayed in equation (8.13), or with mean-reverting no-arbitrage models such as Hull and White (1990) or the Black-Derman-Toy (1990) model. The continuous-time, mean-reverting Heath-Jarrow-Morton (HJM) 1990 model and its discrete version, the LIBOR market model (LMM) of 1997 (credited to three groups of authors: Brace, Gatarek, and Musiela; Miltersen, Sandmann, and Sondermann; and Jamshidian) are generalized models and include the aforementioned models as special cases.

Since many financial assets are successfully modeled with a stochastic process, should we also model financial correlations with a stochastic process? This is mainly an empirical question: Do financial correlations in the real world behave in a way that can be captured with a certain stochastic model? The research in this area has just started, but the first results are promising.

We discussed the conditional correlation modeling approach of Bollerslev (1990), generalized by Engle (2002); sampling correlation values from a stochastic distribution (Hull and White 2010); the Vasicek model applied by Duellman et al. (2008); modeling correlation with a modified Jacobi process (Ma 2009); the Wishart affine stochastic correlation (WASC) model applied by Buraschi et al. (2010; first version 2006); and an extension by Da Fonseca et al. (2008); as well as the approach by Lu and Meissner (2012). All these approaches show that when applying a certain form of stochasticity for financial correlations, correlation properties in reality can be replicated quite well. In addition, the valuation of correlation-dependent structures such as multi-asset options or quanto options (Ma 2009) can be improved if correlation is modeled with a stochastic process.

In conclusion, while stochastic correlation modeling is still in its infancy, first results are promising. Just as other financial variables such as stocks, bonds, interest rates, commodities, volatility, and more are modeled with a stochastic process, it can be expected that in the near future financial correlations will also typically be modeled with a stochastic process.

8.8 SUMMARY

The modeling of financial correlations with a stochastic process is fairly new, but several promising approaches exist. We discussed them in this chapter.

Hull and White (2010) find a simple way to address stochastic correlation. They create a dynamic version of the one-factor Gaussian copula (OFGC) model (see Chapter 6 for details). Hull and White then sample the critical correlation parameter, which indirectly correlates the variables with a beta distribution. The model matches empirical CDX prices in most cases better than a comparable model without stochastic correlation.

In 2002 Robert Engle introduced a dynamic conditional correlation (DCC) concept within the ARCH and GARCH framework. The correlation depends on expectations given at a previous point in time. In addition, the correlation matrix can be made a function of time, constituting a dynamic stochastic correlation model.

A further way to model financial correlations is to use the standard geometric Brownian motion (GBM), which is often applied to model other financial variables such as stocks, exchange rates, commodities, and more. However, the standard GBM suffers from two drawbacks: (1) Mean reversion, a critical property of financial correlations as we derived in Chapter 2, is not incorporated in the GBM, and (2) the model is not bounded, meaning correlation values larger than 1 and smaller than –1 can occur.

An improvement of the GBM for modeling financial correlations is the Vasicek 1977 model, which incorporates mean reversion, or the bounded Jacobi process, which incorporates mean reversion and is also bounded (i.e., the correlation values lie between –1 and +1 if boundary conditions are imposed).

A rigorous, mathematically quite intensive approach to model financial correlations is based on the Wishart affine stochastic correlation (WASC) model. Here a stochastic covariance matrix follows a stochastic process, which is—as in the Heston 1993 model—correlated with the stochastic process of the underlying price matrix. The model has numerous parameters and is able to model several real-world correlation properties well.

In the related stochastic correlation model of Lu and Meissner (2012), which correlates stochastic correlation with stochastic volatility, it is shown that asset modeling can be improved compared to the standard GBM.

PRACTICE QUESTIONS AND PROBLEMS

1. What is a deterministic process? Name two examples.
2. What is a stochastic or random process? Name two examples.

3. Why does it seem like a good idea to model financial correlations as a stochastic process? Name two reasons.
4. How is stochasticity modeled in the dynamic conditional correlation (DCC) concept?
5. The geometric Brownian motion (GBM) is applied to model many financial variables such as stock prices, commodities, and exchange rates. What are two limitations of the GBM model financial correlations?
6. The Vasicek model is an improvement over the GBM to model financial correlations? Why?
7. The bounded Jacobi process seems like a good choice to model financial correlations. What advantage does it have over the GBM and the Vasicek model?
8. In the Buraschi, Porchia, and Trojani (2010) stochastic correlation model, which two stochastic processes are correlated?
9. In the Buraschi, Porchia, and Trojani model, which financial properties can be replicated? Name two.
10. Should we model correlation with a deterministic or a stochastic process?

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CHAPTER 9

Quantifying Market Correlation Risk

The [financial] industry is more technical than ever, and there is as much need to understand the risks of the system as ever.

—Robert Merton

In this chapter we discuss and quantify the correlation risk of financial products whose primary focus is market risk. Let's just clarify what market risk is.

> **MARKET RISK**
>
> The risk of financial loss due to an unfavorable change in the price of a financial security.

We typically differentiate four main types of markets in which market risk can occur: (1) the equity market, (2) the fixed income market, (3) the commodity market, and (4) the foreign exchange rate market. However, other markets can be categorized, such as the energy market, real estate market, weather market, economic variables, and so on. A financial security is a tradable asset in one of these markets, such as stocks, bonds, commodities, exchange rates, and real estate, or futures, options, and swaps on these securities.

9.1 THE CORRELATION RISK PARAMETERS CORA AND GORA

In Chapters 3 and 4 we discussed models to quantify the correlations between one or more financial variables. We will now discuss how to quantify correlation risk (i.e., the risk that the correlations change). We introduce a correlation risk parameter, which we will call Cora.

CORA

A measure of how much a dependent variable changes if the correlation between two or more independent variables changes by an infinitesimally small amount.

Formally,

$$\text{Cora} = \frac{\partial V}{\partial \rho\left(x_{i=1,\ldots,n}\right)} \tag{9.1}$$

where V is the value of a dependent financial variable and $\rho(x_{i=1,\ldots,n})$ is the correlation or correlation matrix of the independent variables $x_{i=1,\ldots,n}$.

So Cora tells us how sensitive a variable V is to changes in correlation ρ. We typically calculate Cora for a portfolio of assets. In this case V can be

- The return/risk ratio μ_P/σ_P of the portfolio. We already derived Cora for the return/risk ratio μ_P/σ_P in the introductory Chapter 1 in section 1.3 for a two-asset portfolio. This was displayed in Figure 1.3, where Cora is the slope of the μ_P/σ_P function.
- The risk of a portfolio, measured by the standard deviation of asset returns.
- The risk of a portfolio measured by the value at risk (VaR) concept. We already derived the Cora in Chapter 1 in section 1.3.3 for a two-asset portfolio. Hence in this case V in equation (9.1) is the VaR. The result was displayed in Figure 1.6. In this chapter we will generalize the result for a portfolio of $n > 2$ assets.

V in equation (9.1) can also be a type of correlation option, or a correlation swap. We already discussed the impact of correlation on the risk of these products in the introductory Chapter 1, section 1.3.2. V can also be a standard option that is valued on a model that includes correlations; see section 9.2.1.

V in equation (9.1) can also be the price of a credit product such as a credit default swap (CDS) or a structured product such as a collateralized debt obligation (CDO). This is because the market price of a CDS or CDO changes when correlation changes; hence there is market price risk. This shows the close relationship between market risk and credit risk: When the market price of an asset decreases (possibly due to a recession), typically the default risk increases, and vice versa, if the default risk of an asset increases (maybe due to bad management), typically the market price decreases.

We already discussed the dependence of a CDS with respect to the correlation between the reference entity and the counterparty in Chapter 1.2, displayed in Figure 1.1. Graphically Cora is the slope of the CDS function. We will discuss correlation risk with respect to credit products in Chapter 10.

In a portfolio context, the x_i in equation (9.1) are the returns of the assets in the portfolio, which are correlated in a correlation matrix $\rho(x_i,\ldots,x_n)$ (see example 9.1). We can calculate Cora for *all* asset returns, but we can also analyze how the risk changes for a *single* pairwise change in the correlation matrix. We will call this Gap-Cora in analogy to the gap analysis in interest rate risk management, where a single interest rate is bumped up or bumped down by a certain amount to see the impact on the present value or on risk.

From Figures 1.2, 1.3, 1.4, 1.6, and 1.7 in Chapter 1 we can see that Cora can be positive or negative. Cora can also be positive and negative within the same function as in Figure 1.2, where Cora is positive for correlation values of -1 to about -0.3 and negative for correlation values of about -0.3 to 1. Conceptually Cora can take values between $-\infty$ and $+\infty$.

We can also look at the sensitivity of Cora, i.e., how much Cora changes. We will call this Gora.

GORA

A measure of how much Cora changes if the correlation between two or more independent variables changes by an infinitesimally small amount. Formally,

$$\text{Gora} = \frac{\partial \text{Cora}}{\partial \rho(x_{i=1,\ldots,n})} = \frac{\partial^2 V}{\partial \rho^2(x_{i=1,\ldots,n})} \tag{9.2}$$

where variables are defined as in equation (9.1).

So Gora tells us how sensitive Cora is to changes in correlation ρ. Mathematically, Gora is the first partial derivative of the Cora function or the second partial derivative of the original function V with respect to correlation. That is, Gora measures the curvature of a function with respect to correlation. As Cora, Gora can take values between $-\infty$ and $+\infty$.

9.2 EXAMPLES OF CORA IN FINANCIAL PRACTICE

Let's first look at a specific type of Cora that is already established in the option markets.

9.2.1 Option Vanna

Measuring the impact of changes in correlation is not totally new. It is already formalized in option theory and called Vanna. It was introduced in the famous stochastic alpha, beta, rho (SABR) model by Hagan et al. (2002). The core equations of the model are

$$\frac{dF}{F^{\beta}} = \sigma\, dW \tag{9.3}$$

$$\frac{d\sigma}{\sigma} = \alpha\, dZ \tag{9.4}$$

and

$$\mathrm{Corr}(dW, dZ) = \rho\, dt \tag{9.5}$$

This model is named after its parameters, α, which is the volatility of volatility; β, which determines the skew of the volatility (i.e., how lopsided or asymmetric the volatility function is); and ρ, which determines the correlation between the Brownian motions dW and dZ of the forward rate F and the volatility σ of forward rate F. The model of equations (9.3) to (9.5) reduces to the zero-drift Brownian motion for $\alpha = \beta = 0$. The model is identical to the constant variance of elasticity (CEV) model of John Cox (1975) for $\alpha = 0$.

The model includes the correlation between the forward rate or price F and volatility σ via equation (9.5). This correlation influences the call price C and put price P, since the call and put prices are functions of F and σ.

We derive the sensitivity of the option price V^* with respect to correlation. Hence, we have

$$\text{Vanna} = \frac{\partial V^*}{\partial \rho(F, \sigma)} \tag{9.6}$$

From equations (9.1) and (9.6), we see that Vanna is a special case of Cora, with the dependent variable being the option price V^* and the correlated variables being the forward price or rate F and the volatility of F, σ. We can replace the functional relationship $\rho(F,\sigma)$ in equation (9.6) and write

$$\text{Vanna} = \frac{\partial\left(\frac{\partial V^*}{\partial F}\right)}{\partial \sigma} = \frac{\partial^2 V^*}{\partial F \partial \sigma} \tag{9.7}$$

From equation (9.7), we see that Vanna calculates how much the delta $\frac{\partial V^*}{\partial F}$ changes if volatility σ changes. The term $\frac{\partial^2 V^*}{\partial F \partial \sigma}$ of equation (9.7) tells us that Vanna is a second-order mathematical derivative: The option price V^* is partially differentiated with respect to F *and* with respect to σ.

9.2.2 Option Cora and Gora

The model of equations (9.6) to (9.7) shows that even when pricing plain-vanilla options, we can include correlations, which determine the option value. Naturally, especially for correlation options (i.e., options whose payoff is at least in part determined by the correlation of two or more variables), correlations are critical. In the introductory Chapter 1, in section 1.3.2, "Trading and Correlation," we already discussed several correlation options. Let's derive Cora for an exchange option.

An exchange option E is the right to exchange asset S_1 for asset S_2 (i.e., the right to give away asset S_1 and receive asset S_2 at option maturity). Therefore, an exchange option has the payoff = max(0, $S_2 - S_1$). The pricing formula is:

$$E = S_2 e^{-q_2 T} N\left(\frac{\ln\left(\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2\right)T}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}\sqrt{T}}\right)$$

$$- S_1 e^{-q_1 T} N\left(\frac{\ln\left(\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right) - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2\right)T}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}\sqrt{T}}\right) \tag{9.8}$$

where

q_2: return of asset 2
q_1: return of asset 1
σ_1: volatility of asset S_1
σ_2: volatility of asset S_2
ρ: correlation coefficient for assets S_1 and S_2
$N(x)$: cumulative standard normal distribution of x

Note that a distinction between a call and a put is not sensible with an exchange option. Interestingly, an interest rate is not an input parameter in equation (9.8). This is because in a risk-neutral framework, both asset S_1 and asset S_2 are expected to grow with the risk-neutral interest rate r, and hence r cancels out.

Differentiating equation (9.8) partially with respect to the correlation parameter ρ requires some stamina, but it can be done. The result is the Cora of an exchange option E:

$$\mathrm{Cora}_E = \frac{\partial E}{\partial \rho} = -\frac{e^{-q_2 T}\sqrt{T} S_2 \sigma_1 \sigma_2 n \left[\dfrac{\ln\left[\dfrac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right] + \dfrac{1}{2} T\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)}{\sqrt{T}\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}} \right]}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}} \tag{9.9}$$

where $n(x)$ is the standard normal distribution of x. For the derivation of equation (9.9), see problem 11 in the questions and problems section at the end of this chapter.

Equation (9.9) tells us how much the exchange option price E changes if the correlation ρ between the assets S_1 and S_2 changes by a very small amount.

From equation (9.9) we observe that the Cora of an exchange option is negative, since all terms in equation (9.9) are positive and there is a negative sign in front of the right term. The negative Cora makes sense, since the lower the correlation between the assets S_1 and S_2, measured by ρ, the higher is the expected payoff $\max(0, S_2 - S_1)$ and the higher the exchange option price E. We already observed Cora in Figure 1.4 in Chapter 1 and saw that the slope of E with respect to ρ (i.e., the Cora) is negative. For a model that calculates Cora of an exchange option, see "Exchange option.xls" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1," cell J19.

Differentiating equation (9.9) again with respect to ρ gives us the Gora that we defined in equation (9.2). It measures the curvature of the option function. In addition, Gora tells us how sensitive Cora is to changes in ρ. In other words, it tells us how instable the correlation hedge is. The higher the Gora, the more often we have to adjust the correlation hedge.

A normal distribution $n(x)$ is conveniently differentiated by using $\frac{\partial n(x)}{\partial x} = -x\,n(x)$. Applying the product rule and chain rule to equation (9.9), we derive the Gora as:

$$\mathrm{Gora}_E = \frac{\partial \mathrm{Cora}_E}{\partial \rho}$$

$$= -\left(e^{-q_2 T} S_2 \sigma_1^2 \sigma_2^2 \left(-4 \ln\left[\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right]^2 + T\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)\left(4 + T\sigma_1^2 - 2T\rho\sigma_1\sigma_2 + T\sigma_2^2\right)\right) n\left[\frac{\ln\left[\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right] + \frac{1}{2}T\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)}{\sqrt{T}\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\right]\right) \Big/ \left(4\sqrt{T}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)^{5/2}\right) \tag{9.10}$$

The programmed Gora of equation (9.10) is in cell J20 of the model "Exchange option.slx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

9.3 CORA AND GORA IN INVESTMENTS

In Chapter 1, section 1.3.1, "Investments and Correlation," we had already briefly discussed the relationship between investments and correlation. We found that a decrease in correlation enhances the effects of diversification (i.e., increases the return/risk ratio of a portfolio). We displayed this in Figure 1.3. We will now formalize this finding. We define the Cora of the return/risk ratio, also called risk-adjusted return of a portfolio, Cora_P, as

$$\mathrm{Cora}_P = \frac{\partial\left(\frac{\mu_P}{\sigma_P}\right)}{\partial \rho(x_{i=1,\ldots,n})} \tag{9.11}$$

where μ_P is the portfolio return mean and σ_P is the standard deviation of the portfolio returns.

The x_i in equation (9.11) are the pairwise correlations of the returns of the assets in the portfolio. Hence equation (9.11) tells us how much the risk-adjusted return of a portfolio μ_P/σ_P changes if all pairwise correlations of all n asset returns in the portfolio $\rho(x_{i=1,\ldots,n})$ change by an infinitesimally small amount.

For a two-asset portfolio, the mean of the portfolio value μ_P is derived by equation (1.1) as $\mu_P = w_A\ \mu_A + w_B\ \mu_B$. The standard deviation σ_P of the two-asset portfolio return is derived in equation (1.5) as $\sigma_P = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \mathrm{Cov}_{AB}}$.

For a two-asset portfolio, we can derive Cora_P in equation (9.11) as $\frac{\partial\left(\frac{\mu_P}{\sigma_P}\right)}{\partial\rho} = -(w_A\mu_A + w_B\mu_B)\frac{1}{2}\left(w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B\sigma_A^2\sigma_B^2\rho\right)^{-1\frac{1}{2}}(2w_A w_B\sigma_A^2\sigma_B^2)$.

However, typically we have more than two assets in a portfolio. In this case, we can use equation $\mu_P = \sum_{i=1}^{n} w_i\mu_i$ and equation (1.9) $\sigma_P = \sqrt{\beta_h C\,\beta_v}$. If we have more than two assets in a portfolio, we have to simulate an increase in every pairwise correlation of the assets' returns and observe the impact on the risk-adjusted return μ_P/σ_P of the portfolio. The magnitude of the simulation can range from a small number such as +1% to a much higher number to stress-test the correlation impact.

We can also analyze the impact of correlation on just the risk of a portfolio, measured by the standard deviation of asset returns, σ_P,

$$\mathrm{Cora}^*_P = \frac{\partial\sigma_P}{\partial\rho(x_{i=1,\ldots,n})} \tag{9.12}$$

To derive the Cora^*_P in equation (9.12) for a two-asset portfolio, we can use equation (1.5), $\sigma_P = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B\sigma_A^2\sigma_B^2\rho}$, and differentiate partially with respect to ρ. We derive $\frac{\partial\sigma_P}{\partial\rho} = \frac{1}{2}\left(w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B\sigma_A^2\sigma_B^2\rho\right)^{-\frac{1}{2}}(2w_A w_B\sigma_A^2\sigma_B^2)$.

Calculating the portfolio return standard deviation σ_P gives some information of the risk of a portfolio. However, σ_P is part of the VaR concept, which includes a time frame and a confidence level and is therefore a more informative risk measure. Calculating Cora for the VaR concept is done in the next section, "Cora in Market Risk Management."

We can also look at the sensitivity of Cora to changes in the correlation ρ of a portfolio; that is, we can calculate the Gora of a portfolio.

$$\mathrm{Gora}_P = \frac{\partial(\mathrm{Cora}_P)}{\partial\rho(x_{i=1,\ldots,n})} = \frac{\partial^2\left(\frac{\mu_P}{\sigma_P}\right)}{\partial\rho^2(x_{i=1,\ldots,n})} \tag{9.13}$$

Equation (9.13) tells us how much the Cora of a portfolio $\text{Cora}_P = \dfrac{\partial\left(\frac{\mu_P}{\sigma_P}\right)}{\partial\rho\left(x_{i=1,\ldots,n}\right)}$ changes if correlation of all assets in the portfolio changes by an infinitesimally small amount. So the Gora of a portfolio tells us how stable the correlation hedge is (see Chapter 11 for correlation hedging). The higher the Gora, the more frequently we have to change the hedge if correlation changes. Graphically, Gora is the curvature of the original μ_P/σ_P function of Figure 1.3.

9.4 CORA IN MARKET RISK MANAGEMENT

Arguably the most important application of Cora and Gora is in risk management. In the introductory Chapter 1, section 1.3.3, we already outlined the basic relationship between correlation and risk. We found that a lower correlation reduces portfolio risk measured by value at risk (VaR), which was displayed in Figure 1.6. VaR measures the maximum loss of a portfolio with respect to a certain probability for a certain time frame. VaR is the most widely applied risk management concept in financial practice. It can be calculated with equation (9.14):

$$VaR_P = \sigma_P \alpha \sqrt{x} \tag{9.14}$$

where VaR_P is the value at risk for portfolio P, and α is the abscise value of a standard normal distribution, corresponding to a certain confidence level. It can be derived as =normsinv(confidence level) in Excel or norminv(confidence level) in MATLAB. α takes the values $-\infty < \alpha < +\infty$. x is the time horizon for the VaR, typically measured in days. σ_P is the volatility of the portfolio P, which includes the correlation between the assets in the portfolio.

We calculate σ_P via equation (9.15):

$$\sigma_P = \sqrt{\beta_h C \beta_v} \tag{9.15}$$

where β_h is the horizontal β vector of invested amounts (price times quantity; "position" in Table 9.1 in example 9.1). β_v is the vertical β vector of invested amounts (also price times quantity, "position" in Table 9.1).[1] C is the covariance matrix of the returns of the assets (Table 9.4).

1. More mathematically, the vector β_h is the transpose of the vector β_v, and vice versa: ${\beta_h}^T = \beta_v$ and ${\beta_v}^T = \beta_h$. Hence we can also write equation (9.15) as $\sigma_p = \sqrt{\beta_h\, C\, {\beta_h}^T}$. See the spreadsheet "Matrix primer.xls," sheet "Matrix Transpose," at www.wiley.com/go/correlationriskmodeling, under "Chapter 1" for details.

EXAMPLE 9.1: DERIVING VaR, CORA$_{VaR}$, AND GORA$_{VaR}$ FOR A 10-ASSET PORTFOLIO

What are the VaR, Cora$_{VaR}$, and Gora$_{VaR}$ of a 10-asset portfolio of unequally weighted stocks AT&T, Citi, Ford, GE, GM, HPQ, IBM, JPM, MSFT, and P&G for a 1-year time horizon for a 99% confidence level? This example and the following results are displayed in the spreadsheet "Var educational.xlsm" at www.wiley.com/go/correlation riskmodeling, under "Chapter 9."

A model that calculates VaR, Cora$_{VaR}$, and Gora$_{VaR}$ for n assets can be found at "Var n asset cora gora.xlsm" at www .wiley.com/go/correlationriskmodeling, under "Chapter 9."

Table 9.1 shows the numerical values of this example.

We first downloaded the stocks' daily closing prices (for example from Yahoo! Finance, http://finance.yahoo.com) from August 1, 2011, to July 31, 2012. We then calculated the daily returns for stock price S as $R = \ln(S_t/S_{t-1})$.[2] We now correlate the daily returns for all 10 stock pairs to find the correlation coefficient ρ for all stock pairs. Mathematically this is done by equation $\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}}$ where x is the return of asset X and y is the return of asset Y, and $\bar{x}$ and $\bar{y}$ are the means of the asset returns of assets X and Y, respectively. Computationally we can use Excel's Correl function or MATLAB's corrcoef function to find ρ. This gives us $n(n-1)/2 = 10 \times 9/2 = 45$ correlation pairs. Table 9.2 shows the correlation matrix.

From Table 9.2 we observe that the return correlations of the 10 stocks are mostly positive.

We now derive the daily standard deviation σ of the stock returns R via equation (1.2) $\sigma_R = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(R_t-\mu_R)^2}$, which are displayed in Table 9.3.

We now build the covariance matrix by using $\mathrm{Cov}_{AB} = \sigma_A\,\sigma_B\,\rho_{AB}$. Table 9.4 shows the result.

2. See Appendix A2 of Chapter 1 why we rather use $\ln(S_t/S_{t-1})$ instead of $(S_t-S_{t-1})/S_{t-1}$ to calculate returns.

TABLE 9.1 Stock Portfolio of 10 Unequally Weighted Stocks, in $ Thousands

	Portfolio	AT&T	Citi	Ford	GE	GM	HPQ	IBM	JPM	Microsoft	P&G
Spot Price		$26.88	$72.50	$10.79	$15.12	$27.05	$58.10	$174.40	$35.58	$24.52	$58.89
Number of Shares		79	12	100	90	38	82	20	35	140	70
Position (price × quantity)	$23,513.82	$2,123.52	$870.00	$1,079.00	$1,360.80	$1,027.90	$4,764.20	$3,488.00	$1,245.30	$3,432.80	$4,122.30
Weight (%)		9%	4%	5%	6%	4%	20%	15%	5%	15%	18%

TABLE 9.2 Correlation Matrix ρ between Returns of 10 Stocks from August 1, 2011, to July 31, 2012

AT&T	Citi	Ford	GE	GM	HPQ	IBM	JPM	Microsoft	P&G
1.00	0.06	0.49	0.65	0.00	0.50	−0.10	0.52	0.50	−0.02
0.06	1.00	0.01	−0.04	−0.14	0.04	−0.06	−0.01	0.01	−0.16
0.49	0.01	1.00	0.68	0.03	0.64	0.02	0.71	0.61	0.11
0.65	−0.04	0.68	1.00	−0.06	0.70	−0.12	0.75	0.65	0.03
0.00	−0.14	0.03	−0.06	1.00	0.02	0.58	−0.01	0.08	0.46
0.50	0.04	0.64	0.70	0.02	1.00	−0.05	0.64	0.60	0.05
−0.10	−0.06	0.02	−0.12	0.58	−0.05	1.00	−0.05	−0.02	0.53
0.52	−0.01	0.71	0.75	−0.01	0.64	−0.05	1.00	0.57	0.05
0.50	0.01	0.61	0.65	0.08	0.60	−0.02	0.57	1.00	0.02
−0.02	−0.16	0.11	0.03	0.46	0.05	0.53	0.05	0.02	1.00

(continued)

(continued)

TABLE 9.3 Standard Deviations of Daily Returns

Stocks in the Portfolio	Standard Deviation σ
AT&T	1.02%
Citi	3.19%
Ford	2.08%
GE	1.69%
GM	2.73%
HPQ	3.00%
IBM	1.42%
JPM	2.65%
Microsoft	1.43%
P&G	1.00%

TABLE 9.4 Covariance Matrix for the Portfolio in Table 9.1 with the Correlation Matrix of Table 9.2 and Standard Deviation in Table 9.3

AT&T	Citi	Ford	GE	GM	HPQ	IBM	JPM	Microsoft	P&G
0.0001031	0.0000182	0.0001026	0.0001113	0.0000013	0.0001529	−0.0000148	0.0001405	0.0000726	−0.0000018
0.0000182	0.0010178	0.0000060	−0.0000212	−0.0001209	0.0000344	−0.0000287	−0.0000123	0.0000025	−0.0000508
0.0001026	0.0000060	0.0004332	0.0002378	0.0000153	0.0004002	0.0000053	0.0003897	0.0001822	0.0000239
0.0001113	−0.0000212	0.0002378	0.0002847	−0.0000258	0.0003546	−0.0000294	0.0003344	0.0001564	0.0000048
0.0000013	−0.0001209	0.0000153	−0.0000258	0.0007477	0.0000142	0.0002234	−0.0000079	0.0000304	0.0001260
0.0001529	0.0000344	0.0004002	0.0003546	0.0000142	0.0009015	−0.0000232	0.0005058	0.0002590	0.0000146
−0.0000148	−0.0000287	0.0000053	−0.0000294	0.0002234	−0.0000232	0.0002006	−0.0000192	−0.0000043	0.0000753
0.0001405	−0.0000123	0.0003897	0.0003344	−0.0000079	0.0005058	−0.0000192	0.0007032	0.0002155	0.0000133
0.0000726	0.0000025	0.0001822	0.0001564	0.0000304	0.0002590	−0.0000043	0.0002155	0.0002059	0.0000034
−0.0000018	−0.0000508	0.0000239	0.0000048	0.0001260	0.0000146	0.0000753	0.0000133	0.0000034	0.0001009

$$
\begin{pmatrix}
0.0001031 & 0.0000182 & 0.0001026 & 0.0001113 & 0.0000013 & 0.0001529 & -0.0000148 & 0.0001405 & 0.0000726 & -0.0000018 \\
0.0000182 & 0.0010178 & 0.0000060 & -0.0000212 & -0.0001209 & 0.0000344 & -0.0000287 & -0.0000123 & 0.0000025 & -0.0000508 \\
0.0001026 & 0.0000060 & 0.0004332 & 0.0002378 & 0.0000153 & 0.0004002 & 0.0000053 & 0.0003897 & 0.0001822 & 0.0000239 \\
0.0001113 & -0.0000212 & 0.0002378 & 0.0002847 & -0.0000258 & 0.0003546 & -0.0000294 & 0.0003344 & 0.0001564 & 0.0000048 \\
0.0000013 & -0.0001209 & 0.0000153 & -0.0000258 & 0.0007477 & 0.0000142 & 0.0002234 & -0.0000079 & 0.0000304 & 0.0001260 \\
0.0001529 & 0.0000344 & 0.0004002 & 0.0003546 & 0.0000142 & 0.0009015 & -0.0000232 & 0.0005058 & 0.0002590 & 0.0000146 \\
-0.0000148 & -0.0000287 & 0.0000053 & -0.0000294 & 0.0002234 & -0.0000232 & 0.0002006 & -0.0000192 & -0.0000043 & 0.0000753 \\
0.0001405 & -0.0000123 & 0.0003897 & 0.0003344 & -0.0000079 & 0.0005058 & -0.0000192 & 0.0007032 & 0.0002155 & 0.0000133 \\
0.0000726 & 0.0000025 & 0.0001822 & 0.0001564 & 0.0000304 & 0.0002590 & -0.0000043 & 0.0002155 & 0.0002059 & 0.0000034 \\
-0.0000018 & -0.0000508 & 0.0000239 & 0.0000048 & 0.0001260 & 0.0000146 & 0.0000753 & 0.0000133 & 0.0000034 & 0.0001009
\end{pmatrix}
\begin{pmatrix} 2,124 \\ 870 \\ 1,097 \\ 1,360 \\ 1,027 \\ 4,764 \\ 3,488 \\ 1,245 \\ 3,432 \\ 4,122 \end{pmatrix}
=
\begin{pmatrix} 1.59 \\ 0.63 \\ 4.16 \\ 3.40 \\ 2.11 \\ 7.08 \\ 1.00 \\ 5.17 \\ 2.81 \\ 0.89 \end{pmatrix}
$$

(*continued*)

(*continued*)

Now we are ready to derive the portfolio standard deviation. We use equation (9.14) $\sigma_P = \sqrt{\beta_h C \beta_v}$, which we already used in Chapter 1 for a two-asset portfolio.

We first derive $C\ \beta_v$, as done on page 193.

We then calculate $\beta_h\ (C\ \beta_v)$:

$$(2{,}124\ \ 870\ \ 1{,}079\ \ 1{,}360\ \ 1{,}027\ \ 4{,}764\ \ 3{,}488\ \ 1{,}245\ \ 3{,}432\ \ 4{,}122)\begin{pmatrix}1.59\\0.63\\4.16\\3.40\\2.11\\7.08\\1.00\\5.17\\2.81\\0.89\end{pmatrix} = 72{,}141$$

Hence we have $\sigma_P = \sqrt{72{,}141} = 268.59$. We are considering a 99% confidence level. Hence α = normsinv(0.99) = 2.3263. We have about 252 trading days in a year. It follows that the portfolio VaR is

$$\text{VaR}_P = \sigma_P\ \alpha \sqrt{X} = 268.59 \times 2.3263 \times \sqrt{252} = 9{,}918.97$$

Interpretation: We are 99% sure that we will not lose more than \$9,918,970 (since numbers are in units of \$1,000) in the next year due to price changes of the stocks in our portfolio. Note that this VaR number includes the correlation between the stocks via σ_P.

Simulation: We now simulate changes in the correlation matrix of Table 9.2 to derive Cora_{VaR}. Cora is defined for an infinitesimally small change in correlation, as we can see from equation (9.1). However, even if we increase the correlation by a larger amount, we will derive the exact value in VaR, since we are performing a numerical simulation. This is different to option theory, where we can apply closed form mathematical derivatives. For example, the delta of an option V with the underlying S, $\partial V/\partial S$, is the first mathematical derivate of V with respect to S. For larger increases in S, the change in V, ∂V, derived by the closed form solution will be imprecise, since the option function V is nonlinear with respect to S.

Figure 9.1 shows the impact of a simulated change in correlation on the VaR, i.e., Cora_{VaR}.

From Figure 9.1 we observe that the higher the increase in the pairwise correlations, the higher VaR is. The impact of a correlation is strong. VaR increases by over 60% for high correlations.

