Symbolic Math Toolbox

For Use with MATLAB®

Computation

Visualization

Programming

The MATH WORKS Inc.

User's Guide

Version 2

How to Contact The MathWorks:

508-647-7000	Phone
508-647-7001	Fax
The MathWorks, Inc. 24 Prime Park Way Natick, MA 01760-1500	Mail
`http://www.mathworks.com`	Web
`ftp.mathworks.com`	Anonymous FTP server
`comp.soft-sys.matlab`	Newsgroup
`support@mathworks.com`	Technical support
`suggest@mathworks.com`	Product enhancement suggestions
`bugs@mathworks.com`	Bug reports
`doc@mathworks.com`	Documentation error reports
`subscribe@mathworks.com`	Subscribing user registration
`service@mathworks.com`	Order status, license renewals, passcodes
`info@mathworks.com`	Sales, pricing, and general information

Symbolic Math Toolbox User's Guide

Printing History:	August 1993	First printing
	October 1994	Second printing
	May 1997	Third printing for Symbolic Math Toolbox 2.0
	September 1998	Updated for Release 11 (online only)

Contents

1 Tutorial

2 Reference

A Compatibility Guide

1

Tutorial

Introduction

The Symbolic Math Toolboxes incorporate symbolic computation into MATLAB®'s numeric environment. These toolboxes supplement MATLAB's numeric and graphical facilities with several other types of mathematical computation:

Facility	Covers
Calculus	Differentiation, integration, limits, summation, and Taylor series
Linear Algebra	Inverses, determinants, eigenvalues, singular value decomposition, and canonical forms of symbolic matrices
Simplification	Methods of simplifying algebraic expressions
Solution of Equations	Symbolic and numerical solutions to algebraic and differential equations
Variable-Precision Arithmetic	Numerical evaluation of mathematical expressions to any specified accuracy
Transforms	Fourier, Laplace, *z*-transform, and corresponding inverse transforms
Special Mathematical Functions	Special functions of classical applied mathematics

The computational engine underlying the toolboxes is the kernel of Maple®, a system developed primarily at the University of Waterloo, Canada, and, more recently, at the Eidgenössiche Technische Hochschule, Zürich, Switzerland. Maple is marketed and supported by Waterloo Maple, Inc.

These versions of the Symbolic Math Toolboxes are designed to work with MATLAB 5 and Maple V Release 4.

There are two toolboxes. The basic Symbolic Math Toolbox is a collection of more than one-hundred MATLAB functions that provide access to the Maple kernel using a syntax and style that is a natural extension of the MATLAB language. The basic toolbox also allows you to access functions in Maple's linear algebra package. The Extended Symbolic Math Toolbox augments this functionality to include access to all nongraphics Maple packages, Maple programming features, and user-defined procedures. With both toolboxes, you can write your own M-files to access Maple functions and the Maple workspace.

The following sections of this Tutorial provide explanation and examples on how to use the toolboxes.

Section	Covers
"Getting Help"	How to get online help for Symbolic Math Toolbox functions
"Getting Started"	Basic symbolic math operations
"Calculus"	How to differentiate and integrate symbolic expressions
"Simplifications and Substitutions"	How to simplify and substitute values into expressions
"Variable-Precision Arithmetic"	How to control the precision of computations
"Linear Algebra"	Examples using the toolbox functions
"Solving Equations"	How to solve symbolic equations
"Integral Transforms"	Fourier, Laplace, and *z*-transforms
"Special Mathematical Functions"	How to access Maple's special math functions
"Using Maple Functions"	How to use Maple's help, debugging, and user-defined procedure functions
"Extended Symbolic Math Toolbox"	Features of the Extended Symbolic Math Toolbox

Chapter 2, "Reference," provides detailed descriptions of each of the functions in the toolboxes.

Getting Help

There are two ways to find information on using Symbolic Math Toolbox functions. One, of course, is to read this manual! The other is to use MATLAB's command line help system. Generally, you can obtain help on MATLAB functions simply by typing

```
help function
```

where *function* is the name of the MATLAB function for which you need help. This is not sufficient, however, for some Symbolic Math Toolbox functions. The reason? The Symbolic Math Toolbox "overloads" many of MATLAB's numeric functions. That is, it provides symbolic-specific implementations of the functions, using the same function name. To obtain help for the symbolic version of an overloaded function, type

```
help sym/function
```

where *function* is the overloaded function's name. For example, to obtain help on the symbolic version of the overloaded function, `diff`, type

```
help sym/diff
```

To obtain information on the numeric version, on the other hand, simply type

```
help diff
```

How can you tell whether a function is overloaded? The help for the numeric version tells you so. For example, the help for the `diff` function contains the section

```
Overloaded methods
    help char/diff.m
    help sym/diff.m
```

This tells you that there are two other `diff` commands that operate on expressions of class `char` and class `sym`, respectively. See the next section for information on class `sym`. For more information on overloaded commands, see Chapter 14 of the *Using MATLAB* guide.

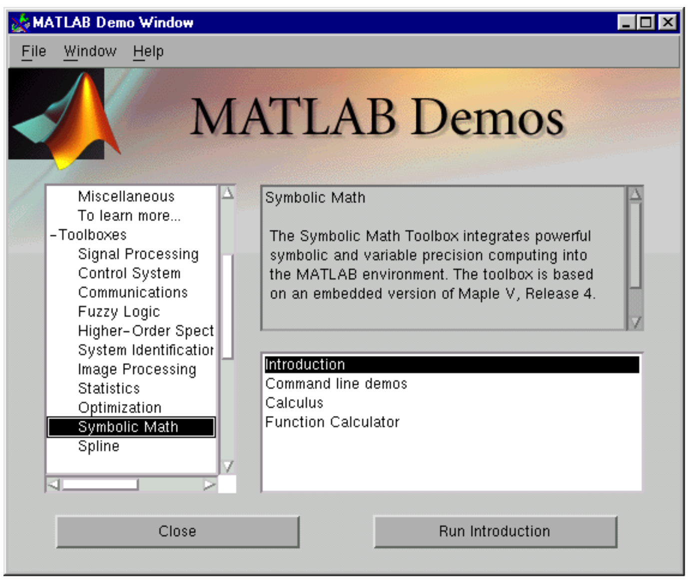
You can use the `mhelp` command to obtain help on Maple commands. For example, to obtain help on the Maple `diff` command, type `mhelp diff`. This returns the help page for the Maple `diff` function. For more information on the `mhelp` command, type `help mhelp` or read the section "Using Maple Functions" in this Tutorial.

Getting Started

This section describes how to create and use symbolic objects. It also describes the default symbolic variable. If you are familiar with version 1 of the Symbolic Math Toolbox, please note that version 2 uses substantially different and simpler syntax.

(If you already have a copy of the Maple V Release 4 library, please see the reference page for `mapleinit` before proceeding.)

To get a quick online introduction to the Symbolic Math Toolbox, type `demos` at the MATLAB command line. MATLAB displays the **MATLAB Demos** dialog box. Select **Symbolic Math** (in the left list box) and then **Introduction** (in the right list box).



Symbolic Objects

The Symbolic Math Toolbox defines a new MATLAB data type called a symbolic object or `sym` (see Chapter 14 in *Using MATLAB* for an introduction to MATLAB classes and objects). Internally, a symbolic object is a data structure that stores a string representation of the symbol. The Symbolic Math Toolbox uses symbolic objects to represent symbolic variables, expressions, and matrices.

Creating Symbolic Variables and Expressions

The `sym` command lets you construct symbolic variables and expressions. For example, the commands

```
x = sym('x')
a = sym('alpha')
```

create a symbolic variable `x` that prints as `x` and a symbolic variable `a` that prints as `alpha`.

Suppose you want to use a symbolic variable to represent the golden ratio

$$\rho = \frac{1 + \sqrt{5}}{2}$$

The command

```
rho = sym('(1 + sqrt(5))/2')
```

achieves this goal. Now you can perform various mathematical operations on `rho`. For example

```
f = rho^2 - rho - 1
f =

(1/2+1/2*5^(1/2))^2-3/2-1/2*5^(1/2)
simplify(f)
```

returns

```
0
```

Now suppose you want to study the quadratic function $f = ax^2 + bx + c$. The statement

```
f = sym('a*x^2 + b*x + c')
```

assigns the symbolic expression $ax^2 + bx + c$ to the variable `f`. Observe that in this case, the Symbolic Math Toolbox does not create variables corresponding to the terms of the expression, *a*, *b*, *c*, and *x*. To perform symbolic math operations (e.g., integration, differentiation, substitution, etc.) on `f`, you need to create the variables explicitly. You can do this by typing

```
a = sym('a')
b = sym('b')
c = sym('c')
x = sym('x')
```

or simply

```
syms a b c x
```

In general, you can use `sym` or `syms` to create symbolic variables. We recommend you use `syms` because it requires less typing.

Symbolic and Numeric Conversions

Let's return to the golden ratio

$$\rho = \frac{1 + \sqrt{5}}{2}$$

The `sym` function has four options for returning a symbolic representation of the numeric value `rho = (1 + sqrt(5)/2)`. The `'f'` option

```
sym(rho,'f')
```

returns a symbolic floating-point representation

```
'1.9e3779b97f4a8'*2^(0)
```

The `'r'` option

```
sym(rho,'r')
```

returns the rational form

```
7286977268806824*2^(-52)
```

This is the default setting for sym. That is, calling sym without a second argument is the same as using sym with the 'r' option.

```
sym(rho)

ans =
7286977268806824*2^(-52)

sym(.25)

ans =
1/4
```

The third option 'e' returns the rational form of rho plus the difference between the theoretical rational expression for rho and its actual (machine) floating-point value in terms of eps (the floating-point relative accuracy)

```
sym(rho,'e')

ans =
7286977268806824*2^(-52)
```

In this case, the theoretical and actual floating-point values for rho are the same. For 1/3, however, we obtain

```
sym(1/3,'e')

ans =
1/3-eps/12
```

The fourth option 'd' returns the decimal expansion of rho up to the number of significant digits specified by digits.

```
sym(rho,'d')

ans =
1.6180339887498949025257388711907
```

The default value of `digits` is 32 (hence, `sym(rho,'d')` returns a number with 32 significant digits), but if you prefer a shorter representation, use the `digits` command as follows.

```
digits(7)
sym(rho,'d')

ans =
1.618034
```

A particularly effective use of `sym` is to convert a matrix from numeric to symbolic form. The command

```
A = hilb(3)
```

generates the 3-by-3 Hilbert matrix.

```
A =

    1.0000    0.5000    0.3333
    0.5000    0.3333    0.2500
    0.3333    0.2500    0.2000
```

By applying `sym` to `A`

```
A = sym(A)
```

you can obtain the (infinitely precise) symbolic form of the 3-by-3 Hilbert matrix.

```
A =

[   1, 1/2, 1/3]
[ 1/2, 1/3, 1/4]
[ 1/3, 1/4, 1/5]
```

Constructing Real and Complex Variables

The `sym` command allows you to specify the mathematical properties of symbolic variables by using the `'real'` option. That is, the statements

```
x = sym('x','real'); y = sym('y','real');
```

or more efficiently

```
syms x y real
z = x + i*y
```

create symbolic variables x and y that have the added mathematical property of being real variables. Specifically this means that the expression

```
f = x^2 + y^2
```

is strictly nonnegative. Hence, z is a (formal) complex variable and can be manipulated as such. Thus, the commands

```
conj(x), conj(z), expand(z*conj(z))
```

return the complex conjugates of the variables

```
x, x – i*y, x^2 + y^2
```

The conj command is the complex conjugate operator for the toolbox. If conj(x) == x returns 1, then x is a real variable.

To clear x of its “real” property, you must type

```
syms x unreal
```

or

```
x = sym('x','unreal')
```

The command

```
clear x
```

does *not* make x a nonreal variable.

Creating Abstract Functions

If you want to create an abstract (i.e., indeterminant) function *f(x)*, type

```
f = sym('f(x)')
```

Then f acts like *f(x)* and can be manipulated by the toolbox commands. To construct the first difference ratio, for example, type

```
df = (subs(f,'x','x+h') – f)/'h'
```

or

```
syms x h
df = (subs(f,x,x+h)–f)/h
```

which returns

```
df =
(f(x+h)-f(x))/h
```

This application of sym is useful when computing Fourier, Laplace, and z-transforms (see the section “Integral Transforms”).

Using sym to Access Maple Functions

Similarly, you can access Maple’s factorial function k!, using sym.

```
kfac = sym('k!')
```

To compute 6! or n!, type

```
syms k n
subs(kfac,k,6), subs(kfac,k,n)

ans =
720

ans =
n!
```

Or, if you want to compute, for example, 12!, simply use the prod function

```
prod(1:12)
```

Example: Creating a Symbolic Matrix

A circulant matrix has the property that each row is obtained from the previous one by cyclically permuting the entries one step forward. We create the circulant matrix A whose elements are a, b, and c, using the commands

```
syms a b c
A = [a b c; b c a; c a b]
```

which return

```
A =
[ a, b, c ]
[ b, c, a ]
[ c, a, b ]
```

Since A is circulant, the sum over each row and column is the same. Let's check this for the first row and second column. The command

```
sum(A(1,:))
```

returns

```
ans =
a+b+c
```

The command

```
sum(A(1,:)) == sum(A(:,2)) % This is a logical test.
```

returns

```
ans =
     1
```

Now replace the (2,3) entry of A with beta and the variable b with alpha. The commands

```
syms alpha beta;
A(2,3) = beta;
A = subs(A,b,alpha)
```

return

```
A =
[     a, alpha,     c]
[ alpha,     c,  beta]
[     c,     a, alpha]
```

From this example, you can see that using symbolic objects is very similar to using regular MATLAB numeric objects.