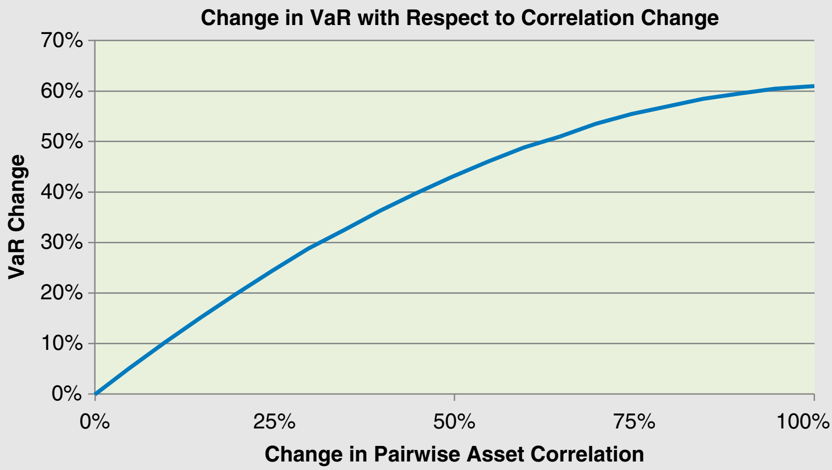

FIGURE 9.1 Cora_{VaR}, the Change in VaR for a Change in All Pairwise Asset Return Correlations in the Portfolio

With respect to Gora, we observe from Figure 9.1 that for small changes in the pairwise correlation, Gora (the change in the slope in Figure 9.1) is quite small. In our example Gora is just –0.11% when calculated for a Cora from 0% to 1% asset correlation change compared to a Cora for 10% to 11% asset correlation change. For higher pairwise asset correlation levels, Gora increases on an absolute level (since the slope of the function in Figure 9.1 changes more). However, correlations are capped at 100%. Therefore, once a correlation has reached 100% it cannot increase further, which caps further increases in the VaR function.

Naturally, VaR has some limitations, especially slim tails and non-additivity. For more on VaR, we recommend Jorion (2006) and Hull (2011).

We will now quantify correlation risk with a real-world example with respect to VaR. We can write

$$\text{Cora}_{\text{VaR}} = \frac{\partial(\text{VaR})}{\partial\rho(x_{i=1,\ldots,n})} \tag{9.16}$$

where the x_i are the pairwise correlations between all asset returns in the portfolio. Equation (9.16) measures how much VaR changes for an infinitesimally small change in all pairwise correlations of all asset returns in the portfolio. There is no closed form solution for equation (9.16), so we have to simulate the change in correlation, possibly with a 1% increase in all pairwise correlations. Let's do this in a real-world example. We first calculate VaR and then Cora_{VaR}.

9.4.1 Gap-Cora

In Figure 9.1 we observe a change in correlation of *all* pairwise correlations in the correlation matrix of Table 9.2. We may also be interested in deriving the correlation exposure of a single asset in our portfolio. In analogy to interest rate risk management, we will call this gap analysis, or Gap-Cora. We can derive two types of Gap-Cora.

The first is $\text{Gap-Cora}_{i,P}$ of a single asset i with respect to all other assets in the portfolio P. Formally,

$$\text{Gap-Cora}_{i,P} = \frac{\partial(\text{VaR})}{\partial\rho(x_{i,j=1,\ldots,n})} \tag{9.17}$$

Equation (9.17) reads: How much does VaR change if the correlation between asset i and all other $j = 1,\ldots, n$ assets in the portfolio P changes by an infinitesimally small amount? We can approximate $\text{Gap-Cora}_{i,P}$ with a 1% increase. For our portfolio, if asset i is AT&T, $\text{Gap-Cora}_{i,P}$ = \$12,364. We interpret this number as: If the correlations between AT&T and all other assets in the portfolio increase by 1%, VaR increases by \$12,364. The interested reader can confirm this number in the spreadsheet at "Var educational.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 9."

We can also stress-test $\text{Gap-Cora}_{i,P}$. For example, we can increase the correlation by a random number, for example by one standard deviation of the vector of correlations between AT&T and the other assets in the portfolio, which comes out to 35.74%. In this case $\text{Gap-Cora}_{\text{AT\&T},P}$ = \$432,752. We interpret this number as: If the correlation between AT&T and all other assets in the portfolio increases by 35.74%, VaR increases by \$432,752.

We can also derive the exposure of the correlation between a single asset with respect to another single asset. Formally,

$$\text{Gap-Cora}_{i,j} = \frac{\partial(\text{VaR})}{\partial\rho(x_{i,j})} \tag{9.18}$$

Equation (9.18) reads: How much does VaR change, if the correlation between asset i and asset j changes by an infinitesimally small amount? We can approximate $\text{Gap-Cora}_{i,j}$ with a 1% increase. For our portfolio, if asset i is AT&T, and asset j is Citi, $\text{Gap-Cora}_{i,j}$ = \$823. We interpret this number as: If the correlation between AT&T and Citi increases by 1%, VaR increases by \$823. The interested reader can confirm this number in the spreadsheet "Var educational.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 9."

We can also stress-test Gap-Cora$_{i,j}$. For example, we can increase the correlation between AT&T and Citi by a random number. We could calculate the historical correlation of AT&T and Citi and derive the standard deviation of this historical correlation. Let's assume this standard deviation is 50%. In this case Gap-Cora$_{\text{AT\&T,Citi}}$ = \$41,051. We interpret this number as: If the correlation between AT&T and Citi increases by 50%, VaR increases by \$41,051. The interested reader can again confirm this number with the spreadsheet "Var educational.xlsm" at www.dersoft.com/vareducational.xlsm.

9.5 GORA IN MARKET RISK MANAGEMENT

As defined in equation (9.2), Gora is the second partial derivative of a function with respect to correlation. For VaR we define

$$\text{Gora}_{\text{VaR}} = \frac{\partial \text{Cora}_{\text{VaR}}}{\partial \rho(x_{i=1,\ldots,n})} = \frac{\partial^2(\text{VaR})}{\partial \rho^2(x_{i=1,\ldots,n})} \tag{9.19}$$

Equation (9.19) reads: How much does Cora of VaR change if the correlation of all assets in the portfolio changes, or what is the curvature of the original VaR function?

From Figure 1.6 in Chapter 1 and Figure 9.1 we observe that the curvature of the VaR function is negative (since the slope decreases for increasing correlation). Hence Gora of VaR is negative. Let's derive Gora of VaR numerically and interpret the result.

There are several ways to simulate Gora for market risk.

We can calculate the change in VaR for a y% increase between the correlation of all assets in the portfolio and a y% decrease between the correlation of all assets in the portfolio and then take the average. Formally,

$$\text{Gora}_{\text{VaR}} \approx \left(\frac{[\text{VaR}(+y\%\text{ in }\rho(x_{i=1,\ldots,n})) - \text{VaR}] + [\text{VaR}(-y\%\text{ in }\rho(x_{i=1,\ldots,n})) - \text{VaR}]}{2}\right) \tag{9.20}$$

For our example 9.1, Gora$_{\text{VaR}}$ for a $y = 10\%$ simulation comes out to $\text{Gora}_{\text{VaR}} \approx \left(\frac{1{,}053.60 + (-1{,}179.78)}{2}\right) = -63.09$.

Since this number is in units of \$1,000, we interpret it as follows. As an approximation, for a 1% increase in all correlations in the portfolio, the Cora will reduce by \$6,309; hence we have to reduce our correlation hedge by this amount. The reader can verify this number in the spreadsheet "Var educational.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 9," cell P19 on sheet "Portfolio VAR simulation."

We can also simulate Gora of VaR by comparing different Coras. For example, we could calculate Cora for an increase in correlation from 10% to 11%, and Cora for an increase in correlation from 0% to 1%, and then look at the difference. Formally, $\text{Gora}^{*}_{\text{VaR}} \approx \text{Cora}_{\text{VaR}}(\rho = 10\% \rightarrow \rho = 11\%) - \text{Cora}_{\text{VaR}}(\rho = 0\% \rightarrow \rho = 1\%)$, where ρ is the pairwise correlation of all assets in the portfolio. We expect $\text{Gora}^{*}_{\text{VaR}}$ to be negative since the slope of Cora for an increase in correlation from 10% to 11% is lower than Cora for an increase in correlation from 0% to 1%, as we can see in Figure 9.1. For our example 9.1, we derive $\text{Gora}^{*}_{\text{VaR}} \approx 99.84 - 110.34 = -10.5$. Since this number is again in units of \$1,000, we can interpret this as: Approximately, if all correlations in the portfolio increase by 10%, we have to reduce our correlation hedge by \$10,500. The reader can verify this number in the spreadsheet "Var educational.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 9," cell T17 on sheet "Portfolio VAR simulation."

9.6 SUMMARY

In this chapter we discussed how to quantify market correlation risk. **Market risk** is the risk of an unfavorable change in the market price or rate in four main markets: (1) equity market, (2) fixed income market, (3) commodity market, and (4) foreign exchange market. However, other markets such as real estate, energy, and weather can be categorized. **Market correlation risk** is the risk that the correlations between the prices in one market or between these markets change.

We can quantify market correlation risk with two measures.

1. **Cora** measures how much a dependent financial variable changes if the correlation between independent variables changes. For example, Cora can measure how much value at risk (VaR) in a portfolio changes if the correlation of the assets in the portfolio changes. However, the dependent financial variable can be any financial variable that is exposed to correlation risk: the return/risk ratio of a portfolio, the price of an option, a credit default swap (CDS), a collateralized debt obligation (CDO), and many more. Graphically,

Cora is the slope of the variable's function with respect to correlation. If we hedge correlation risk, Cora tells us the magnitude of the correlation hedge.

2. **Gora** measures how much Cora changes. Hence it tells how much we have to adjust our correlation hedge. Mathematically, Gora is the second mathematical derivative of the variable's function with respect to correlation.

Arguably, Cora and Gora are most critical in risk management. We find that one of the most widely applied market risk measures of a portfolio, value at risk (VaR), is highly sensitive to Cora; that is, VaR is highly sensitive to changes in correlation of the assets in the portfolio. The sensitive of VaR to Gora is only moderate.

Cora and Gora can be extended in numerous ways: We can calculate Cora not only for a correlation change of *all* assets in the portfolio, but for a change in (1) the correlation between *one* particular asset with all other assets or (2) the correlation between two specific assets. This provides the risk manager with the correlation risk of specific assets, possibly critical assets in the portfolio. The same exercise can be done for Gora.

Cora and Gora can also be applied to stress testing. In this case, we can simulate the correlation change between the independent variables by a large amount to observe correlation risk in crisis scenarios.

PRACTICE QUESTIONS AND PROBLEMS

1. When we talk about market risk, which four markets are typically included?
2. Name several other markets not included in the four markets mentioned in question 1.
3. What is market correlation risk?
4. We can measure market correlation risk with Cora. What information does Cora give us?
5. What is Cora mathematically?
6. Name three applications of Cora in finance.
7. Measuring correlation risk is not totally new. In option theory, a Vanna exists. What information does Vanna give us?
8. What is the relationship between Vanna and Cora?
9. What information does Gora give us?
10. What is Gora mathematically?

11. Okay, here is a tough one: Differentiate the price function of an exchange option

$$E = S_2 e^{-q_2 T} N\left(\frac{\ln\left(\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2)T}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}\sqrt{T}}\right)$$
$$- S_1 e^{-q_1 T} N\left(\frac{\ln\left(\frac{S_2 e^{-q_2 T}}{S_1 e^{-q_1 T}}\right) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2)T}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}\sqrt{T}}\right)$$

with respect to the correlation coefficient ρ. Try doing this yourself first. After rearranging, you can just use the chain rule. If you give up, look at "Exchange option cora.docx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 9," for the answer.

12. Arguably, the most important application of correlation risk management is in risk management. In practice, the risk of a portfolio is often measured with the value at risk (VaR) concept. Is VaR sensitive to changes in the correlation of the assets in the portfolio? That is, what is the Cora of VaR?

13. What is the Gora of VaR?

14. Cora and Gora can be extended in many ways. Name two.

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CHAPTER 10

Quantifying Credit Correlation Risk

A key aspect of any credit risk VaR model is credit correlation.

—John Hull

In this chapter we discuss and quantify the correlation risk of financial products whose primary focus is credit risk. Let's just clarify what credit risk is.

> **CREDIT RISK**
>
> The risk of a financial loss due to an adverse change in the credit quality of a debtor.

There are principally two types of credit risk: (1) migration risk and (2) default risk. Figure 10.1 gives an overview of credit risk.

In Figure 10.1, migration risk refers to a migration from one credit state to another, for example a downward migration from AAA to B. An upward migration from B to AAA can also hurt an investor, if she is short a bond or if the investor is paying fixed in a credit default swap (CDS); see Figure 10.2. Default risk is a special case of migration risk for a migration of the debtor into the default state. Default risk exists only for a long credit position, for example being long a bond or long a tranche in a collateralized debt obligation (CDO).

However, migration risk and default risk have different dynamics. For example, if a bond migrates to a lower state for example from B to CCC, the

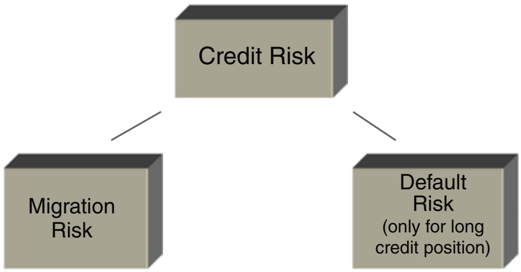

FIGURE 10.1 Types of Credit Risk

bond investor just suffers a paper loss and will receive his principal investment back at maturity if the bond does not default. However, if a bond defaults and stays in default, the bond investor will not receive his principal investment back, just the recovery rate of the bond.

What is credit correlation risk?

> ### CREDIT CORRELATION RISK
>
> The risk that credit quality correlations between two or more counterparties change unfavorably.

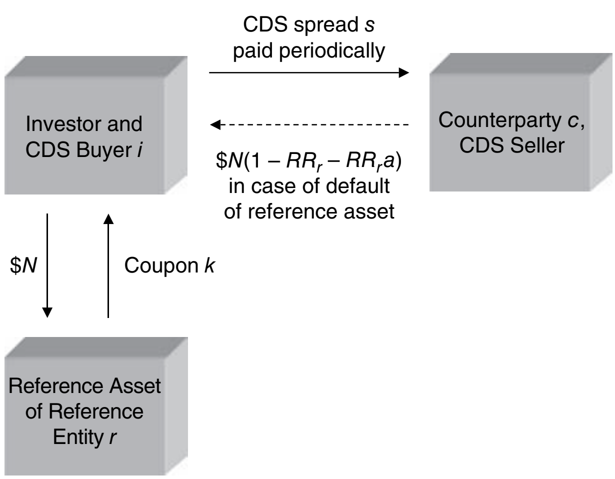

FIGURE 10.2 A CDS Used as Insurance to Hedge the Credit Risk of the Reference Asset

All loan portfolios of financial institutions as well as all structured products such as collateralized debt obligations (CDOs) and mortgage-backed securities (MBSs) are exposed to credit correlation risk. In addition, all portfolios that apply derivatives as a hedge also include credit correlation risk. Let's explain credit correlation risk with a portfolio of a bond and a credit default swap (CDS) that is used as a hedge.

10.1 CREDIT CORRELATION RISK IN A CDS

Let's just clarify what a credit default swap (CDS) is.

CREDIT DEFAULT SWAP (CDS)

A financial product in which the credit risk of an underlying asset is transferred from the CDS buyer to the CDS seller.

We have already briefly discussed some aspects of credit correlation risk of CDSs in the introductory Chapter 1, section 1.1. We will expand this discussion now. In a CDS, often the CDS buyer owns the underlying reference asset. In this case, the CDS can be viewed as insurance against the credit risk of the underlying asset: If the reference asset r defaults, the counterparty c (CDS seller) will compensate the investor and default swap buyer i. Figure 10.2 displays this graphically.

Figure 10.2 shows the CDS in the case of cash settlement. N is the notional amount, RR_r is the recovery rate of the reference asset, and $RR_r a$ is the accrued interested of the reference asset from the time of default to the next coupon date.[1]

Let's outline the credit correlation risk between these entities.

1. **Credit correlation between the counterparty c and the reference asset r.** We briefly discussed this critical correlation in the introductory Chapter 1, section 1.2. We found that the credit correlation between

1. We see from Figure 10.2 that the accrued interest $RR_r a$ is deducted from the settlement amount. This is because it is assumed that the coupon of the reference bond will be paid from the reference entity at the next coupon date. This may not happen though, since the reference asset is in a state of default. Hence, in some CDS contracts, the accrued interest is excluded.

the counterparty and the reference asset significantly influences the CDS price; see Figure 1.2. We will discuss the counterparty–reference asset correlation relationship in detail in section 10.2.

2. **Credit correlation between the investor *i* and the reference entity *r*, and the impact for the counterparty *c*.**

   This case is relevant if the investor pays the CDS spread *s* periodically, which is typically the case in practice. If the investor paid the CDS spread up front, the counterparty would have no credit risk with the investor and hence no correlation exposure with respect to the correlation between the investor and the reference entity. When the CDS spread is paid periodically, we have four cases:

   a. The credit correlation between the investor *i* and the reference entity *r* is **negative** with the credit quality of the investor decreasing and the credit quality of the reference asset increasing. This is the worst-case scenario for the counterparty: An increase in the credit quality of the reference entity increases the present value of the CDS for the counterparty since the counterparty is now receiving an above-market spread (this is beneficial from a profit perspective but negative from a risk perspective since a higher present value means more credit exposure). In addition, the decrease in the credit quality of the investor means higher credit risk for the counterparty with respect to the investor.

   b. The credit correlation between the investor *i* and the reference entity *r* is **negative** with the credit quality of the investor increasing and the credit quality of the reference asset decreasing. From a risk perspective, this is the best-case scenario for the counterparty because an increasing credit quality of the investor means a higher probability that the investor can pay the credit spread *s*. In addition, a lower credit quality of the reference asset means the present value of the CDS decreases (possibly getting negative). Hence the credit exposure of the counterparty with respect to the investor will decrease.

   c. The credit correlation between the investor *i* and the reference entity *r* is **positive** with the credit quality of the investor increasing and the credit quality of the reference asset also increasing. In this scenario there is a compensation effect: A higher credit quality of the investor makes future payments of the credit spread *s* more likely. However, the higher credit quality of the reference asset increases the present value of the CDS for the counterparty, which increases the counterparty's credit exposure to the investor.

   d. The credit correlation between the investor *i* and the reference entity *r* is **positive** with the credit quality of the investor decreasing and the credit quality of the reference asset also decreasing. In this scenario there is again a compensation effect: A lower credit quality of the investor makes future payments of the credit spread s less likely. However, a lower credit quality of the reference asset increases the present value of the CDS for the counterparty, which increases the counterparty's credit exposure to the investor.

In conclusion, the correlation between the investor and the reference asset should be included when deriving a CDS spread, since it impacts the credit exposure of the counterparty. For more details, see section 10.3.1, especially the last figure in that section, captioned "CDS Spread Dependence on Investor–Reference Asset Default Intensity Correlation."

3. **The correlation between the investor *i* and the counterparty *c*.**

   The default correlation between the investor i and the counterparty c is not a critical correlation to be considered in the default swap pricing process. If the investor goes into bankruptcy, he does not have to be too concerned if the counterparty also enters bankruptcy. The same logic applies to the counterparty with respect to the investor. For more details on the correlation between the investor i and the counterparty c (see section 10.3.1.5, "Results," point 4).

In the following section we discuss how to derive the CDS spread, including the critical default correlation between the reference entity r and the counterparty c.

10.2 PRICING CDSs, INCLUDING REFERENCE ENTITY–COUNTERPARTY CREDIT CORRELATION

The most critical credit correlation for pricing CDSs is the one between the reference entity r and the counterparty c. See Figure 10.2 for the role the reference entity r and the counterparty c play in a CDS. The correlation between the counterparty and the reference asset has received quite a bit of media attention recently, since financial institutions had sold CDSs on their own home countries; see, for example, Risk (2010) and European Central Bank (2009). In this case the counterparty–reference asset credit correlation should be high and consequently the CDS spread s low. In the following pages we derive a closed form solution for the CDS spread s, including reference asset–counterparty credit correlation.[2]

2. The following is a short version of the 2013 paper by G. Meissner, S. Rooder, and K. Fan, "The Impact of Different Correlation Approaches on Pricing CDS with Counterparty Credit Risk," *Journal of Quantitative Finance*, March 2013.

We use the following notation:

λ_t^r: default intensity of reference entity r, during time t to $t+1$ (hence λ_t^r: the default probability of r from t to $t+1$ conditional on survival until t).
Λ_t^r: cumulative default probability of reference entity r until t.
$\Delta\tau_t$: time between nodes $t-1$ and t, expressed in years.
s_t: annual default swap spread to be paid at time t.
τ_t: time between time 0 and time t, expressed in years.
Td_t: time of default, measured between time $t-1$ and default time, expressed in years.
N: notional amount of the swap.
r_t: risk-free interest rate from time 0 to time $t+1$.
RR_r: exogenous recovery rate of the reference entity.
$S_f(T_{dt})$: fair value of the default swap from the time the CDS was issued until the time of reference asset default without the possibility of counterparty default. $S_f(T_{dt})$ includes the notional amount N.
a: accrued interest on the reference asset from the last coupon date until the default date, hence $a = k\ T_d$, where k = coupon of the reference asset.

As displayed in Figure 10.2, we assume that the obligation that the counterparty has in case of default of the reference entity r is $N(1 - RR_r - RR_r a)$, where N is the notional amount of the swap, RR_r is the recovery rate of the reference asset issuer of the reference bond, and $RR_r a$ is the accrued interest of the reference bond from the last coupon date until default.

In analogy of the default intensity of the reference asset r, λ_t^r, we define the default intensity of the counterparty λ_t^c as:

λ_t^c: default intensity of counterparty (i.e., default swap seller) c, during time t to $t+1$ (hence λ_t^c is the default probability of c from t to $t+1$ conditional on survival until t).
Λ_t^c: cumulative default probability of counterparty c until t.
RR_c: exogenous recovery rate of the counterparty.

10.2.1 The Model

We now build a quadruple CDS payoff tree and a quadruple CDS premium tree, and then use swap evaluation techniques to find the fair CDS spread s. In both trees we have four scenarios:

1. Both counterparty c and reference entity r default, $\lambda(r \cap c)$.
2. Both counterparty c and reference entity r do not default, $\lambda(\bar{r} \cap \bar{c})$.

3. The reference entity r defaults but not the counterparty c, $\lambda\,(r \cap \bar{c})$.
4. The reference entity r survives but the counterparty c defaults, $\lambda\,(c \cap \bar{r})$.

We now build the payoff tree and the CDS spread tree and assign cash flows to each of these four scenarios.

10.2.1.1 The CDS Payoff Tree The payoff is the amount of cash that the counterparty c pays to the investor i in case of default of the reference entity r (see Figure 10.2 for the role of these three entities in a CDS).

We assume that if both the reference entity and the counterparty default, $\lambda(r \cap c)$, the standard payoff in case of default of the reference entity will be reduced by the recovery rate of the counterparty. Hence the payoff will be $N(1 - RR_r - RR_r a)RR_c$. There will be no payoff if neither the reference entity nor the counterparty default, $\lambda\,(\bar{r} \cap \bar{c})$. There will be the standard payoff $N(1 - RR_r - RR_r a)$ if only the reference entity defaults, $\lambda\,(r \cap \bar{c})$. We assume that if only the counterparty defaults, $\lambda\,(c \cap \bar{r})$, the counterparty will pay the time t fair value of the default swap, $S_f(t)$, without counterparty default risk, multiplied by the recovery rate of the counterparty, hence $S_f(t)\,RR_c$. Graphically we derive Figure 10.3.

From Figure 10.3 we observe that the payoff tree continues only if both the reference asset and the counterparty survive, $\lambda(r \cap c)$. In all other cases the CDS terminates. Including discount factors $e^{-r\tau}$, we derive

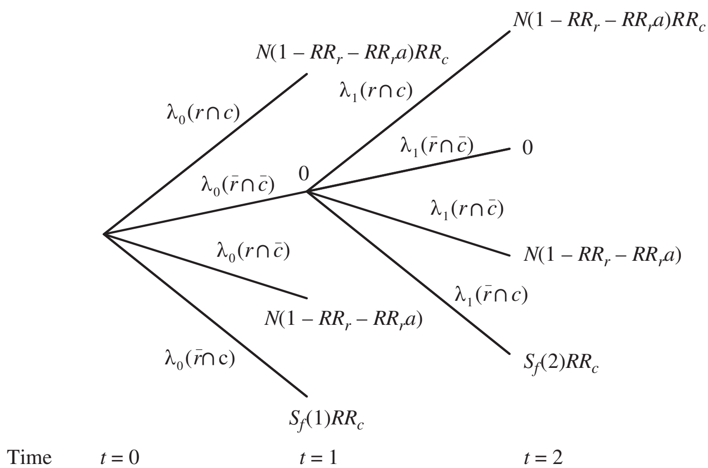

FIGURE 10.3 Two-Period Payoff Tree of a Default Swap, Including Counterparty Default Risk
Source: Meissner et al. (2013).

from Figure 10.3 the present value of the payoff of a two-period default swap as

$$
\begin{aligned}
&[\lambda_0(r \cap c)N(1 - RR_r - RR_r a)RR_c + \lambda_0(\bar{r} \cap \bar{c})N0 + \lambda_0(r \cap \bar{c})N(1 - RR_r \\
&- RR_r a) + \lambda_0\,(c \cap \bar{r})S_f(1)RR_c]e^{-r_0\tau_1} \\
&+ \lambda_0(\bar{r} \cap \bar{c})[\lambda_1(r \cap c)N(1 - RR_r - RR_r a)RR_c + \lambda_1(\bar{r} \cap \bar{c})N0 \\
&+ \lambda_1(r \cap \bar{c})N(1 - RR_r - RR_r a) + \lambda_1(c \cap \bar{r})(S_f(2)RR_c)]e^{-r_1\tau_2}
\end{aligned}
\tag{10.1}
$$

Generalizing equation (10.1) for T periods, we derive

$$
\begin{aligned}
&\sum_{t=1}^{T}\{[\lambda_{t-1}(r \cap c)N(1 - RR_r - RR_r a)RR_c + \lambda_{t-1}(r \cap \bar{c})N(1 - RR_r - RR_r a) \\
&+ \lambda_{t-1}(c \cap \bar{r})S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r} \cap \bar{c})\}
\end{aligned}
\tag{10.2}
$$

Equation (10.2) requires the critical inputs default intensity of the reference asset, λ_r, and default intensity of the counterparty, λ_c. These can be derived with a structural Merton (1974) based model (see Chapter 5, section 5.2.1), which requires the inputs asset value and debt value. Alternatively, we can derive λ_r and λ_c with a reduced form model, which abstracts from asset and debt values of the underlying entity and uses market prices as bonds or swaps in a stochastic model to derive the default probabilities λ_r and λ_c. Alternatively, λ_r and λ_c can be derived with a term structure model as the LIBOR market model (LMM),[3] which requires the inputs forward default intensity and default volatility. In the model presented here, λ_r and λ_c are simulated with an LMM model.

10.2.1.2 The CDS Spread Tree In most credit default swap contracts, the CDS spread s is paid in arrears (i.e., at the end of each period). In case of default, the default swap buyer typically has to pay the accrued spread amount from the last spread payment date to the default date. In addition, the solvent party still has to honor its obligations to the defaulting party. Hence we associate the following spread payments for the four default scenarios:

1. If both the reference asset and the counterparty default, $\lambda(r \cap c)$, the default swap buyer will make a final accrual payment, which is capped at

3. For an intuitive explanation of the LMM model, see Hull (2011, Chapter 31).

the payoff level in default: min[$s\ N\ \Delta\tau\ Td$, $N(1 - RR_r - RR_r a)RR_c$]. This scenario nets the obligations in the case of $s\ N\ \Delta\tau\ Td \geq N(1 - RR_r - RR_r a)RR_c$ and gives a payoff of $N(1 - RR_r - RR_r a)RR_c - s\ N\ \Delta\tau\ Td$ in the case of $N(1 - RR_r - RR_r a)RR_c \geq s\ N\ \Delta\tau\ Td$. This guarantees that the investor does not pay more CDS premium *s* than she receives as a payoff in the case of both *r* and *c* defaulting.

2. If both the reference asset and the counterparty survive, $\lambda(\bar{r} \cap \bar{c})$, the spread payment will be the standard $s\ N\ \Delta\tau$. Only in this case will the spread payment tree continue.
3. If the reference asset defaults but not the counterparty, $\lambda(r \cap \bar{c})$, the spread payment will be $s\ N\ \Delta\tau\ Td$.
4. If the reference asset survives but the counterparty defaults, $\lambda(c \cap \bar{r})$, the spread payment $s\ N\ \Delta\tau\ Td$ will be capped at the fair value of the swap times the recovery value: min[s N $\Delta\tau_t Td_t$, $S_f(t)RR_c$]. This again guarantees that the investor does not pay more CDS premium *s* than he receives as a payoff in case the counterparty *c* defaults.

Applying these cash flows, we derive the swap spread payment tree as seen in Figure 10.4.

From Figure 10.4 we get for the present value of the CDS spread payments

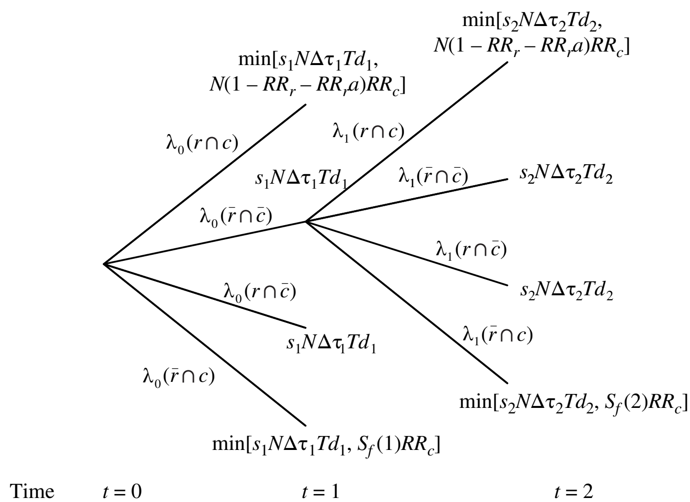

FIGURE 10.4 Two-Period CDS Spread Tree *s* of a Default Swap, Including Counterparty Default Risk
Source: Meissner et al. (2013).

$\{\lambda_0(r \cap c)\min[sN\Delta\tau_1 Td_1, N(1 - RR_r - RR_r a)RR_c] + \lambda_0(\bar{r} \cap \bar{c})s_1 N\Delta\tau_1$

$Td_1 + \lambda_0(r \cap \bar{c})s_1 N\Delta\tau_1\, Td_1 + \lambda_0(c \cap \bar{r})\min[sN\Delta\tau_1 Td_1, S_f(1)RR_c]\}e^{-r_0\tau_1}$

$+ \lambda_0(\bar{r} \cap \bar{c})\{\lambda_1(r \cap c)\min[s_2 N\Delta\tau_2 Td_2, N(1 - RR_r - RR_r a)RR_c]$

$+ \lambda_1(\bar{r} \cap \bar{c})s_2 N\Delta\tau_2 Td_2 + \lambda_1(r \cap \bar{c})s_2 N\Delta\tau_2\, Td_2 + \lambda_0(c \cap \bar{r})\min[s_2 N$

$\Delta\tau_2 Td_2, S_f(2)RR_c]\}e^{-r_1\tau_2}$

Assuming a constant swap spread s (i.e., $s_1 = s_2 = s_3, \ldots$), generalizing for T periods, and simplifying the notation by using $\min[s\ N\ \Delta\tau_t Td_t,\ N(1 - RR_r - RR_r a)RR_c] \equiv \min[x_t]$ and $\min[s\ N\ \Delta\tau_t Td_t,\ S_f(t)RR_c] \equiv \min[y_t]$, we derive

$$\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\min[x_t]+\lambda_{t-1}(\bar{r}\cap\bar{c})sN\Delta\tau_t Td_t+\lambda_{t-1}(r\cap\bar{c})sN\Delta\tau_t Td_t$$
$$+\lambda_{t-1}(c\cap\bar{r})\min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\} \tag{10.3}$$

As mentioned before, the default intensity of reference entity r, λ_t^r, and default intensity of counterparty c (i.e., default swap seller), λ_t^c, are inputs that will be modeled with a LIBOR market model (LMM) process.

10.2.1.3 Combining the CDS Payoff Tree and the CDS Spread Payment Tree

We derive the value of the CDS from the viewpoint of the CDS buyer by subtracting equation (10.3) from equation (10.2):

$$\begin{aligned}
&\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})N(1-RR_r-RR_r a)\\
&+\lambda_{t-1}(c\cap\bar{r})S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\\
&-\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)(\min[x_t]\\
&+\lambda_{t-1}(\bar{r}\cap\bar{c})sN\Delta\tau_t Td_t+\lambda_{t-1}(r\cap\bar{c})sN\Delta\tau_t Td_t\\
&+\lambda_{t-1}(c\cap\bar{r})\min[y_t]]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}
\end{aligned} \tag{10.4}$$

Setting equation (10.4) to zero and solving for the fair default swap spread s, which gives the credit default swap a value of zero, we derive

$$s=\frac{\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)N(1-RR_r-RR_r a)RR_c+\lambda_{t-1}(r\cap\bar{c})N(1-RR_r-RR_r a)\\ \quad+\lambda_{t-1}(c\cap\bar{r})S_f(t)RR_c]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}}{\begin{array}{l}\sum_{t=1}^{T}\{[\lambda_{t-1}(r\cap c)\min[x_t]/s+\lambda_{t-1}(\bar{r}\cap\bar{c})N\,\Delta\tau_t\,Td_t+\lambda_{t-1}(r\cap\bar{c})N\,\Delta\tau_t Td_t\\ \quad+\lambda_{t-1}(c\cap\bar{r})\min[y_t]/s]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c})\}\end{array}} \tag{10.5}$$

Equation (10.5) is a convenient and practical result. It is a closed form solution for valuing a CDS, including counterparty default risk and the correlation between reference asset and counterparty default. In addition, equation (10.5) is versatile in excluding counterparty default risk, as well as counterparty–reference asset correlation. This enables the user to isolate both counterparty risk and counterparty–reference asset correlation.