The Default Symbolic Variable

When manipulating mathematical functions, the choice of the independent variable is often clear from context. For example, consider the expressions in the table below.

Mathematical Function	MATLAB Command
$f = x^n$	`f = x^n`
$g = \sin(at+b)$	`g = sin(a*t + b)`
$h = J_v(z)$	`h = besselj(nu,z)`

If we ask for the derivatives of these expressions, without specifying the independent variable, then by mathematical convention we obtain $f' = nx^n$, $g' = a\cos(at+b)$, and $h' = J_v(z)(v/z) - J_{v+1}(z)$. Let's assume that the independent variables in these three expressions are x, t, and z, respectively. The other symbols, n, a, b, and v, are usually regarded as "constants" or "parameters." If, however, we wanted to differentiate the first expression with respect to n, for example, we could write

$$\frac{d}{dn}f(x) \text{ or } \frac{d}{dn}x^n$$

to get $x^n \ln x$.

By mathematical convention, independent variables are often lower-case letters found near the end of the Latin alphabet (e.g., x, y, or z). This is the idea behind `findsym`, a utility function in the toolbox used to determine default symbolic variables. Default symbolic variables are utilized by the calculus, simplification, equation-solving, and transform functions. To apply this utility to the example discussed above, type

```
syms a b n nu t x z
f = x^n; g = sin(a*t + b); h = besselj(nu,z);
```

This creates the symbolic expressions `f`, `g`, and `h` to match the example. To differentiate these expressions, we use `diff`.

```
diff(f)
```

returns

```
ans =
x^n*n/x
```

See the section “Differentiation” for a more detailed discussion of differentiation and the `diff` command.

Here, as above, we did not specify the variable with respect to differentiation. How did the toolbox determine that we wanted to differentiate with respect to `x`? The answer is the `findsym` command.

```
findsym(f,1)
```

which returns

```
ans =
x
```

Similarly, `findsym(g,1)` and `findsym(h,1)` return `t` and `z`, respectively. Here the second argument of `findsym` denotes the number of symbolic variables we want to find in the symbolic object `f`, using the `findsym` rule (see below). The absence of a second argument in `findsym` results in a list of all symbolic variables in a given symbolic expression. We see this demonstrated below. The command

```
findsym(g)
```

returns the result

```
ans =
a, b, t
```

---

findsym Rule: The default symbolic variable in a symbolic expression is the letter that is closest to `'x'` alphabetically. If there are two equally close, the letter later in the alphabet is chosen.

Here are some examples:

Expression	Variable Returned By findsym
x^n	x
sin(a*t+b)	t
besselj(nu,z)	z
w*y + v*z	y
exp(i*theta)	theta
log(alpha*x1)	x1
y*(4+3*i) + 6*j	y
sqrt(pi*alpha)	alpha

Creating Symbolic Math Functions

There are two ways to create functions:

- Use symbolic expressions
- Create an M-file

Using Symbolic Expressions

The sequence of commands

```
syms x y z
r = sqrt(x^2 + y^2 + z^2)
t = atan(y/x)
f = sin(x*y)/(x*y)
```

generates the symbolic expressions `r`, `t`, and `f`. You can use `diff`, `int`, `subs`, and other Symbolic Math Toolbox functions to manipulate such expressions.

Creating an M-File

M-files permit a more general use of functions. Suppose, for example, you want to create the `sinc` function `sin(x)/x`. To do this, create an M-file in the `@sym` directory.

```
function z = sinc(x)
%SINC The symbolic sinc function
%     sin(x)/x. This function
%     accepts a sym as the input argument.
if is equal(x,sym(0))
   z = 1;
else
   z = sin(x)/x;
end
```

You can extend such examples to functions of several variables. For a more detailed discussion on object-oriented programming, see Chapter 14 of the *Using MATLAB* guide.

Calculus

The Symbolic Math Toolboxes provide functions to do the basic operations of calculus; differentiation, limits, integration, summation, and Taylor series expansion. The following sections outline these functions.

Differentiation

Let's create a symbolic expression.

```
syms a x
f = sin(a*x)
```

Then

```
df = diff(f)
```

differentiates `f` with respect to its symbolic variable (in this case `x`), as determined by `findsym`.

```
df =
cos(a*x)*a
```

To differentiate with respect to the variable `a`, type

```
dfa = diff(f,a)
```

which returns df/da

```
dfa=
cos(a*x)*x
```

Mathematical Function	MATLAB Command
$f = x^n$ $f' = nx^{n-1}$	`f = x^n` `diff(f)` or `diff(f,x)`
$g = acos(at+b)$ $g' = acos(at+b)$	`g = acos(a*t + b)` `diff(g)` or `diff(g,t)`
$h = J_v(z)$ $h' = J_v(z)(v/z) - J_{v+1}(z)$	`h = besselj(nu, z)` `diff(h)` or `diff(h,z)`

To calculate the second derivatives with respect to x and a, respectively, type

```
diff(f,2) % or diff(f,x,2)
```

which returns

```
ans =
-sin(a*x)*a^2
```

and

```
diff(f,a,2)
```

which returns

```
ans =
-sin(a*x)*x^2
```

Define a, b, x, n, t, and theta in the MATLAB workspace, using the sym command. The table below illustrates the diff command.

f	diff(f)
x^n	x^n*n/x
sin(a*t+b)	a*cos(a*t+b)
exp(i*theta)	i*exp(i*theta)

To differentiate the Bessel function of the first kind, besselj(nu,z), with respect to z, type

```
syms nu z
b = besselj(nu,z);
db = diff(b)
```

which returns

```
db =
-besselj(nu+1,z)+nu/z*besselj(nu,z)
```

The `diff` function can also take a symbolic matrix as its input. In this case, the differentiation is done element-by-element. Consider the example

```
syms a x
A = [cos(a*x),sin(a*x);-sin(a*x),cos(a*x)]
```

which returns

```
A =
[  cos(a*x),  sin(a*x)]
[ -sin(a*x),  cos(a*x)]
```

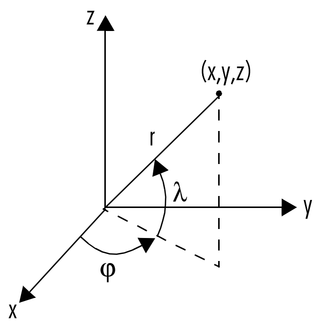
The command

```
dy = diff(A)
```

returns

```
dy =
[ -sin(a*x)*a,  cos(a*x)*a]
[ -cos(a*x)*a, -sin(a*x)*a]
```

You can also perform differentiation of a column vector with respect to a row vector. Consider the transformation from Euclidean (x, y, z) to spherical (r, λ, φ) coordinates as given by $x = r \cos \lambda \cos \varphi$, $y = r \cos \lambda \sin \varphi$, and $z = r \sin \lambda$. Note that λ corresponds to elevation or latitude while φ denotes azimuth or longitude.



To calculate the Jacobian matrix, J, of this transformation, use the `jacobian` function. The mathematical notation for J is

$$J = \frac{\partial(x, y, x)}{\partial(r, \lambda, \varphi)}$$

For the purposes of toolbox syntax, we use `l` for λ and `f` for φ. The commands

```
syms r l f
x = r*cos(l)*cos(f); y = r*cos(l)*sin(f); z = r*sin(l);
J = jacobian([x; y; z], [r l f])
```

return the Jacobian

```
J =

[    cos(l)*cos(f), -r*sin(l)*cos(f), -r*cos(l)*sin(f)]
[    cos(l)*sin(f), -r*sin(l)*sin(f),  r*cos(l)*cos(f)]
[           sin(l),         r*cos(l),                0]
```

and the command

```
detJ = simple(det(J))
```

returns

```
detJ =
-cos(l)*r^2
```

Notice that the first argument of the `jacobian` function must be a column vector and the second argument a row vector. Moreover, since the determinant of the Jacobian is a rather complicated trigonometric expression, we used the `simple` command to make trigonometric substitutions and reductions (simplifications). The section “Simplifications and Substitutions” discusses simplification in more detail.

A table summarizing `diff` and `jacobian` follows.

Mathematical Operator	MATLAB Command
$f(x) = \exp(ax + b)$	`syms a b x` `f = exp(a*x + b)`
$\frac{df}{dx}$	`diff(x)` or `diff(f,x)`
$\frac{df}{da}$	`diff(f,a)`
$\frac{d^2 f}{db^2}$	`diff(f,b,2)`

$r = u^2 + v^2$
$t = arctan(v/u)$
	`syms r t u v`
	`r = u^2 + v^2`
	`t = atan(v/u)`

$J = \frac{\partial(r, t)}{\partial(u, v)}$	`J = jacobian([r:t],[u,v])`

Limits

The fundamental idea in calculus is to make calculations on functions as a variable "gets close to" or approaches a certain value. Recall that the definition of the derivative is given by a limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists. The Symbolic Math Toolbox allows you to compute the limits of functions in a direct manner. The commands

```
syms h n x
dc = limit( (cos(x+h) – cos(x))/h,h,0 )
```

which return

```
dc =
-sin(x)
```

and

```
limit( (1 + x/n)^n,n,inf )
```

which returns

```
ans =
exp(x)
```

illustrate two of the most important limits in mathematics: the derivative (in this case of cos x) and the exponential function. While many limits

$$\lim_{x \to a} f(x)$$

are “two sided” (that is, the result is the same whether the approach is from the right or left of a), limits at the singularities of $f(x)$ are not. Hence, the three limits,

$$\lim_{x \to 0} \frac{1}{x}, \quad \lim_{x \to 0^-} \frac{1}{x}, \text{ and } \lim_{x \to 0^+} \frac{1}{x}$$

yield the three distinct results: undefined, $-\infty$, and $+\infty$, respectively.

In the case of undefined limits, the Symbolic Math Toolbox returns `NaN` (not a number). The command

```
limit(1/x,x,0) % Equivalently, limit(1/x)
```

returns

```
ans =
NaN
```

The command

```
limit(1/x,x,0,'left')
```

returns

```
ans =
-inf
```

while the command

```
limit(1/x,x,0,'right')
```

returns

```
ans =
inf
```

Observe that the default case, `limit(f)` is the same as `limit(f,x,0)`. Explore the options for the `limit` command in this table. Here, we assume that `f` is a function of the symbolic object `x`.

Mathematical Operation	MATLAB Command
$\lim_{x \to 0} f(x)$	`limit(f)`
$\lim_{x \to a} f(x)$	`limit(f,x,a)` or `limit(f,a)`
$\lim_{x \to a^-} f(x)$	`limit(f,x,a,'left')`
$\lim_{x \to a^+} f(x)$	`limit(f,x,a,'right')`

Integration

If `f` is a symbolic expression, then

```
int(f)
```

attempts to find another symbolic expression, `F`, so that `diff(F) = f`. That is, `int(f)` returns the indefinite integral or antiderivative of `f` (provided one exists in closed form). Similar to differentiation,

```
int(f,v)
```

uses the symbolic object `v` as the variable of integration, rather than the variable determined by `findsym`. See how `int` works by looking at this table.

Mathematical Operation	MATLAB Command
$\int x^n dx = \frac{x^{n+1}}{n+1}$	`int(x^n)` or `int(x^n,x)`
$\int_0^{\pi/2} \sin(2x)dx = 1$	`int(sin(2*x),0,pi/2)` or `int(sin(2*x),x,0,pi/2)`

$g = \cos(at+b)$
$\int g(t)dt = \sin(at+b)/a$
	`g = cos(a*t + b)`
	`int(g)` or
	`int(g,t)`

$\int J_1(z)dz = -J_0(z)$	`int(besselj(1,z))` or `int(besselj(1,z),z)`

In contrast to differentiation, symbolic integration is a more complicated task. A number of difficulties can arise in computing the integral. The antiderivative, `F`, may not exist in closed form; it may define an unfamiliar function; it may exist, but the software can't find the antiderivative; the software could find it on a larger computer, but runs out of time or memory on the available machine. Nevertheless, in many cases, MATLAB can perform symbolic integration successfully. For example, create the symbolic variables

```
syms a b theta x yn x1 u
```

This table illustrates integration of expressions containing those variables.

f	int(f)
`x^n`	`x^(n+1)/(n+1)`
`y^(-1)`	`log(y)`
`n^x`	`1/log(n)*n^x`

f	int(f)
`sin(a*theta+b)`	`–cos(a*theta+b)/a`
`exp(–x1^2)`	`1/2*pi^(1/2)*erf(x1)`
`1/(1+u^2)`	`atan(u)`
`besselj(nu,z)`	`int(besselj(nu,z),z)`

The last example shows what happens if the toolbox can't find the antiderivative; it simply returns the command, including the variable of integration, unevaluated.

Definite integration is also possible. The commands

```
int(f,a,b)
```

and

```
int(f,v,a,b)
```

are used to find a symbolic expression for

$$\int_a^b f(x)dx \text{ and } \int_a^b f(v)dv$$

respectively.

Here are some additional examples.

f	a, b	int(f,a,b)
`x^7`	`0, 1`	`1/8`
`1/x`	`1, 2`	`log(2)`
`log(x)*sqrt(x)`	`0, 1`	`–4/9`
`exp(–x^2)`	`0, inf`	`1/2*pi^(1/2)`
`bessel(1,z)`	`0, 1`	`–besselj(0,1)+1`

For the Bessel function (besselj) example, it is possible to compute a numerical approximation to the value of the integral, using the double function. The command

```
a = int(besselj(1,z),0,1)
```

returns

```
a =
-besselj(0,1)+1
```

and the command

```
a = double(a)
```

returns

```
a =
  0.23480231344203
```

Integration with Real Constants

One of the subtleties involved in symbolic integration is the “value” of various parameters. For example, it would seem evident that the expression

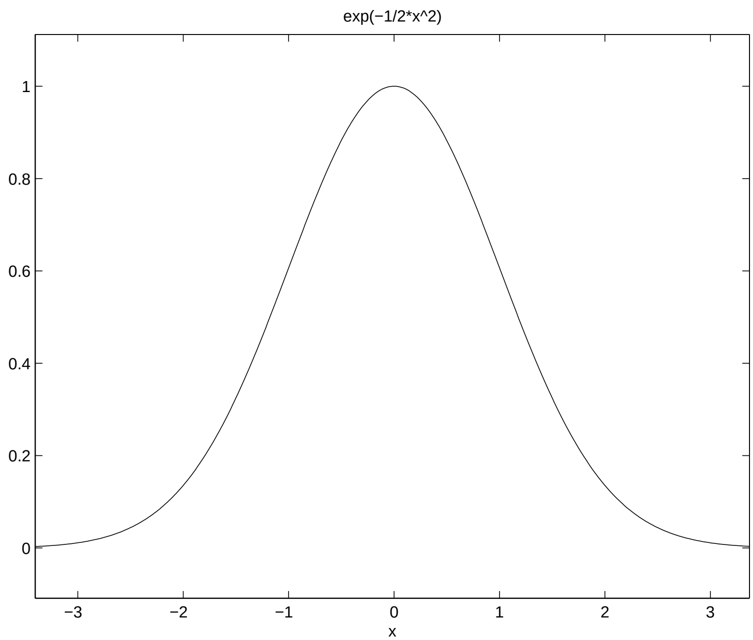
$$e^{-(kx)^2}$$

is the positive, bell shaped curve that tends to 0 as x tends to $\pm\infty$ for any real number k. An example of this curve is depicted below with

$$k = \frac{1}{\sqrt{2}}$$

and generated, using these commands.

```
syms x
k = sym(1/sqrt(2));
f = exp(-(k*x)^2);
ezplot(f)
```



The Maple kernel, however, does not, *a priori,* treat the expressions k^2 or x^2 as positive numbers. To the contrary, Maple assumes that the symbolic variables x and k as *a priori* indeterminate. That is, they are purely formal variables with no mathematical properties. Consequently, the initial attempt to compute the integral

$$\int_{-\infty}^{\infty} e^{-(kx)^2} dx$$

in the Symbolic Math Toolbox, using the commands

```
syms x k;
f = exp(-(k*x)^2);
int(f,x,-inf,inf) % Equivalently, inf(f,-inf,inf)
```

result in the output

```
Definite integration: Can't determine if the integral is
convergent.
Need to know the sign of --> k^2
Will now try indefinite integration and then take limits.

Warning: Explicit integral could not be found.
ans =
int(exp(-k^2*x^2),x= -inf..inf)
```

In the next section, you well see how to make k a real variable and therefore k^2 positive.

Real Variables via sym

Notice that Maple is not able to determine the sign of the expression `k^2`. How does one surmount this obstacle? The answer is to make `k` a real variable, using the `sym` command. One particularly useful feature of `sym`, namely the `real` option, allows you to declare `k` to be a real variable. Consequently, the integral above is computed, in the toolbox, using the sequence

```
syms k real % Be sure that x has been declared a sym.
int(f,x,-inf,inf)
```

which returns

```
ans =
signum(k)/k*pi^(1/2)
```

Notice that `k` is now a symbolic object in the MATLAB workspace and a real variable in the Maple kernel workspace. By typing

```
clear k
```

you only clear `k` in the MATLAB workspace. To ensure that `k` has no formal properties (that is, to ensure `k` is a purely formal variable), type

```
syms k unreal
```

This variation of the `syms` command clears `k` in the Maple workspace. You can also declare a sequence of symbolic variables w, y, x, z to be real, using

```
syms w x y z real
```

In this case, all of the variables in between the words `syms` and `real` are assigned the property `real`. That is, they are real variables in the Maple workspace.

Mathematical Operation	**MATLAB Commands**
$f(x) = e^{-kx}$	`syms k x` `f = exp(-k*x)`
$\int f(x)dx$	`int(f)` or `int(f,x)`
$\int f(k)dk$	`int(f,k)`
$\int_0^1 f(x)dx$	`int(f,x,0,1)` or `int(f,0,1)`
$g(x) = e^{-(kx)^2}$	`syms k real` `g = exp(-(k*x)^2)`
$\int_{-\infty}^{\infty} g(x)dx$	`int(g,x,-inf,inf)` or `int(g,-inf,inf)`

Symbolic Summation

You can compute symbolic summations, when they exist, by using the `symsum` command. For example, the p-series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

adds to $\pi^2/6$, while the geometric series $1 + x + x^2 + \ldots$ adds to $1/(1-x)$, provided $|x| < 1$. Three summations are demonstrated below.

```
syms x k
s1 = symsum(1/k^2,1,inf)
s2 = symsum(x^k,k,0,inf)

s1 =

1/6*pi^2

s2 =

-1/(x-1)
```

Taylor Series

The statement

```
T = taylor(f,8)
```

returns

```
T =
1/9+2/81*x^2+5/1458*x^4+49/131220*x^6
```

which is all the terms up to, but not including, order eight ($O(x^8)$) in the Taylor series for $f(x)$.

$$\sum_{n=0}^{\infty} (x-a)^n \frac{f^{(n)}(a)}{n!}$$

Technically, `T` is a MacLaurin series, since its basepoint is `a = 0`.

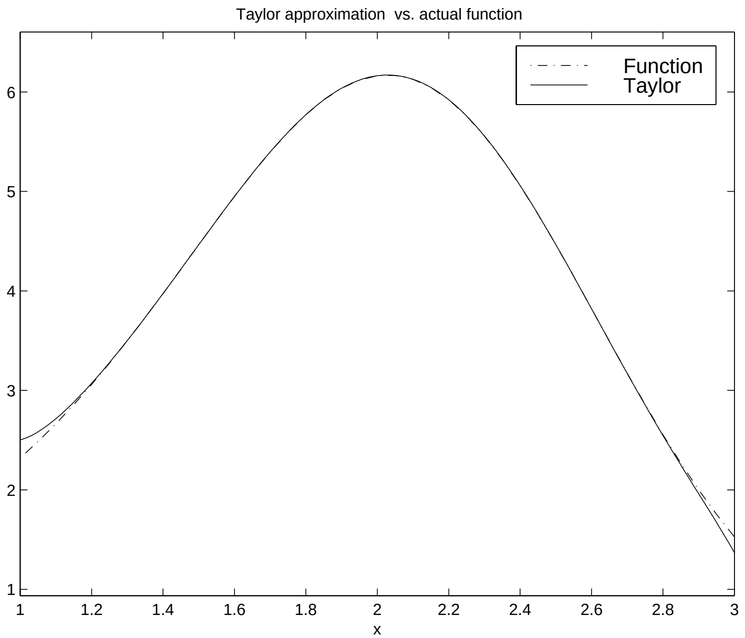
These commands

```
syms x
g = exp(x*sin(x))
t = taylor(g,12,2)
```

generate the first 12 nonzero terms of the Taylor series for `g` about `x = 2`.

Let's plot these functions together to see how well this Taylor approximation compares to the actual function g.

```
xd = 1:0.05:3; yd = subs(g,x,xd);
ezplot(t, [1,3]); hold on;
plot(xd, yd, 'r-.')
title('Taylor approximation vs. actual function');
legend('Function','Taylor')
```



Special thanks to Professor Gunnar Bäckstrøm of UMEA in Sweden for this example.

Then the command

```
pretty(T)
```

prints T in a format resembling typeset mathematics.

```
                 2            4      49     6
1/9 + 2/81 x  + 5/1458 x  + ------ x
                              131220
```

Extended Calculus Example

The function

$$f(x) = \frac{1}{5 + 4\cos(x)}$$

provides a starting point for illustrating several calculus operations in the toolbox. It is also an interesting function in its own right. The statements

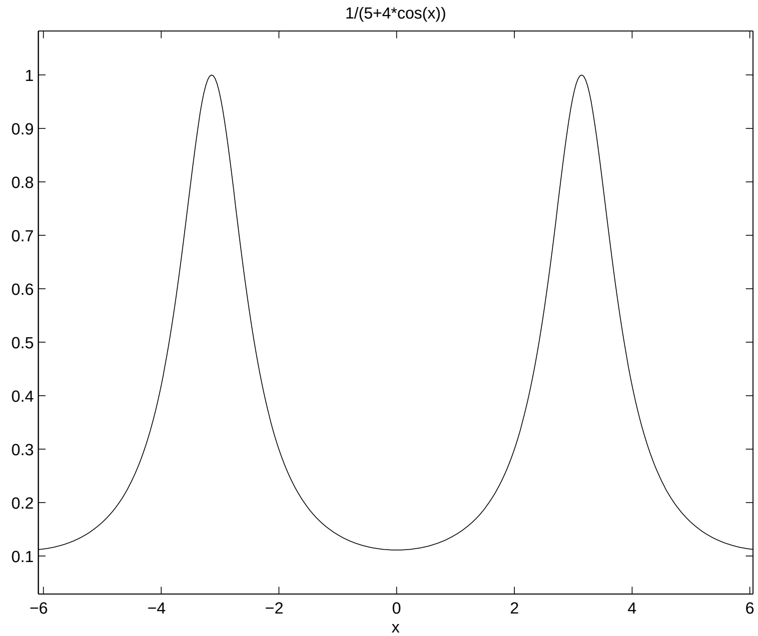
```
syms x
f = 1/(5+4*cos(x))
```

store the symbolic expression defining the function in `f`.

Plotting Symbolic Functions

The Symbolic Math Toolbox offers a set of easy-to-use commands for plotting symbolic expressions, including planar curves (`ezplot`), contours (`ezcontour` and `ezcontourf`), surfaces (`ezsurf`, `ezsurfc`, `ezmesh`, and `ezmeshc`), polar coordinates (`ezpolar`), and parametrically defined curves (`ezplot` and `ezplot3`) and surfaces (`ezsurf`). See Chapter 2, “Reference” for a detailed description of these functions. The rest of this section illustrates the use of `ezplot` to graph functions of the form *y=f(x)*.

The function `ezplot(f)` produces the plot of *f(x)* as shown below.



The `ezplot` function tries to make reasonable choices for the range of the *x*-axis and for the resulting scale of the *y*-axis. Its choices can be overridden by an additional input argument, or by subsequent `axis` commands. The default domain for a function displayed by `ezplot` is $-2\pi \le x \le 2\pi$. To alter the domain, type

```
ezplot(f,[a b])
```

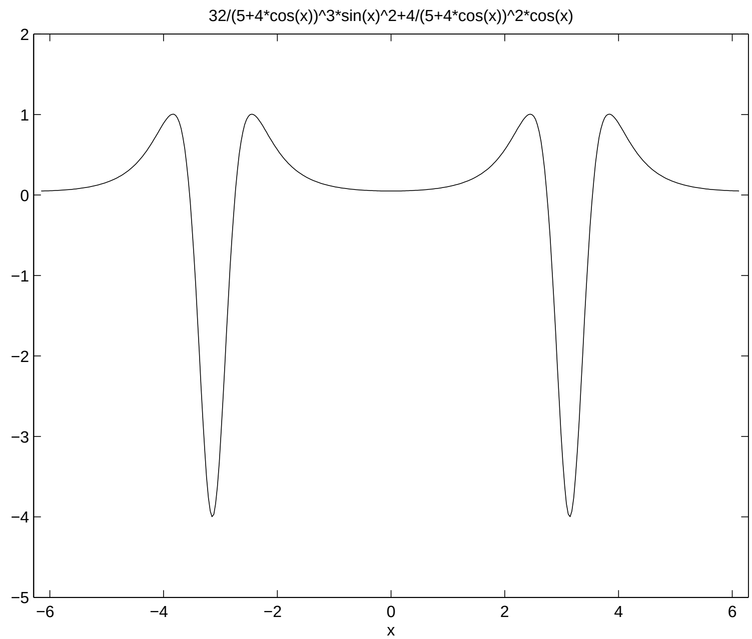
This produces a graph of $f(x)$ for $a \le x \le b$.

Let's now look at the second derivative of the function `f`.

```
f2 = diff(f,2)

f2 =
32/(5+4*cos(x))^3*sin(x)^2+4/(5+4*cos(x))^2*cos(x)
```

Equivalently, we can type `f2 = diff(f,x,2)`. The default scaling in `ezplot` cuts off part of `f2`'s graph. Set the axes limits manually to see the entire function.

```
ezplot(f2)
axis([-2*pi 2*pi -5 2])
```



From the graph, it appears that the values of $f''(x)$ lie between -4 and 1. As it turns out, this is not true. We can calculate the exact range for `f` (i.e., compute its actual maximum and minimum).