- To exclude counterparty default risk, we can apply equation (10.5) and set the default intensity of the counterparty λ^c to zero. In this case all terms except $\lambda_{t-1}(r\cap\bar{c})N(1-RR_r-RR_r a)$ and $\lambda_{t-1}(r\cap\bar{c})N\,\Delta\tau_{t-1}$ drop out, and $\lambda(\bar{r}\cap\bar{c})$ becomes $\lambda(\bar{r})$.
- To include counterparty default risk, which is, however, not correlated to the reference asset, we apply equation (10.5) and set the dependence parameter of the particular correlation approach ρ to zero.[4]
- To use the full version of the model (i.e., include counterparty risk, which is correlated to the reference asset), we use equation (10.5) with $\lambda^c \neq 0$ and apply a correlation concept and $\rho \neq 0$.

10.2.1.4 Testing the Impact of Different Dependence Approaches on the CDS Spread Besides λ_r and λ_c, which are modeled with an LMM process, equation (10.5) also requires the critical input joint default correlation $\lambda(r\cap c)$. We can use different dependence approaches, which we have already discussed in this book, to derive the joint default probability. We will test the impact of five different dependency approaches and apply them to equation (10.5) to study the impact on the CDS spread. The five dependency approaches are discussed next.

4. An exception is the Student's *t*. Here tail dependence exists, even for $\rho = 0$.

1. Correlating Brownian motions via the Heston (1993) model.

   We discussed this model in Chapter 4, section 4.1. In the Heston model the Brownian motions dz_1 and dz_2 are correlated by the identity

$$dz_1 = \sqrt{\rho_1}\, dz_2 + \sqrt{1 - \rho_1} dz_3 \tag{4.4}$$

   where dz_2 and dz_3 are $n \sim (0, 1)$ and independent.

   In this model we will correlate the Brownian motions dz_1 and dz_2 of two LIBOR market model (LMM) processes. One LMM process models the default intensity of the reference asset λ_r, one LMM process models the default intensity of the counterparty λ_c.

2. The binomial correlation approach of Lucas (1995), which we analyzed in Chapter 4, section 4.2.

   Here a variable takes the value 1 if entity r defaults and 0 otherwise. Equally, a variable takes the value 1 if entity c defaults and 0 otherwise. Applying the binomial approach to entities r and c, hence rewriting equation (4.8), we have

$$\lambda(r \cap c) = \lambda^r \lambda^c + \rho_2 \sqrt{[\lambda^r - (\lambda^r)^2][\lambda^c - (\lambda^c)^2]} \tag{4.8a}$$ [5]

3. The one-factor copula approach by Vasicek 1987, which we discussed in Chapter 6.

   We will test three different versions.

   a. The one-factor Gaussian copula. The core equation is

$$x_i = \sqrt{\rho}\, M + \sqrt{1 - \rho}\, Z_i \tag{6.1}$$

   where M and Z_i are independent and $n \sim (0,1)$. As a result, the latent variable x_i is $n \sim (0,1)$.

   b. The Student's t copula. The core equation is

$$\overline{x}_i = \sqrt{\rho}\, M + \sqrt{1 - \rho}\, Z_i \tag{6.7}$$

   where M and Z_i are independent and $n \sim (0,1)$. $x_i = \overline{x}_i \sqrt{W}$ where W follows an inverse gamma distribution. It follows that the latent variable x_i is Student's t distributed.

5. In Chapter 4 we used the notation of Lucas (1995), where $P(XY)$ is the joint default probability of X and Y. In this chapter we use a more statistical notation, i.e., $\lambda(r \cap c)$ is the joint default probability of r and c.

c. The double-t copula. The core equation is

$$x_i = \sqrt{\rho}\, M_s + \sqrt{1-\rho}\, Z_{i,S} \tag{6.8}$$

where M_s and $Z_{i,S}$ are independent and follow a Student's t distribution. Since the Student's t distribution is not stable under convolution, the latent variable x_i in equation (6.8) is not Student's t distributed.

Once we have generated λ_r and λ_c with an LMM process, and $\lambda(r \cap c)$ by one of the dependency approaches 1 to 3c, we can find the other dependency inputs of equation (10.5) via basic statistics:

$$\begin{aligned}
\lambda\,(r \cap \bar{c}) &= \lambda^r - \lambda(r \cap c) \\
\lambda\,(c \cap \bar{r}) &= \lambda^c - \lambda(r \cap c) \\
\lambda\,(\bar{r} \cap \bar{c}) &= 1 - \lambda(r \cup c) = 1 - [\lambda^r + \lambda^c - \lambda\,(r \cap c)]
\end{aligned}$$

10.2.1.5 Results We now present the results when applying different dependence approaches to equation (10.5). We first do a naive comparison of the dependency approaches; that is, we plot the dependence parameter of each approach on the abscise and derive the CDS spread.

Figure 10.5 displays the resulting CDS spread for a relatively low default environment. Figure 10.5 shows the very different CDS prices that result from different dependence parameters of a particular correlation approach. Hence dependence parameters cannot be compared directly, but must be viewed within their correlation context.

From Figure 10.5 we observe that just correlating the noise terms of the LMM processes in a low volatility environment has no noticeable effect on the CDS spread. The binomial approach displays the strongest correlation; that is, it results in the lowest CDS spread relative to its dependence parameter. Of the three one-factor copula approaches, the Student's t exhibits the highest correlation. The double-t correlation is lower (i.e., it produces a higher spread) than the Student's t since the t-distribution for the idiosyncratic factor Z_i in the double-t generates fatter tails for Z_i. Hence the dependence of x_i on the idiosyncratic factor Z_i increases, reducing correlation between the entities i.

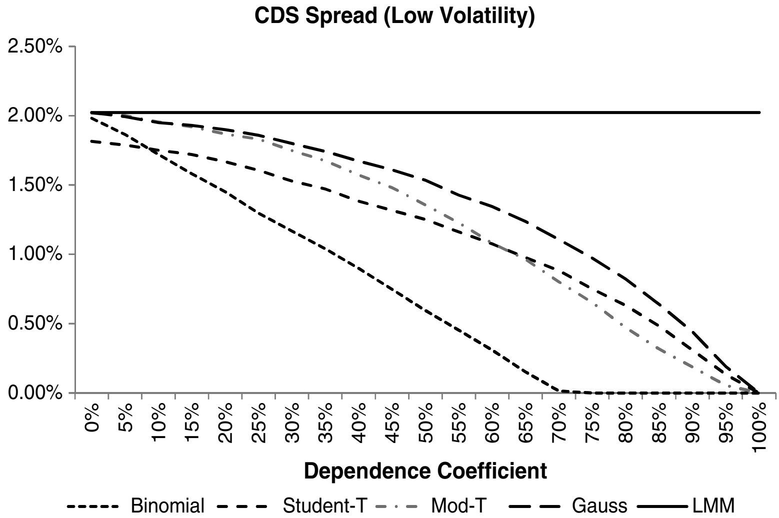

FIGURE 10.5 Three-Year CDS Spread Derived by Equation (10.5) with Respect to Different Correlation Approaches
The default probabilities λ_r and λ_c are derived with an LMM model (forward reference entity volatility 2% and 3%, forward counterparty volatility 3% and 4% for year 2 and 3 respectively, reference entity default intensity constant at 2%, counterparty default intensity constant at 4%); 100,000 simulations per dependency parameter, which takes about 25 seconds on a Core i5 PC. Upper and lower 95% confidence intervals are below 1.5%.

Figure 10.6 displays the CDS spread in a high volatility environment to show how the models compare in a stressed market environment.

Figure 10.6 shows a slight impact of higher correlation in the LMM approach on the CDS spread. The high volatility also smooths the binomial correlation function, similar to the effect that higher volatility has on the delta function of an option. The nonzero CDS spread for 100% correlation is due to the fact that sampling from the LMM process in case of high volatility produces some simulations in which the reference asset default rates are higher than counterparty default rates. This means that if the reference asset defaults, the counterparty can survive. Hence the CDS has some value and the CDS spread is nonzero.

For a model that derives the CDS spread, including reference asset–counterparty default correlation, see "CDS with default correlation.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 10."

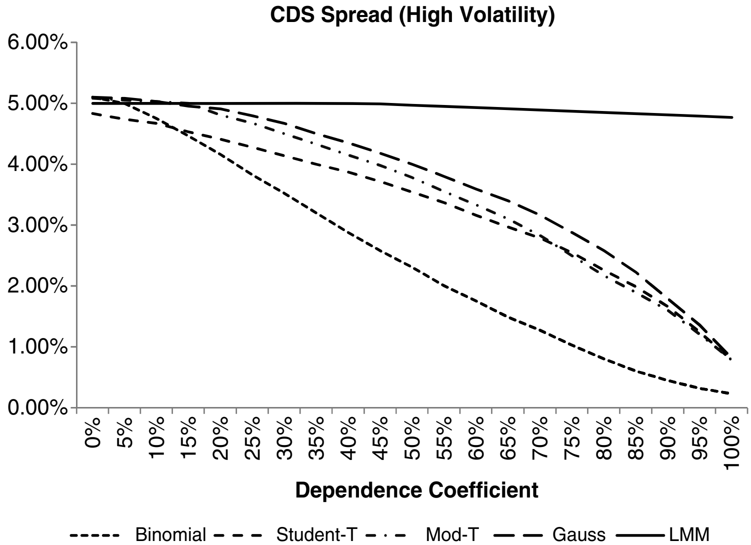

FIGURE 10.6 Three-Year CDS Spread with Respect to Different Correlation Approaches
Forward volatility for the reference entity and counterparty 50% for years 2 and 3 respectively (reference entity default intensity constant at 5%, counterparty default intensity constant at 7%); 100,000 simulations per dependency parameter. Upper and lower 95% confidence intervals are below 1.5%.

10.3 PRICING CDSs, INCLUDING THE CREDIT CORRELATION OF ALL THREE ENTITIES

From Figure 10.2, we observe the three entities in a CDS:

1. The investor and default swap buyer *i*.
2. The counterparty or default swap seller *c*.
3. The reference entity *r* that issued the reference asset.

In section 10.1 we outlined the correlation properties between the three entities in a CDS. We concluded that the credit correlation between the counterparty *c* and the reference asset *r* is critical. In addition, the credit correlation between the investor *i* and the reference asset *r* is important to the counterparty *c*. The correlation between the investor *i* and the counterparty *c* is less critical, but can be included in the valuation of the fair CDS spread *s*.

We will now present a model that includes the credit correlation between all three entities in a CDS. The only other study to include the default correlation of all three entities to value a CDS is Brigo and Capponi (2008). They apply a trivariate copula function in a reduced from, continuous time setting to evaluate the CDS spread s. The approach presented here is more practical since it is a discrete time model, in which the user can alter the cash flows if desired.

In the following pages we outline this model, which is an extension of the model presented in section 10.2.1.[6]

In analogy to the definitions of the reference asset r and the counterparty c in section 10.2.1, we define:

λ_t^i: default intensity of investor (i.e., default swap buyer) i; hence λ_t^i is the default probability of i from t to $t + 1$ conditional on survival until t.
Λ_t^i: cumulative default probability of counterparty i until t.
RR_i: exogenous recovery rate of the investor.

10.3.1 The Model

In the following we will build a CDS payoff tree and a CDS spread tree, and then use swap evaluation techniques to find the fair CDS spread s. The trees will include the correlation between all three entities in the CDS.

10.3.1.1 The CDS Payoff Tree We assume that if both reference entity and the counterparty default but the investor survives, $\lambda(r \cap c \cap \bar{i})$, the standard payoff in case of default of the reference asset $N(1 - RR_r - RR_r a)$ will be reduced by the recovery rate of the counterparty. Hence the payoff will be $N(1 - RR_r - RR_r a)RR_c$. There will be the standard payoff $N(1 - RR_r - RR_r a)$ if only the reference entity defaults and the counterparty and the investor survive, $\lambda(r \cap \bar{c} \cap \bar{i})$. We assume that if only the counterparty defaults, $\lambda(c \cap \bar{r} \cap \bar{i})$, the counterparty will pay the time t value of the default swap, $S_f(t)$, multiplied by the recovery rate of the counterparty, hence $S_f(t)\,RR_c$. If no entity defaults, $\lambda(\bar{r} \cap \bar{c} \cap \bar{i})$, there will be no payoff. Only in this case will the CDS stay alive and enter into the second-period octuple tree.

This brings us to Figure 10.7.

Displaying Figure 10.7 mathematically for multiple points in time t, we get

$$\begin{aligned}&\sum_{t=1}^{T}[\lambda_t(r \cap c \cap \bar{i})(N(1 - RR_r - RR_r a)RR_c)\\&+\lambda_t(r \cap \bar{c} \cap \bar{i})(N(1 - RR_r - RR_r a))\\&+\lambda_t(\bar{r} \cap c \cap \bar{i})(S_f(t)RR_c)]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r} \cap \bar{c} \cap \bar{i})\end{aligned} \tag{10.6}$$

6. Here we present a short version of the 2012 paper by G. Meissner, D. Mesarch, and O. Olkov, "The Valuation of Credit Default Swaps (CDSs) Including Investor-Counterparty-Reference Entity Default Correlation," forthcoming in the *Journal of Risk*.

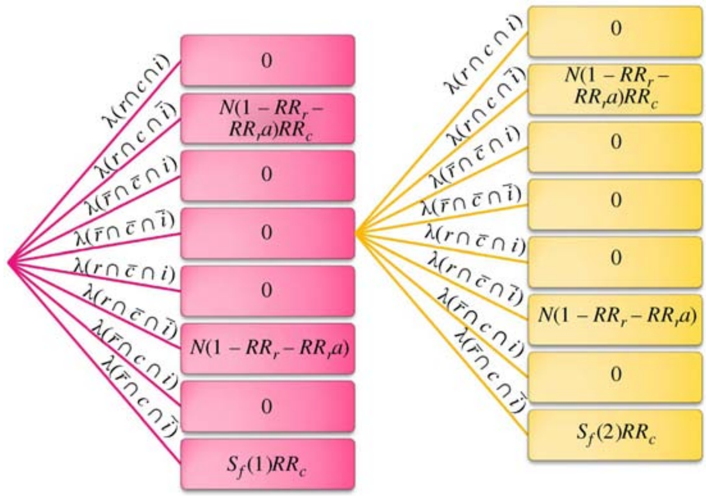

FIGURE 10.7 Two-Period CDS Payoff Tree with Associated Cash Flows

where T is the maturity date of the CDS and $(t + 1) - t$ is the length of each time step.

Equation (10.7) assumes a zero payoff in many default scenarios. A user can easily adjust payoffs in Figure 10.7 and equation (10.6) if the CDS contract specifies otherwise.

10.3.1.2 The CDS Spread Payment Tree We apply in our model that in most CDS contracts the CDS spread is paid in arrears (i.e., at the end of each period). In addition, in case of default, the default swap buyer typically has to pay the accrued spread amount a from the last spread payment date to the default date. Also, the U.S. bankruptcy law requires that the solvent party still has to honor its obligations to the defaulting party. Hence we associate the following spread payments for the four specific default scenarios:

1. If the investor survives but both the reference asset and the counterparty default, $\lambda(r \cap c \cap \bar{i})$, the CDS buyer will make a final accrual spread payment, which is capped at the payoff level in default: $\min[s\, N\, \Delta\tau\, Td,\, N(1 - RR_r - RR_r a)RR_c]$. This guarantees that the investor will not pay more CDS spread than the payout she receives. Specifically, this scenario nets the obligations in case of $s\, N\, \Delta\tau\, Td \geq N(1 - RR_r - RR_r a)RR_c$ and gives a payoff of $N(1 - RR_r - RR_r a)RR_c - s\, N\, \Delta\tau\, Td$ in case of $N(1 - RR_r - RR_r a)RR_c \geq s\, N\, \Delta\tau\, Td$.
2. If all three entities survive, $\lambda(\bar{r} \cap \bar{c} \cap \bar{i})$, the spread payment with be the standard $s\, N\, \Delta\tau$. Only in this case will the CDS spread payment tree continue.

3. If the reference asset defaults but the counterparty and investor survive, $\lambda(r \cap \bar{c} \cap \bar{i})$, the spread payment will be $s\ N\ \Delta\tau\ Td$.
4. If the counterparty defaults but the reference asset and the investor survive, $\lambda\,(c \cap \bar{r} \cap \bar{i})$, the spread payment $s\ N\ \Delta\tau\ Td$ will be capped at the fair value of the swap times the recovery value: min[$s\ N\ \Delta\tau_t Td_t$, $S_f(t)\ RR_c$]. This guarantees, as in scenario 1, that the investor will not pay more CDS spread than the payout she receives.

Applying these cash flows, we derive the swap spread payment tree as seen in Figure 10.8.

Displaying Figure 10.8 mathematically for multiple t, we derive

$$\begin{aligned}&\sum_{t=1}^{T}[\lambda_t(r \cap c \cap \bar{i})\min(sN\Delta tTd_t, N(1-RR_r-RR_r a)RR_c)\\&+\lambda_t(\bar{r} \cap \bar{c} \cap \bar{i})(sN\Delta t)+\lambda_t(r \cap \bar{c} \cap \bar{i})(sN\Delta tTd_t)\\&+\lambda_t(\bar{r} \cap c \cap \bar{i})\min(sN\Delta tTd_t, S_f(t)RR_c)]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r} \cap \bar{c} \cap \bar{i})\end{aligned} \tag{10.7}$$

where $t = 1,\ldots, T$ are the CDS spread payment points in time, and $(t+1)-t$ is the length of each time period between spread payments s according to the CDS contract.

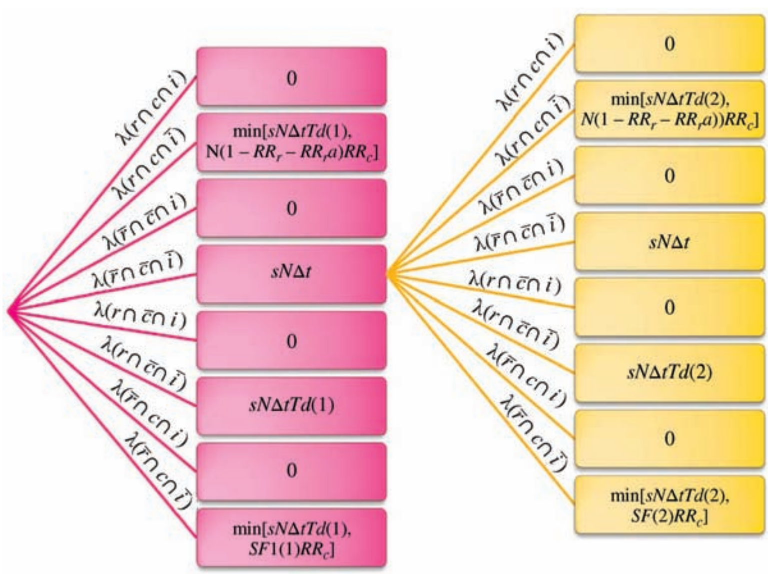

FIGURE 10.8 Two-Period CDS Spread Tree with Associated Cash Flows

We have again assumed that in many default scenarios the CDS spread payment is zero. The user can easily alter this in case the CDS contract specifies otherwise.

10.3.1.3 Combining the CDS Payoff Tree and the CDS Spread Payment Tree We derive the value of the CDS from the viewpoint of the CDS buyer by subtracting equation (10.7) from equation (10.6).

$$\begin{aligned}&\sum_{t=1}^{T}[\lambda_t(r\cap c\cap\bar{i})(N(1-RR_r-RR_r a)RR_c)\\&+\lambda_t(r\cap\bar{c}\cap\bar{i})(N(1-RR_r-RR_r a))\\&+\lambda_t(\bar{r}\cap c\cap\bar{i})(S_f(t)RR_c)]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c}\cap\bar{i})\end{aligned}$$

$$\begin{aligned}-&\left\{\sum_{t=1}^{T}[\lambda_t(r\cap c\cap\bar{i})\min(sN\Delta tTd_t, N(1-RR_r-RR_r a)RR_c)\right.\\&+\lambda_t(\bar{r}\cap\bar{c}\cap\bar{i})(sN\Delta t)+\lambda_t(r\cap\bar{c}\cap\bar{i})(sN\Delta tTd_t)\\&\left.+\lambda_t(\bar{r}\cap c\cap\bar{i})\min(sN\Delta tTd_t, S_f(t)RR_c)]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c}\cap\bar{i})\right\}\end{aligned}\tag{10.8}$$

Setting equation (10.8) to zero and solving for the CDS spread s, which gives the CDS a value of zero, we derive

$$s=\frac{\begin{aligned}&\sum_{t=1}^{T}[\lambda_t(r\cap c\cap\bar{i})(N(1-RR_r-RR_r a)RR_c)\\&+\lambda_t(r\cap\bar{c}\cap\bar{i})(N(1-RR_r-RR_r a))\\&+\lambda_t(\bar{r}\cap c\cap\bar{i})(S_f(t)RR_c)]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c}\cap\bar{i})\end{aligned}}{\begin{aligned}&\sum_{t=1}^{T}[\lambda_t(r\cap c\cap\bar{i})\min(sN\Delta tTd_t, N(1-RR_r-RR_r a)RR_c)/s\\&+\lambda_t(\bar{r}\cap\bar{c}\cap\bar{i})(N\Delta t)+\lambda_t(r\cap\bar{c}\cap\bar{i})(N\Delta tTd_t)\\&+\lambda_t(\bar{r}\cap c\cap\bar{i})\min(sN\Delta tTd_t, S_f(t)RR_c)/s]e^{-r_{t-1}\tau_t}\prod_{u=0}^{t-2}\lambda_u(\bar{r}\cap\bar{c}\cap\bar{i})\end{aligned}}\tag{10.9}$$

Equation (10.9) is a closed form solution for valuing a CDS, including default correlation of all involved parties: the investor i (CDS buyer), counterparty c (CDS seller), and reference asset r.

Underlying equation (10.9) is the set of basic probability equations:

$$\begin{aligned}
&\lambda(r \cap c \cap \bar{i}) = \lambda(r \cap c) - \lambda(r \cap c \cap i)\\
&\lambda(r \cap \bar{c} \cap i) = \lambda(r \cap i) - \lambda(r \cap c \cap i)\\
&\lambda(\bar{r} \cap c \cap i) = \lambda(c \cap i) - \lambda(r \cap c \cap i)\\
&\lambda(r \cap \bar{c} \cap \bar{i}) = \lambda(r) - \lambda(r \cap c) - \lambda(r \cap i) + \lambda(r \cap c \cap i)\\
&\lambda(\bar{r} \cap c \cap \bar{i}) = \lambda(c) - \lambda(r \cap c) - \lambda(c \cap i) + \lambda(r \cap c \cap i)\\
&\lambda(\bar{r} \cap \bar{c} \cap i) = \lambda(i) - \lambda(r \cap i) - \lambda(c \cap i) + \lambda(r \cap c \cap i)\\
&\lambda(\bar{r} \cap \bar{c} \cap \bar{i}) = 1 - \lambda(r \cup c \cup i)\\
&= 1 - [\lambda(r) + \lambda(c) + \lambda(i) - \lambda(r \cap c) - \lambda(r \cap i) - \lambda(c \cap i) + \lambda(r \cap c \cap i)]
\end{aligned} \tag{10.10}$$

Equation set (10.10) can be graphically displayed as shown in Figure 10.9.

The equation set (10.10) is quite versatile. We can eliminate the default risk of an entity by simply setting the default intensity of that entity $\lambda(\cdot)$ to zero. We can include the default intensity of the entities but without default correlation between two of the entities by setting the joint correlation coefficient ρ of the specific correlation approach to zero. The same logic applies to any of the three entities: We can include their default intensities $\lambda(\cdot)$ in the CDS valuation without correlating them by setting the entries in the correlation matrix ρ_M to zero.

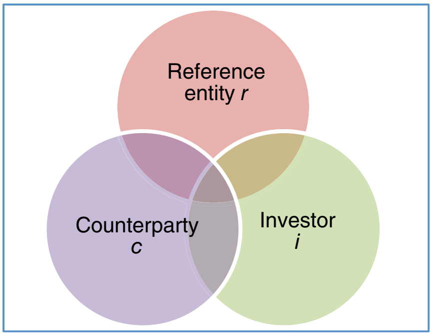

FIGURE 10.9 Graphical Representation of Equation Set (10.10)

10.3.1.4 Input Parameters of the Model The critical input variables that the model requires, $\lambda(r)$, $\lambda(c)$, $\lambda(i)$, $\lambda(r \cap c)$, $\lambda(r \cap i)$, $\lambda(c \cap i)$, and $\lambda(r \cap c \cap i)$, can be derived from historical data and input into the model. However, we can also model $\lambda(r)$, $\lambda(c)$, and $\lambda(i)$ with a term structure approach such as Cox-Ingersoll-Ross (CIR), Heath-Jarrow-Morton (HJM), or the LIBOR market model (LMM). For details see Meissner et al. (2012).

In addition, different approaches to model the joint probabilities can be applied. The bivariate default probabilities $\lambda(r \cap c)$, $\lambda(r \cap i)$, and $\lambda(c \cap i)$ can be modeled by the binomial correlation model (Lucas 1995) or the Heston model (1993) approach, which correlates Brownian motions. We can also apply a bivariate copula to model $\lambda(r \cap c)$, $\lambda(r \cap i)$, and $\lambda(c \cap i)$ and a trivariate copula to model $\lambda(r \cap c \cap i)$ as done in Brigo and Pallavicini (2008).

10.3.1.5 Results We display four main results with respect to the CDS spread s.

1. We first investigate the impact of the default intensity of the investor $\lambda(i)$ on the CDS spread, which is displayed in Figure 10.10.

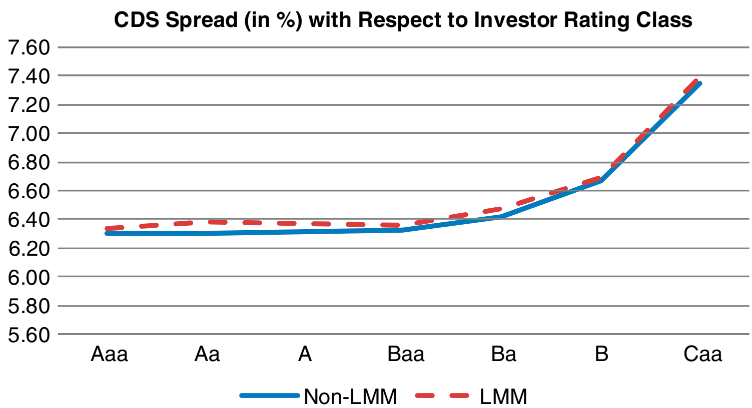

FIGURE 10.10 CDS Spread for Different Investor Rating Classes Derived by Equation (10.9)
CDS maturity: 5 years; rating class reference asset: B, which in 2011 represented $\lambda(r) = 5.27\%$; rating class counterparty: Aa, which in 2011 represented $\lambda(c) = 0.1\%$; rating class investor: abscise of Figure 10.10; default correlation between entities: 3%; recovery rate of reference asset, counterparty, and investor: 20%; coupon of reference asset: 9%; coupon frequency: semiannual; interest rates constant at 3%. For the LMM process: forward volatility of default intensity of all entities: constant at 15%; forward volatility of interest rates: constant at 9%.
Source: Meissner et al. (2013).

Figure 10.10 shows an expected result. The lower the rating class of the investor, the higher the CDS spread that the investor has to pay. This is because the counterparty will incur a loss if the investor defaults and the present value of the CDS is positive for the counterparty. The model also displays little difference between the CDS spreads when the input parameters are input directly (non-LMM) or when the input parameters are being modeled by an LMM process.

2. The most critical correlation in a CDS is the default intensity correlation between reference asset r and the counterparty c, $\lambda(r \cap c)$. We discussed this relationship already in Chapter 1, section 1.2, Figure 1.1. We concluded that the CDS spread is highly sensitive to the default intensity correlation $\lambda(r \cap c)$. This correlation risk also constitutes *wrong-way risk* (WWR). Wrong-way risk means that if the exposure (to the reference entity r) increases, it is more unlikely that the insurance provider (the counterparty) can honor its obligation; see Chapter 12, sections 12.4 and 12.5 on wrong-way risk.
3. Another important correlation of the CDS is the correlation between the investor i and the reference asset r, $\lambda(i \cap r)$. Regarding this correlation, we discussed the four possible scenarios in section 10.1. We concluded that a negative credit correlation with credit quality of the reference entity r increasing and the credit quality of the investor i deteriorating (case 2a), is the worst-case scenario for the counterparty c. A negative credit correlation, however, can also mean best-case scenario if credit quality of the reference entity r decreases and the credit quality of the investor i increases (case 2b). For a positive credit correlation between the reference entity r investor i, there are offsetting effects (cases 2c and 2d).

   The model of equation (10.10) derives the sensitivity of the CDS spread with respect to investor–reference asset counterparty default intensity correlation: $\lambda(r \cap i)$ as shown in Figure 10.11.

   From Figure 10.11 we observe a slightly negative dependence of the CDS spread with respect to the investor–reference asset default intensity correlation. This is due to the fact that the worst-case scenario 2a (see preceding point) for the counterparty can occur for high negative correlation and the counterparty wants to be compensated for this risk with a higher CDS spread.
4. The model shows that the CDS spread has close to zero sensitivity with respect to investor i–counterparty c default intensity correlation, $\lambda(c \cap i)$. This is because the possible effects net:
   - For negative correlation, $\lambda(i)$ may increase, while $\lambda(c)$ can decrease. Both effects tend to increase s. A decrease in $\lambda(i)$ and an increase in $\lambda(c)$ both tend to decrease s.

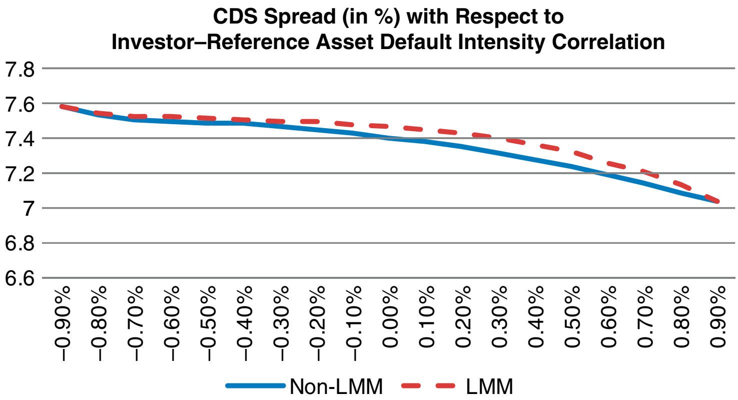

FIGURE 10.11 CDS Spread Dependence on Investor–Reference Asset Default Intensity Correlation
Rating class investor: Baa; all other input parameters as in Figure 10.10.
Source: Meissner et al. (2013).

- For positive correlation, both $\lambda(i)$ and $\lambda(c)$ may increase. An increase in $\lambda(i)$ tends to increase s, whereas an increase in $\lambda(c)$ tends to decrease s. For positive correlation both $\lambda(i)$ and $\lambda(c)$ may also decrease. Whereas a decrease in $\lambda(i)$ tends to decrease s, a decrease in $\lambda(c)$ tends to increase s. Since both the investor and the counterparty have credit risk with respect to each other, these effects net.

It is generally questionable whether an entity should consider its own default intensity and subsequently its own default correlation with other entities. This bilateral counterparty risk or debt value adjustment (DVA) is appealing from a mathematical perspective since it creates congruence and symmetry in pricing. However, if company *a* takes into account its own default intensity, the debt value of company *a* decreases since in the case of its own default, company *a* pays only the recovery rate on its debt. This arguably artificially increases the debt/equity ratio and can possibly increase *a*'s credit rating. We will discuss the aspect of DVA in detail in Chapter 12, section 12.7.

A code that prices a CDS with the default intensity correlation between all three entities can be found at "LMM pricing code.docx," at www.wiley.com/go/correlationriskmodeling, under "Chapter 10."

10.3.2 Cora for CDSs

As discussed in section (9.1), Cora measures how much a dependent variable changes if the correlation between one or more independent variables changes. For a CDS, we have four Coras. The most critical Cora is the change in the CDS value for a change in the reference asset–counterparty default correlation:

$$\text{CDSCora}_1 = \frac{\partial \text{CDS}}{\partial \lambda(r \cap c)} \tag{10.11}$$

Equation (10.11) reads: How much does the value of a CDS change if the default intensity correlation between the reference asset r and the counterparty c, $\lambda(r \cap c)$, changes by a very small amount? For the role r and c play in a CDS, see again Figure 10.2. We already analyzed this correlation in the introductory Chapter 1, Figure 1.2. We concluded that the impact of the default correlation between the reference entity r and the counterparty c on the CDS is significant. For the extreme case of the default correlation $\lambda(r \cap c) = 1$, the CDS is worthless, since if the reference entity defaults, so will the insurance seller c.[7]

We can derive a second Cora for a CDS as displayed in equation (10.12):

$$\text{CDSCora}_2 = \frac{\partial \text{CDS}}{\partial \lambda(r \cap i)} \tag{10.12}$$

Equation (10.12) reads: How much does the value of CDS change if the default correlation between the reference asset r and the investor i, $\lambda(r \cap i)$, changes by a very small amount? The CDSCora_2 function is displayed in Figure 10.11. We observe that CDSCora_2 values are in a relatively narrow range from 7 to 7.6 percent of the notional amount. Therefore the counterparty does not have to change her correlation hedge much if the correlation between the reference entity r and the investor i, $\lambda(r \cap i)$, changes.