The actual maxima and minima of $f''(x)$ occur at the zeros of $f'''(x)$. The statements

```
f3 = diff(f2);
pretty(f3)
```

compute *f'''(x)* and display it in a more readable format.

```
                  3
             sin(x)                sin(x) cos(x)            sin(x)
    384 --------------- + 96 --------------- – 4 ---------------
                      4                      3                  2
        (5 + 4 cos(x))        (5 + 4 cos(x))       (5 + 4 cos(x))
```

We can simplify and this expression using the statements

```
f3 = simple(f3);
pretty(f3)
```

```
                                   2                         2
            sin(x) (96 sin(x)  + 80 cos(x) + 80 cos(x)  – 25)
          4 -------------------------------------------------
                                                4
                                  (5 + 4 cos(x))
```

Now use the `solve` function to find the zeros of *f'''(x)*.

```
z = solve(f3)
```

returns a 5-by-1 symbolic matrix

```
z =
[                                                      0]
[       atan((–255–60*19^(1/2))^(1/2),10+3*19^(1/2))]
[      atan(–(–255–60*19^(1/2))^(1/2),10+3*19^(1/2))]
[  atan((–255+60*19^(1/2))^(1/2)/(10–3*19^(1/2)))+pi]
[ –atan((–255+60*19^(1/2))^(1/2)/(10–3*19^(1/2)))–pi]
```

each of whose entries is a zero of *f'''(x)*. The command

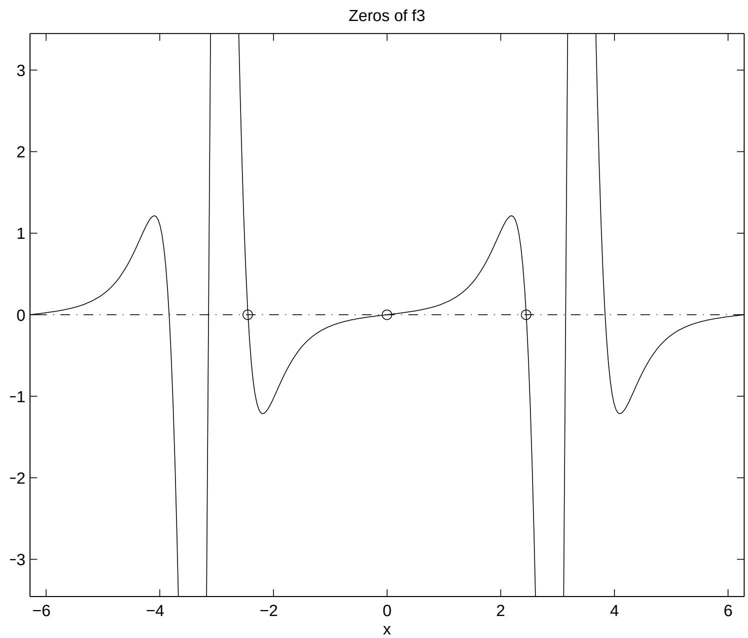
```
format; % Default format of 5 digits
zr = double(z)
```

converts the zeros to double form.

```
zr =

         0
         0+ 2.4381i
         0- 2.4381i
    2.4483
   -2.4483
```

So far, we have found three real zeros and two complex zeros. However, a graph of f3 shows that we have not yet found all its zeros.

```
ezplot(f3)
hold on;
plot(zr,0*zr,'ro')
plot([-2*pi,2*pi], [0,0],'g-.');
title('Zeros of f3')
```



This occurs because $f'''(x)$ contains a factor of $\sin(x)$, which is zero at integer multiples of π. The function, `solve(sin(x))`, however, only reports the zero at $x = 0$.

We can obtain a complete list of the real zeros by translating `zr`

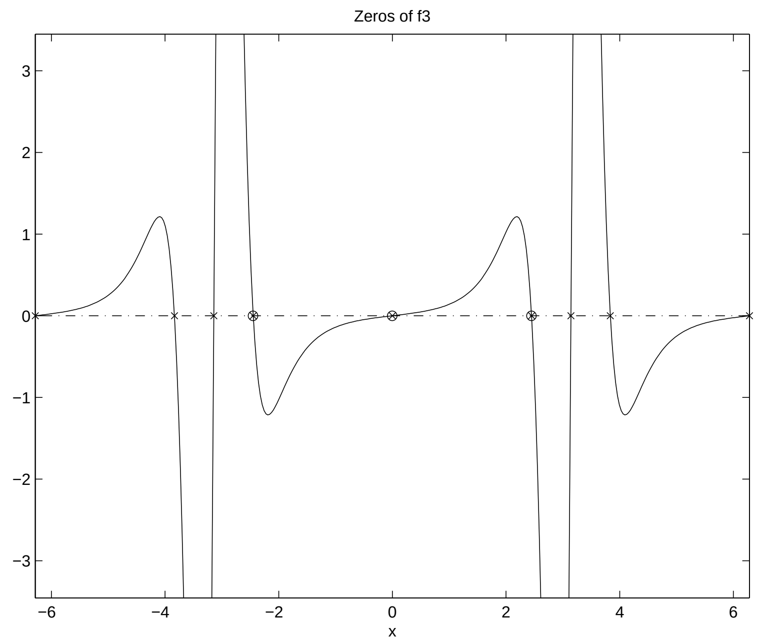
```
zr = [0 zr(4) pi 2*pi–zr(4)]
```

by multiples of 2π

```
zr = [zr–2*pi zr zr+2*pi];
```

Now let's plot the transformed `zr` on our graph for a complete picture of the zeros of `f3`.

```
plot(zr,0*zr,'kX')
```



The first zero of $f'''(x)$ found by `solve` is at $x = 0$. We substitute 0 for the symbolic variable in `f2`

```
f20 = subs(f2,x,0)
```

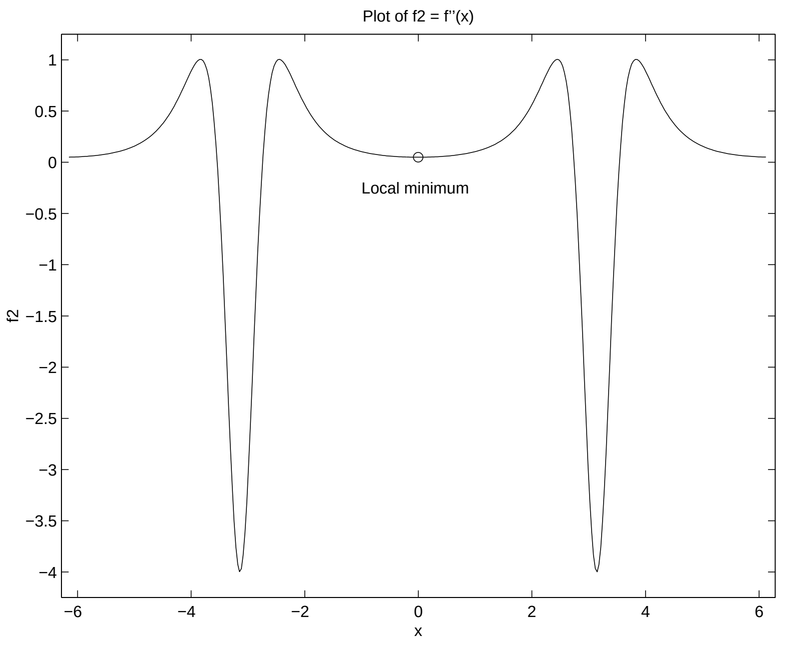
to compute the corresponding value of $f''(0)$.

```
f20 =
    0.0494
```

A look at the graph of $f''(x)$ shows that this is only a local minimum, which we demonstrate by replotting `f2`.

```
clf
ezplot(f2)
axis([-2*pi 2*pi -4.25 1.25])
ylabel('f2');
title('Plot of f2 = f''''(x)')
hold on
plot(0,double(f20),'ro')
text(-1,-0.25,'Local minimum')
```

The resulting plot



indicates that the global minima occur near $x = -\pi$ and $x = \pi$. We can demonstrate that they occur exactly at $x = \pm\pi$, using the following sequence of commands. First we try substituting $-\pi$ and π into $f'''(x)$.

```
simple([subs(f3,x,–pi),subs(f3,x,pi)])
```

The result

```
ans =
[ 0, 0]
```

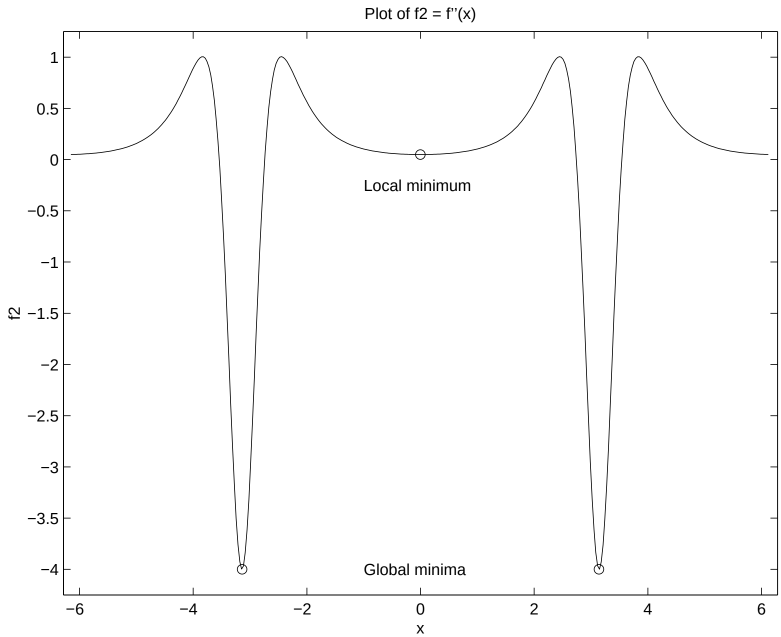
shows that $-\pi$ and π happen to be critical points of $f'''(x)$. We can see that $-\pi$ and π are global minima by plotting `f2(–pi)` and `f2(pi)` against `f2(x)`.

```
m1 = double(subs(f2,x,-pi)); m2 = double(subs(f2,x,pi));
plot(-pi,m1,'go',pi,m2,'go')
text(-1,-4,'Global minima')
```

The actual minima are `m1`, `m2`

```
ans =
[ –4, –4]
```

as shown in the plot on the following page.



The foregoing analysis confirms part of our original guess that the range of *f''(x)* is `[-4, 1]`. We can confirm the other part by examining the fourth zero of *f'''(x)* found by `solve`. First extract the fourth zero from `z` and assign it to a separate `variable`

```
s = z(4)
```

to obtain

```
s =
atan((-255+60*19^(1/2))^(1/2)/(10-3*19^(1/2)))+pi
```

Executing

```
sd = double(s)
```

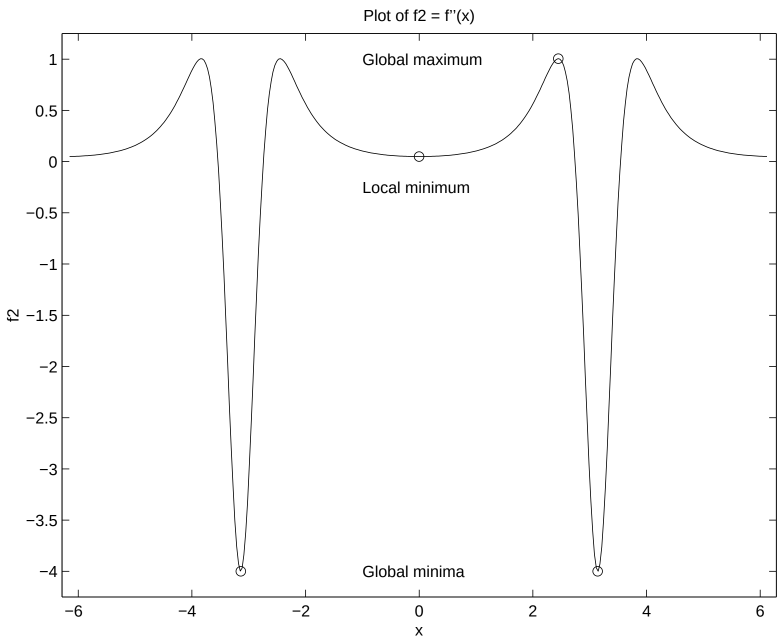
displays `the zero's` corresponding numeric value.

```
sd =
2.4483
```

Plotting the point (s, f2(s)) against f2, using

```
M1 = double(subs(f2,x,s));
plot(sd,M1,'ko')
text(-1,1,'Global maximum')
```

visually confirms that s is a maximum.



The maximum is M1 = 1.0051.

Therefore, our guess that the maximum of $f''(x)$ is [–4, 1] was close, but incorrect. The actual range is [–4, 1.0051].

Now, let's see if integrating $f''(x)$ twice with respect to x recovers our original function $f(x) = 1/(5 + 4 \cos x)$. The command

```
g = int(int(f2))
```

returns

```
g =
-8/(tan(1/2*x)^2+9)
```

This is certainly not the original expression for *f(x)*. Let's look at the difference *f(x)* – *g(x)*.

```
d = f - g
pretty(d)

                  1                  8
            [ ------------ + ---------------- ]
              5 + 4 cos(x)              2
                              tan(1/2 x)  + 9
```

We can simplify this using `simple(d)` or `simplify(d)`. Either command produces

```
ans =
1
```

This illustrates the concept that differentiating *f(x)* twice, then integrating the result twice, produces a function that may differ from *f(x)* by a linear function of *x*.

Finally, integrate *f(x)* once more.

```
F = int(f)
```

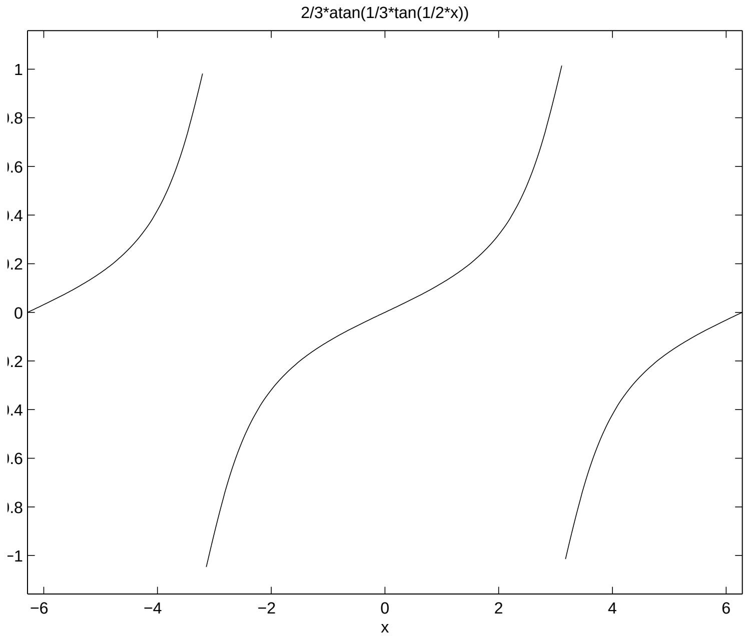
The result

```
F =
2/3*atan(1/3*tan(1/2*x))
```

involves the arctangent function.

Though *F(x)* is the antiderivative of a continuous function, it is itself discontinuous as the following plot shows.

```
ezplot(F)
```

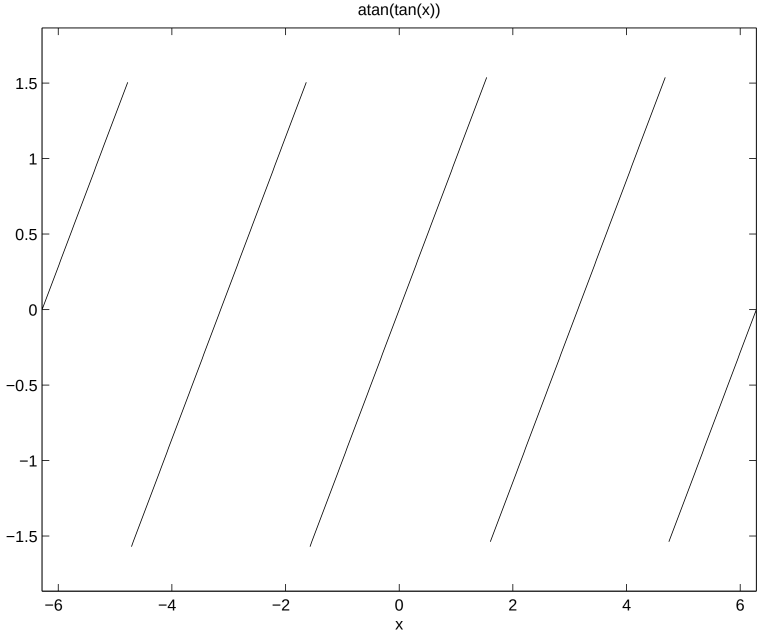


Note that $F(x)$ has jumps at $x = \pm\pi$. This occurs because $\tan x$ is singular at $x = \pm\pi$.

In fact, as

```
ezplot(atan(tan(x)))
```

shows, the numerical value of `atan(tan(x))` differs from `x` by a piecewise constant function that has jumps at odd multiples of $\pi/2$.



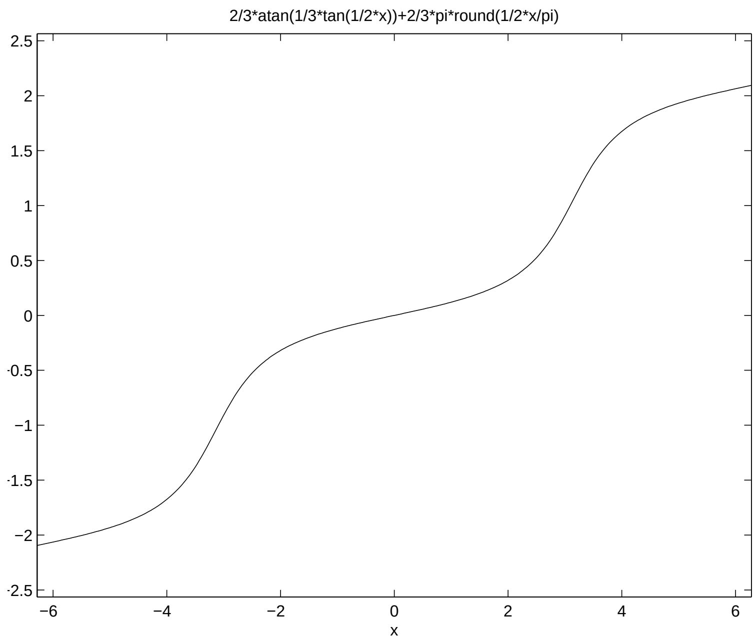
To obtain a representation of *F(x)* that does not have jumps at these points, we must introduce a second function, *J(x)*, that compensates for the discontinuities. Then we add the appropriate multiple of *J(x)* to *F(x)*.

```
J = sym('round(x/(2*pi))');
c = sym('2/3*pi');
F1 = F+c*J
F1 =
2/3*atan(1/3*tan(1/2*x))+2/3*pi*round(1/2*x/pi)
```

and plot the result.

```
ezplot(F1,[-6.28,6.28])
```

This representation does have a continuous graph.



Notice that we use the domain [–6.28, 6.28] in `ezplot` rather than the default domain [-2π, 2π]. The reason for this is to prevent an evaluation of $F1 = 2/3 \operatorname{atan}(1/3 \tan 1/2\, x)$ at the singular points $x = -\pi$ and $x = \pi$ where the jumps in F and J do not cancel out one another. The proper handling of branch cut discontinuities in multivalued functions like $\arctan x$ is a deep and difficult problem in symbolic computation. Although MATLAB and Maple cannot do this entirely automatically, they do provide the tools for investigating such questions.

Simplifications and Substitutions

There are several functions that simplify symbolic expressions and are used to perform symbolic substitutions.

Simplifications

Here are three different symbolic expressions.

```
syms x
f = x^3-6*x^2+11*x-6
g = (x-1)*(x-2)*(x-3)
h = x*(x*(x-6)+11)-6
```

Here are their prettyprinted forms, generated by

```
pretty(f), pretty(g), pretty(h)
```

```
 3     2
x - 6 x  + 11 x - 6

(x - 1) (x - 2) (x - 3)

x (x (x - 6) + 11) - 6
```

These expressions are three different representations of the same mathematical function, a cubic polynomial in x.

Each of the three forms is preferable to the others in different situations. The first form, f, is the most commonly used representation of a polynomial. It is simply a linear combination of the powers of x. The second form, g, is the factored form. It displays the roots of the polynomial and is the most accurate for numerical evaluation near the roots. But, if a polynomial does not have such simple roots, its factored form may not be so convenient. The third form, h, is the Horner, or nested, representation. For numerical evaluation, it involves the fewest arithmetic operations and is the most accurate for some other ranges of x.

The symbolic simplification problem involves the verification that these three expressions represent the same function. It also involves a less clearly defined objective—which of these representations is "the simplest"?

This toolbox provides several functions that apply various algebraic and trigonometric identities to transform one representation of a function into another, possibly simpler, representation. These functions are `collect`, `expand`, `horner`, `factor`, `simplify`, and `simple`.

collect

The statement

```
collect(f)
```

views `f` as a polynomial in its symbolic variable, say `x`, and collects all the coefficients with the same power of `x`. A second argument can specify the variable in which to collect terms if there is more than one candidate. Here are a few examples:

f	collect(f)
`(x–1)*(x–2)*(x–3)`	`x^3–6*x^2+11*x–6`
`x*(x*(x–6)+11)–6`	`x^3–6*x^2+11*x–6`
`(1+x)*t + x*t`	`2*x*t+t`

expand

The statement

```
expand(f)
```

distributes products over sums and applies other identities involving functions of sums. For example,

f	expand(f)
`a*(x + y)`	`a*x + a*y`
`(x-1)*(x-2)*(x-3)`	`x^3-6*x^2+11*x-6`
`x*(x*(x-6)+11)-6`	`x^3-6*x^2+11*x-6`
`exp(a+b)`	`exp(a)*exp(b)`
`cos(x+y)`	`cos(x)*cos(y)-sin(x)*sin(y)`
`cos(3*acos(x))`	`4*x^3 - 3*x`

horner

The statement

```
horner(f)
```

transforms a symbolic polynomial `f` into its Horner, or nested, representation. For example,

f	horner(f)
`x^3-6*x^2+11*x-6`	`-6+(11+(-6+x)*x)*x`
`1.1+2.2*x+3.3*x^2`	`11/10+(11/5+33/10*x)*x`

factor

If `f` is a polynomial with rational coefficients, the statement

```
factor(f)
```

expresses `f` as a product of polynomials of lower degree with rational coefficients. If `f` cannot be factored over the rational numbers, the result is `f` itself. For example,

f	factor(f)
`x^3–6*x^2+11*x–6`	`(x–1)*(x–2)*(x–3)`
`x^3–6*x^2+11*x–5`	`x^3–6*x^2+11*x–5`
`x^6+1`	`(x^2+1)*(x^4–x^2+1)`

Here is another example involving `factor`. It factors polynomials of the form `x^n + 1`. This code

```
syms x;
n = 1:9;
x = x(ones(size(n)));
p = x.^n + 1;
f = factor(p);
[p; f].'
```

returns a matrix with the polynomials in its first column and their factored forms in its second:

```
[                  x+1,                             x+1 ]
[                x^2+1,                           x^2+1 ]
[                x^3+1,                 (x+1)*(x^2-x+1) ]
[                x^4+1,                           x^4+1 ]
[                x^5+1,         (x+1)*(x^4-x^3+x^2-x+1) ]
[                x^6+1,             (x^2+1)*(x^4-x^2+1) ]
[                x^7+1, (x+1)*(1-x+x^2-x^3+x^4-x^5+x^6) ]
[                x^8+1,                           x^8+1 ]
[                x^9+1,       (x+1)*(x^2-x+1)*(x^6-x^3+1) ]
```

As an aside at this point, we mention that `factor` can also factor symbolic objects containing integers. This is an alternative to using the `factor` function in MATLAB's `specfun` directory. For example, the following code segment

```
one = '1'
for n = 1:11
   N(n,:) = sym(one(1,ones(1,n)));
end
[N factor(N)]
```

displays the factors of symbolic integers consisting of 1s.

```
[                       1,                         1 ]
[                      11,                      (11) ]
[                     111,                 (3)*(37) ]
[                    1111,               (11)*(101) ]
[                   11111,               (41)*(271) ]
[                  111111,    (3)*(7)*(11)*(13)*(37) ]
[                 1111111,             (239)*(4649) ]
[                11111111,    (11)*(73)*(101)*(137) ]
[               111111111,        (3)^2*(37)*(333667) ]
[              1111111111,     (11)*(41)*(271)*(9091) ]
[             11111111111,          (513239)*(21649) ]
```

simplify

The `simplify` function is a powerful, general purpose tool that applies a number of algebraic identities involving sums, integral powers, square roots and other fractional powers, as well as a number of functional identities involving trig functions, exponential and log functions, Bessel functions, hypergeometric functions, and the gamma function. Here are some examples.

f	simplify(f)
`x*(x*(x–6)+11)–6`	`x^3–6*x^2+11*x–6`
`(1–x^2)/(1–x)`	`x + 1`
`(1/a^3+6/a^2+12/a+8)^(1/3)`	`((2*a+1)^3/a^3)^(1/3)`
`syms x y positive` `log(x*y)`	 `log(x) + log(y)`
`exp(x) * exp(y)`	`exp(x+y)`

`besselj(2,x) – ...`
`2*besselj(1,x)/x +`
`besselj(0,x)`
	`0`

`gamma(x+1)–x*gamma(x)`	`0`
`cos(x)^2 + sin(x)^2`	`1`

simple

The `simple` function has the unorthodox mathematical goal of finding a simplification of an expression that has the fewest number of characters. Of course, there is little mathematical justification for claiming that one expression is “simpler” than another just because its ASCII representation is shorter, but this often proves satisfactory in practice.

The `simple` function achieves its goal by independently applying `simplify`, `collect`, `factor`, and other simplification functions to an expression and keeping track of the lengths of the results. The `simple` function then returns the shortest result.