We can derive a third function of the Cora in a CDS.

$$\text{CDSCora}_3 = \frac{\partial \text{CDS}}{\partial \lambda(i \cap c)} \tag{10.13}$$

7. There is one exception, though. If the default intensity of the counterparty c is smaller than the default intensity of the reference entity r, $\lambda(c) < \lambda(r)$, then even in the case of perfect default intensity correlation, some Monte Carlo simulations will result in the reference entity defaulting but the counterparty surviving. (The reader should keep in mind that we are correlating default *intensities*, not actual defaults.) In this case the CDS has some value. This is especially the case of high default intensity volatility. The reader can verify this with the model "CDS with default correlation.xlsm," at www.wiley.com/go/correlationriskmodeling, under "Chapter 10."

However, we concluded in subsection 10.3.1.5, point 4 that CDSCora_3 is close to 0, since the effects of a change in the correlation between the investor i and the counterparty c net.

Last, we can derive the sensitivity of the CDS value with a change in the joint default correlation of *all* entities in a CDS. This is expressed in equation (10.14).

$$\text{CDSCora}_4 = \frac{\partial \text{CDS}}{\partial \lambda(r \cap c \cap i)} \tag{10.14}$$

Equation (10.14) reads: How much does the value of CDS change if the default correlation between the reference asset r, the counterparty c, and the investor i, $\lambda(r \cap c \cap i)$ changes by a very small amount? The default intensity correlation $\lambda(r \cap c \cap i)$ can be simulated by a trivariate copula as in Brigo and Pallavicini (2008) or can be derived by Monte Carlo simulation. The numerical values for CDSCora_4 are complex and depend on the default intensity input parameter values $\lambda(r)$, $\lambda(c)$, and $\lambda(i)$; the volatilities of $\lambda(r)$, $\lambda(c)$, and $\lambda(i)$; and the correlation $\lambda(r \cap c \cap i)$. Different sensitivities of the CDS spread result for different combinations of the input parameters.

10.3.3 Gora for CDSs

In section 9.1 and equation (9.2) we defined Gora as the changes of Cora for a small change in the correlation of two or more variables. Since we have four different Coras for a CDS, we have four different Goras. The most critical Cora for a CDS is CDSCora_1 in equation (10.11). Therefore the most critical Gora is the Gora with respect to CDSGora_1. Hence we derive

$$\text{CDSGora}_1 = \frac{\partial \text{CDSCora}_1}{\partial \lambda(r \cap c)} = \frac{\partial^2 \text{CDS}}{\partial \lambda^2(r \cap c)} \tag{10.15}$$

Equation (10.15) reads: How much does the Cora of a CDS change if the default intensity correlation between the reference asset r and the counterparty c, $\lambda(r \cap c)$, changes by a infinitesimally small amount? From the last term in equation (10.15) we see that CDSGora_1 is the curvature of the original CDS function with respect to the default intensity correlation $\lambda(r \cap c)$. CDSGora tells us how stable the correlation hedge of the CDS is. The higher CDSGora is, the higher is the change of Cora and the more often we have to adjust the correlation hedge.

There is no closed form solution for the CDSGora_1. However, we can easily simulate it by numerically differentiating the CDS spread function, which we displayed in Figure 1.2 twice with respect to correlation. Doing so, we derive Figure 10.12.

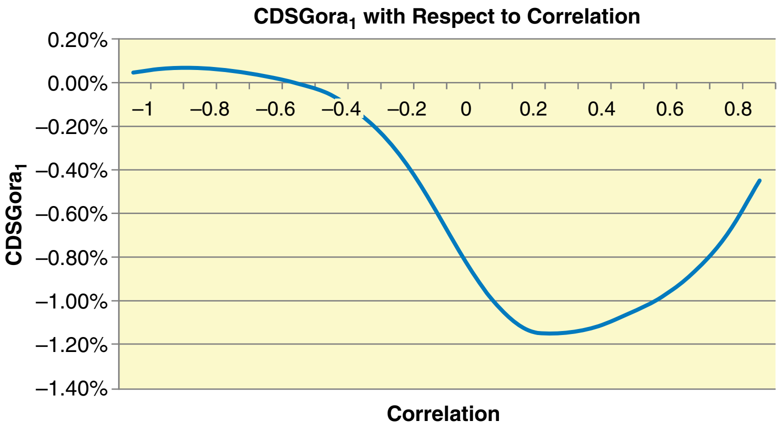

FIGURE 10.12 CDSGora_1 with Respect to the Default Intensity Correlation between the Reference Entity r and the Counterparty c, $\lambda(r \cap c)$
Source: Meissner et al. (2013).

From Figure 10.12 we observe that CDSGora_1 is slightly positive and slightly decreasing for correlation values from -0.9 to about -0.55. This means that in this area the necessity to change the hedge reduces (see Chapter 11 for hedging correlation risk). At about a correlation value of -0.55 the value of the CDSGora_1 is close to zero, so the investor does not have to adjust the correlation hedge much. For a correlation of -0.55 to $+0.3$, we observe that CDSGora_1 decreases. This means that the investor is more exposed to changes in the reference asset–counterparty correlation and has to adjust the hedge often. For an increasing correlation in the range of about $+0.2$ to $+1$, CDSGora_1 increases and approaches zero. This means that the investor is less exposed to changes in correlation and has to adjust the correlation hedge less often for changes in correlation.

There are also the CDSGoras resulting from the CDSCoras in equations (10.12) to (10.14). From equation (10.12) we derive

$$\text{CDSGora}_2 = \frac{\partial \text{CDSCora}_2}{\partial \lambda(r \cap i)} = \frac{\partial^2 \text{CDS}}{\partial \lambda^2(r \cap i)} \tag{10.16}$$

Equation (10.16) reads: How much does the Cora of a CDS change if the default intensity correlation between the reference asset r and the investor i, $\lambda(r \cap i)$, changes by a infinitesimally small amount? From the last term in equation (10.16) we see that CDSGora_2 is the curvature of the original CDS function with respect to the default intensity correlation $\lambda(r \cap i)$. We have displayed the $CDSCora_2$ in Figure 10.11. We observe that $CDSCora_2$ decreases for increasing correlation. Therefore the counterparty has to increase his correlation hedge if the default intensity correlation $\lambda(r \cap i)$ increases. Since the $CDSCora_2$ function in Figure 10.11 is quite monotonously decreasing (i.e., has fairly low curvature), the counterparty does not have to change the degree of his correlation hedge much if the default intensity correlation $\lambda(r \cap i)$ increases.

In the previous section we concluded in the $CDSCora_3$ function that there is no significant influence of the correlation $\lambda(r \cap i)$ on the CDS spread; that is, the $CDSCora_3$ function is close to horizontal. Therefore the $CDSGora_3$ function is close to zero.

With respect to $CDSCora_4$, we concluded that the numerical values for $CDSCora_4$ are complex and depend on the default intensity input parameter values $\lambda(r)$, $\lambda(c)$, and $\lambda(i)$; the volatilities of $\lambda(r)$, $\lambda(c)$, and $\lambda(i)$; and the correlation $\lambda(r \cap c \cap i)$. Different combinations of input parameters result in sensitivities. Therefore, $CDSGora_4$ values are also complex and give different results for different input parameter values.

10.4 CORRELATION RISK IN A COLLATERALIZED DEBT OBLIGATION (CDO)

In Chapters 5 and 6 we discussed the valuation of CDOs in detail. Here we will derive the correlation risk parameters Cora and Gora in a one-factor Gaussian copula (OFGC) framework. The underlying CDO will be a synthetic CDO. The tranches are the same as in the U.S. CDX index.

In Figure 10.13 we recognize the three parties in a CDO: The protection buyer buys CDSs typically to hedge credit exposure. The special purpose vehicle (SPV) is an intermediary that manages the CDO. The investor invests in a particular tranche and assumes the credit risk.

10.4.1 Types of Risk in a CDO

There are two main factors that determine the value of a CDO: the default probability of assets in the CDO and the default correlation between the assets. Consequently, the two main risks when hedging CDOs are credit risk and correlation risk, as shown in Figure 10.14.

Correlation risk in a CDO tranche is the risk that the correlation between the assets in a CDO tranche and consequently the value of the CDO tranche changes unfavorably. We will now discuss correlation risk, which is measured by Cora and Gora.

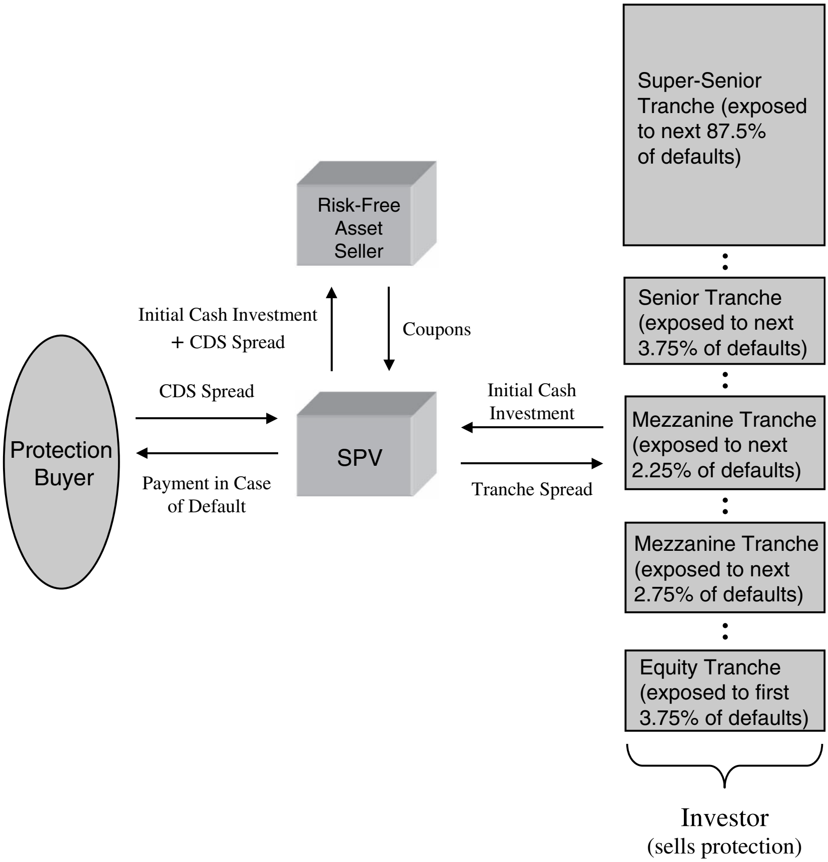

FIGURE 10.13 A Synthetic CDO with Tranches as in the U.S. CDX Index

FIGURE 10.14 Main Risks of a CDO

10.4.2 Cora of a CDO

We already displayed the dependency of the tranche spreads in a CDO with respect to correlation in Chapter 1, Figure 1.7. We display it again here.

From Figure 10.15, we observe that the investor in the 0%–3% equity tranche investor is *long correlation*. This means that as the correlation between the assets increases, so does the present value of the equity tranche for the investor. This is because the investor receives a fixed spread of the tranche, for example LIBOR + 500 basis points. When the market equity tranche spread decreases with increasing correlation, the investor then receives a spread that is higher than the market spread.

Formally, the Cora of a tranche x in a CDO is displayed in equation 10.17:

$$\text{Cora}(\text{Tranche}\,x) = \frac{\partial s(\text{Tranche}\,x)}{\partial \rho} \tag{10.17}$$

Equation (10.17) reads: How much does the spread s of tranche x change if the correlation between all assets ρ in the CDO changes by an infinitesimally small amount?

Differentiating the functions in Figure 10.15 gives the Cora of the tranches in a CDO, which is displayed in Figure 10.16.

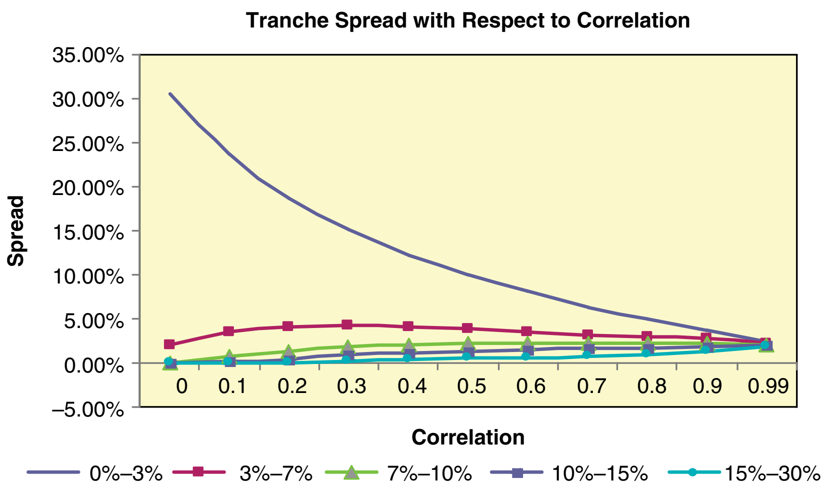

FIGURE 10.15 Tranche Spread with Respect to Correlation in the One-Factor Gaussian Copula (OGFC) Model; 125 Credits, 1% Default Intensity Rate, 5-Year Maturity, 30,000 Monte Carlo Simulations
Source: Meissner et al. (2013).

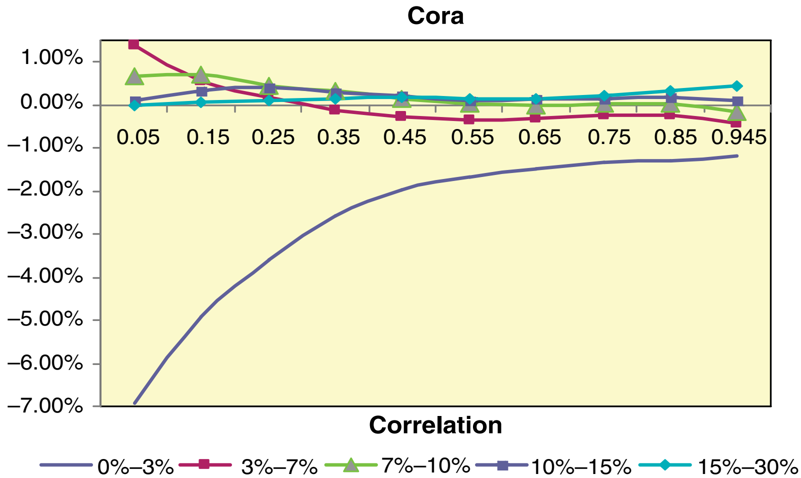

FIGURE 10.16 Cora for Different Tranches in a CDO; 125 Credits, Default Intensity 1%, 5-Year Maturity, 30,000 Monte Carlo Simulations
Source: Meissner et al. (2013).

From Figure 10.16 we observe that Cora is fairly constant and close to zero for the 3%–7%, 7%–10%, 10%–15%, and 15%–30% tranches. Therefore the investor in these tranches has little correlation risk and consequently does not have to hedge correlation risk much. However, the 0%–3% equity tranche is highly sensitive to correlation changes: If correlation increases, the value of Cora decreases (on an absolute basis), and the investor can reduce his correlation hedge. Conversely, if correlation decreases, the value of the Cora increases (on an absolute basis), and the investor has to increase the correlation hedge.

10.4.3 Gora of a CDO

Differentiating the tranche functions in Figure 10.17 gives the Gora for tranches in the CDO. Formally,

$$\text{Gora}(\text{Tranche } x) = \frac{\partial \text{Cora}(\text{Tranche } x)}{\partial \rho} = \frac{\partial^2 (\text{Tranche } x)}{\partial \rho^2} \tag{10.18}$$

Equation (10.18) reads: How much does the Cora of tranche x change if the correlation between all assets ρ in the CDO changes by an infinitesimally small amount? From the last term in equation (10.18) we see that the Gora of a tranche is the curvature of the original tranche functions (displayed in

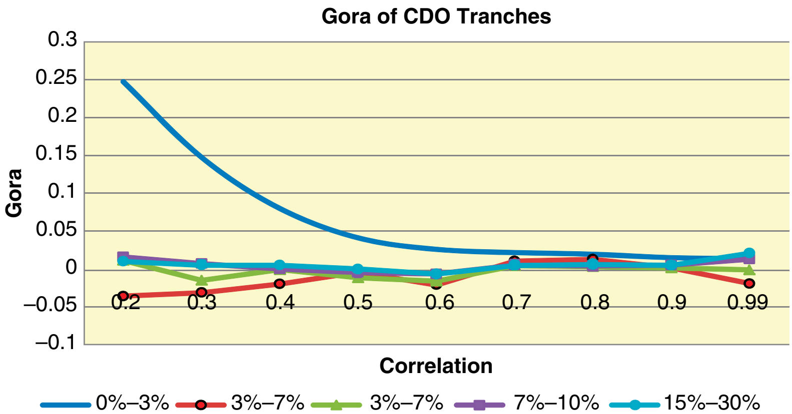

FIGURE 10.17 Gora of Tranches of a CDO; 125 Credits, Default Intensity 1%, 5-Year Maturity, 30,000 Monte Carlo Simulations

Figure 10.15) with respect to the correlation coefficient ρ. Graphically, the Goras of the tranches in the CDO are displayed in Figure 10.17.

From Figure 10.17 we observe that the Gora of the 3%–7%, 7%–10%, 10%–15%, and 15%–30% tranches is close to zero. Therefore, the investor does not have to change his correlation hedge much. This is sensible since in Figure 10.16 we observed a Cora of close to zero for these tranches; hence there is little need to hedge correlation risk at all. However, the 0%–3% equity tranche shows a high Gora, which means a high necessity to change the hedge amount Cora. In particular, Gora is high for low correlation levels. This means that for low correlation levels the investor has to adjust the correlation hedge often for changes in the level of correlation.

10.5 SUMMARY

In this chapter we discussed how to quantify credit correlation risk. **Credit risk** is the risk of financial loss due to an adverse change in the credit quality of a debtor. There are two main types of credit risk: (1) **migration risk**, which is the risk of an unfavorable change in the credit quality of a debtor, and (2) **default risk**, which is a special case of migration risk and occurs only if an investor is long credit (i.e., has bought a bond or is receiving fixed in a CDS).

Credit correlation risk is the risk that credit quality correlations between two or more counterparties change unfavorably.

All loan portfolios of financial institutions, as well as all structured products such as collateralized debt obligations (CDOs) and mortgage-backed securities (MBSs), are exposed to credit correlation risk. In addition, all derivatives used as a hedge also include credit correlation risk. For example, a credit default swap (CDS) used as a hedge includes three parties: (1) the CDS buyer, (2) the CDS seller (counterparty), and (3) the underlying asset.

Therefore, there are three types of credit correlation risk in a CDS: (1) If the default correlation between the counterparty and the reference asset increases, the CDS value will decrease, with a paper loss for the investor; (2) if the default intensity of the underlying asset decreases and the default intensity of the investor increases, the counterpart will have higher credit risk exposure; and (3) the default intensity correlation between the investor and the counterparty is of minor importance, because the investor and the counterparty do not care too much if they themselves default at the same time their counterparty defaults.

Models that value a CDS, including the reference asset–counterparty default correlation risk, can be derived in a rigorous way. In addition, models that include the default intensity correlation of all three entities in a CDS are available.

Formally, Cora and Gora of a CDS measure the sensitivity of a CDS value change with respect to changes in correlation between two entities in a CDS. Since we have three entities in a CDS, principally three Coras and three Goras for a CDS exist. We can also derive the Cora and Gora of a CDS for the default intensity correlation of all three entities in a CDS.

The correlation risk of CDOs has received great attention during the global financial crisis of 2007 to 2009. CDOs and their correlation properties were called toxic. However, once understood, the correlation risk in a CDO is quite intuitive. In addition, the correlation risk can be quantified with Cora and Gora and hedged accordingly.

PRACTICE QUESTIONS AND PROBLEMS

1. What is credit risk?
2. Which two types of credit risk exist? What is the relationship between these two types of credit risk?
3. What is credit correlation risk?
4. Name three financial products that are exposed to credit correlation risk.
5. A CDS that is used as a hedge has three parties: (1) the investor (CDS buyer), (2) the counterparty (CDS seller), and (3) the underlying asset. The default correlation between which two entities is most significant for the valuation of a CDS?

6. For the counterparty, the default correlation between the investor and the underlying asset is also of importance. Which is the worst-case scenario for the counterparty from a risk perspective?
7. When valuing a CDS, we can also include the default intensity correlation between all three entities. Draw a Venn diagram that displays the default intensity correlation's properties.
8. What information does the Cora of a CDS give us?
9. Since there are three entities in a CDS, there are principally three Coras. Name them and interpreted them. Which one is the most critical?
10. What does the Gora of a CDS tell us?
11. What are the two main risks in a CDO?
12. The value of a CDO and its tranches depends critically on the correlation of the assets in the CDO. Draw a graph showing the equity tranche value, mezzanine tranche value, and a senior tranche value with respect to correlation.
13. What does the Cora of a tranche in a CDO tell us?
14. Which tranche in a CDO has the highest correlation risk (i.e., the highest Cora)?
15. CDOs and their correlation properties are sometime termed "toxic." Do you agree with this view?

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CHAPTER 11
Hedging Correlation Risk

Only if it was possible to delta-hedge correlation risk…would it make sense to use a full-blown stochastic correlation model.

—Lorenzo Bergomi

In this chapter we discuss why hedging financial correlation risk is more challenging than hedging other financial risks such as market risk or credit risk. However, we will show two methods that can be applied to hedge financial correlation risk. At the end of the chapter, we will discuss in which situations it is better to hedge with options and in which situations it is better to hedge with futures.

11.1 WHAT IS HEDGING?

Let's first clarify what hedging is.

> **HEDGING**
>
> Reducing risk. More precisely, entering into a second trade to reduce the risk of an original trade.

If the original trade is a simple transaction such as being long a bond, there are three main ways to hedge the market risk (risk of an unfavorable change in the price) and the credit risk (migration risk and default risk) (see

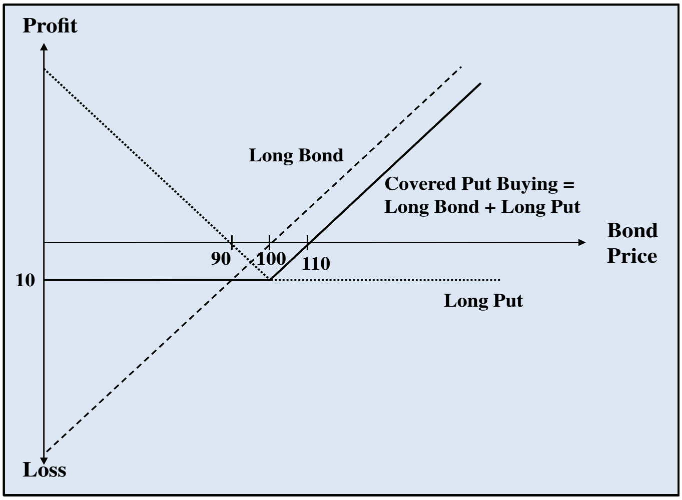

FIGURE 11.1 An Investor Hedging a Long Bond Position (Dashed Line) with a Put Option (Dotted Line). The Overall Payoff Is the Covered Put Buying Line.

Figure 10.1). Let's assume that an investor has bought a bond of Greece. To reduce the risk, she can:

1. Simply sell the bond. This is beneficial since all types of risk such as market risk, credit risk, operational risk, liquidity risk, and correlation risk with other assets are then eliminated. The severe drawback, though, is that the investor may have to sell at a low price. Any loss realized will not be recovered if the Greek bond price improves at a later point in time.
2. Hedge the bond with a derivative such as a forward, future, or swap,[1] or an option. Let's look at an example of a hedge for the long bond position with a put option. The investor has bought a bond at $100. She is now worried about a possible price deterioration and buys a put option with a strike of $100 as insurance, paying a premium of $10. This is also referred to as covered put buying or a married put. The overall payoff is displayed in Figure 11.1.

1. Forwards, futures, and swaps are closely related. A forward is the agreement between two parties to conduct a trade at a certain price in the future. A swap is just a series of forward contracts. Forwards and swaps are typically traded over the counter (OTC), not on an exchange. A future trades on an exchange and is just a standardized forward (i.e., the maturity date, notional amount, underlying, etc. are standardized). See Hull (2011) or Meissner (1997) for more on derivatives.

The investor has now limited her downside risk to a loss of \$10 (the premium cost). The catch is the reduced upside potential, which is lowered by the put premium of \$10.

3. The investor can also hedge the long Greek bond position by selling a product that is correlated with the price movement of the Greek bond. For example, let's assume that historically the correlation between Spanish bonds and Greek bonds has been positive. The investor decides to sell a bond of Spain to offset the price risk of the long Greek bond. However, in this case the investor has correlation risk, in particular the risk that the correlation between Greek bonds and Spanish bonds is not positive in the future. It could happen that the Greek bond declines in price and the Spanish bond increases in price, leading to a loss in both the underlying and the hedge.

In the second case we applied a derivative, an option, to hedge a nonderivative, a bond position. In trading practice, the reverse operation of case 2 is also applied. We can hedge the risk of an option using a nonderivative. Let's discuss this case.

Typically when hedging a derivative such as an option, the underlying asset is bought or sold to offset an unfavorable change in the option value. For example, if a trader has bought a call option on the Apple (AAPL) stock, the trader is vulnerable to a price decline in AAPL, since in this case the call value decreases. To hedge this risk, the trader typically sells the underlying AAPL stock, or more precisely, the trader sells the delta amount of AAPL. This delta is given from a model, for example the Black-Scholes-Merton option pricing model. Let's look at an example of a delta hedge.

EXAMPLE 11.1: DELTA HEDGING

An option trader at Goldman Sachs buys a call on IBM. The call option premium is \$10,000 (e.g., the trader is buying 1,000 calls with a call premium of $C_0 = \$10$). IBM trades at $S_0 = 100$. The trader decides to delta hedge the IBM price risk of the option. The delta, derived from an option pricing model such as the Black-Scholes-Merton model, comes out to 51%. Formally:

$$\Delta_C = \frac{\partial C}{\partial S} \approx \frac{0.51}{1} \tag{11.1}$$

where

Δ_C: delta of the call

(*continued*)

(*continued*)

C: call price

S: price of the underlying stock IBM

∂: partial derivatives operator

Equation (11.1) reads: How much does the call price C change if S changes by an infinitesimally small amount, assuming all other variables influencing the call price are constant? For practical purposes, the change in S can be approximated by a change of 1, as done in equation (11.1).

How much IBM stock does the option trader have to sell to stay delta neutral, meaning the option trade has no price risk with respect to the IBM stock?

The option trader has to sell IBM stock in the delta amount, hence 51% of the option premium of \$10,000. Therefore the option trader sells 51 shares at \$100 each and receives \$5,100. The option trader now has no IBM price risk.[2] Let's show this.

If IBM increases by 1%, following equation (11.1) the call price increases by 0.51%. Therefore the profit on the call is

$$C_1 \times 1{,}000 - C_0 \times 1{,}000 = \$10.051 \times 1{,}000 - \$10 \times 1{,}000 = \$51$$

The loss on the hedge is

$$S_1 \times 51 - S_0 \times 51 = \$101 \times 51 - \$100 \times 51 = \$51$$

Hence the option trade is hedged against price risk of IBM. What the option trader gains on the call is lost on the hedge, and vice versa.

11.2 WHY IS HEDGING FINANCIAL CORRELATIONS CHALLENGING?

Hedging correlation risk is more difficult than hedging a bond, a stock, or an option, for two main reasons.

1. Hedging correlation risk involves two or more assets, since the correlation is measured between at least two assets.

2. This is true for small changes in the IBM price. If the IBM price changes by a large amount, the delta changes and has to be adjusted.

2. Hedging financial correlation risk is challenging because there is principally no underlying instrument that trades in the market and that can be bought or sold as a hedge.

However, the correlation market is evolving. We have already discussed four ways to trade correlation in Chapter 1, section 1.3.2, "Trading and Correlation." We will now discuss how to use the correlation products, which already exist in financial practice, to hedge correlation risk.

11.3 TWO EXAMPLES TO HEDGE CORRELATION RISK

In the following, we present two methods for hedging financial correlation risk. The first method is to hedge with correlation-dependent options, and the second is to hedge with a correlation swap.

11.3.1 Hedging CDS Counterparty Risk with a Correlation-Dependent Option

Any financial product can be used for two main purposes:

1. Speculation (i.e., trying to generate a profit).
2. Hedging (i.e., reducing risk).

In Chapter 1, section 1.3.2, "Trading and Correlation," we discussed the speculative aspect of correlation options and showed how the correlation influences the prices of certain assets such as exchange options and quanto options.

These correlation-dependent options can also be used to hedge correlation risk. Let's show how the counterparty credit risk in a credit default swap (CDS) can be hedged with an option whose value depends on correlation.

EXAMPLE 11.2: HEDGING CDS COUNTERPARTY RISK WITH AN OPTION ON THE BETTER OF TWO

Let's start with an investor who has invested in a Spanish bond and has decided to hedge the default risk of Spain with a credit default swap (CDS) from BNP Paribas. We have already discussed this CDS in Chapter 1, section 1.2, and a similar CDS in Chapter 10, section 10.2.

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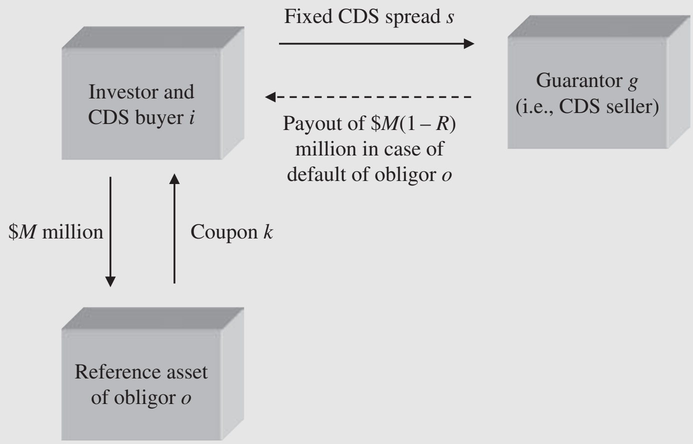

FIGURE 11.2 An Investor Hedging Spanish Bond Exposure with a CDS from BNP Paribas

The investor in the CDS of Figure 11.2 has default correlation risk between the reference entity (Spain) and the CDS seller (BNP Paribas). As discussed in Chapter 1, section 1.3.2, the higher the default correlation between the reference entity (Spain) and the CDS seller (PNB Paribas), the lower the CDS spread s for the investor. The worst-case scenario for the investor is the default of both the reference entity and the counterparty. In this case the investor loses his entire investment. The higher the default correlation is, the more likely it is that the reference entity and the counterparty default together, hence the lower the CDS spread is. This is displayed in Figure 11.3.

For a model that derives the CDS spread s, including reference entity–counterparty default correlation, see "CDS with default correlation.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 11." How can we hedge this correlation risk? We could use one of many existing correlation dependent options, such as:

- Option on the better of two. Payoff = $\max(S_1, S_2)$.
- Option on the worse of two. Payoff = $\min(S_1, S_2)$.
- Call on the maximum of two. Payoff = $\max[0,(\max(S_1, S_2)) - K]$.
- Exchange option (as a convertible bond). Payoff = $\max(0, S_2 - S_1)$.
- Spread call option. Payoff = $\max[0, (S_2 - S_1) - K]$.

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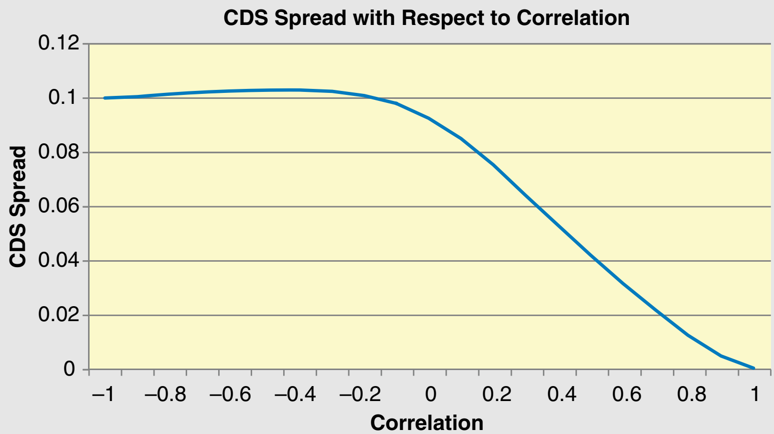

FIGURE 11.3 CDS Spread s of a Hedged Bond Exposure Displayed in Figure 11.2 with Respect to Default Correlation between the Reference Entity (Spain) and the Counterparty (BNP Paribas)

- Option on the better of two or cash. Payoff $= \max(S_1, S_2, \text{cash})$.
- Dual-strike call option. Payoff $= \max(0, S_1 - K_1, S_2 - K_2)$.
- Portfolio of basket option. Payoff $= \left[\sum_{i=1}^{n} n_i S_i - K, 0\right]$, where n_i is the weight of assets i.

All of these options are candidates for hedging correlation risk, since their values are dependent upon the correlation between the underlying variables. Let's look at an "option on the better of two," with a payoff of $\max(S_1, S_2)$. As the payoff shows, in this option the investor can choose to receive either the underlying S_1 or S_2 at option maturity. The lower the correlation between S_1 and S_2, the more valuable is an option on the better of two. If the correlation between S_1 and S_2 would be 1, this would just be a zero-strike call option on the underlying with the higher starting value. For a model that derives the value of an option on the better of two, see "Option on the better of two.xlsm" at www.wiley.com/go/correlation-riskmodeling, under "Chapter 11." Figure 11.4 shows the value of an option on the better of two.

In Figure 11.4, the option price is standardized; that is, a zero correlation is set to a zero option value. Figure 11.4 shows the strong impact of correlation on the option price. The option price fluctuates by about 12% for correlation levels from –0.9 to +0.9. Figure 11.4 shows

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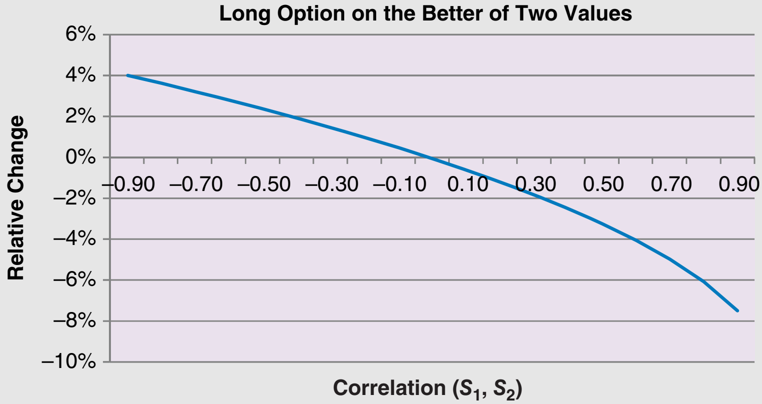

FIGURE 11.4 The Relative Change of the Value of a Long Position in an "Option on the Better of Two" with Respect to Correlation between the Underlying Assets S_1 and S_2

a long position of an option on the better of two. In this case the trader is short correlation (i.e., benefits if correlation decreases). A trader who wants to be long correlation has to *sell* an option on the better of two. In this case, the payoff reverses and we have an option function as displayed in Figure 11.5.

From Figures 11.4 and 11.5, we observe that the profit of the option buyer is the loss of the option seller with respect to correlation changes, and vice versa.

We can now use the short option position in Figure 11.5 to hedge the long correlation risk of the CDS displayed in Figure 11.3. Ideally, S_1 and S_2 in the short option on the better of two are the default probabilities of Spain and BNP Paribas. These could quite well be approximated with the CDS spread of Spain and BNP Paribas. If we combine Figures 11.3 and 11.5, we derive Figure 11.6.

From Figure 11.6 we observe that one of the main objectives is achieved. The low value of the original position (graph with no squares in Figure 11.6) for high correlation is increased. However, there are very few free lunches in finance.[3] The cost of the hedge is the lower

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3. One of these free lunches is diversification, which is related to low correlation of the assets in a portfolio. This increases the return/risk ratio as discussed in Chapter 1, section 1.3.1.

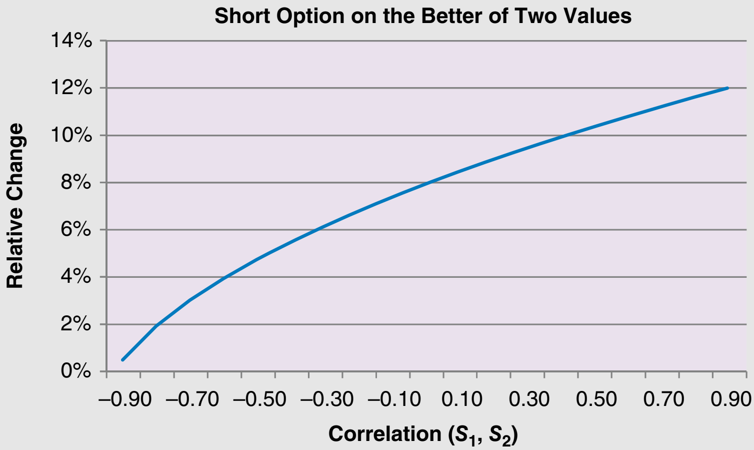

FIGURE 11.5 The Relative Change of the Value of a Short Position in an "Option on the Better of Two" with Respect to Correlation between the Underlying Assets S_1 and S_2

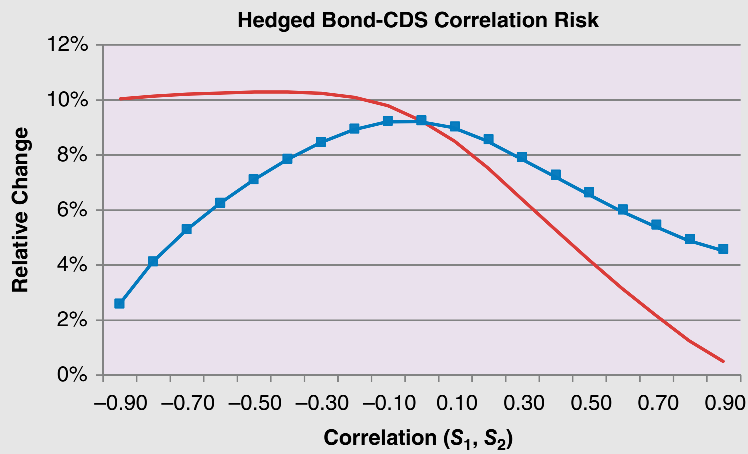

FIGURE 11.6 CDS Correlation Exposure (from Figure 11.3), and Hedged Exposure (i.e., Added Figures 11.3 and 11.5) (Function with Squares)

value of the hedged position (graph with squares) for negative correlation values.

If the investor wants to further increase the overall value for high correlation, he can increase the notional amount of the option on the

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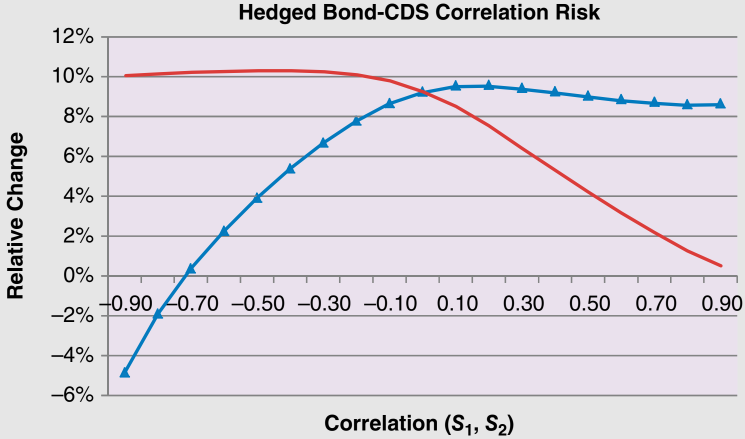

FIGURE 11.7 CDS Correlation Exposure (from Figure 11.3), and Hedged Exposure (i.e., Added Figures 11.3 and 11.5) (Function with Triangles) with Twice the Notional Amount in the Hedge

better of two hedge. This leads to an overall position as displayed in Figure 11.7.

From Figure 11.7 we observe that for high correlation values, the value of the overall hedged position (graph with triangles) is higher than in Figure 11.6. The catch is the now lower value of the overall hedged position for negative correlation compared to Figure 11.6.

11.3.2 Hedging VaR Correlation Risk with a Correlation Swap

We discussed correlation swaps in Chapter 1, section 1.3.2; see equations (1.6) and (1.7). Let's now apply correlation swaps to hedge value at risk (VaR) correlation risk. We analyzed VaR correlation risk in Chapter 1, section 1.3.3, and in more detail in Chapter 9, section 9.4. Let's hedge the VaR correlation risk of the 10-asset portfolio in example 9.1. We derived VaR as a function of the pairwise correlation of the assets in the portfolio in Figure 9.1.

We can now hedge this VaR correlation risk with a correlation swap. The dependence of a correlation swap with respect to the pairwise change in correlation of the assets is displayed in Figure 11.8.

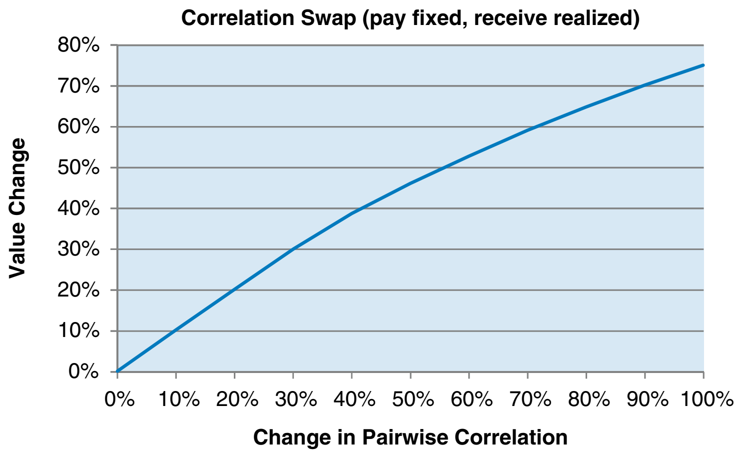

FIGURE 11.8 Change in Value of a Long Correlation Swap (Pay Fixed and Receive Realized; See Figure 1.5 for Details) with Respect to Pairwise Change in the Correlation between All Assets in a 10-Asset Portfolio. The Returns and Consequently the Correlation between the Returns of the 10 Assets in the Portfolio are the Same as in the VaR Example 9.1.

The fixed rate in the correlation swap in Figure 11.8 is set at 23.14%. This is the actual correlation of the assets in the portfolio following equation (1.6) $\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j}$ and example 9.1. For a simple spreadsheet showing the present value of a correlation swap, see the spreadsheet "Correlation swap" at www.wiley.com/go/correlationriskmodeling, under "Chapter 11." A model valuing interest rate swaps, which can be applied to correlation swaps if a correlation term structure is available, can be found at "Interest rate swap.xls"; see www.wiley.com/go/correlationriskmodeling, under "Chapter 11." In order to hedge the VaR correlation risk in Figure 9.1, we have to reverse the swap in Figure 11.8 (i.e., receive fixed and pay realized). This is displayed in Figure 11.9.

From Figure 11.8 and 11.9 we observe that the present values of the long correlation swap and the short correlation swap are reversed. Hence, what the correlation buyer (who is long correlation) gains, the correlation seller loses for a change in correlation, and vice versa.

We can now hedge the VaR correlation risk in Figure 9.1 with the short correlation swap in Figure 11.9. We derive Figure 11.10.

From Figure 11.10 we see that VaR correlation risk is eliminated for small changes in the all pairwise correlations of the assets in the portfolio.

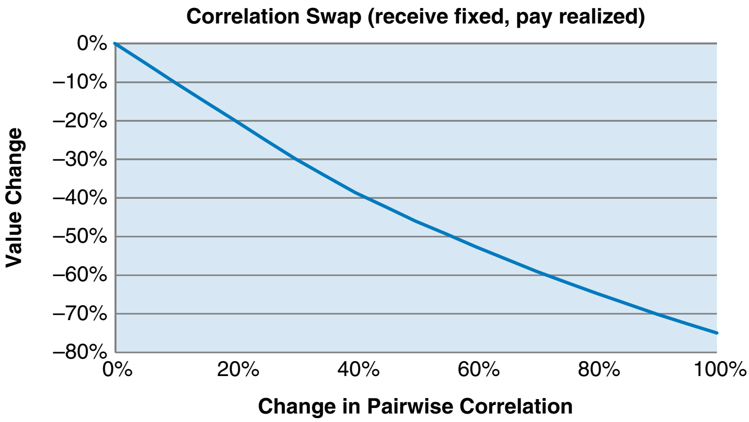

FIGURE 11.9 Change in Value of a Short Correlation Swap (Receive Fixed and Pay Realized; See Figure 1.5 for Details) with Respect to Pairwise Change in the Correlation of a 10-Asset Portfolio. The Returns and Consequently the Correlation between the Returns of the 10 Assets in the Portfolio Are the Same as in the VaR Example 9.1, in Chapter 9.

For larger changes, the hedged VaR now has a negative dependence on correlation. This means as correlations increase strongly, the hedged VaR actually decreases. This hedge can be fine-tuned: A lower or higher notional can be applied in the correlation swap hedge or a different fixed rate in the

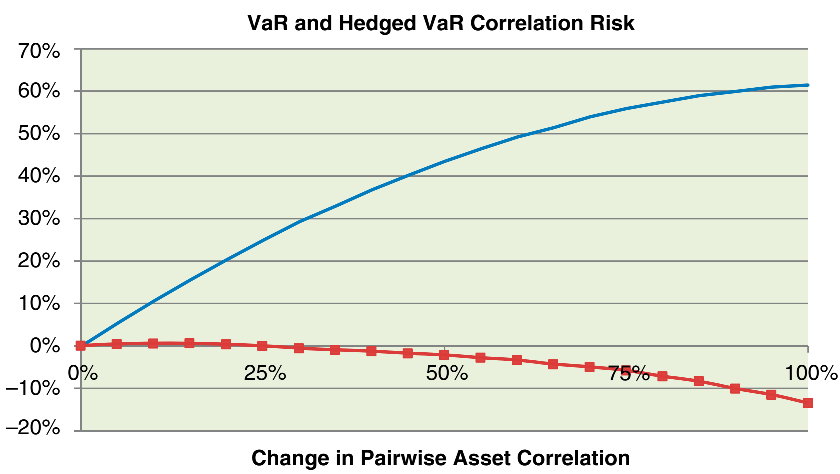

FIGURE 11.10 VaR with Respect to Correlation (from Figure 9.1), and Hedged VaR (Graph with Squares, which is a Combination of Figures 9.1 and 11.9)

correlation swap can be used. This will lead to a slightly different overall hedge function.

11.4 WHEN TO USE OPTIONS AND WHEN TO USE FUTURES TO HEDGE

Two key aspects have to be considered when choosing whether to hedge with an option or a futures contract.[4]

1. We have to analyze which contract matches the price function of the underlying instrument. For example, in Figure 11.9 we observed that hedging VaR correlation risk can be achieved well with a correlation swap. The overall correlation risk function is close to zero and only decreases for high correlation levels. Therefore a correlation swap is a good hedging instrument for VaR correlation risk, at least in our given, real-world example.
2. In contrast to an option, which has an option premium, a futures contract has no up-front premium. One benefit of a long option position is that the loss is limited to the option premium, which is typically quite low. However, in a futures contract the loss can be significant if the underlying has moved strongly in the undesired direction. Let's look at the implication of this in an example. Let's assume an investor wants to hedge a long Greek bond position. When should the investor use a swap such as a credit default swap (CDS), and when should the investor use a put option?
   - If the investor is quite certain that the Greek bond will decline further (but she does not want to sell and realize a loss), she can hedge with a swap or a future. This way no option premium is wasted.
   - The investor wants to hedge her Greek bond price risk, but is somewhat uncertain about whether the Greek bond price will actually decline. In this case a long put option is warranted, since a profit is generated if the Greek bond increases in price; see Figure 11.1.

In conclusion, the more confident an investor is that the undesirable event will occur (a price decline of the Greek bond in the preceding example), the more appropriate it is to hedge with a future or a swap. The less confident an

4. See footnote 1 for the close relationship of futures, forwards, and swaps.

investor is that the undesirable event will occur, the more appropriate it is to hedge with an option.

11.5 SUMMARY

In finance, hedging means reducing risk or, more precisely, entering into a second trade to reduce the risk of an original trade. In this chapter we discussed ways to hedge correlation risk. Hedging can be principally done in three ways:

1. Close the original position (e.g., if a bond was purchased, sell the bond). The drawback is that if the position has created a paper loss, this paper loss is realized.
2. Use a derivative to hedge the position. This is an efficient way to hedge. Typically the hedge amount has to be adjusted when the underlying price changes, called dynamic delta hedging.
3. Enter into a position that is negatively correlated with the original trade. Here the investor has correlation risk, though: the risk that both the original trade and the hedge create a loss.

Hedging correlation risk is more difficult than hedging equity risk, bond price risk, or the risk of a derivatives position, for two reasons: (1) Hedging correlation risk involves two or more assets, since the correlation is measured between at least two assets. (2) There is principally no underlying instrument that trades in the market and that can be bought or sold as a hedge.

However, the correlation market is evolving. There are numerous products that are sensitive to correlation, such as correlation options, correlation swaps, and options on correlation swaps. Each of them can be used to hedge correlation risk. An investor who wants to hedge his correlation risk will have to find the correlation hedge that matches the correlation exposure of the original trade best.

A general question with respect to hedging is when to use a future or forward (or swap, which is just a series of forwards) and when to use options. Let's assume an investor has a CDS and is exposed to correlation risk with respect to the reference entity and the counterparty. The investor should use a forward (or a swap such as a correlation swap) if she is very certain—as certain as she can be—that the hedge is needed. This way no option premium is wasted. However, the investor should use options in the hedge (e.g., an option on a correlation swap) if the investor wants to hedge the correlation risk, but is not that certain that the correlation risk will occur. If the correlation risk does not occur, the loss on the hedge is just the option premium.

PRACTICE QUESTIONS AND PROBLEMS

1. What is hedging?
2. Name the three main ways to hedge.
3. Name two reasons why it is more difficult to hedge correlation risk compared to equity risk or currency risk.
4. Can it be a good idea not to hedge an exposure as a correlation exposure?
5. In a delta hedge, the delta amount of the exposure is sold or bought. Give an example of delta hedging.
6. Delta hedges are typically not constant. Give an example of dynamic delta hedging.
7. Name several instruments that can hedge correlation risk.
8. How can we determine whether a certain correlation hedge is a good hedge?
9. When an investor has a perfect hedge, doesn't this mean that the profit potential is zero?
10. Generally, when should we hedge with forwards, futures, and swaps, and when should we hedge with options?

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CHAPTER 12

Correlation and Basel II and III

Bank supervisors play an important role in encouraging the proper balance of risk-taking by developing prudent standards and enforcing sound practices at banks.

—Alan Greenspan

In this chapter, we discuss the correlation concepts in the Basel accord, which are designed to address the risk management failures that led to the Great Recession of 2007 to 2009. In particular, we address credit value at risk (CVaR), which derives the correlated maximum loss due to credit risk.

We also address a key topic in today's financial markets, credit value adjustment (CVA). CVA is an adjustment to mitigate credit counterparty risk and includes two types of correlations: (1) general wrong-way risk and (2) specific wrong-way risk.

In addition, we address the new concepts of debt value adjustment (DVA) and funding value adjustment (FVA) and their implementation in the Basel accord.

First, let's look at some basics.

12.1 WHAT ARE THE BASEL I, II, AND III ACCORDS? WHY DO MOST SOVEREIGNS IMPLEMENT THE ACCORDS?

We briefly introduced the Basel accords in Chapter 1, section 1.3.5. We expand the discussion in this chapter, especially the correlation aspects of the accords.

The Basel accords are developed by the Basel Committee for Banking Supervision (BCBS), which is a subcommittee of the Bank for International

Settlements (BIS). The Basel I accord was implemented in 1988; the Basel II started development in 1999 and was implemented in 2006. The Basel III accord was initiated in 2008 and is intended to be implemented by 2018. Basel III is designed to particularly address the banking failures in the global financial crisis of 2007 to 2009. The objective of the accords is to "improve the banking sector's ability to absorb shocks arising from financial and economic stress" and "to reduce the risk of spillover from the financial sector to the real economy."[1]

The Basel accords do not have international legal authority. However, most sovereigns (about 100 for Basel II) have implemented legislation to enforce the accords. The reason for the implementation of the accords is simple: It increases the creditworthiness of the country and its banking system, which leads to a higher credit rating by the agencies Moody's, Standard & Poor's, and Fitch. This in turn leads to an increase in international trade and capital flows at a lower cost of capital.

One of the most critical aspects of the new Basel III accord is the way credit value at risk (CVaR) is calculated, since sharp increases in credit risk and consequently defaults were one of the main reasons of the global financial crisis of 2007 to 2009. Correlations play a key role in the derivation of portfolio credit risk. Let's discuss how the Basel accords quantify portfolio credit value at risk (CVaR) and which correlation concept the Basel accord applies.

12.2 BASEL II AND III's CREDIT VALUE AT RISK (CVaR) APPROACH

In Chapter 1, section 1.3.3, and Chapter 9, section 9.4, we defined market value at risk (VaR). It measures the maximum loss of a portfolio with respect to market risk with a certain probability for a certain time frame. Analogously, we can define credit value at risk (CVaR).

CREDIT VALUE AT RISK (CVaR)

The maximum loss of a portfolio due to credit risk with a certain probability for a certain time frame.

Credit risk can be considered the most critical of all types of risk. It is estimated that financial institutions allocate about 60% of the regulatory capital to credit risk, about 15% to market risk, and about 25% to operational risk.

1. BCBS, "Basel III: A Global Regulatory Framework for More Resilient Banks and Banking Systems," June 2011, 1, www.bis.org/publ/bcbs189.htm.

We defined credit risk in Chapter 10 as the risk of a financial loss due to an adverse change in the credit quality of a debtor, and mentioned the two types of credit risk: (1) migration risk and (2) default risk.

To value CVaR, it is tempting to just take the market VaR equation (1.8) $\text{VaR}_P = \sigma_P\ \alpha\ \sqrt{x}$ and transfer it to CVaR. However, there are two main problems when using equation (1.8) for CVaR:

1. The portfolio variance, defined in equation (1.9), $\sigma_P = \sqrt{\beta_h C \beta_v}$, would require input data as standard deviations of relative credit rating changes, and the correlation coefficient between the changes. However, these data for credit risk are rare, since credit rating changes for most entities rarely occur, often only once a year or even not at all.
2. The value for α in equation (1.8) would assume a normal distribution of relative credit rating changes. However, credit rating changes are typically not normally distributed and depend on the current credit rating, past credit rating changes, country, sector, seniority, coupon, yield, and so on. See Meissner (2005) for a further discussion.

Since credit data are much scarcer than market data, in practice a much more granular approach is used to derive CVaR. Basel II uses the one-factor Gaussian copula (OFGC) model, which we discussed in detail in Chapter 6 for valuing CDO tranches. Let's apply the OFGC to value CVaR.

We start with the core equation of the OFGC, which we discussed in Chapter 6,

$$x_i = \sqrt{\rho} M + \sqrt{1-\rho} Z_i \tag{12.1}$$

where

- ρ: Default correlation parameter for the companies in the portfolio, $0 \leq \rho \leq 1$. ρ is assumed identical and constant for all company pairs in the portfolio.
- M: Systematic market factor, which impacts all companies in the portfolio. M can be thought of as the general economic environment, for example, the return of the S&P 500. M is a random drawing from a standard normal distribution, formally $M = n \sim (0, 1)$. M is the same as ε in Chapter 4, section 4.1.
- Z_i: Idiosyncratic factor of asset i. Z_i expresses ith company's individual strength, possibly measured by company i's stock price return. As M, Z_i is also a random drawing from a standard normal distribution.

x_i: The value for x_i results from equation (12.1) and is interpreted as a "Default indicator variable" for company i. The lower i, the earlier is the default time T for company i. x_i is by construction standard normal.

Solving equation (12.1) for Z_i, we derive

$$Z_i = \frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}$$

Taking cumulative values, we get

$$N(Z_i) = N\left(\frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{12.2}$$

where $N(x)$ is the cumulative standard normal distribution at x. Since we use the standard normal cumulative distribution N, this approach is called *Gaussian* copula.

We now equate the individual default probability of entity i at time T, $PD_i(T)$, which is given or estimated from the market data with the model-simulated barrier $N(x_i)$, which includes the default correlation via the x_i: $PD_i(T) = N(x_i)$. Solving for x_i, we derive $x_i = N^{-1}(PD_i(T))$, where N^{-1} is the inverse of N. See Chapter 4, Figure 4.3 for details of the mapping procedure $N^{-1}(PD_i(T))$. Inputting $x_i = N^{-1}(PD_i(T))$ into equation (12.2), we get

$$N(Z_i) = N\left(\frac{N^{-1}[PD_i(T)] - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{12.3}$$

Now the strong assumption is made that all entities i have the same default probability PD at a certain time T. Hence we can drop the index i and get

$$N(Z) = N\left(\frac{N^{-1}[PD(T)] - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{12.4}$$

For a large homogeneous portfolio (LHP) with identical pairwise correlation ρ and identical default correlation $PD(T)$, the right side of equation (12.4) is approximately the percentage of entities in the portfolio defaulting at T. For example, if there is no correlation between the entities (i.e., $\rho = 0$), then equation (12.4) reduces to $N[N^{-1}(PD(T))] = PD(T)$. In this case, if the individual default probability is $PD(T) = 10\%$, we can assume that approximately 10% of the entities will default by T.

We now replace the market factor M with a confidence level X. M is standard normal. Therefore, for a certain abscise value $N^{-1}(Y)$ of M, we have:

$$\Pr(M \le N^{-1}(Y)) = \int_{-\infty}^{N^{-1}(Y)} n(M)dM = N(N^{-1}(Y)) = Y \tag{12.5}$$

Equation (12.5) reads: The probability of M being smaller than or equal to $N^{-1}(Y)$ is the surface of a normal distribution from $-\infty$ to $N^{-1}(Y)$, where $n(M)$ is the normal distribution of M. This can be written as $N(N^{-1}(Y))$, since $N(N^{-1}(Y))$ is the cumulative normal distribution from $-\infty$ to $N^{-1}(Y)$. N is the inverse of N^{-1}; therefore $N(N^{-1}(Y)) = Y$.

Graphically, we can express this as shown in Figure 12.1.

Replacing M in equation (12.4) with $N^{-1}(Y)$, we get

$$N\left(\frac{N^{-1}[PD(T)] - \sqrt{\rho}N^{-1}(Y)}{\sqrt{1-\rho}}\right) \tag{12.6}$$

The term (12.6) tells us the probability Y of the percentage of defaults in the portfolio being bigger than $N\left(\frac{N^{-1}[PD(T)] - \sqrt{\rho}N^{-1}(Y)}{\sqrt{1-\rho}}\right)$. We are interested in probability of defaults *smaller* than Y. This is $1 - Y$. Let's set $1 - Y = X$,

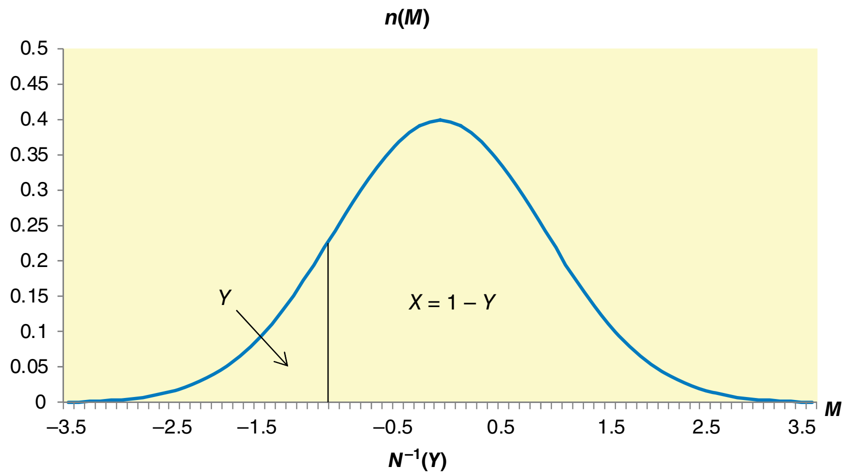

FIGURE 12.1 Graphical Representation of a Normally Distributed Default Distribution with $\Pr(N^{-1}(Y)) \le M = Y$, and the Confidence Level X

where X is a certain confidence level. Replacing Y with $1 - X$, and using $N^{-1}(1 - X) = -N^{-1}(X)$, we derive

$$\mathrm{CVaR}(X,T) = N\left(\frac{N^{-1}[PD(T)] + \sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right) \tag{12.7}$$

where

CVaR(X,T): credit value at risk for the confidence level X for the time horizon T
N: cumulative normal distribution
N^{-1}: inverse of the cumulative normal distribution
$PD(T)$: average probability of default of the assets in the portfolio for the time horizon T
ρ: pairwise correlation coefficient of the assets in the portfolio (ρ is assumed constant for all asset pairs)

Equation (12.7) reads: We are X% certain that regarding our loan portfolio we will not lose more than $N\left(\frac{N^{-1}[PD(T)] + \sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right)$ due to (correlated) default risk, for the time horizon T.

Equation (12.7) is an important result, which was first published by Vasicek in 1987.[2] It is derived from the one-factor Gaussian copula (OFGC) model of equation (12.1); see above. Equation (12.7) is currently used in the Basel II accord as the basis to value credit risk in a portfolio. It takes into consideration default risk, not migration risk (see Figure 10.1). CVaR is also called credit at risk or worst-case default rate (WCDR).

Let's look at an example of equation (12.7).

EXAMPLE 12.1: CALCULATING CREDIT VALUE AT RISK (CVaR)

Suppose JPMorgan has given loans to several companies in the amount of $100,000,000. The average 1-year default probability of the companies is 1%. The copula default correlation coefficient between the companies is 5%. What is the 1-year CVaR on a 99.9% confidence level?

(continued)

2. O. Vasicek, "Probability of Loss on a Loan Portfolio," KMV Working paper, 1987. Results published in *Risk* magazine with the title "Loan Portfolio Value," December 2002.

It is

$$\mathrm{CVaR}(0.999,1) = N\left(\frac{N^{-1}(0.01) + \sqrt{0.05}\,N^{-1}(0.999)}{\sqrt{1-0.05}}\right) = 4.67\%$$

We can derive $N^{-1}(x)$ via Excel's =normsinv(x) or MATLAB's norminv(x) function. $N(x)$ is =normsdist(x) in Excel and normdist(x) in MATLAB.

Interpretation: JPMorgan is 99.9% sure that it will not lose more than 4.67% of its loan exposure of $100,000,000 due to (correlated) default risk of its debtors for a 1-year time horizon. In dollar amounts and including a recovery rate of 40%, we derive that JPMorgan is 99.9% sure that it will not lose more than $100,000,000 × 0.0467 × (1 − 0.4) = $2,802,000.

12.2.1 Properties of Equation (12.7)

Equation (12.7) has some interesting properties.

- We observe that for zero default correlation between the debtors in the portfolio $\rho = 0$, it follows that CVaR $= PD(T)$. So for a 99.9% confidence level, we are 99.9% sure that we will not lose more than the average default rate $PD(T)$. This is reasonable because there is no effect from correlation and the maximum loss is just the average default probability of the debtors.
- CVaR is a function of the default probability PD for the time horizon T, the confidence level X, and the pairwise default correlation ρ. T is typically set to one year and the confidence level used is typically 99.9%. In this case we get a relationship between CVaR and $PD(T)$ and ρ as displayed in Figure 12.2.

From Figure 12.2 we observe that CVaR is a positive function with respect to $PD(T)$ and ρ. The positive relationship between CVaR and $PD(T)$, $\frac{\partial \mathrm{CVaR}}{\partial PD(T)} > 0$ is obvious: The higher the default probability $PD(T)$, the higher is the maximum loss CVaR. The positive relationship between CVaR and the default correlation between the debtors ρ, $\frac{\partial \mathrm{CVaR}}{\partial \rho} > 0$, is also plausible: The higher the default correlation, the higher is the probability that many debtors default and the higher is the maximum loss CVaR. This is especially the case for high default probability $PD(T)$. For value of $PD(T)$ higher than 20% and ρ close to or higher than 80%, the maximum loss CVaR is close to 100% of the total loan exposure; see Figure 12.2.

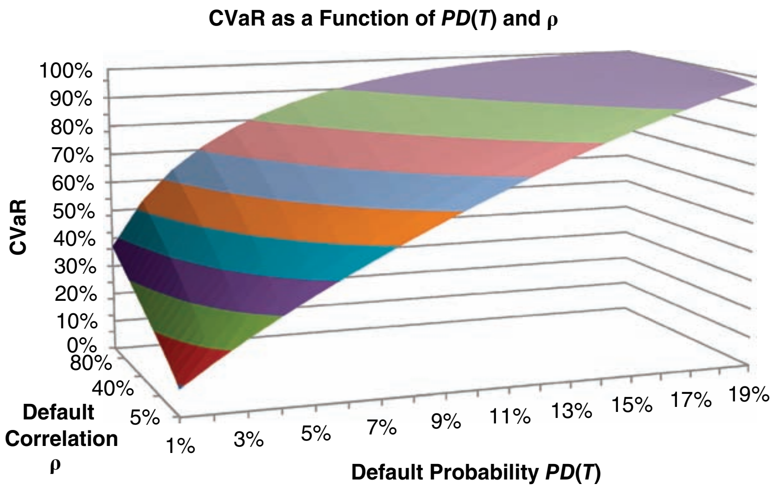

FIGURE 12.2 CVaR as a Function of the Default Probability $PD(T)$ where $T = 1$ and the default correlation between the debtor's asset ρ. The confidence level is $X = 99.9\%$.