The `simple` function has several forms, each returning different output. The form

```
simple(f)
```

displays each trial simplification and the simplification function that produced it in the MATLAB command window. The `simple` function then returns the shortest result. For example, the command

```
simple(cos(x)^2 + sin(x)^2)
```

displays the following alternative simplifications in the MATLAB command window

```
simplify:
1

radsimp:
cos(x)^2+sin(x)^2

combine(trig):
1

factor:
cos(x)^2+sin(x)^2

expand:
cos(x)^2+sin(x)^2

convert(exp):
(1/2*exp(i*x)+1/2/exp(i*x))^2-1/4*(exp(i*x)-1/exp(i*x))^2

convert(sincos):
cos(x)^2+sin(x)^2


convert(tan):
(1-tan(1/2*x)^2)^2/(1+tan(1/2*x)^2)^2+4*tan(1/2*x)^2/
(1+tan(1/2*x)^2)^2

collect(x):
cos(x)^2+sin(x)^2
```

and returns

```
ans =
1
```

This form is useful when you want to check, for example, whether the shortest form is indeed the simplest. If you are not interested in how `simple` achieves its result, use the form

```
f = simple(f)
```

This form simply returns the shortest expression found. For example, the statement

```
f = simple(cos(x)^2+sin(x)^2)
```

returns

```
f =
1
```

If you want to know which simplification returned the shortest result, use the multiple output form

```
[F, how] = simple(f)
```

This form returns the shortest result in the first variable and the simplification method used to achieve the result in the second variable. For example, the statement

```
[f, how] = simple(cos(x)^2+sin(x)^2)
```

returns

```
f =
1

how =
combine(trig)
```

The `simple` function sometimes improves on the result returned by `simplify`, one of the simplifications that it tries. For example, when applied to the examples given for `simplify`, `simple` returns a simpler (or at least shorter) result in two cases:

f	simplify(f)	simple(f)
`(1/a^3+6/a^2+12/a+8)^(1/3)`	`((2*a+1)^3/a^3)^(1/3)`	`(2*a+1)/a`
`syms x y positive` `log(x*y)`	`log(x)+log(y)`	`log(x*y)`

In some cases, it is advantageous to apply `simple` twice to obtain the effect of two different simplification functions. For example, the statements

```
f = (1/a^3+6/a^2+12/a+8)^(1/3);
simple(simple(f))
```

return

```
1/a+2
```

The first application, `simple(f)`, uses `radsimp` to produce `(2*a+1)/a`; the second application uses `combine(trig)` to transform this to `1/a+2`.

The `simple` function is particularly effective on expressions involving trigonometric functions. Here are some examples:

f	simple(f)
`cos(x)^2+sin(x)^2`	`1`
`2*cos(x)^2-sin(x)^2`	`3*cos(x)^2-1`
`cos(x)^2-sin(x)^2`	`cos(2*x)`
`cos(x)+(-sin(x)^2)^(1/2)`	`cos(x)+i*sin(x)`
`cos(x)+i*sin(x)`	`exp(i*x)`
`cos(3*acos(x))`	`4*x^3-3*x`

Substitutions

There are two functions for symbolic substitution: `subexpr` and `subs`.

subexpr

These commands

```
syms a x
s = solve(x^3+a*x+1)
```

solve the equation `x^3+a*x+1 = 0 for x`.

```
s =
[      (-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)-
          1/3*a/(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)              ]
[     -1/2*(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)+
           1/6*a/(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)+
           1/2*i*3^(1/2)*((-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)+
          1/3*a/(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3))             ]
[     -1/2*(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)+
         1/6*a/(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)-
         1/2*i*3^(1/2)*((-1/2+1/18*(12*a^3+81)^(1/2))^(1/3)+
          1/3*a/(-1/2+1/18*(12*a^3+81)^(1/2))^(1/3))             ]
```

Use the pretty function to display s in a more readable form.

```
pretty(s)

s =
          [                              1/3        a                      ]
          [                           %1    - 1/3 -----                    ]
          [                                          1/3                   ]
          [                                        %1                      ]
          [                                                                ]
          [          1/3        a            1/2 /  1/3        a  \ ]
          [ - 1/2 %1    + 1/6 ----- + 1/2 i 3    |%1    + 1/3 -----| ]
          [                      1/3              |              1/3| ]
          [                    %1                 \            %1   / ]
          [                                                                ]
          [          1/3        a            1/2 /  1/3        a  \ ]
          [ - 1/2 %1    + 1/6 ----- - 1/2 i 3    |%1    + 1/3 -----| ]
          [                      1/3              |              1/3| ]
          [                    %1                 \            %1   / ]

                                              3        1/2
%1 :=                    - 1/2 + 1/18 (12 a   + 81)
```

The pretty command inherits the %n (n, an integer) notation from Maple to denote subexpressions that occur multiple times in the symbolic object. The subexpr function allows you to save these common subexpressions as well as the symbolic object rewritten in terms of the subexpressions. The subexpressions are saved in a column vector called sigma.

Continuing with the example

```
r = subexpr(s)
```

returns

```
sigma =
-108+12*(12*a^3+81)^(1/2)

r =
[                              1/6*sigma^(1/3)-2*a/sigma^(1/3)]
[     -1/12*sigma^(1/3)+a/sigma^(1/3)+
                1/2*i*3^(1/2)*(1/6*sigma^(1/3)+2*a/sigma^(1/3))]
[    -1/12*sigma^(1/3)+a/sigma^(1/3)-
                1/2*i*3^(1/2)*(1/6*sigma^(1/3)+2*a/sigma^(1/3))]
```

Notice that `subexpr` creates the variable `sigma` in the MATLAB workspace. You can verify this by typing `whos`, or the command

```
sigma
```

which returns

```
sigma =
-108+12*(12*a^3+81)^(1/2)
```

subs

Let's find the eigenvalues and eigenvectors of a circulant matrix A.

```
syms a b c
A = [a b c; b c a; c a b];
[v,E] = eig(A)

v =

[ -(a+(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2)-b)/(a-c),
          -(a-(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2)-b)/(a-c),  1]
[ -(b-c-(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2))/(a-c),
          -(b-c+(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2))/(a-c),  1]
[ 1,
         1,                                             1]


E =

[ (b^2-b*a-c*b-
   c*a+a^2+c^2)^(1/2),                      0,           0]
[                   0,   -(b^2-b*a-c*b-
                         c*a+a^2+c^2)^(1/2),           0]
[                   0,                      0,       b+c+a]
```

Suppose we want to replace the rather lengthy expression

```
(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2)
```

throughout v and E. We first use subexpr:

```
v = subexpr(v,'S')
```

which returns

```
S =
(b^2-b*a-c*b-c*a+a^2+c^2)^(1/2)

v =
[ -(a+S-b)/(a-c), -(a-S-b)/(a-c),            1]
[ -(b-c-S)/(a-c), -(b-c+S)/(a-c),            1]
[             1,             1,            1]
```

Next, substitute the symbol S into E with

```
E = subs(E,S,'S')

E =
[      S,      0,      0]
[      0,     -S,      0]
[      0,      0, b+c+a]
```

Now suppose we want to evaluate v at a = 10. We can do this using the subs command:

```
subs(v,a,10)
```

This replaces all occurrences of a in v with 10

```
[ -(10+S-b)/(10-c), -(10-S-b)/(10-c),                 1]
[  -(b-c-S)/(10-c),  -(b-c+S)/(10-c),                 1]
[                1,                1,                 1]
```

Notice, however, that the symbolic expression represented by S is unaffected by this substitution. That is, the symbol a in S is not replaced by 10. The subs command is also a useful function for substituting in a variety of values for several variables in a particular expression. Let's look at S. Suppose that in addition to substituting a = 10, we also want to substitute the values for 2 and 10 for b and c, respectively. The way to do this is to set values for a, b, and c in the workspace. Then subs evaluates its input using the existing symbolic and double variables in the current workspace. In our example, we first set

```
a = 10; b = 2; c = 10;
subs(S)

ans =
64^(1/2)
```

To look at the contents of our workspace, type whos, which gives

```
Name      Size         Bytes  Class

  A         3x3           878  sym object
  E         3x3           888  sym object
  S         1x1           186  sym object
  a         1x1             8  double array
  ans       1x1           140  sym object
  b         1x1             8  double array
  c         1x1             8  double array
  v         3x3           982  sym object
```

a, b, and c are now variables of class double while A, E, S, and v remain symbolic expressions (class sym).

If you want to preserve a, b, and c as symbolic variables, but still alter their value within S, use this procedure.

```
syms a b c
subs(S,{a,b,c},{10,2,10})

ans =
64^(1/2)
```

Typing whos reveals that a, b, and c remain 1-by-1 sym objects.

The subs command can be combined with double to evaluate a symbolic expression numerically. Suppose we have

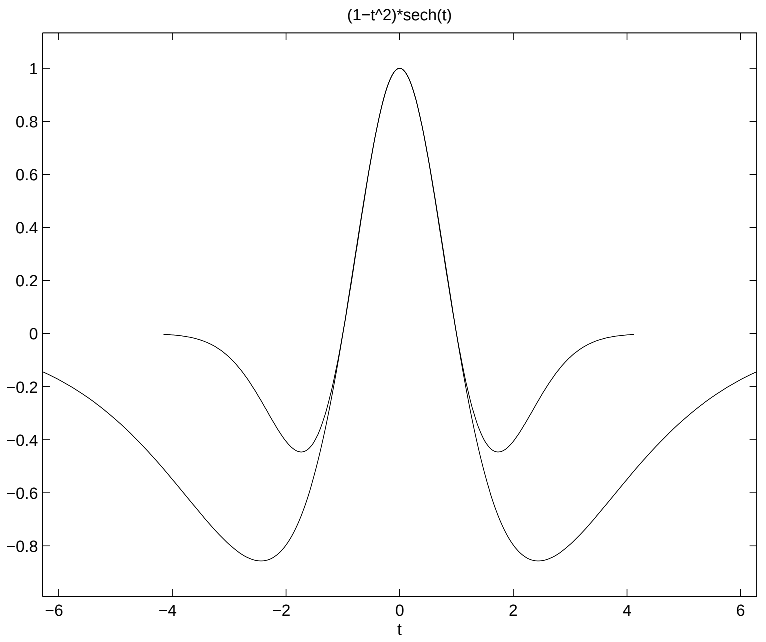
```
syms t
M = (1-t^2)*exp(-1/2*t^2);
P = (1-t^2)*sech(t);
```

and want to see how M and P differ graphically.

One approach is to type

```
ezplot(M); hold on; ezplot(P)
```

but this plot

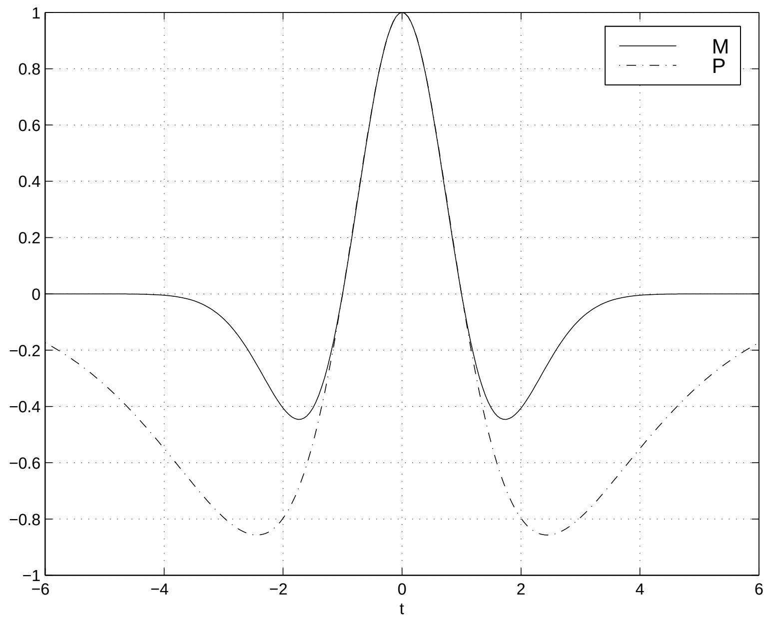


does not readily help us identify the curves.

Instead, combine subs, double, and plot

```
T = -6:0.05:6;
MT = double(subs(M,t,T));
PT = double(subs(P,t,T));
plot(T,MT,'b',T,PT,'r-.')
title(' ')
legend('M','P')
xlabel('t'); grid
```

to produce a multicolored graph that indicates the difference between M and P.



Finally the use of subs with strings greatly facilitates the solution of problems involving the Fourier, Laplace, or *z*-transforms. See the section "Integral Transforms" for complete details.

Variable-Precision Arithmetic

Overview

There are three different kinds of arithmetic operations in this toolbox.

- Numeric MATLAB's floating-point arithmetic
- Rational Maple's exact symbolic arithmetic
- VPA Maple's variable-precision arithmetic

For example, the MATLAB statements

```
format long
1/2+1/3
```

use numeric computation to produce

```
0.83333333333333
```

With the Symbolic Math Toolbox, the statement

```
sym(1/2)+1/3
```

uses symbolic computation to yield

```
5/6
```

And, also with the toolbox, the statements

```
digits(25)
vpa(1/2+1/3)
```

use variable-precision arithmetic to return

```
.8333333333333333333333333
```

The floating-point operations used by numeric arithmetic are the fastest of the three, and require the least computer memory, but the results are not exact. The number of digits in the printed output of MATLAB's double quantities is controlled by the `format` statement, but the internal representation is always the eight-byte floating-point representation provided by the particular computer hardware.

In the computation of the numeric result above, there are actually three roundoff errors, one in the division of 1 by 3, one in the addition of 1/2 to the result of the division, and one in the binary to decimal conversion for the printed output. On computers that use IEEE floating-point standard arithmetic, the resulting internal value is the binary expansion of 5/6, truncated to 53 bits. This is approximately 16 decimal digits. But, in this particular case, the printed output shows only 15 digits.