For a model that displays the CVaR, see "CVaR.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 12."

12.3 BASEL II's REQUIRED CAPITAL (RC) FOR CREDIT RISK

Basel II uses equation (12.7) as a basis to calculate the capital charge for credit risk. However, the capital charge is reduced by the expected loss, which is measured by $PD(T)$. The rationale is that banks cover the expected loss with their own provisions as the interest rate that they charge. (Naturally, low-rated debtors have to pay a higher interest rate on their loans than highly rated debtors.) Therefore the required capital RC for credit risk in the Basel II accord is

$$\mathrm{RC} = \mathrm{EAD} \times (1 - R) \times [\mathrm{CVaR} - PD(T)] \tag{12.8}$$

where

RC is the required capital by Basel II for credit risk in a portfolio
EAD is the exposure at default (for loans EAD is equal to the loan amount)

R is the recovery rate (rate that is recovered from the defaulted loan)
CVaR is the credit value at risk derived by equation (12.7)
$PD(T)$ is the average probability of default of the debtors in the portfolio for time horizon T

Let's look at equation (12.8) in an example. Let's expand example 12.1.

EXAMPLE 12.2: CALCULATING REQUIRED CAPITAL (RC) FOR CREDIT RISK

Suppose JPMorgan has given loans to several companies in the amount of \$100,000,000. The average 1-year default probability PD of the companies is 1%. The copula default correlation coefficient between the companies is 5%. What is the 1-year capital charge of Basel II on a 99.9% confidence level assuming the recovery rate is 40%?

Answer:

We had already derived CVaR in example 12.1 as CVaR = 4.67%. Following equation (12.8), the required capital charge of Basel II is

$$\text{RC} = \$100{,}000{,}000 \times (1 - 0.4) \times (4.67\% - 1\%) = \$2{,}202{,}000$$

Credit value at risk CVaR is typically calculated for a 1-year time horizon. If a different time horizon is used, Basel II adds a maturity adjustment (MA). In this case the equation (12.8) changes to

$$\text{RC} = \text{EAD} \times (1 - R) \times [\text{CVaR} - PD(T)] \times \text{MA} \qquad (12.8\text{a})$$

where $\text{MA} = \frac{1 + (M - 2.5) \times b}{1 - 1.5 \times b}$ and M is the maturity date. b is a constant set at $b = [0.11852 - 0.05478 \times \ln(PD(T))]^2$.

12.3.1 The Default Probability–Default Correlation Relationship

Interestingly, the correlation coefficient ρ is not an exogenous input in equations (12.8) or (12.8a), but ρ is a function of the default probability

$PD(T)$, $\rho = f(PD(T))$. In particular, the Basel accord sets[3]

$$\rho = 0.12\frac{1-\exp(-50\times PD(T))}{1-\exp(-50)} + 0.24\left(1-\frac{1-\exp(-50\times PD(T))}{1-\exp(-50)}\right) \tag{12.9}$$

Equation (12.9) can be approximated well with equation (12.9a), as Hull (2012) points out:

$$\rho = 0.12[1+\exp(-50\times PD(T))] \tag{12.9a}$$

Indeed, equations (12.9) and (12.9a) are identical to at least four decimal places for any value of $PD(T)$.

Equation (12.9) or (12.9a) is displayed in Figure 12.3.

What is the rationale for making correlation ρ a function of the default probability $PD(T)$ as displayed in Figure 12.3? It is assumed that highly rated companies with a low default probability have a higher correlation of default since they are mostly prone to systematic factors such as a recession, in which they default together. However, companies with a high default probability are more affected by their own idiosyncratic factors and less by systematic risk; hence they are assumed to be less correlated. This is replicated in equations (12.9) and (12.9a) and Figure 12.3.

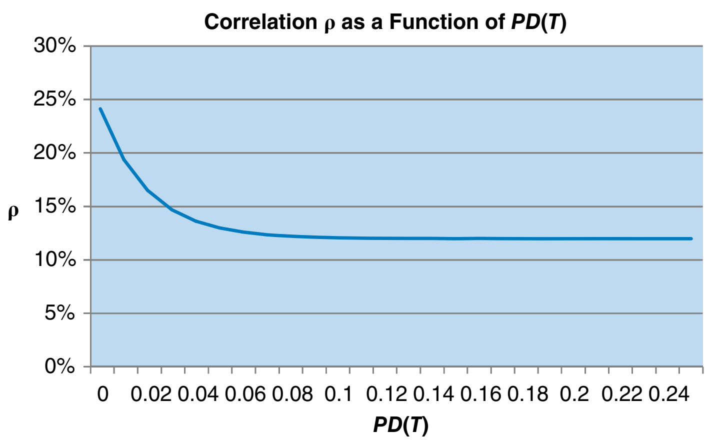

FIGURE 12.3 Correlation between the Debtors in a Portfolio ρ as a Function of the Average Default Probability of the Debtors in the Portfolio $PD(T)$ in the Basel II Accord

3. See BCBS, "Basel III: A Global Regulatory Framework," p. 39.

12.4 CREDIT VALUE ADJUSTMENT (CVA) APPROACH WITHOUT WRONG-WAY RISK (WWR) IN THE BASEL ACCORD

Credit value adjustment (CVA) has become an important part of correlated credit risk modeling in the recent past. Most investment and commercial banks have CVA quant groups who analyze CVA risk and CVA desks, where CVA risk is traded and hedged. The importance of CVA is highlighted by Basel II, which reports that two-thirds of the credit risk losses during the global financial crisis were caused by CVA volatility rather than actual defaults. In addition, the derivatives portfolios of investment banks are typically quite large. When Lehman Brothers defaulted in September 2008, it had 1.5 million derivatives transactions with 8,000 different counterparties, stressing the importance of managing derivatives credit risk.

What is CVA? A broad definition of CVA is:

> **CREDIT VALUE ADJUSTMENT (CVA)**
>
> An adjustment to address counterparty credit risk.

The focus of CVA is typically narrower than this definition, as it often refers to counterparty credit risk in a *derivatives* transaction. This narrower CVA framework is applied in the Basel accord:

> In addition to the default risk capital requirements for counterparty credit risk determined based on the standardized or internal ratings-based (IRB) approaches for credit risk, a bank must add a capital charge to cover the risk of mark-to-market losses on the expected counterparty risk (such losses being known as credit value adjustments, CVA) to OTC derivatives.[4]

We will concentrate on this derivatives aspect of CVA here. With respect to a derivative, CVA is the difference between the price of a credit-risky derivative and the price of a default-free derivative, as displayed in equation (12.10).

4. Ibid., p. 31.

$$\text{Credit-Risky Derivative} = \text{Default-Free Derivative} - \text{CVA} \tag{12.10}$$

CVA is by definition ≥ 0; see also equations (12.11) and (12.12) below. Hence from equation (12.10) it follows that a credit-risky derivative has a lower price than a derivative without credit risk. This is because the buyer of the credit-risky derivative (often referred to as the dealer) lowers the price of the derivative since he assumes the credit risk of the counterparty (the derivatives seller). In particular, if the counterparty defaults, the buyer of the derivative will not receive the payout of the derivative. CVA is an adjustment since the derivatives buyer adjusts (lowers) the price of the derivative due to credit risk.

CVA is an integral part of the Basel III accord. Figure 12.4 shows CVA and the associated wrong-way risk (WWR), which will be discussed in section 12.5.

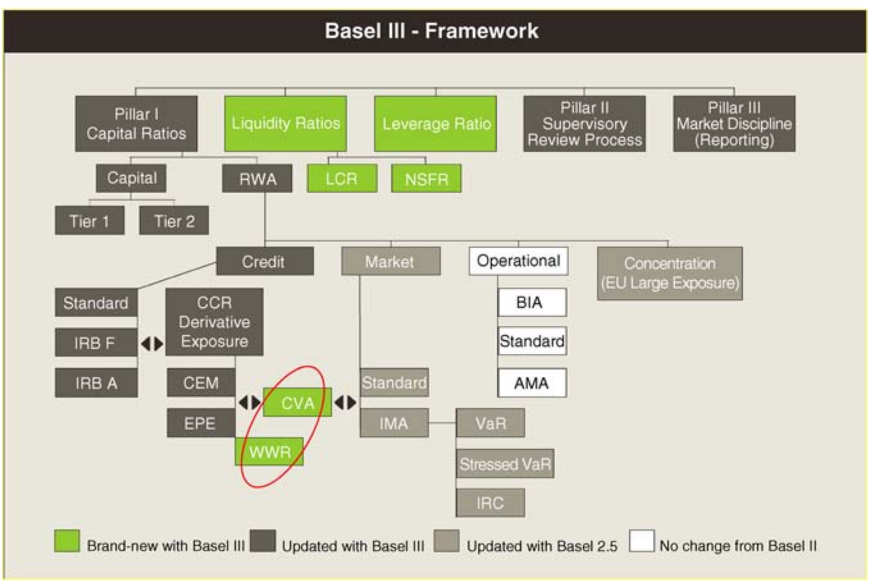

FIGURE 12.4 Credit Value Adjustment (CVA) and Wrong-Way Risk (WWR) in the Basel III Framework
Source: Moody's Analytics, 2011.

From Figure 12.4 we observe that CVA has a market risk component and a credit risk component. We formalize this in equation (12.11).

$$\mathrm{CVA}_{a,c} = f(\underbrace{D^{+}_{a,c}}_{\text{Market risk}}, \underbrace{PD_c}_{\text{Credit risk}}) \tag{12.11}$$

where

$\mathrm{CVA}_{a,c}$: credit value adjustment of entity a with respect to the counterparty c

${D^{+}}_{a,c}$: netted, positive derivatives portfolio value of entity a with counterparty c

PD_c: default probability of counterparty c

In equation (12.11), we take only the *positive* netted derivative portfolio value, $D^{+}_{a,c} = \max\left(\sum D_{a,c}, 0\right)$, into consideration. This is because entity a has credit exposure only if the netted derivatives portfolio between a and c is positive for a. (In simple terms, a has credit exposure with respect to c only if c is a's debtor.) Figure 12.5 shows this property.

Figure 12.5 shows that for positive credit exposure, the credit exposure is identical with the netted portfolio value, also called portfolio marked-to-market (MtM) value. Credit risk is the risk that the credit exposure changes. For example, the credit risk for entity a with respect to c would increase if the credit exposure increases due to an increase in the market value of the derivatives $D^{+}_{a,c}$ or an increase in the default probability of c, PD_c.

Equation (12.11) also shows that CVA can be viewed as a derivative itself. It is a complex derivative since it has two underlyings, D^{+} and PD_c,

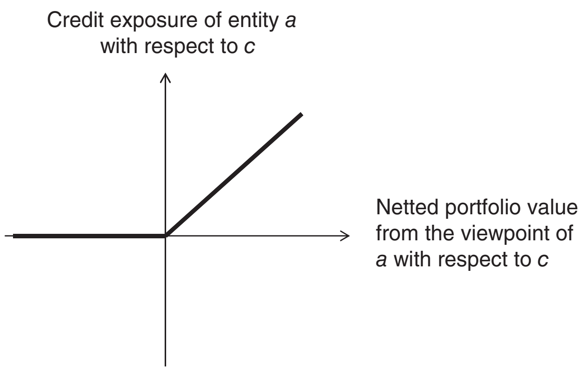

FIGURE 12.5 Credit Exposure of Entity a with Respect to Counterparty c

which may be correlated! If they are not correlated, we can multiply the market risk component and the credit risk component. Adding a recovery rate of the counterparty c, R_c, we can write:

$$\mathrm{CVA}_{a,c} = (D^{+}_{a,c} \times PD_c)(1 - R_c) \tag{12.12}$$

Let's look at an example of equation (12.12).

EXAMPLE 12.2: CALCULATING CVA, ASSUMING NO CORRELATION BETWEEN MARKET RISK AND CREDIT RISK

Entity a has a derivatives portfolio with counterparty c, which has a present value of +\$100,000,000 for a. The default probability of c for a 1-year time horizon is 5%. The recovery rate of counterparty c in case of default of c is expected to be 30%. What is the CVA from the viewpoint of a with respect to counterparty c for a 1-year time horizon?

Following equation (12.12), it is $\mathrm{CVA}_{a,c} = \$100{,}000{,}000 \times 0.05 \times (1 - 0.3) = \$3{,}500{,}000$.

Equation (12.12) is the basis for calculating CVA in the Basel III accord, when the correlation between market risk and credit risk is assumed to be zero. However, this is a simplistic assumption, which we will alter in the next section.

12.5 CREDIT VALUE ADJUSTMENT (CVA) WITH WRONG-WAY RISK IN THE BASEL ACCORD

As mentioned earlier, equation (12.12) assumes that market risk of the derivative D and credit risk PD are not correlated. However, this is not a realistic assumption. Market risk and credit risk are clearly related. For example, if the equity market declines (maybe due to a recession), the default probabilities of companies typically increase (since debt-to-equity ratios increase). Conversely, if the default probability of a company increases (maybe due to bad management or increased competition), the stock price of the company will decline.

The Basel accord recognizes the correlation between market risk and credit risk. The Basel accord defines two types of wrong-way risk (WWR), *general wrong-way risk* and *specific wrong-way risk*. Let's look at general wrong-way risk first.

GENERAL WRONG-WAY RISK (WWR)

Exists when the probability of default of counterparties is positively correlated with general market risk factors.[5]

Following the Basel II accord, general market risk factors are interest rates, equity prices, foreign exchange rates, commodity prices, real-estate prices, and more.

Let's discuss an example of general wrong-way risk regarding the market risk factor interest rates, which can be positively correlated with default probability. We will explain general wrong-way risk with the practical example of a long bond position, which is displayed in Figure 12.6.

In Figure 12.6 only the bond investor has credit risk with respect to bond issuer. This is because in case of default of the issuer, the bond investor will not receive the coupon payments, and, most important, will just receive the recovery rate of the principal investment of \$1,000,000. The bond issuer does not have credit exposure to the bond investor, since the bond issuer has received all contractual payments (i.e., the initial investment of \$1,000,000 at t_0).

A bond price B is mainly a function of the market interest rate level i and the default probability of the issuer PD_c; hence $B = f(i, PD_c, \ldots)$. There is a negative relationship between the bond price B and market rates i: The higher the market interest rates i, the lower is the bond price B, since the coupon of the bond price is now lower compared to the market interest rate i; formally: $\frac{\partial B}{\partial i} \leq 0$. There is also a negative relationship between the bond price B and the default probability of the issuer PD_c: $\frac{\partial B}{\partial PD_c} \leq 0$.

The relationship of B, i, and PD_c constitutes general wrong-way risk: In a weakening economy, typically interest rates i decrease and default probabilities

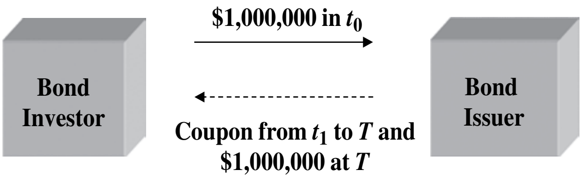

FIGURE 12.6 Cash Flows of a Standard Bond Purchase with Maturity T

5. BCBS, "Annex 4 (to Basel II)," 2003, 211, www.bis.org/bcbs/cp3annex.pdf.

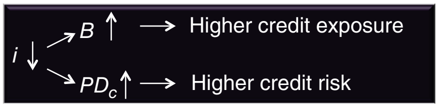

FIGURE 12.7 General Wrong-Way Risk
Decreasing interest rates i lead to higher credit exposure via a higher bond price B. Decreasing interest rates i in a recession also mean increasing default probability PD_c of the bond issuer. Hence, the higher the credit exposure, the higher is the credit risk (i.e., the higher the risk that the issuer can't meet its obligation to pay coupons and principal).

such as PD_c increase. However, from the relationship $\frac{\partial B}{\partial i} \leq 0$, decreasing interest rates also mean a higher bond price (i.e., higher credit exposure of the bond buyer with respect to the bond issuer). But a higher default probability PD_c also means a lower probability that the issuer will be able to pay the coupons and the principal amount. Hence the higher the credit exposure, the more likely it is that the bond issuer can't pay coupons and principal, which constitutes general wrong-way risk. Graphically this is displayed in Figure 12.7.

Let's now look at *specific* wrong-way risk.

SPECIFIC WRONG-WAY RISK (WWR)

Exists if the exposure to a specific counterparty is positively correlated with the counterparty's probability of default due to the nature of the transaction with the counterparty.[6]

We can formalize specific wrong-way risk with equation (12.13),

$$\frac{\partial PD}{\partial D^+} > 0 \tag{12.13}$$

Equation (12.13) reads: If the credit exposure, expressed by the netted positive derivatives value D^+ increases, credit risk, expressed as the default probability PD, also tends to increase. This is clearly not a good situation to be in. In simple words: The higher the credit exposure, the higher is the credit risk (i.e., the risk that the debtor can't meet its obligations).

6. See BCBS, "Basel III: A Global Regulatory Framework," p. 38.

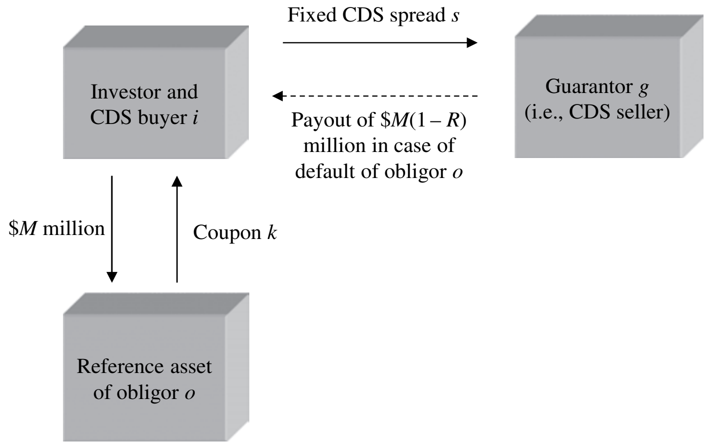

FIGURE 12.8 Cash Flows of an Investor i, Who Has Credit Exposure to an Obligor o, which is Hedged with a Credit Default Swap (CDS) with the Guarantor g. R = Recovery Rate.

Let's look at an example of specific wrong-way risk. We had already briefly mentioned an example of specific wrong-way risk in Chapter 1, section 1.2, in Figure 1.1. Let's discuss it in detail now.

Figure 12.8 shows the cash flows between the three entities in a CDS.

In Figure 12.8 the terminology and notation of the Basel accord are applied. In most literature the guarantor g is called counterparty c and the obligor o is called reference entity r.

In Figure 12.8, the investor has specific wrong-way risk, if there is a positive correlation between the default probability of the obligor o and the guarantor g (i.e., the CDS seller). This means that the higher the default probability of the obligor PD_o is, the higher is also the default probability of the guarantor PD_g.

In particular, if the default probability of the obligor increases, the market spread of the CDS increases. Therefore the present value for the CDS buyer increases, since his fixed spread s is now lower than the market spread. If the CDS is marked-to-market, this is nice from a profit perspective, but from a risk perspective it means that the credit exposure for the CDS buyer i increases.

Also, with increasing default probability of the guarantor, the credit risk increases, since it is less likely that the guarantor can pay the payoff in default. Hence we have increased credit exposure together with increased credit risk, constituting specific wrong-way risk. Figure 12.9 shows the wrong-way risk dilemma.

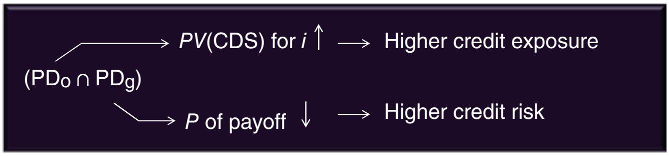

FIGURE 12.9 Specific Wrong-Way Risk
Specific wrong-way risk of the hedged bond position of Figure 12.8 exists if the default correlation between the obligor PD_o and the guarantor PD_g is positive. Let's assume PD_o and PD_g both increase; that is, $(PD_o \cap PD_g)\uparrow$. Hence the present value of the CDS, PV(CDS) for i increases, which means higher credit exposure for i. In addition, the increasing probability of default of the guarantor means that the probability P of the future payoff from the guarantor decreases. Hence we have increasing credit exposure together with increasing credit risk, constituting specific wrong-way risk (WWR).

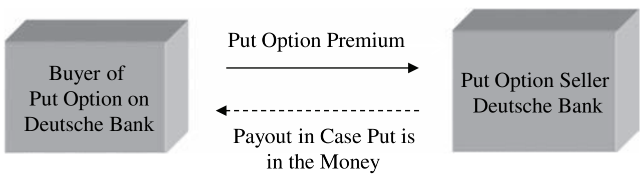

FIGURE 12.10 Example of Specific Wrong-Way Risk: Deutsche Bank Selling a Put on Its Own Stock

A further example of specific wrong-way risk (which is mentioned in the Basel III accord[7]) is if a company sells put options on its own stock. This is displayed in Figure 12.10.

Selling a put on its own stock constitutes specific wrong-way risk since the lower the stock price, the more the put is in the money (i.e., the higher is the credit exposure for the put option buyer with respect to the put option seller, Deutsche Bank). But the lower the Deutsche Bank stock price is, the higher is typically also the default probability of Deutsche Bank. This means that the higher the credit exposure (when the put is deeper in the money), the higher is the credit risk (the probability that Deutsche Bank defaults), constituting specific wrong-way risk.

12.5.1 How Do Basel II and III Quantify Wrong-Way Risk?

Basel II and III have a simple approach to address general wrong-way risk and specific wrong-way risk. A multiplier α is applied to increase the derivatives

7. Ibid., p. 38.

exposure $D^+_{a,c}$. The multiplier α is set to 1.4, which means the credit exposure $D^+_{a,c}$ is increased by 40%, compared to assuming credit exposure $D^+_{a,c}$ and credit risk PD_c are independent, as was expressed in equation (12.12). Banks that use their own internal models are allowed to use a α of 1.2, meaning the credit exposure is increased by 20% to capture wrong-way risk. Banks report an actual alpha of 1.07 to 1.1; hence the α of 1.2 to 1.4 that Basel III requires is conservative.

Currently models are developed to quantify wrong-way risk in a more rigorous way. See, for example, Hull and White (2011) or Cepedes et al. (2010).

12.6 HOW DO THE BASEL ACCORDS TREAT DOUBLE DEFAULTS?

The Basel accords recognize the credit risk reduction when a CDS is used as a hedge, as displayed in Figure 12.8. In particular, the investor will lose the investment to the obligor only if *both* the obligor and the guarantor default. Under the Basel accord, banks may use two approaches—the substitution approach and the double default approach—to address double default.[8] Let's discuss them.

12.6.1 Substitution Approach

For hedged credit exposures as in Figure 12.8, the Basel II accord allows that the exposure to the original obligor is replaced with the exposure of the guarantor. Hence from rewriting equation (12.7) we derive

$$\text{CVaR}_{hs}(X,T)=N\left[\min\left(\frac{N^{-1}(PD_o)+\sqrt{\rho_o}N^{-1}(X)}{\sqrt{1-\rho_o}}\right),\left(\frac{N^{-1}(PD_g)+\sqrt{\rho_g}N^{-1}(X)}{\sqrt{1-\rho_g}}\right)\right] \tag{12.14}$$

where

$\text{CVaR}_{hs}(X,T)$: credit value at risk for hedged exposures using the substitution approach in the Basel accord for the confidence level X, and

8. BCBS, "International Convergence of Capital Measurement and Capital Standard: A Revised Framework," November 2005, www.bis.org/publ.bcbs118.pdf; and BCBS, "The Application of Basel II to Trading Activities and the Treatment of Double Default Effects," 2005, www.bis.org/publ.bcbs116.pdf.

time horizon T; X is set at 99.9% in the Basel II approach; compare with equation (12.7) for unhedged exposures.

PD_o: probability of default of the obligor.

PD_g: probability of default of the guarantor.

ρ_o: copula default correlation coefficient between all assets in the portfolio of the obligor.

ρ_g: copula default correlation coefficient between all assets in the portfolio of the guarantor.

X: confidence level; X is set at 99.9% in the Basel II accord.

Other variables are defined as in equation (12.7).

The Basel accord interprets ρ_o in equation (12.14) as "the sensitivity of the obligor to the systematic risk factor [M]."[9] Strictly speaking, ρ_o is the default correlation coefficient between all asset pairs in the portfolio of the obligor o. As discussed earlier, this is a conditional correlation on the market factor M, as seen in the core equation (12.1) $x_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$ of the one-factor Gaussian copula model. It is reasonable to approximate the conditional correlation between the obligor's assets on the market factor M as the correlation of the obligor to the market factor M. The same logic applies to ρ_g.

Importantly, ρ_o in equation (12.14) is derived in the Basel accord with equation (12.9) and therefore takes values between 12% and 24% as shown in Figure (12.3). From equation (12.14) we also observe that the substitution approach is more valuable the lower the CVaR of the guarantor (second term in the min function) compared to the CVaR of the obligor (first term in the min function). Since in reality typically the default probability of the guarantor PD_g is lower than the default probability of the obligor PD_o, regulatory capital relief is often achieved when the substitution approach is applied.

12.6.2 Double Default Approach

The Basel II accord also allows banks to address credit risk that is hedged with a credit derivative, as displayed in Figure 12.8, with the double default approach. This approach is quantified with the bivariate normal distribution M_2. We have already discussed the bivariate normal distribution in Chapter 4 (see Figure 4.4). A bivariate normal distribution has three input parameters: the variables X and Y and the correlation parameter between X and Y, ρ:

$$M_2 = f(X, Y, \rho) \tag{12.15}$$

9. BCBS, "The Application of Basel II," p. 49.

To reduce the capital charge for hedged exposures, the Basel accord defines the variables X and Y as the credit value at risk (CVaR) values of the obligor o and the guarantor g, which we derived in equation (12.7). These are correlated with a correlation factor, which correlates the CVaR of the obligor and the guarantor and includes wrong-way risk. Let's have a look at this correlation factor.

From equation (12.1) we derive the default indicator variable x for the obligor x_o and the guarantor x_g:

$$x_o = \sqrt{\rho_o}M + \sqrt{1-\rho_o}Z_o \tag{12.1a}$$

$$x_g = \sqrt{\rho_g}M + \sqrt{1-\rho_g}Z_g \tag{12.1b}$$

The correlation between x_o and x_g in equations (12.1a) and (12.1b) is $\sqrt{\rho_o\rho_g}$. This can be seen easily: If ρ_o and ρ_g both are 1, x_o and x_g are equal to M in every simulation and hence are perfectly correlated. If ρ_o and ρ_g both $= 0$, x_o and x_g are determined solely by their idiosyncratic variables Z_o and Z_g, hence are uncorrelated. Even if either ρ_o *or* ρ_g is 0, the correlation between x_o and x_g is 0. Let's assume ρ_o is zero. Hence, the obligor is uncorrelated to the systematic market factor M. Since all correlation is conditioned on M, there is also zero correlation between x_o and M, and therefore also zero correlation between x_o and x_g. For values of $0 < \sqrt{\rho_o\rho_g} < 1$, x_o and x_g are partially correlated.

Basel II now adds a correlation factor for wrong-way risk between the obligor and guarantor, $\rho^*(1 - \rho_o)(1 - \rho_g)$. Altogether, the correlation between the obligor o and the guarantor g is set to

$$\rho_{og} \equiv \sqrt{\rho_o\rho_g} + \rho^*\sqrt{(1-\rho_o)(1-\rho_g)} \tag{12.16}$$

where

ρ_{og}: copula default correlation between (the assets of) the obligor o and the guarantor g.

$\equiv$ means "is set to" or "defined as".

$\sqrt{\rho_o\rho_g}$: default correlation (between the assets) of x_o and x_g without wrong-way risk (WWR); is the correlation induced by systematic risk (since it correlates x_o and x_g by indirectly conditioning them on the common market factor M).

ρ^*: correlation coefficient for wrong-way risk.

$\rho^*\sqrt{(1-\rho_o)(1-\rho_g)}$: correlation term to address wrong-way risk.

Other variables defined as in equation (12.14).

Equation (12.16) reminds us of the Pearson correlation approach. From equations (1A.4) and (1A.5) in the appendix of Chapter 1, we have

$$E(XY) = E(X)E(Y) + \rho\ \sigma(X)\sigma(Y) \tag{1A.5a}$$

However, equations (12.15) and (1A.5a) are fundamentally different. From $\rho = 0$ in equation (1A.5a) it follows that $E(XY) = E(X)E(Y)$, which means that X and Y are uncorrelated. From $\rho^* = 0$ in equation (12.16) it follows that $\rho_{og} \equiv \sqrt{\rho_o \rho_g}$. However, this is not a case of uncorrelatedness. The correlation between the obligor o and the guarantor g, ρ_{og}, will be zero only if either ρ_o or ρ_g is zero, since from equations (12.1a) and (12.1b), ρ_o and ρ_g are the correlation parameters that conditionally correlate on the common factor M.

We are now ready to derive the double default approach for hedged credit exposures in the Basel accord. We apply the bivariate equation (12.15) and define the variable X as the CVaR of the obligor o and Y as the CVaR of the guarantor g; see equation (12.7) for CVaR. We solve equation (12.16) for the correlation coefficient $\rho* = \frac{\rho_{og} - \sqrt{\rho_o \rho_g}}{\sqrt{(1-\rho_o)(1-\rho_g)}}$. Hence we derive

$$\mathrm{CVaR}_{hDD}(X,T) = M_2\left(\frac{N^{-1}(PD_o) + \sqrt{\rho_o}N^{-1}(X)}{\sqrt{1-\rho_o}}, \frac{N^{-1}(PD_g) + \sqrt{\rho_g}N^{-1}(X)}{\sqrt{1-\rho_g}}; \frac{\rho_{og} - \sqrt{\rho_o \rho_g}}{\sqrt{(1-\rho_o)(1-\rho_g)}}\right) \tag{12.17}$$

where

$\mathrm{CVaR}_{hDD}(X,T)$: credit value at risk for hedged exposures using the double default approach in the Basel accord for a confidence level of X and time horizon T. X is set at 99.9% in the Basel accord.

M_2: bivariate cumulative normal distribution.

ρ_o: copula default correlation coefficient between all assets in the portfolio of the obligor, derived by equation (12.9); hence ρ_o takes values between 0.12 and 0.24.

ρ_g: copula default correlation coefficient between all assets in the portfolio of the guarantor; ρ_g is set to 0.7 in the Basel accord.

ρ_{og}: copula default correlation coefficient between the obligor and the guarantor; ρ_{og} is set to 0.5 in the Basel accord.

Other variables are defined as in equation (12.14).

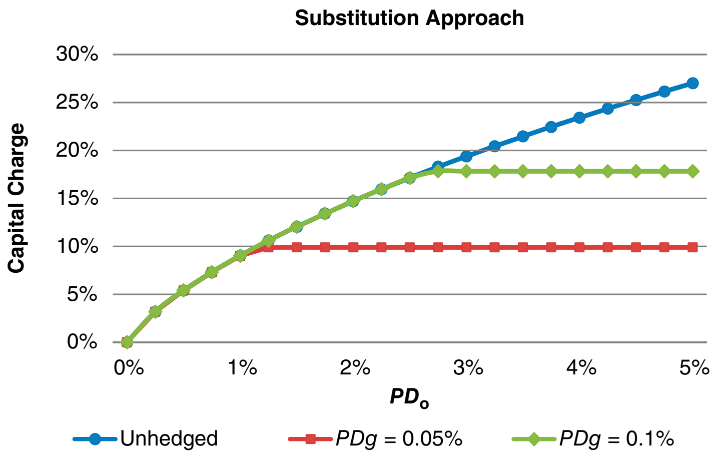

FIGURE 12.11 Basel Accord Capital Charge for Credit Risk When Applying the Substitution Approach of Equation (12.14)
The asset correlation of the obligor $\rho_o = 0.12$, the asset correlation of the guarantor $\rho_g = 0.7$, and the default correlation between the obligor and the guarantor $\rho_{og} = 0.5$.

From equation (12.16) we can expect a much lower CVaR compared to an unhedged exposure of equation (12.7) since a joint probability M_2 is typically much lower than a single probability N.

Let's compare the three scenarios with respect to credit value at risk (CVaR).

1. Unhedged capital charge CVaR for credit risk derived in equation (12.7). CVaR is the basis for calculating the required capital of equation (12.8).
2. A hedged CVaR, displayed in Figure 12.8, applying the substitution approach of equation (12.14), which reduces CVaR.
3. A hedged CVaR, displayed in Figure 12.8, applying the double default approach of equation (12.17), which also reduces CVaR.

Figure 12.11 shows the reduction in capital charge if the substitution approach is applied.

From Figures 12.11 and 12.12 we observe the significant capital charge reduction in the Basel accord when a credit exposure is hedged. Comparing Figures 12.11 and 12.12, we also see that the double default approach typically allows a lower capital charge than the substitution approach does.

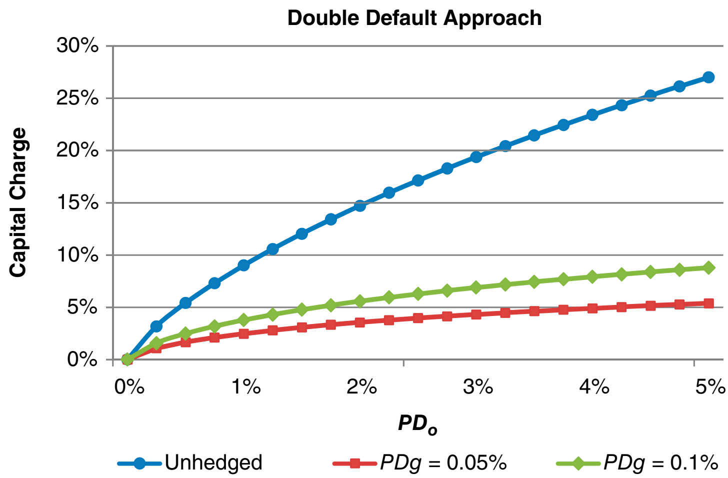

FIGURE 12.12 Basel Accord Capital Charge for Credit Risk When Applying the Double Default Approach of Equation (12.17)
As in Figure 12.11, $\rho_o = 0.12$, $\rho_g = 0.7$, and $\rho_{og} = 0.5$.

The substitution approach has been criticized for its lack of mathematical foundation and a lack of sensitivity to high risk exposure (since the high risk exposure is substituted for the guarantor's risk exposure). The double default approach, also called the asymptotic single risk factor (ASRF) approach following a paper by Gordy (2003) has a more rigorous mathematical foundation and is sensitive to both high-risk (obligor) and low-risk (guarantor) debtors.

For a spreadsheet that derives the Basel III capital charge for hedged credit exposure, see the spreadsheet "Basel double default.xlsm" at www.wiley.com/go/correlationriskmodeling, under "Chapter 12."

12.7 DEBT VALUE ADJUSTMENT (DVA): IF SOMETHING SOUNDS TOO GOOD TO BE TRUE...

Let's first clarify: Credit value at risk (CVaR) derived in equation (12.7) addresses counterparty credit risk in a portfolio with relatively fixed exposures such as bonds and loans. Credit value adjustment (CVA) derived in equations (12.12) and (12.13) is a specific capital charge that typically addresses counterparty credit risk in a derivatives transaction.

There have been two recent developments related to CVA: debt value adjustment (DVA) and funding value adjustment (FVA). Let's discuss them.

What is DVA?

DEBT VALUE ADJUSTMENT (DVA)

Allows an entity to adjust the value of its portfolio by taking its own default probability into consideration.

The Basel accord prefers the term *CVA liability* instead of DVA. However, we will refer to it as DVA.

In Figure 12.5 we displayed credit exposure and concluded that credit exposure can only be bigger or equal zero. Credit exposure for entity a with counterparty c exists if the counterparty c is a net debtor to a. If we allow recognizing negative credit exposure or debt exposure, Figure 12.5 would change to Figure 12.13.

This debt exposure of a with respect to c could theoretically be taken into consideration when evaluating a portfolio. In particular, debt exposure could be recognized in derivatives transactions. This debt exposure in derivatives transactions is the netted negative derivatives portfolio value of entity a with respect to c, $D^-_{a,c}$. This is weighted, i.e. reduced by the probability of default of entity a. Including a recovery rate of a, we derive in analogy to equation (12.12) for CVA, which is: $\text{CVA}_{a,c} = (D^+_{a,c} \times PD_c)(1 - R_c)$,

$$\text{DVA}_{a,c} = (D^-_{a,c} \times PD_a)(1 - R_a) \tag{12.18}$$

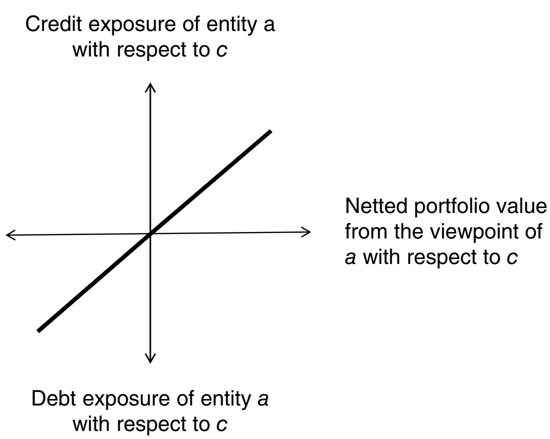

FIGURE 12.13 Debt Exposure when the Netted Portfolio Value of Entity a is Negative with Respect to Entity c (i.e., a is a net debtor for c)

where

$DVA_{a,c}$: debt value adjustment of entity a with respect to entity c
$D^{-}_{a,c}$: netted negative derivatives portfolio value of a with respect to c (i.e., a is a debtor to c)
PD_a: default probability of entity a
R_a: recovery rate of entity a

Importantly, let's now consider that in the event of default of entity a, only the recovery rate of a's debt is paid out. If this is accounted for, this decreases a's debt and increases the book value (defined as assets minus debt) of a. If we apply this concept to a derivatives portfolio, the derivatives portfolio value increases and equation (12.10) expands to

$$\text{Value of Derivatives Portfolio} = \text{Default-Free Value} - \text{CVA} + \text{DVA} \tag{12.19}$$

However, there are two critical problems with DVA:

1. An entity such as a would benefit from its own increasing default probability PD_a, since a higher default probability would increase DVA via equation (12.18), which in turn increases the value of the derivatives portfolio via equation (12.19).
2. Entity a could realize the DVA benefit only if it actually defaults.

Both properties defy financial logic. Therefore the Basel accord has principally refrained from allowing DVA to be recognized. In 2008 several financial firms had actually reported huge increases in their derivatives portfolios due to DVA. This is no longer possible.

12.8 FUNDING VALUE ADJUSTMENT (FVA)

A further recent development relating to CVA and DVA is funding value adjustment (FVA). What is FVA?

FUNDING VALUE ADJUSTMENT (FVA)

An adjustment to the price of a transaction due to the cost of funding for the transaction or the related hedge.

Funding cost had not been a major issue in derivatives pricing in the past. However, in 2008, when interest rates especially for poor credits increased sharply, funding cost could no longer be ignored.

There has been quite a spirited debate in 2012 and 2013 about whether the cost of funding should be taken into consideration when pricing a derivative. Hull and White as well as Duffie (Risk 2012(a) and 2012(b)) argue that adding funding costs violates the risk-neutral derivatives pricing principle. It would lead to arbitrage opportunities, since the same derivative would have different prices. However, derivatives traders argue that their treasury departments charge them the funding costs. Hence funding costs exist in reality and cannot just be ignored. The funding cost should be priced in and passed through to the end user. See "The FVA Debate" in *Risk*, July 2012, and "Traders v. Theorists" in *Risk*, September 2012, for further details.

Let's look at the issue of cost of funding. The cost of funding of an entity is mainly a function of the default probability of the entity. Hence we have

$$\mathrm{FVA}_a = f(PD_a, \ldots,) \tag{12.20}$$

There is a positive relationship between FVA and PD, since the higher the default probability, the higher is the cost of funding:

$$\frac{\partial \mathrm{FVA}_a}{\partial PD_a} > 0 \tag{12.21}$$

If the cost of FVA is taken into account, the value of a derivatives portfolio is reduced. Hence equation (12.19) then changes to

$$\text{Value of Derivatives Portfolio} = \text{Default-Free Value} - \text{CVA} + \text{DVA} - \text{FVA} \tag{12.22}$$

As we see from equations (12.18) and (12.21), both DVA and FVA increase if the probability of default increases; hence credit quality decreases.

In a 2012 response to the Basel proposal, the International Swaps and Derivatives Association (ISDA) has suggested that "CVA liability [i.e. DVA] should be deducted only to the extent that it exceeds the increase in FVA."[10] In this case there would be no benefit (i.e., no increase in the value of a derivatives portfolio) if the default probability of an entity increases, as we can see from equation (12.22), (since DVA is added only up to the amount that FVA is subtracted).

12.9 SUMMARY

In this chapter we discussed the way correlation risk is addressed in the Basel II and Basel III frameworks. The Basel committee has recognized the significance of correlation risk and has suggested several approaches to managing correlation risk.

Correlation risk is a critical factor in managing credit risk. In the Basel II and III accords, credit risk of a portfolio is quantified with the credit value at risk (CVaR) concept. CVaR measures the maximum loss of a portfolio due to credit risk with a certain probability for a certain time frame. Basel II and Basel III derive CVaR on the basis of the one-factor Gaussian copula (OFGC) correlation model, which we discussed in Chapter 6.

The required capital to be set aside for credit risk is the CVaR minus the average probability of default of the debtors in the portfolio. This is because the Basel committee assumes that banks cover the expected loss (approximated as the average probability of default) with their own provisions such as the interest rate that they charge.

Interestingly, the Basel committee requires an inverse relationship between the default correlation of the debtors in a portfolio with respect to the default probability of the debtors: The lower the default probability of debtors in a portfolio, the higher is the default correlation between the debtors. This is reasonable, since debtors with a low default probability are more prone to default for systematic reasons; that is, they more often default together in a recession. Conversely, low rated debtors with a high default probability are more affected by their own idiosyncratic factors and less by systematic risk. Hence the default risk of low rated debtors is assumed to be less correlated. This is supported by empirical data.

10. See ISDA, "ISDA and Industry Response to BCBS Paper on Application of Own Credit Risk Adjustments to Derivatives," 2012, www2.isda.org/functional-areas/risk-management/page/3.

A further aspect, which has become a critical factor in credit risk management, is credit value adjustment (CVA). CVA is a capital charge to address credit risk, mainly in derivatives transactions. CVA has a market risk component (the netted derivatives value) and a credit risk component (the probability of default of the counterparty). Importantly, these market risk and credit risk components are typically correlated! This results in the correlation concept of wrong-way risk (WWR). The Basel committee defines two types of wrong-way risk:

1. *General wrong-way risk* arises when the probability of default of counterparties is positively correlated with general market risk factors.
2. *Specific wrong-way risk* exists if the exposure to a specific counterparty is positively correlated with the counterparty's probability of default due to the nature of the transaction with the counterparty.

The Basel committee requires financial institutions to address wrong-way risk: Financial institutions have to increase their credit exposure value (calculated without wrong-way risk) by 40%. Financial institutions that use their own internal models can apply a 20% increase. This is conservative, since banks report a numerical value for wrong-way risk of 1.07 to 1.1.

The Basel committee also realizes the risk reduction that is achieved when a credit exposure is hedged with a credit default swap (CDS). The Basel committee allows banks to address the credit risk reduction of a CDS in two ways: (1) the *substitution approach*, which allows banks to use the typically lower default probability of the guarantor (CDS seller) in the credit exposure calculation, and (2) the *double default approach*, which derives the joint probability of the obligor and the guarantor defaulting. This joint default probability is typically much lower than the individual default probability of the obligor, lowering the overall credit exposure value.

The concept of CVA has recently been extended by the concepts debt value adjustment (DVA) and funding value adjustment (FVA). Debt value adjustment (DVA) allows an entity to adjust the value of a position (such as a loan or a derivative) in a portfolio by taking its own default probability into consideration. If an entity applies DVA (i.e., takes its own default probability into consideration), this actually reduces the credit exposure of the entity. This is highly controversial and has been banned by the Basel committee.

Funding value adjustment (FVA) is an adjustment to the price of a transaction, typically a derivative, due to the cost of funding the transaction. FVA has been quite controversially debated in 2012 and 2013. Finance professors argue that it creates arbitrage opportunities, since different FVA values lead to different derivatives prices. However, traders argue that the funding costs are substantial and have to be included in the transaction price.

PRACTICE QUESTIONS AND PROBLEMS

1. What information does credit value at risk (CVaR) give us?
2. Why don't we just apply the market value at risk (VaR) concept to value credit risk?
3. Which correlation concept underlies the CVaR concept of the Basel II and III approach?
4. In the Basel committee CVaR approach, what follows for the relationship between the CVaR value and the average probability of default, if we assume the correlation between all assets in the portfolio is zero?
5. Suppose Deutsche Bank has given loans to several companies in the amount of \$500,000,000. The average 1-year default probability of the companies is 2%. The copula default correlation coefficient between the companies is 3%. What is the 1-year CVaR on a 99.9% confidence level?
6. In the Basel committee CVaR model, the default correlation is an inverse function of the average probability of the default of the assets in the portfolio. Explain the rationale for this relationship.
7. In the Basel committee approach, the required capital to be set aside for credit risk is the CVaR minus the average probability of default. Explain why.
8. CVA is an important concept of credit risk. What is CVA? Why is it important?
9. Why can CVA be considered a complex derivative?
10. How can CVA without correlation between market risk and credit risk be calculated?
11. Including the correlation between market risk and credit, the concept of wrong-way risk (WWR) arises. What is general wrong-way risk, and what is specific wrong-way risk?
12. Name two examples of specific wrong-way risk.
13. How does the Basel committee address wrong-way risk?
14. What is DVA? Should DVA be allowed to be applied in financial practice?
15. What is FVA? Should FVA be included in the pricing of derivatives?

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CHAPTER 13

The Future of Correlation Modeling

Solving the right problem numerically beats solving the wrong problem analytically every time.

—Richard Martin

In this chapter we discuss new developments in financial modeling that can be extended to correlation modeling. We address the application of graphical processing units (GPUs), which allow fast parallel execution of numerically intensive code without the need for mathematical solvency. We also discuss some new artificial intelligence approaches such as neural networks, genetic algorithms, as well as fuzzy logic, Bayesian mathematics, and chaos theory.

13.1 NUMERICAL FINANCE: SOLVING FINANCIAL PROBLEMS NUMERICALLY WITH THE HELP OF GRAPHICAL PROCESSING UNITS (GPUs)

Some problems in finance are quite complex so that a closed form solution is not available. For example, path-dependent options such as American-style options principally have to be evaluated on a binomial or multinominal tree, since we have to check at each node of the tree if early exercise is rational. In risk management, especially in credit risk management, thousands of correlated default risks have to be evaluated. While there are simple approximate measures to model counterparty risk in a portfolio such as the Gaussian copula model (see Chapter 6), it is more rigorous to model counterparty risk on a multifactor approach using numerical methods such as Monte Carlo simulation.

In the recent past, the increase of computer power has made numerical finance an alternative to analytical solutions. Let's define it:

NUMERICAL FINANCE

Attempts to solve financial problems with numerical methods (such as Monte Carlo simulation), without the need of mathematical solvency.

Other terms for numerical finance are statistical finance, computational finance, and also econophysics. More narrowly defined, econophysics is the combination of physical concepts and economics. However, the economic concepts include stochastic processes and their uncertainty, which are also an essential part of finance.

Why waste good technology on science and medicine?
—Lighthearted phrase of gamers on GPU technology

13.1.1 GPU Technology

Graphical processing units (GPUs) are the basis for a technology that alters memory in a parallel execution of commands to instantaneously produce high-resolution three-dimensional images. The GPU technology was derived in the computer gaming industry, where gamers request high-resolution, instant response for their three-dimensional activities at low cost. This caught the attention of the financial industry, which is paying millions of dollars to receive real-time response for valuing complex financial transactions and risk management sensitivities.

Hence, over time, financial software providers have started to rewrite their mathematical code to make it applicable for the GPU environment. Companies such as Murex, SciComp, Global Valuation Limited, Hanweck Associates, BNP Paribas, and many others have implemented GPU-based infrastructures to numerically solve complex derivatives transactions and calculate risk parameters. The academic environment has also responded. More than 600 universities worldwide offer courses in GPU programming.

13.1.2 A GPU Model for Valuing Portfolio Counterparty Risk

Albanese et al. (2011) display a detailed approach that applies GPU technology to model counterparty risk in a portfolio. These portfolios typically consist of thousands of counterparties whose default and transition probabilities are correlated. To evaluate this complex counterparty risk in a rigorous way, numerical methods are necessary. The model consists of the following steps:

1. The portfolio data (i.e., the contracts of the loans, swaps, credit default swaps [CDSs], foreign exchange [FX] contracts, options, futures, etc.) with each counterparty are netted[1] and input into the model.
2. The stochastic processes of the underlying input variables such as interest rates, exchange rates, default intensities, and CDS contracts are generated.
3. Parallelized pricing of the contracts is done using multithreading[2] technology in a GPU framework.
4. The model is calibrated using a large number of liquid assets and derivatives. It is tested whether backward induction or Monte Carlo forward induction gives better calibration results.
5. Correlations are integrated in the model via dynamic multifactor copulas; see Chapter 6, section 6.5.
6. GPU technology allows real-time 3-D visualization of the output.

Figure 13.1 shows the model in graphical form.

13.1.3 Benefits of GPUs

Let's have a look at the benefits of GPUs.

- *Speed.* The main benefit of the GPU technology is speed. Thousands of cores, executing in parallel, are implemented in a GPU, whereas the central processing unit (CPU) of a standard PC has just multiple cores (as of 2013). Most software providers claim that the GPUs are 20 to

1. In most sovereign states, legislation allows netting with a specific counterparty. This is the process of adding together the positive present value (PV) and the negative PV of all deals with the counterparty. Only if this netted portfolio PV is positive does counterparty risk exist. See Chapter 12, section 12.4 for details.
2. Multithreading is a technology in which the processor switches between different programming instructions (threads), which allows the parallel execution of multiple commands.

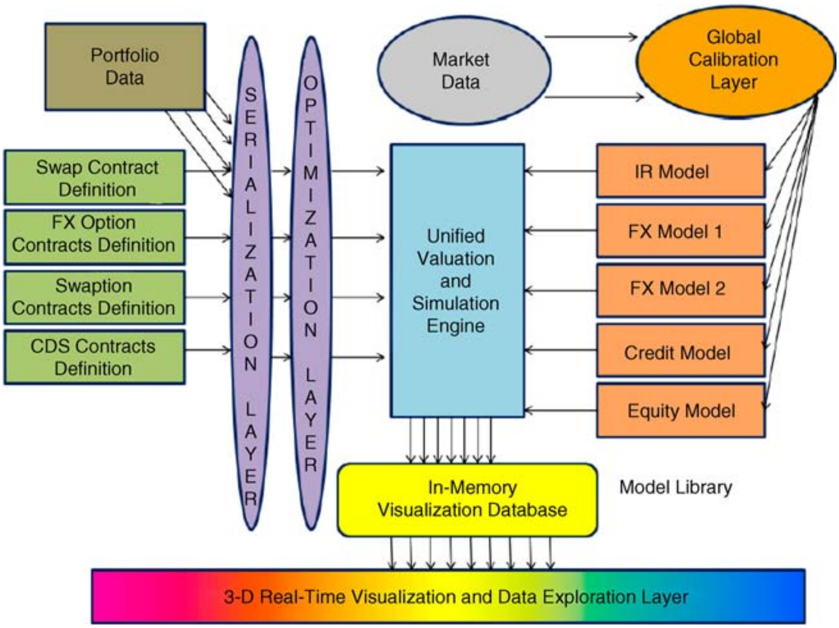

FIGURE 13.1 A GPU Model to Quantify Counterparty Credit Risk
Source: Albanese et al. (2011).

100 times faster than their rival CPUs. In addition, CPU applications and fast GPU technology have been combined in the recent past to implement standard CPU applications at high speed easily.

- *Increasing user-friendliness.* Special languages to execute financial code in the GPU framework, such as CUDA (compute unified device architecture), as well as OpenGL (open graphics library) and OpenCg (*C* stands for the computer language and *g* for graphics), have been developed, specifically designed for parallel computing in the GPU framework.
- *Structural efficiency.* CPU technology typically depends on a compiler that translates the source code into an executable language. The GPU languages such as CUDA, OpenGL, and OpenCg may be more difficult to program, but do not require compiling. Therefore manipulation of the language code, for example stress testing or optimization, can be easily and quickly achieved.
- *More precision?* Advocates of the GPU technology claim that "you can eat the cake and have it, too," meaning GPUs are faster and at the same time more accurate. The faster speed is evident. However, the question of whether parallel iterative search procedures are more accurate than

standard mathematical techniques such as finite difference methods, vector-splitting techniques, and Fourier transforms to solve complex financial problems is debatable.

13.1.4 Limitations of GPUs

Let's outline the limitations of GPUs.

- Although there have been efforts to combine CPU and GPU technologies, GPUs still have a different architecture and require their own distinct structure of programming and specialized programming languages such as CUDA, OpenGL, and OpenCg.
- GPUs provide efficient and fast solutions for problems that are complex but can be represented in matrices, since it is easy to execute matrix multiplication and manipulation. However, GPUs may not be well suited for nonlinear, path-dependent structures.

Altogether, the GPU technology is a promising new approach to derive fast, real-time results for complex financial problems for which analytical solutions are questionable or unavailable. However, one can argue that if mathematical techniques are available, although slower, they should have preference over brute-force nonanalytical iterative search procedures.

13.2 NEW DEVELOPMENTS IN ARTIFICIAL INTELLIGENCE AND FINANCIAL MODELING

In this section, we discuss new developments in artificial intelligence (AI) such as neural networks and genetic algorithms. We also investigate whether related areas such as Bayesian probabilities, fuzzy logic, and chaos theory are promising tools for financial modeling.

What is artificial intelligence? Typically it is not a good idea to define a term and use that same term in the definition. But let's do it anyway: Artificial intelligence is the attempt to create artificial intelligence. Less ironical: Artificial intelligence is the science of creating intelligent machines. One of the most successful concepts to create artificial intelligence is neural networks.

13.2.1 Neural Networks

Neural networks are the most widely applied artificial intelligence concept in finance. Neural networks are typically used in trading and investing, trying to forecast prices and volatilities of stocks, bonds, exchange rates, and the like.

Several companies such as Tradecicion or BrainMaker offer specific neural network software to make financial predictions. Neural networks are also applied in derivatives pricing and credit risk management. Two types of neural networks currently exist:

1. A biological neural network consists of living systems with neurons (nerve cells), whose electrical or chemical information is transmitted to other neurons via synapses.
2. An artificial neural network (ANN), our main interest, is a nonliving system that tries to mimic the functioning of the human brain, in particular the ability to learn.

The learning process in an ANN can be achieved by different methods. One of the most popular is *backpropagation* (an abbreviation for "backward propagation of errors"), which applies weighting factors. Each weighting factor has a resistance attached to it. Numerous simulations (epochs) of different combinations of weighting factors are run. If the neural network output is close to the target value, the weighting factors are strengthened; that is, the resistance is turned down. If certain weighting factors produce bad results, the weighting factors are weakened; that is, the resistance is turned up.

Meissner and Kawano (2001) show that a neural network can improve option pricing. The applied neural network is the popular multilayer perceptron (MLP) network, which can be mathematically expressed as

$$y_{NN} = \sum_{h=1}^{H} \beta_h T \sum_{i=1}^{n} w_{ih} x_i \tag{13.1}$$

where y_{NN}: output of the neural network (the option price)

β_h: weighting factor of layer node h_j, which reflects the strength between h_j and the output y_{NN} (see Figure 13.2)

T: transfer function, usually a simple hyperbolic function such as the tangent function (this function standardizes weighted input variables to values between −1 and +1)

w_{ih}: weighting factor of input x_i, which reflects the strength between input x_i and the hidden layer node h (see Figure 13.2)

x_i: input variable i (the spot/strike S/K, option maturity T, implied volatility σ, and interest rate r)

Graphically, an MLP neural network can be expressed as in Figure 13.2.

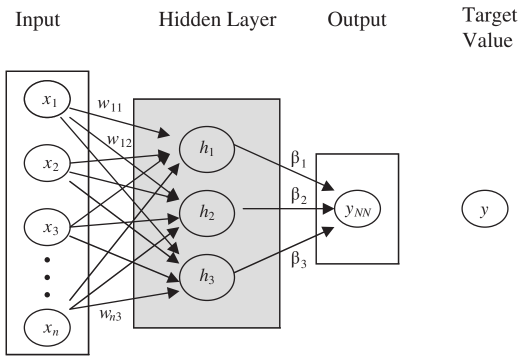

FIGURE 13.2 Sample Structure of an MLP Neural Network with Various Input Variables x_i, $i = 1...n$, One Hidden Layer with Three Units h_j, $j = 1...3$, and One Output Variable y_{NN}.
Source: Meissner and Kawano (2001).

Using backwardation techniques together with a GARCH (see Chapter 8, section 8.3) generated volatility, Meissner and Kawano (2001) show that a neural network can learn to produce the volatility smile of options, which is observed in reality. Hence the neural network gives better option pricing results than the standard Black-Scholes-Merton model.

Other research on neural networks and option pricing exists, such as Rubinstein (1985); Freisleben and Ripper (1997); White, Hatfield, and Dorsey (1998); Yao, Yili, and Tan (2000); and Gradojevic, Gençay, and Kukolj (2009). However, most neural networks in finance are applied to simply forecasting stock prices, option prices, mutual funds values and the like, often based on finding technical analysis patterns. For a nice overview on neural networks applied in finance, although a bit outdated, see Fadlalla and Lin (2001).

13.2.1.1 Limitations of Neural Networks Naturally neural networks have their drawbacks. First, they are a black box; that is, the mathematical algorithm that optimizes the output is hidden. Second, neural networks often have a fairly slow convergence rate. Third, and most important, neural networks can get stuck at local optima, not deriving the general optimum. These reasons have limited the usage of neural networks in reality.

To overcome these drawbacks, neural networks in the recent past have been combined with other disciplines, for example fuzzy logic, genetic algorithms, or Bayesian statistics, to improve the neural network performance.

13.2.2 Fuzzy Logic

Fuzzy logic is an exciting field. It alters the traditional concept of reasoning. In traditional logic a statement can either be true, typically assigned the value 1, or false, assigned the value 0. Fuzzy logic, however, argues that there can be a "partial truth" to a statement, assigning truth values ranging *between* 0 and 1.

For example, a professor asks the question: "What are the prime numbers between 10 and 20? The correct answer is the set {11, 13, 17, 19}. If a student gives the answer {11, 13, 17}, traditional logic would argue that the answer is false. However, fuzzy logic would argue that the answer is partially true, actually 75% true.

Fuzzy logic has some real-world applications. For example, it is applied in Japan to improve the punctuality of trains: If a train is one minute late, fuzzy logic can assign a punctuality value of close to 1 (totally punctual), making no action necessary. The later a train arrives, fuzzy logic can assign gradually lower punctuality values, initiating action for a train with low punctually values to have priority access to stations so it catches up to its scheduled time. Hence fuzzy logic often applies flexible if-then statements to construct specific, flexible commands to address a problem.

Fuzzy logic can tolerate imprecise information, but lacks learning ability. Therefore the learning ability of neural networks is often combined with fuzzy logic. In addition, genetic algorithms are often introduced, creating genetic fuzzy neural networks (GFNNs).

13.2.3 Genetic Algorithms

Genetic algorithms are based on phenomena found in evolution, such as selection, crossover, and mutation.

Each individual or element in a genetic algorithm is defined as a vector with parameters and weights. *Selection* can be done by various methods:

- *Roulette wheel selection.* Here a random selection of potentially useful solutions for recombining the individuals is performed.
- *Crossover method.* The next generation is created by different "crossings" (loosely speaking, combinations) of the parents' organism strings to produce a variety of child organisms.
- *Mutation.* A change in the genetic property of an individual is typically quantified as single point mutation, meaning only a single nucleotide parameter is replaced by a new parameter. This replacement is performed at a certain mutation rate. For example, if the mutation rate is

specified as 30%, this means that in 30% of the simulations a mutation will take place.

13.2.3.1 Genetic Fuzzy Neural Networks In finance, numerous studies on genetic fuzzy neural networks (GFNNs) exist. They are mainly applied to forecast stock prices or support trading decisions; see Yu and Zhang (2005); Yang, Wu, and Lin (2008); Li and Xiong (2006); Kuo, Chen, and Hwang (2001); or Huang (2008). After learning, most of the GFNN models were able to outperform the benchmark index.

> ***My book was criticized before it was published, which is a compliment.***
>
> **—Benoit Mandelbrot**

13.2.4 Chaos Theory

Another interesting field with potential applications in finance is chaos theory.

Chaos theory is a field that studies the behavior of dynamic systems. It is mainly applied in meteorology to explain and forecast weather dynamics, but is also used to explain phenomena in physics, engineering, biology, economics, and finance. The definition of a chaotic system is not unique. However, the five main criteria of a chaotic system are:

1. Strong dependence on initial conditions.

   This insight was derived in the famous 1972 paper by the mathematician and meteorologist Edward Lorenz: "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?" Lorenz argued that a butterfly in Brazil can influence weather conditions, and even cause a tornado in a region very far away. Earlier, in 1961, Lorenz had found that changing the number of decimal places from three to six improved the modeling of weather predictions significantly. This strong dependence on initial conditions is one reason for the next criterion of a chaotic system.
2. Predictions are possible for only a short period of time.

   Clearly this condition applies to modeling weather. Weather can be predicted quite well for a short period of time, for example for 12 hours or 24 hours. However, the longer the time horizon of the prediction, for example three days or five days, the less predictable weather becomes.

For an ever longer time horizon, for example 15 or 30 days, a reliable forecast is virtually impossible, since the weather system is chaotic in the long term; that is, it is unpredictable.

3. Self-similarity.

   Chaotic systems typically display self-similarity. This means that an object is similar or exactly the same as the parts of that object viewed on a different scale. For example, a coastline viewed from far away often displays the same patterns as the same coastline viewed from very close. Self-similarity is a property of *fractals*, which were introduced by Benoît Mandelbrot in 1975. Mandelbrot discovered fractals in many natural phenomena such as mountains, rivers, and blood vessels, and also in music, paintings, and architecture. Interestingly, Mandelbrot claimed to have found fractals also in stock price movements. In 2004, he wrote a book with Richard Hudson, *The (Mis)Behavior of Markets: A Fractal View of Risk, Ruin, and Reward*, in which also the Gaussian distribution of stock price returns is questioned.

4. Regime changes.

   Chaotic systems do not behave chaotically all the time; they can, when moving on an *attractor*, be somewhat predictable. An attractor is a set toward which a variable, according to a mathematical algorithm, evolves over time. Loosely speaking, an attractor attracts a variable to move toward or around the attractor. An attractor can simply be a point, a curve, or a surface. Complex attractors with fractal structures or with noninteger dimensions are called *strange attractors*. The form of a strange attractor may resemble a butterfly or the number 8, as seen in Figure 13.3. Figure 13.3 shows a two-dimensional attractor.

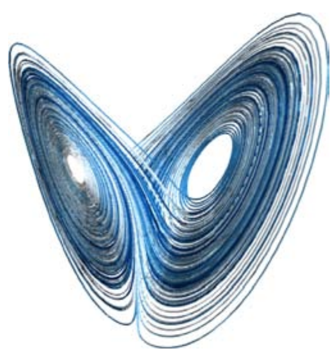

FIGURE 13.3 Example of an Attractor
A variable will often stay in one set of rings until a regime change occurs, and then the variable will move toward a new regime (i.e., a different set of rings with different dynamics).

5. Deterministic nature.

   A chaotic system is mathematically deterministic. It does not include random factors, such as a random drawing from a certain distribution, which is often applied in finance; compare equations (4.1) and (4.2), equations (8.33) to (8.35), and equation (12.1). However, the deterministic nature of chaos theory is not a sufficient condition for good predictability of the system in the long run.

13.2.4.1 Chaos Theory and Finance Can chaos theory explain and predict phenomena in finance?

Criterion 1 of chaos theory, the strong dependence on initial conditions, is not critical in finance. Low levels of a stock price in t_0 do tend to have higher volatilities. For example, a stock price moving by \$5 from \$100 to \$105 has a 5% increase. A stock price moving by \$5 from \$10 to \$15 has a 50% increase. In general, though, it is of minor importance at what level the stock price that we are trying to forecast is in t_0.

Criterion 2, the possible predictability in the short term, but limited predictability in the long run, is also not present in finance. It can even be argued that in this sense finance is antichaotic. Stock market prediction in the short term, for a day or a week, is often more difficult than longer-term predictions, for example yearly predictions. This is because longer-term stock market movements are based on long-term economic conditions such as recessions or periods of prosperity.

Criterion 3, self-similarity, can be found in finance. For example, the performance of the Dow Jones Industrial Average shows a similar pattern when observed monthly, by week, or intraday.

From Figures 13.4 to Figures 13.6, we observe the self-similarity property for the Dow's performance. Especially the weekday (Figure 13.5) and intraday performance (Figure 13.6) are similar. A strong performance in the beginning of the time period (Tuesday and 10.30 to 11.00) then weakens and is followed by a negative performance toward the end (Thursday and 14.00 to 14.30) with an uptick at the end (Friday and 15.00 to 16.00). This is also similar to the monthly performance in Figure 13.4.

Criterion 4, the different regimes in chaos theory, translate well to finance. We just call them trends. Long-term upward trends during an economic expansion or long-term downward trends in a recession exist, as well as short-term intraday upward and downward trends. These trends are exploited by traders: "The trend is your friend."

Criterion 5, the deterministic nature of chaos theory, is not an adequate property for financial modeling. As mentioned numerous times in this book [see Chapter 4 and equations (4.1) and (4.2) or equations (8.33) to (8.35)], financial variables can be well modeled with a stochastic process.