The symbolic operations used by rational arithmetic are potentially the most expensive of the three, in terms of both computer time and memory. The results are exact, as long as enough time and memory are available to complete the computations.

Variable-precision arithmetic falls in between the other two in terms of both cost and accuracy. A global parameter, set by the function `digits`, controls the number of significant decimal digits. Increasing the number of digits increases the accuracy, but also increases both the time and memory requirements. The default value of `digits` is 32, corresponding roughly to floating-point accuracy.

The Maple documentation uses the term “hardware floating-point” for what we are calling “numeric” or “floating-point” and uses the term “floating-point arithmetic” for what we are calling “variable-precision arithmetic.”

Example: Using the Different Kinds of Arithmetic

Rational Arithmetic

By default, the Symbolic Math Toolbox uses rational arithmetic operations, i.e., Maple’s exact symbolic arithmetic. Rational arithmetic is invoked when you create symbolic variables using the `sym` function.

The `sym` function converts a double matrix to its symbolic form. For example, if the double matrix is

```
A =
1.1000     1.2000     1.3000
2.1000     2.2000     2.3000
3.1000     3.2000     3.3000
```

its symbolic form, `S = sym(A)`, is

```
S =
[11/10,  6/5, 13/10]
[21/10, 11/5, 23/10]
[31/10, 16/5, 33/10]
```

For this matrix A, it is possible to discover that the elements are the ratios of small integers, so the symbolic representation is formed from those integers. On the other hand, the statement

```
E = [exp(1) sqrt(2); log(3) rand]
```

returns a matrix

```
E =
2.71828182845905   1.41421356237310
1.09861228866811   0.21895918632809
```

whose elements are not the ratios of small integers, so `sym(E)` reproduces the floating-point representation in a symbolic form.

```
[3060513257434037*2^(-50), 3184525836262886*2^(-51)]
[2473854946935174*2^(-51), 3944418039826132*2^(-54)]
```

Variable-Precision Numbers

Variable-precision numbers are distinguished from the exact rational representation by the presence of a decimal point. A power of 10 scale factor, denoted by `'e'`, is allowed. To use variable-precision instead of rational arithmetic, create your variables using the `vpa` function.

For matrices with purely double entries, the `vpa` function generates the representation that is used with variable-precision arithmetic. Continuing on with our example, and using `digits(4)`, applying `vpa` to the matrix `S`

```
vpa(S)
```

generates the output

```
S =
[1.100, 1.200, 1.300]
[2.100, 2.200, 2.300]
[3.100, 3.200, 3.300]
```

and with `digits(25)`

```
F = vpa(E)
```

generates

```
F =
[2.718281828459045534884808, 1.414213562373094923430017]
[1.098612288668110004152823, .2189591863280899719512718]
```

Converting to Floating-Point

To convert a rational or variable-precision number to its MATLAB floating-point representation, use the `double` function.

In our example, both `double(sym(E))` and `double(vpa(E))` return `E`.

Another Example

The next example is perhaps more interesting. Start with the symbolic expression

```
f = sym('exp(pi*sqrt(163))')
```

The statement

```
double(f)
```

produces the printed floating-point value

```
2.625374126407687e+17
```

Using the second argument of `vpa` to specify the number of digits,

```
vpa(f,18)
```

returns

```
262537412640768744.
```

whereas

```
vpa(f,25)
```

returns

```
262537412640768744.0000000
```

We suspect that `f` might actually have an integer value. This suspicion is reinforced by the 30 digit value, `vpa(f,30)`

```
262537412640768743.999999999999
```

Finally, the 40 digit value, `vpa(f,40)`

```
262537412640768743.99999999999925007259444
```

shows that `f` is very close to, but not exactly equal to, an integer.

Linear Algebra

Basic Algebraic Operations

Basic algebraic operations on symbolic objects are the same as operations on MATLAB objects of class double. This is illustrated in the following example.

The Givens transformation produces a plane rotation through the angle t. The statements

```
syms t;
G = [cos(t) sin(t); -sin(t) cos(t)]
```

create this transformation matrix:

```
G =
[  cos(t),  sin(t) ]
[ -sin(t),  cos(t) ]
```

Applying the Givens transformation twice should simply be a rotation through twice the angle. The corresponding matrix can be computed by multiplying G by itself or by raising G to the second power. Both

```
A = G*G
```

and

```
A = G^2
```

produce

```
A =
[cos(t)^2-sin(t)^2,   2*cos(t)*sin(t)]
[ -2*cos(t)*sin(t), cos(t)^2-sin(t)^2]
```

The simple function

```
A = simple(A)
```

uses a trigonometric identity to return the expected form by trying several different identities and picking the one that produces the shortest representation:

```
A =
[ cos(2*t), sin(2*t)]
[–sin(2*t), cos(2*t)]
```

A Givens rotation is an orthogonal matrix, so its transpose is its inverse. Confirming this by

```
I = G.' *G
```

which produces

```
I =
[cos(t)^2+sin(t)^2,                 0]
[                0, cos(t)^2+sin(t)^2]
```

and then

```
I = simple(I)
I =
[1, 0]
[0, 1]
```

Linear Algebraic Operations

Let's do several basic linear algebraic operations.

The command

```
H = hilb(3)
```

generates the 3-by-3 Hilbert matrix. With `format short`, MATLAB prints

```
H =
1.0000    0.5000    0.3333
0.5000    0.3333    0.2500
0.3333    0.2500    0.2000
```

The computed elements of `H` are floating-point numbers that are the ratios of small integers. Indeed, `H` is a MATLAB array of class `double`. Converting `H` to a symbolic matrix

```
H = sym(H)
```

gives

```
[  1, 1/2, 1/3]
[1/2, 1/3, 1/4]
[1/3, 1/4, 1/5]
```

This allows subsequent symbolic operations on `H` to produce results that correspond to the infinitely precise Hilbert matrix, `sym(hilb(3))`, not its floating-point approximation, `hilb(3)`. Therefore,

```
inv(H)
```

produces

```
[  9,  -36,   30]
[-36,  192, -180]
[ 30, -180,  180]
```

and

```
det(H)
```

yields

```
1/2160
```

We can use the backslash operator to solve a system of simultaneous linear equations. The commands

```
b = [1 1 1]'
x = H\b    % Solve Hx = b
```

produce the solution

```
[ -3]
[-24]
[ 30]
```

All three of these results, the inverse, the determinant, and the solution to the linear system, are the exact results corresponding to the infinitely precise, rational, Hilbert matrix. On the other hand, using `digits(16)`, the command

```
V = vpa(hilb(3))
```

returns

```
[                1., .5000000000000000, .3333333333333333]
[.5000000000000000, .3333333333333333, .2500000000000000]
[.3333333333333333, .2500000000000000, .2000000000000000]
```

The decimal points in the representation of the individual elements are the signal to use variable-precision arithmetic. The result of each arithmetic operation is rounded to 16 significant decimal digits. When inverting the matrix, these errors are magnified by the matrix condition number, which for `hilb(3)` is about 500. Consequently,

```
inv(V)
```

which returns

```
[ 9.000000000000082, -36.00000000000039,  30.00000000000035]
[-36.00000000000039,  192.0000000000021, -180.0000000000019]
[ 30.00000000000035, -180.0000000000019,  180.0000000000019]
```

shows the loss of two digits. So does

```
det(V)
```

which gives

```
.462962962962958e-3
```

and

```
V\b
```

which is

```
[ 3.000000000000041]
[-24.00000000000021]
[ 30.00000000000019]
```

Since `H` is nonsingular, the null space of `H`

```
null(H)
```

and the column space of H

```
colspace(H)
```

produce an empty matrix and a permutation of the identity matrix, respectively. To make a more interesting example, let's try to find a value s for H(1,1) that makes H singular. The commands

```
syms s
H(1,1) = s
Z = det(H)
sol = solve(Z)
```

produce

```
H =
[  s, 1/2, 1/3]
[1/2, 1/3, 1/4]
[1/3, 1/4, 1/5]

Z =
1/240*s–1/270

sol =
8/9
```

Then

```
H = subs(H,s,sol)
```

substitutes the computed value of sol for s in H to give

```
H =
[8/9, 1/2, 1/3]
[1/2, 1/3, 1/4]
[1/3, 1/4, 1/5]
```

Now, the command

```
det(H)
```

returns

```
ans =
0
```

and

```
inv(H)
```

produces an error message

```
??? error using ==> inv
Error, (in inverse) singular matrix
```

because `H` is singular. For this matrix, `Z = null(H)` and `C = colspace(H)` are nontrivial.

```
Z =
[    1]
[   -4]
[10/3]

C =
[  0,      1]
[  1,      0]
[6/5, -3/10]
```

It should be pointed out that even though `H` is singular, `vpa(H)` is not. For any integer value `d`, setting

```
digits(d)
```

and then computing

```
det(vpa(H))
inv(vpa(H))
```

results in a determinant of size `10^(-d)` and an inverse with elements on the order of `10^d`.

Eigenvalues

The symbolic eigenvalues of a square matrix `A` or the symbolic eigenvalues and eigenvectors of `A` are computed, respectively, using the commands

```
E = eig(A)
[V,E] = eig(A)
```

The variable-precision counterparts are

```
E = eig(vpa(A))
[V,E] = eig(vpa(A))
```

The eigenvalues of A are the zeros of the characteristic polynomial of A, `det(A–x*I)`, which is computed by

```
poly(A)
```

The matrix H from the last section provides our first example.

```
H =
[8/9, 1/2, 1/3]
[1/2, 1/3, 1/4]
[1/3, 1/4, 1/5]
```

The matrix is singular, so one of its eigenvalues must be zero. The statement

```
[T,E] = eig(H)
```

produces the matrices T and E. The columns of T are the eigenvectors of H.

```
T =

[     1, 28/153+2/153*12589^(1/2),  28/153–2/153*12589^(12)]
[    –4,                         1,                          1]
[  10/3, 92/255–1/255*12589^(1/2), 292/255+1/255*12589^(12)]
```

Similarly, the diagonal elements of E are the eigenvalues of H.

```
E =

[0,                          0,                          0]
[0, 32/45+1/180*12589^(1/2),                          0]
[0,                          0, 32/45–1/180*12589^(1/2)]
```

It may be easier to understand the structure of the matrices of eigenvectors, T, and eigenvalues, E, if we convert T and E to decimal notation. We proceed as follows. The commands

```
Td = double(T)
Ed = double(E)
```

return

```
Td =
    1.0000    1.6497   -1.2837
   -4.0000    1.0000    1.0000
    3.3333    0.7051    1.5851
Ed =
    0         0         0
    0    1.3344         0
    0         0    0.0878
```

The first eigenvalue is zero. The corresponding eigenvector (the first column of `Ts`) is the same as the basis for the null space found in the last section. The other two eigenvalues are the result of applying the quadratic formula to

```
x^2–64/45*x+253/2160
```

which is the quadratic factor in `factor(poly(H))`.

```
syms x
g = simple(factor(poly(H))/x);
solve(g)
```

Closed form symbolic expressions for the eigenvalues are possible only when the characteristic polynomial can be expressed as a product of rational polynomials of degree four or less. The Rosser matrix is a classic numerical analysis test matrix that happens to illustrate this requirement. The statement

```
R = sym(gallery('rosser'))
```

generates

```
 R =
[ 611   196  –192   407    –8   –52   –49    29]
[ 196   899   113  –192   –71   –43    –8   –44]
[–192   113   899   196    61    49     8    52]
[ 407  –192   196   611     8    44    59   –23]
[  –8   –71    61     8   411  –599   208   208]
[ –52   –43    49    44  –599   411   208   208]
[ –49    –8     8    59   208   208    99  –911]
[  29   –44    52   –23   208   208  –911    99]
```

The commands

```
p = poly(R);
pretty(factor(p))
```

produce

```
                         2                        2                           2
[x (x – 1020) (x   – 1020 x + 100)(x   – 1040500) (x – 1000)   ]
```

The characteristic polynomial (of degree 8) factors nicely into the product of two linear terms and three quadratic terms. We can see immediately that four of the eigenvalues are 0, 1020, and a double root at 1000. The other four roots are obtained from the remaining quadratics. Use

```
eig(R)
```

to find all these values

```
[               0]
[            1020]
[510+100*26^(1/2)]
[510–100*26^(1/2)]
[  10*10405^(1/2)]
[ –10*10405^(1/2)]
[            1000]
[            1000]
```

The Rosser matrix is not a typical example; it is rare for a full 8-by-8 matrix to have a characteristic polynomial that factors into such simple form. If we change the two “corner” elements of R from 29 to 30 with the commands

```
S = R;  S(1,8) = 30;  S(8,1) = 30;
```

and then try

```
p = poly(S)
```

we find

```
p =
40250968213600000+51264008540948000*x–
   1082699388411166000*x^2+4287832912719760*x^–3–
   5327831918568*x^4+82706090*x^5+5079941*x^6–
   4040*x^7+x^8
```

We also find that `factor(p)` is `p` itself. That is, the characteristic polynomial cannot be factored over the rationals.

For this modified Rosser matrix

```
F = eig(S)
```

returns

```
F =
[ -1020.053214255891516593189425260 0]
[ -.1705352972876899857520087460775 7]
[   .2180398054830160686085756442498 1]
[   999.9469178604427675532028922860 2]
[   1000.120698293384133571281707545 4]
[   1019.524355263201635832493327829 1]
[   1019.993550129162925734809180817 3]
[   1020.420188201504727818545749884 0]
```

Notice that these values are close to the eigenvalues of the original Rosser matrix. Further, the numerical values of `F` are a result of Maple's floating-point arithmetic. Consequently, different settings of `digits` do not alter the number of integers to the right of the decimal place.