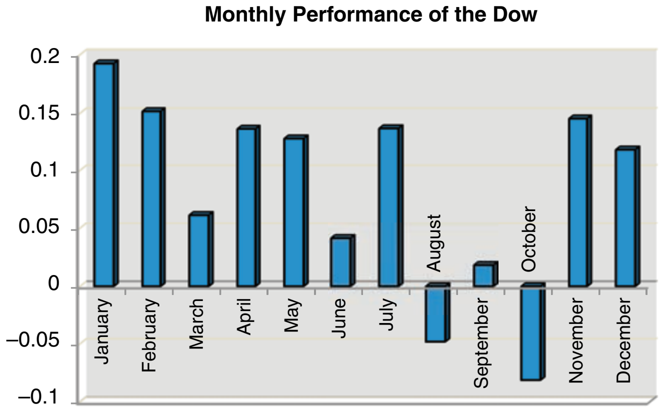

FIGURE 13.4 Monthly Performance (Percentage Change) of the Dow (Data from 1968 to 2001)

13.2.4.2 Conclusion: Can We Apply Chaos Theory to Model and Forecast Financial Variables? From our analysis, we find that chaos criteria 3 and 4, the self-similarity principle and the regime changes, are also found in finance. However, criteria 1, 2, and 5, the dependence on initial conditions, the short-term but not long-term predictability, and the deterministic nature of chaos theory, are typically not properties in finance. The critical question is whether criteria 3 and 4 are sufficient to support financial trading decisions.

Some companies believe so. Financial Chaos Theory, a consulting firm, and Tetrahex, which sells fractal finance software, are using chaos theory to offer trade support. They apply fractal dimensions (for example the dimension 2.5) and try to find buy and sell signals based by deriving technical indicators

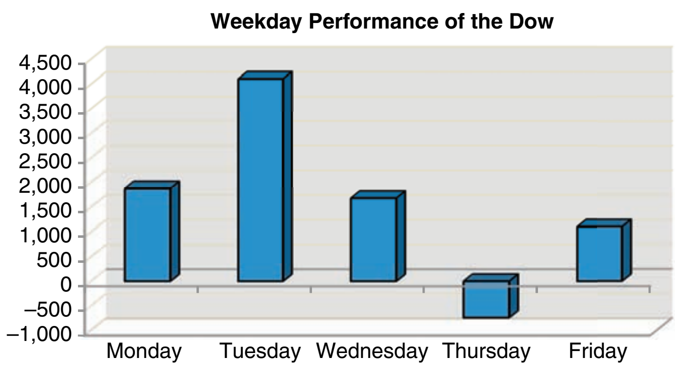

FIGURE 13.5 Performance by Day of the Dow (Data from 1968 to 2001)

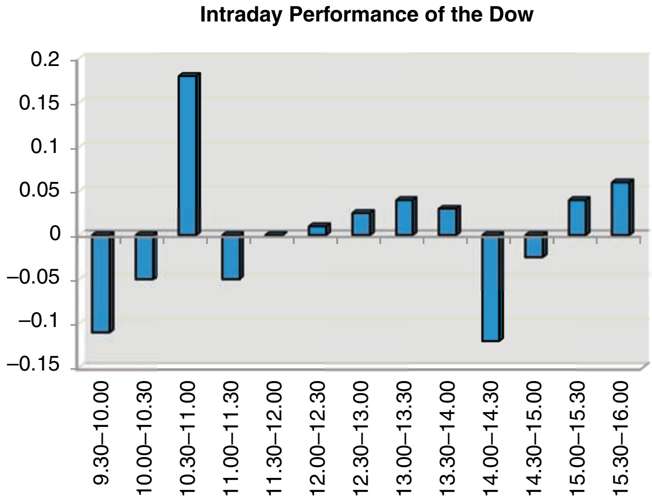

FIGURE 13.6 Intraday Performance of the Dow (Data from 1968 to 2001)

such as moving average convergence/divergence (MACD), which gives buy and sell signals when different moving averages cross, as well as identifying trends.

Some research has shown some explanatory power for financial variables using chaotic dynamics, such as Trippi (1994) and Peters (1996).

So far no attempt has been made to apply chaos theory for correlation modeling. Since only some of the chaos criteria are appropriate for financial modeling, it is questionable whether chaos theory is a useful tool for correlation modeling. But we are always happy to be proven wrong.

13.2.5 Bayesian Probabilities

Bayesian statistics and related probabilities are a further concept with potential application to financial modeling. The Bayesian approach reinterprets and extends the classical probability reasoning. Bayesian probabilities were founded by the English mathematician Thomas Bayes in the eighteenth century and popularized by the French mathematician and astronomer Pierre Laplace.

At the heart of the Bayesian approach is the Bayes theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{13.2}$$

It is important to note that the Bayesian theorem (13.2) is algebraically identical with the classical frequentist probability theory (called "frequentist" since it draws its conclusion from the frequency of data). In fact, we had derived equation (13.2) in the appendix of Chapter 1, in equations (1A.1) to (1A.3). However, the Bayesian theory reinterprets the variables in equation (13.2).

In frequentist theory, $P(A)$ and $P(B)$ are the probabilities of events A and B, respectively. $P(A|B)$ is the probability of A, conditionally on B occurring or having occurred. $P(B|A)$ is the probability of B, conditionally on the probability of *occurring or having occurred*. However, in Bayesian theory the variables in equation (13.2) have the following interpretations:

- $P(A)$ is a *prior* initial probability. $P(A)$ is the hypothesis before accounting for evidence. $P(A)$ can be a personal *subjective belief*, rather than an objectively derived probability.
- $P(B)$ is the probability of the evidence B, which will influence the critical outcome $P(A|B)$.
- $P(A|B)$ is the *posterior* probability. $P(A|B)$ is the probability of A given that the evidence B is observed.
- $P(B|A)/P(B)$ is the support that B provides for deriving $P(A|B)$.

From these definitions, two main properties follow:

1. In classical probability theory, a hypothesis is a proposition that is either true or false, so the probability of the proposition is 0 or 1. In Bayesian logic, a probability is assigned to a hypothesis (above the hypothesis $P(A)$) that can take truth values *between* 0 and 1. This adds flexibility to the process of deriving the conditional probabilities. It is related to fuzzy logic, which also applies the concept of partial truth (see section 13.2.2).
2. Bayesian logic is a *dynamic* theory. If new evidence is found incorporating the values for $P(B)$ and $P(B|A)$, this evidence is incorporated and a revised outcome $P(A|B)$ is derived. Hence Bayesian theory shows how a subjective belief changes in time following new evidence.

Let's look at an example of Bayesian probability.

EXAMPLE 13.1: A NUMERICAL EXAMPLE OF BAYESIAN PROBABILITIES

An analyst performs some approximate studies and believes that the default probability of Ford Motor Company is $P(A) = 10\%$. The economy, as in 2008, is in a severe recession and systemic risk exists (see Chapter 1, section 1.4). The analyst finds evidence that there is a positive default correlation between General Motors (GM) and Ford.

(*continued*)

In particular, the conditional default probability of GM defaulting if Ford defaults is $P(B|A) = 20\%$. In addition, the default probability of GM is $P(B) = 15\%$. What is the default probability of Ford, applying this new evidence? Using Bayesian equation (3.2), this new posterior probability $P(A|B)$ is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.2 \times 0.1}{0.15} = 0.1333 = 13.33\%$$

Hence, the initially assumed default probability of 10% for Ford increases to 13.33% when new evidence is applied via Bayes' theorem.

Note that in order to apply Bayes' rule of equation (13.2), a codependence of A and B has to be sensible. This means that A has to be dependent on B, and B has to be dependent on A as well, since if $P(B|A) \geq 0$, the result $P(A|B)$ will also be nonzero, assuming $P(A)$ and $P(B)$ are also nonzero.

For example, let $P(A)$ be the default probability of company A, which issues a catastrophe bond. Let $P(B)$ be the probability of the catastrophe (for example a tsunami) occurring. $P(A|B)$ is the probability of the company default dependent on the catastrophe occurring. $P(A|B) \geq 0$, since if the catastrophe occurs, the default probability of company A increases, as it may not be able to pay the insurance claims. However, $P(B|A) = 0$, since the probability of the catastrophe occurring does not depended on the default probability of the company. Hence in this case, Bayes' rule cannot be applied (see Burke and Meissner 2011 for the valuation of cat bonds).

Bayesian methods have been successfully applied in numerical financial applications such as financial forecasting, risk modeling, and inferring financial data such as volatilities and derivatives prices; see Gamerman and Lopes (2006), Rachef et al. (2008), Hore et al. (2009), or Quintana et al. (2009). While Bayesian methods are useful in fairly simple, one-parameter estimation problems, the drawback of Bayesian methods lies in their limitations in solving complex statistical problems. However, numerical approximations such as maximum entropy, transformation group analysis, reference analysis, or sequential Monte Carlo simulations based on particle filtering have enhanced Bayesian methods in the past; see Hore et al. (2009) for details.

So far no attempt has been made to use Bayesian methods to model or infer correlations. However, with its flexibility and assisted by advanced numerical methods, Bayesian methods should be able to enhance correlation modeling in the future.

13.3 SUMMARY

In this chapter we discussed new developments in financial modeling, which can be applied to correlation modeling.

One new methodology to solve complex financial problems is to apply **GPU technology**. Graphical processing units (GPUs) have their roots in the computer gaming industry, where gamers receive instant response for their three-dimensional activities at low cost. Not surprisingly, the GPU technology has been adapted to solve complex financial problems such as evaluating the correlation credit risk of large portfolios or the sensitivities of complex derivatives. The GPU technology implies that the financial problems are solved numerically, for example with Monte Carlo simulation without the need of mathematical solvency. Advantages of the GPU technology are speed and the special code, which does not require compiling. One of the disadvantages of the GPU technology is the difficulty to handle nonlinear processes. Some quants are generally apprehensive about solving financial problems numerically without an underlying mathematical algorithm.

Neural networks are by far the most widely applied artificial intelligence concept in finance. Neural networks try to mimic the functioning of the human brain and have therefore the ability to learn. The most popular learning technique is backpropagation. If the neural network output is close to the target value, the network strengthens the algorithm that led to successful outcome by increasing weighting factors. Conversely, if an output is far away from the target value, the network weakens the algorithm. Neural networks have been applied in finance to improve option pricing and in trading to forecast stock prices, option prices, volatilities, and other variables. Limitations of neural networks lie in the facts that the algorithm is hidden and that neural networks can get stuck at local but not general optima.

Fuzzy logic is another interesting field with potential applications for finance. Fuzzy logic alters the traditional concept of truth, in which a statement is either entirely true or entirely false, by introducing the concept of *partial truth*. If the question is "What is the natural number set from 1 and 4, including 1 and 4?" the answer is of course the set {1, 2, 3, 4}. If the answer given is {1, 2, 3}, then this would be incorrect under traditional reasoning. However, fuzzy logic would argue that the answer is at least partially true. Fuzzy logic can handle imprecise information well, but lacks the ability to learn; therefore it is a good idea to combine fuzzy logic with neural networks. In addition, generic algorithms are often introduced to fuzzy neural networks.

Genetic algorithms apply phenomena found in evolution such as selection, crossover, and mutation. Each of these phenomena is expressed as a mathematical algorithm. Selection can be modeled either randomly by sampling from a statistical distribution or deterministically by applying parametric inputs. Crossover produces the next generation by different combinations of parents' organisms. Mutation (i.e., the change in the genetic property) can be achieved by introducing a mutation rate. A high mutation rate parameter means a high rate of mutation from the parents to the next generation and a low parameter means a low rate of mutation.

Several financial studies exist that combine genetic algorithms, fuzzy logic, and neural networks, termed **GFNN** models. Most GFNN models are able to outperform a benchmark index.

A further interesting field with potential applications to finance is **chaos theory**. It was developed in the 1960s and 1970s and is typically applied to weather modeling but has also been applied in physics, biology, economics, and finance.

Chaos theory has several criteria: (1) strong dependence on initial conditions, as shown in the famous butterfly analogy that the flap of a butterfly's wings in Brazil can potentially cause a tornado in a different continent; (2) predictions are possible for only a short period of time; (3) self-similarity (i.e., patterns found when viewing a system from afar can also be found when viewing the system up close); (4) regime changes; and (5) deterministic nature. Investigating whether these criteria apply in finance, we find that the self-similarity principle and the regime changes translate well to finance. However, the high dependence on initial conditions, the short-term but not long-term predictability, and the deterministic nature of chaos theory do not apply in finance. Nevertheless, some companies provide trading models based on chaos theory, applying foremost the self-similarity principle to forecast short-term and long-term stock price patterns.

Bayesian statistics and its redefined probabilities are a further concept with potential application to financial modeling. The Bayesian approach reinterprets and extends the classical probability reasoning. Bayesian logic introduces a *prior probability*, which is the hypothesis before accounting for evidence. This prior probability can be based on personal beliefs. In addition, a *posterior probability* is derived when additional evidence is considered. As a consequence, the probability assigned to a hypothesis can have *partial truth values*, just as in fuzzy logic. In addition, Bayesian logic is a dynamic theory, since the posterior probability is revised if new evidence is found. Bayesian methods have been successfully applied in financial forecasting, risk modeling, and financial inference. No attempts have been made so far to apply Bayesian methods to model financial correlations.

PRACTICE QUESTIONS AND PROBLEMS

1. GPU technology originated in the gaming industry and has been modified to solve complex financial problems. What is the general approach of graphical processing units (GPUs) to solve financial problems?
2. What are the advantages of applying the GPU technology in finance?
3. What are the disadvantages of the GPU technology in finance?
4. What is a general concern when applying GPU technology to finance?
5. Neural networks mimic the human brain and therefore have the ability to learn. How do they learn?
6. What are the limitations of neural networks?
7. Fuzzy logic is cool since it alters the traditional logic of a statement being either true or false. What logic does fuzzy logic apply?
8. Which three main concepts of evolution do genetic algorithms apply? Explain them briefly.
9. A chaotic system has several properties; name four.
10. Which properties of chaos theory translate well to finance, and which do not?
11. Which concept does Bayesian logic share with fuzzy logic?
12. What are prior probabilities and posterior probabilities in the Bayesian theory?

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Glossary

A

accrued interest The accumulated interest of an investment from the last payment date.

American-style option Option that can be exercised at any time before or at the option maturity date.

antithetic variable technique A method to reduce computations when simulating trials (as in the Monte Carlo method) by changing the sign of the random sample.

arbitrage A risk-free profit, achieved by simultaneously buying and selling equivalent securities on different markets. In trading practice, often defined more widely as a strategy which tries to exploit price anomalies.

asset swap A swap based on the fixed rate of an asset. Typically that fixed rate is swapped into LIBOR plus a spread.

asset swap spread Spread over LIBOR that is paid in an asset swap. Reflects the credit quality of the issuer.

association A measure of association is a fairly new statistical term synonymous with measure of dependence (*see also* dependence).

attachment point The number or the amount of defaults necessary to trigger a payoff in a basket default swap or a tranche of a CDO.

autocorrelation The degree of which a variable is correlated to its past values.

B

back-testing Testing how well current value at risk (VaR), credit value at risk (CVaR), or other methods would have performed in the past.

Bank for International Settlements (BIS) An international organization that fosters cooperation among central banks and other agencies in pursuit of monetary and financial stability.

banking book Constitutes the account where a bank's conventional transactions such as loans, bonds, and deposits are recorded. These transactions are typically not marked-to-market (compare trading book).

bankruptcy Refers to a party not honoring its obligations to its creditors and whose assets are therefore administered by a trustee.

barrier option A type of option that can be knocked in or knocked out, depending on whether the underlying asset has reached a predetermined price. A barrier option is cheaper than a standard option.

Basel Committee Committee of the BIS, established in 1975. It functions as a supervisory authority, establishing the regulatory framework for financial institutions.

Basel III A set of guidelines developed after the global financial crisis of 2007 to 2009 to strengthen the global financial system and make it more shock resilient.

basis The difference between the spot price and the futures price of a security. In the credit default swap market, the difference between the credit default swap premium and the asset swap spread.

basis point One hundredth of one percentage point (i.e., 0.01%).

basis risk The risk of the basis changing.

bid The price a buyer is willing to pay for a security.

binomial correlation measure Measures the correlation between two variables with binomial outcomes such as default or no default. It is a limiting case of the Pearson correlation model.

binomial model A model in which the price of a security can move only two (bi) ways, typically up or down.

BIS *See* Bank for International Settlements (BIS).

Black-Scholes-Merton model A mathematical model suggested by Fischer Black and Myron Scholes and separately by Robert Merton in 1973 to find a theoretical price for European-style options on an underlying security that pays no dividends.

bottom-up correlation model A model that derives correlations on an individual level and aggregates them to an overall correlation measure (compare top-down correlation model).

Brady bonds Bonds issued by emerging countries in the early 1990s, which were guaranteed by U.S. Treasury bonds.

C

calibration The process of finding values for the input parameters of a model so that the model's output matches market values.

call option The right but not the obligation to buy an underlying asset at the strike price on a certain date (European style) or during a certain period (American style).

cancelable credit default swap A swap in which one or both parties have the right to terminate the swap.

cap A contract that gives the cap owner the right to pay a fixed interest rate (strike) and receive a LIBOR rate.

capital adequacy Capital requirements set by the Basel Committee for Banking Supervision of the BIS for different types of risk.

capital asset pricing model (CAPM) A model that demonstrates the relationship between risk and return.

cash settlement Type of settlement of derivatives where a cash amount is paid to the profiteer. *See also* physical settlement.

CDO *See* collateralized debt obligation (CDO).

collateral An asset pledged by a debtor as a guarantee for repayment.

concentration risk The risk of financial loss due to a concentrated exposure to a particular group of counterparties.

collateralized debt obligation (CDO) A tranched debt structure in with the credit risk of an underlying portfolio is transferred from the CDO seller (originator) to the CDO investor.

concentration risk The risk of financial loss due to a concentrated exposure to a particular group of counterparties.

conditionally independent default (CID) correlation model A model that does not derive the correlation between variables directly, but indirectly by conditioning on a common (market) factor.

continuously compounded interest rate An interest rate where interest is compounded in infinitesimally short time units (*see also* instantaneous interest rate).

control variate technique A method to reduce computations when simulating trials (as in the Monte Carlo method) applicable for two similar derivatives.

convertible A bond issued by a company that can be converted into shares of that company during the life of the bond.

convertible arbitrage A long position in a convertible security and a short position in the underlying stock.

convexity The second (partial) derivative of a function. Measures the curvature of the function.

copula A function that joins multiple univariate distribution functions to form a single multivariate distribution function.

Cora A measure of how much a dependent variable changes if the correlation between two or more independent variables changes by an infinitesimally small amount.

correlation Used quite broadly in practice, referring to the comovement of assets. Defined more narrowly in statistics, referring to the linear strength of a relationship derived in the Pearson correlation framework.

correlation coefficient A standardized statistical measure in the Pearson correlation framework that measures the strength of a linear relationship. Takes values between -1 and $+1$. Defined as the covariance divided by the product of the standard deviations of the two variables.

correlation desk A term for an area where traders perform correlation trading.

correlation risk The risk of financial loss due to the adverse movement in correlation between two or more variables.

correlation trading The attempt to generate a profit by anticipating the change in the correlation between two or more variables.

counterparty A partner in a financial transaction.

counterparty risk The risk of the counterparty not honoring its obligation.

covariance A statistical measure within the Pearson correlation framework that measures the linear strength between two variables.

covered call writing A short call option position and a long position in the underlying security.

credit correlation risk The risk that credit quality correlations between two or more counterparties change unfavorably.

credit default swap (CDS) A financial product in which the credit risk of an underlying asset is transferred from the CDS buyer to the CDS seller.

credit default swap premium Price of a credit default swap. Also termed credit default swap spread, fee, or fixed rate.
credit derivative A future, swap, or option that transfers credit risk from one counterparty to another.
credit event The ISDA 1999 documentation defines six credit events: bankruptcy, failure to pay, obligation acceleration, obligation default, repudiation/moratorium, and restructuring.
CreditMetrics A transition-matrix-based model developed by JPMorgan to value portfolio credit risk.
credit rating An assessment of the credit quality of a debtor, expressed in categories from AAA to D.
credit risk The risk of a financial loss due to an adverse change in the credit quality of a debtor. Consists of credit migration risk and default risk.
Credit Risk$^+$ An actuarially based model developed by Credit Suisse Financial Products to value portfolio credit risk.
credit risk premium *See* credit spread.
credit spread Also referred to as credit risk premium. The excess in yield of a security with credit risk over a comparable security without credit risk.
credit triangle An approximate relationship of the credit default swap premium s, the default probability λ, and the recovery rate R (see equation 5.6).
credit value adjustment (CVA) An adjustment to address counterparty credit risk. Often applied to derivative transactions.
credit value at risk (CVaR) The maximum loss of a portfolio due to credit risk with a certain probability for a certain time frame. Also called credit at risk or worst-case default rate (WCDR).
CVaR *See* credit value at risk (CVaR).

D

debt value adjustment (DVA) Allows an entity to adjust the value of its portfolio by taking its own default probability into consideration.
default Occurs when a party has not honored its legal obligation to its creditors.
default correlation A measure of joint default probability of two or more entities within a short time frame.
default intensity The probability of default for a short period of time conditional on no earlier default. Identical with hazard rate.
default probability The likelihood that a debt instrument or counterparty will default within a certain time. *See also* default intensity.
default risk The risk that a debtor may be unable to honor its financial obligation.
delta The change in the value of a derivative for an infinitesimally small change in the price (or rate) of the underlying security.
dependence Two events are statistically dependent if the occurrence of one affects the outcome of another.
derivative A security whose value is at least in part derived by the price of an underlying asset.

discount factor The number that a cash flow occurring at a future date is multiplied by, to bring it to its present value.

discount rate The interest rate that is used in the discount factor.

distance to default A term derived by Moody's KMV displaying the difference between the value of assets and the value of liabilities at a certain future point in time. Mathematically identical with the risk-neutral d_2 in the Merton model.

drift rate The average change of a variable in a stochastic process.

duration A measure of the relative change in the value of a bond with respect to a change in its yield to maturity. Also measures the average time that an investor has to wait to get the investment back.

E

economic capital Capital to protect against loss; often measured by value at risk.

efficient market hypothesis A hypothesis that asset prices include all relevant information. Past asset price patterns are irrelevant.

European-style option An option that can be exercised only on the maturity date.

excess yield The difference between the yield of a risky bond and the yield of a risk-free bond.

exchange option An option to exchange one asset for another.

exotic option An option whose payoff, evaluation, and hedging are different, typically more complex than those of standard options.

expected default frequency (EDF) A term from Moody's KMV's model for the probability of default. Real-world representation of the risk-neutral $N(-d_2)$ in the Merton model.

F

finite difference method A method to solve differential equations by transferring the differential equations into difference equations and solving these iteratively.

first passage time model A type of structural model. In first passage time models, bankruptcy occurs when the asset value drops below a predefined, usually exogenous barrier, allowing for bankruptcy before the maturity of the debt.

floating rate An interest rate that periodically changes according a certain reference rate such as LIBOR.

floor Opposite of a cap. A contract that gives the floor owner the right to receive a fixed interest rate (strike) and pay a LIBOR rate.

forward A transaction in which the price is fixed today, but settlement takes place at a future date.

funded transaction A transaction in which the buyer pays an up-front premium to buy a security (compare unfunded transaction).

funding value adjustment (FVA) An adjustment to the price of a transaction due to the cost of funding for the transaction or the related hedge.

future A standardized forward that trades on an exchange. Standardized are the notional, price, maturity quality, deliverability, type of settlement, trading hours, and so forth.

G

gamma Second partial derivative of the option function with respect to the underlying price. A measure for the curvature of the option function. Gamma is the change in the delta of an option for an infinitesimally small change in the price of the underlying.

generalized Wiener process A process in which a variable has a constant, expected growth rate. Superimposed on this growth rate is a stochastic volatility term.

general wrong-way risk (WWR) Exists when the probability of default of counterparties is positively correlated with general market risk factors (BIS definition).

geometric Brownian motion A process in which the relative change of a variable follows a generalized Wiener process (see equation 4.1).

Gora Second partial derivative of a function with respect to correlation. A measure of how much Cora changes if the correlation between two or more independent variables changes by an infinitesimally small amount.

H

haircut A discount to the value of securities held as collateral, reflecting the price uncertainty of the security.

hazard rate *See* default intensity.

hedging Reducing risk. More precisely, entering into a second trade to reduce the risk of an original trade.

Heston 1993 model One of the most rigorous and useful correlation models for finance. Correlates the Brownian motions of two or more variables.

I

implied volatility Volatility that is implied by observed option prices.

in arrears Refers to a later date at which a payment is made.

instantaneous interest rate An interest rate that is applied to an infinitesimally short period of time (*see also* continuously compounded interest rate).

interest rate swap An exchange of interest rate payments on a predetermined notional amount and in reference to predetermined interest rate indexes.

intrinsic value The payoff when the option is exercised. For a call, the intrinsic value is the maximum of the spot price minus the strike price and zero. For a put, the intrinsic value is the maximum of the strike price minus the spot price and zero.

investment grade bond A bond with a rating of BBB or higher.

J

junk bond A high yield bond with a rating lower than BBB.

K

kurtosis Fourth moment of a distribution; a measure of the fatness of the tails of the distribution.

L

LIBOR market model (LMM) A term structure model in which interest rates are conveniently expressed as discrete forward rates.

liquidity premium A premium that lowers an asset price due to asset illiquidity.

lognormal distribution A distribution with a fat right tail. A variable follows a lognormal distribution if the logarithm of the variable is normal. Often applied for stock price behavior, as in the Black-Scholes-Merton model.

London Interbank Offered Rate (LIBOR) An interest rate paid by highly rated borrowers; fixed daily in London.

long position A trading position that generates a profit if the underlying instrument increases in price (opposite of a short position).

M

market price of risk *See* Sharpe ratio.

market risk The risk of financial loss due to an unfavorable change in the price of a financial security.

marking to market The adjustment of a transaction price or an account value to reflect profits and losses.

Markov process A stochastic memoryless process. Hence only present information, not past information, is relevant.

martingale process A stochastic process with a zero drift rate. Hence the expected future value of a variable is the current value.

maturity The date on which a transaction or a financial instrument is due to end.

mean reversion The tendency for a price or a rate to revert back toward its long-term mean.

migration probability The probability of a firm's credit rating moving to another rating state.

migration risk The risk that the credit rating changes unfavorably.

Monte Carlo simulation A technique for approximating the price of a derivative by randomly sampling the evolution of the underlying security.

N

netting Offsetting contracts with positive and negative values with another counterparty.

normal distribution Also called Gaussian distribution or bell curve. A standard, popular probability distribution forming a symmetrical curve. Suffers from the inability to replicate fat tails found in practice.

notional amount Also called principal amount. Dollar amount of a security or transaction.

numeraire The price of a security in which other securities are measured.

numerical finance Attempts to solve financial problems with numerical methods (such as Monte Carlo simulation), without the need for mathematical solvency.

O

off balance sheet Refers to a transaction that does not have to be included on the balance sheet of the party concerned.

one-factor Gaussian copula (OFGC) A shortcut of the standard copula function. Variables are not correlated directly, but indirectly by conditioning on a common (market) factor.

operational risk The risk of direct or indirect loss resulting from inadequate or failed internal processes, people, and systems or from external events (BIS definition).

over the counter (OTC) Refers to a transaction dealt directly between counterparties, hence not on an exchange.

P

Pearson correlation model Measures the strength of a linear dependence. It is the most popular correlation model in statistics and is also widely applied in finance but suffers from a variety of problems (see Chapter 3, section 3.2.1).

percentile Value of a distribution under which the percentile value falls. For example, the 95th percentile is the value under which 95% of values in the distribution are found.

physical settlement Type of settlement of derivatives in which physical delivery and payment of the underlying asset take place (*see also* cash settlement).

premium The price of a financial transaction (*see also* credit spread).

present value Current value of discounted future cash flows.

principal component analysis A method of trying to find the critical factors (components) that explain the variation of a large number of possibly correlated variables.

put option The right but not the obligation to sell an underlying asset at the strike price on a certain date (European style) or during a certain period (American style).

Q

quantile An integer, indicating certain essentially equally sized intervals of a cumulative distribution function. For example, the 2-quantile is the median, and the 100-quantiles are the percentiles (*see also* percentile).

quanto option An option that allows buyers to exchange their payoff in a foreign currency into their home currency at a fixed exchange rate.

R

random variable A variable that can take a set of different values, each associated with a probability.

random walk A term expressing the process of a random variable. The randomness is often generated by a drawing from a standard normal distribution.

recovery rate The percentage of the notional amount that a creditor receives in case of default.

reduced form model A type of model that does not include the asset-liability structure of the firm to generate default probabilities. Rather, reduced form models use bonds and credit default swaps (CDSs) as main inputs to model the bankruptcy process.

reference obligation The obligation that, if in default, triggers the default swap payment.

repo Repurchase agreement. A securitized loan; in a repo, a security is sold with a guarantee that it will repurchased at a later date at a fixed, typically higher, price.

risk averse An attitude toward risk that causes an investor to prefer an investment with a certain expected return to an investment with the same expected return but greater uncertainty.

risk-free rate An interest rate that can be achieved without risk. Typically the interest rate for securities issued by an AAA-rated government or firm.

risk management The process of identifying, quantifying, and, if desired, reducing risk.

risk-neutral An attitude toward risk that leads an investor to be indifferent between investment A with a certain expected return and investment B with the same expected return but higher uncertainty.

S

Sharpe ratio Also termed market price of risk; return of a risky asset minus the return of the risk-free asset, divided by the standard deviation of the risky asset.

short position A trading position that generates a profit if the underlying instrument decreases in price (opposite of long position).

short selling Selling a security that is borrowed, in anticipation of a decline of that security.

short squeeze A term for traders buying a security to increase the price since they know the security has to be bought back by (short) sellers.

skewness Third moment of a distribution function. Measure of the asymmetry of a distribution.

smile effect A term referring to the higher implied volatilities of out-of-the-money options and in-the-money options compared to at-the-money options.

special purpose corporation (SPC) *See* special purpose vehicle (SPV).

special purpose entity (SPE) *See* special purpose vehicle (SPV).

special purpose vehicle (SPV) Legal entity, separate from the parent entity, typically highly rated. Often functions as an intermediary in structured financial transactions.

specific wrong-way risk (WWR) Exists when the exposure to a specific counterparty is positively correlated with the counterparty's probability of default due to the nature of the transaction with the counterparty (BIS definition).

spot price The price of a security for immediate (in practice often two days) delivery.

stochastic process A process with an unknown outcome.

stress testing Testing how a portfolio behaves under extreme market movements.

strike price For a call option, the price at which the underlying security may be bought; for a put option, the price at which the underlying security may be sold.

structural model A type of model that derives the probability of default by analyzing the capital structure of a firm, especially the value of the firm's assets compared to the value of the firm's debt.

swap The agreement between two parties to exchange a series of cash flows.

swaption Also called swap option; an option on a swap. A payer swaption allows the owner to pay a fixed swap rate and to receive a floating rate. A receiver swaption allows the owner to receive a fixed swap rate and to pay a floating rate.

synthetic structure A financial structure in which exposure is assumed synthetically. For example, in a synthetic CDO, the SPV assumes credit risk by selling CDSs.

systematic risk Also called market risk or common risk. Risk associated with the movement of a market or market segment as opposed to risk associated with a specific security. Systematic risk cannot be diversified away.

systemic risk The risk of a financial market or an entire financial system collapsing.

T

term structure model A stochastic, binomial, or multinomial discrete or continuous model, generating the process of short-term interest rates.

theta The change in price of a derivative for an infinitesimally small change in time.

time value The portion of an option's premium that is attributed to uncertainty. Time value equals the option price minus the intrinsic value.

top-down correlation model A model that abstracts from individual correlations, but rather models correlations on an aggregate level (compare bottom-up correlation model).

trading book Comprises instruments that are explicitly held with trading intent or in order to hedge other positions (compare banking book).

tranches Segments of deals or structures, typically with different risk levels.

transition matrix A matrix showing the probability of a firm moving to other rating categories within a certain time frame.

U

underlying The security that a derivative is based on and that at least in part determines the price of the derivative.

unexpected loss A loss amount exceeding value at risk (VaR) or credit value at risk (CVaR).

unfunded transaction A transaction in which the buyer does not pay an up-front premium as in a swap or a futures contract.

unsystematic risk Also called idiosyncratic risk or specific risk. Risk that can be largely eliminated by diversification.

V

value at risk (VaR) The maximum loss in a certain time frame, with a certain probability, due to a certain type of risk.

vega First partial derivative of the option function with respect to implied volatility. A measurement of the sensitivity of the value of an option to changes in implied volatility.

volatility The standard deviation of percentage price changes (returns); see Chapter 1, section 1.3.1.

vulnerable option An option whose price includes the possibility of default of the option seller.

W

Wiener process A process in which the movement of a variable for a certain time interval is determined by a random drawing from a standard normal distribution, multiplied by the square root of the time interval.

wrong-way risk (WWR) Two types exist: General wrong-way risk exists when the probability of default of counterparties is positively correlated with general market risk factors. Specific wrong-way risk exists when the exposure to a specific counterparty is positively correlated with the counterparty's probability of default due to the nature of the transaction with the counterparty.

Y

yield curve Shows the relationship between yields and their maturities.

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