It is also possible to try to compute eigenvalues of symbolic matrices, but closed form solutions are rare. The Givens transformation is generated as the matrix exponential of the elementary matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The Symbolic Math Toolbox commands

```
syms t
A = sym([0 1; -1 0]);
G = expm(t*A)
```

return

```
[  cos(t),  sin(t)]
[ -sin(t),  cos(t)]
```

Next, the command

```
g = eig(G)
```

produces

```
g =
[ cos(t)+(cos(t)^2–1)^(1/2)]
[ cos(t)–(cos(t)^2–1)^(1/2)]
```

We can use `simple` to simplify this form of `g`. Indeed, repeated application of `simple`

```
for  j = 1:4
   [g,how] = simple(g)
end
```

produces the best result

```
g =
[ cos(t)+(–sin(t)^2)^(1/2)]
[ cos(t)–(–sin(t)^2)^(1/2)]

how =
simplify

g =
[ cos(t)+i*sin(t)]
[ cos(t)–i*sin(t)]

how =
radsimp

g =
[   exp(i*t)]
[ 1/exp(i*t)]
```

```
how =
convert(exp)

g =
[  exp(i*t)]
[ exp(-i*t)]

how =
simplify
```

Notice the first application of `simple` uses `simplify` to produce a sum of sines and cosines. Next, `simple` invokes `radsimp` to produce `cos(t) + i*sin(t)` for the first eigenvector. The third application of `simple` uses `convert(exp)` to change the sines and cosines to complex exponentials. The last application of `simple` uses `simplify` to obtain the final form.

Jordan Canonical Form

The Jordan canonical form results from attempts to diagonalize a matrix by a similarity transformation. For a given matrix A, find a nonsingular matrix V, so that inv(V)*A*V, or, more succinctly, J = V\A*V, is "as close to diagonal as possible." For almost all matrices, the Jordan canonical form is the diagonal matrix of eigenvalues and the columns of the transformation matrix are the eigenvectors. This always happens if the matrix is symmetric or if it has distinct eigenvalues. Some nonsymmetric matrices with multiple eigenvalues cannot be diagonalized. The Jordan form has the eigenvalues on its diagonal, but some of the superdiagonal elements are one, instead of zero. The statement

```
J = jordan(A)
```

computes the Jordan canonical form of A. The statement

```
[V,J] = jordan(A)
```

also computes the similarity transformation. The columns of V are the generalized eigenvectors of A.

The Jordan form is extremely sensitive to perturbations. Almost any change in A causes its Jordan form to be diagonal. This makes it very difficult to compute the Jordan form reliably with floating-point arithmetic. It also implies that A must be known exactly (i.e., without round-off error, etc.). Its elements must be integers, or ratios of small integers. In particular, the variable-precision calculation, jordan(vpa(A)), is not allowed.

For example, let

```
A = sym([12,32,66,116;–25,–76,–164,–294;
         21,66,143,256;–6,–19,–41,–73])
A =
[   12,   32,   66,  116]
[  –25,  –76, –164, –294]
[   21,   66,  143,  256]
[   –6,  –19,  –41,  –73]
```

Then

```
[V,J] = jordan(A)
```

produces

```
V =
[   4,  -2,   4,   3]
[  -6,   8, -11,  -8]
[   4,  -7,  10,   7]
[  -1,   2,  -3,  -2]

J =
[ 1, 1, 0, 0]
[ 0, 1, 0, 0]
[ 0, 0, 2, 1]
[ 0, 0, 0, 2]
```

Therefore A has a double eigenvalue at 1, with a single Jordan block, and a double eigenvalue at 2, also with a single Jordan block. The matrix has only two eigenvectors, V(:,1) and V(:,3). They satisfy

```
A*V(:,1) = 1*V(:,1)
A*V(:,3) = 2*V(:,3)
```

The other two columns of V are generalized eigenvectors of grade 2. They satisfy

```
A*V(:,2) = 1*V(:,2) + V(:,1)
A*V(:,4) = 2*V(:,4) + V(:,3)
```

In mathematical notation, with $\mathbf{v}_j$ = v(:,j), the columns of V and eigenvalues satisfy the relationships

$$(A - \lambda_2 I)\boldsymbol{v}_4 = \boldsymbol{v}_3$$

$$(A - \lambda_1 I)\boldsymbol{v}_2 = \boldsymbol{v}_1$$

Singular Value Decomposition

Only the variable-precision numeric computation of the singular value decomposition is available in the toolbox. One reason for this is that the formulas that result from symbolic computation are usually too long and complicated to be of much use. If A is a symbolic matrix of floating-point or variable-precision numbers, then

```
S = svd(A)
```

computes the singular values of A to an accuracy determined by the current setting of digits. And

```
[U,S,V] = svd(A);
```

produces two orthogonal matrices, U and V, and a diagonal matrix, S, so that

```
A = U*S*V';
```

Let's look at the n-by-n matrix A with elements defined by

```
A(i,j) = 1/(i–j+1/2)
```

There are several interesting ways to generate this matrix A. They are described below. For n = 5, the matrix is

```
[   2     –2    –2/3    –2/5    –2/7]
[2/3       2      –2    –2/3    –2/5]
[2/5    2/3        2      –2    –2/3]
[2/7    2/5     2/3        2      –2]
[2/9    2/7     2/5     2/3        2]
```

It turns out many of the singular values of these matrices are close to π. When n = 16, the MATLAB floating-point computation svd(A) results in

```
3.14159265358979
3.14159265358979
3.14159265358979
3.14159265358979
3.14159265358976
3.14159265358767
3.14159265349961
3.14159265052655
3.14159256925492
3.14159075458606
3.14155754359918
3.14106044663470
3.13504054399745
3.07790297231120
2.69162158686066
1.20968137605669
```

The first four singular values appear equal to π to the available precision.

The most obvious way of generating the 5-by-5 matrix `A` whose *i*-, *j*-th element 1/(*i*–*j*+1/2) is

```
for i=1:n
    for j=1:n
      A(i,j) = sym(1/(i–j+1/2));
   end
end
```

MATLAB's matrix-based computational paradigm, however, permits other, more efficient, approaches. Consider the famous (among MATLAB afficianados) "Tony's Trick."

```
m = ones(n,1); i=(1:n)'; j=1:n;
A = sym(1./(i(:,m)–j(m,:)+1/2));
```

The most efficient way to generate this matrix is with the purely numeric statements

```
[J,I] = meshgrid(1:n);
A = sym(1./(I – J+1/2));
```

Since the elements of `A` are the ratios of small integers, `vpa(A)` produces a variable-precision representation, which is accurate to `digits` precision. Hence

```
S = svd(vpa(A))
```

computes the desired singular values to full accuracy. With `n = 16` and `digits(30)`, the result is

```
S =
[ 1.20968137605668985332455685357 ]
[ 2.69162158686066606774782763594 ]
[ 3.07790297231119748658424727354 ]
[ 3.13504054399744654843898901261 ]
[ 3.14106044663470063805218371924 ]
[ 3.14155754359918083691050658260 ]
[ 3.14159075458605848728982577119 ]
[ 3.14159256925492306470284863102 ]
[ 3.14159265052654880815569479613 ]
[ 3.14159265349961053143856838564 ]
[ 3.14159265358767361712392612384 ]
[ 3.14159265358975439206849907220 ]
[ 3.14159265358979270342635559051 ]
[ 3.14159265358979323325290142781 ]
[ 3.14159265358979323843066846712 ]
[ 3.14159265358979323846255035974 ]
```

There are two ways to compare `S` with `pi`, the floating-point representation of π. In the vector below, the first element is computed by subtraction with variable-precision arithmetic and then converted to a `double`. The second element is computed with floating-point arithmetic.

```
[double(pi*ones(16,1)-S)  pi-double(S)]
```

The results are

```
1.9319e+00   1.9319e+00
4.4997e–01   4.4997e–01
6.3690e–02   6.3690e–02
6.5521e–03   6.5521e–03
5.3221e–04   5.3221e–04
3.5110e–05   3.5110e–05
1.8990e–06   1.8990e–06
8.4335e–08   8.4335e–08
3.0632e–09   3.0632e–09
9.0183e–11   9.0183e–11
2.1196e–12   2.1196e–12
3.8846e–14   3.8636e–14
5.3504e–16   4.4409e–16
5.2097e–18            0
3.1975e–20            0
9.3024e–23            0
```

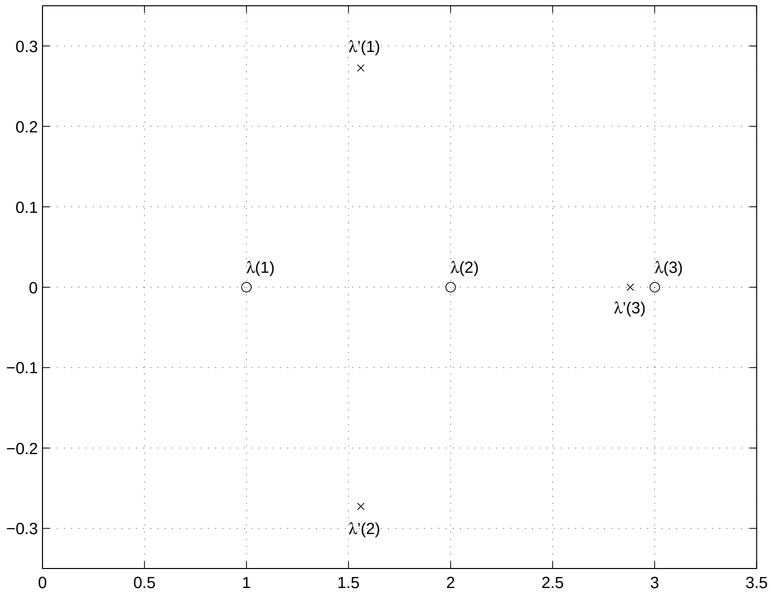
Since the relative accuracy of `pi` is `pi*eps`, which is `6.9757e–16`, either column confirms our suspicion that four of the singular values of the 16-by-16 example equal π to floating-point accuracy.

Eigenvalue Trajectories

This example applies several numeric, symbolic, and graphic techniques to study the behavior of matrix eigenvalues as a parameter in the matrix is varied. This particular setting involves numerical analysis and perturbation theory, but the techniques illustrated are more widely applicable.

In this example, we consider a 3-by-3 matrix A whose eigenvalues are 1, 2, 3. First, we perturb A by another matrix E and parameter t: $A \rightarrow A + tE$. As t

increases from 0 to 10^{-6}, the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ change to $\lambda_1' \approx 1.5596 + 0.2726i$, $\lambda_2' \approx 1.5596 - 0.2726i$, $\lambda_3' \approx 2.8808$.



This, in turn, means that for some value of $t = \tau$, $0 < \tau < 10^{-6}$, the perturbed matrix $A(t) = A + tE$ has a double eigenvalue $\lambda_1 = \lambda_2$.

Let's find the value of `t`, called τ, where this happens.

The starting point is a MATLAB test example, known as `gallery(3)`.

```
A = gallery(3)
A =
  –149        –50      –154
   537        180       546
   –27         –9       –25
```

This is an example of a matrix whose eigenvalues are sensitive to the effects of roundoff errors introduced during their computation. The actual computed eigenvalues may vary from one machine to another, but on a typical workstation, the statements

```
format long
e = eig(A)
```

produce

```
e =
   0.99999999999642
   2.00000000000579
   2.99999999999780
```

Of course, the example was created so that its eigenvalues are actually 1, 2, and 3. Note that three or four digits have been lost to roundoff. This can be easily verified with the toolbox. The statements

```
B = sym(A);
e = eig(B)'
p = poly(B)
f = factor(p)
```

produce

```
e =
[1,  2,  3]

p =
x^3-6*x^2+11*x-6

f =
(x-1)*(x-2)*(x-3)
```

Are the eigenvalues sensitive to the perturbations caused by roundoff error because they are "close together"? Ordinarily, the values 1, 2, and 3 would be regarded as "well separated." But, in this case, the separation should be viewed on the scale of the original matrix. If A were replaced by A/1000, the eigenvalues, which would be .001, .002, .003, would "seem" to be closer together.

But eigenvalue sensitivity is more subtle than just "closeness." With a carefully chosen perturbation of the matrix, it is possible to make two of its eigenvalues coalesce into an actual double root that is extremely sensitive to roundoff and other errors.

One good perturbation direction can be obtained from the outer product of the left and right eigenvectors associated with the most sensitive eigenvalue. The following statement creates

```
E = [130,–390,0;43,–129,0;133,–399,0]
```

the perturbation matrix

```
E =
130   –390      0
 43   –129      0
133   –399      0
```

The perturbation can now be expressed in terms of a single, scalar parameter `t`. The statements

```
syms x t
A = A+t*E
```

replace `A` with the symbolic representation of its perturbation.

```
A =
[–149+130*t, –50–390*t, –154]
[  537+43*t, 180–129*t,  546]
[ –27+133*t,  –9–399*t,  –25]
```

Computing the characteristic polynomial of this new `A`

```
p = poly(A)
```

gives

```
p =
x^3+(–t–6)*x^2+(492512*t+11)*x–6–1221271*t
```

Prettyprinting

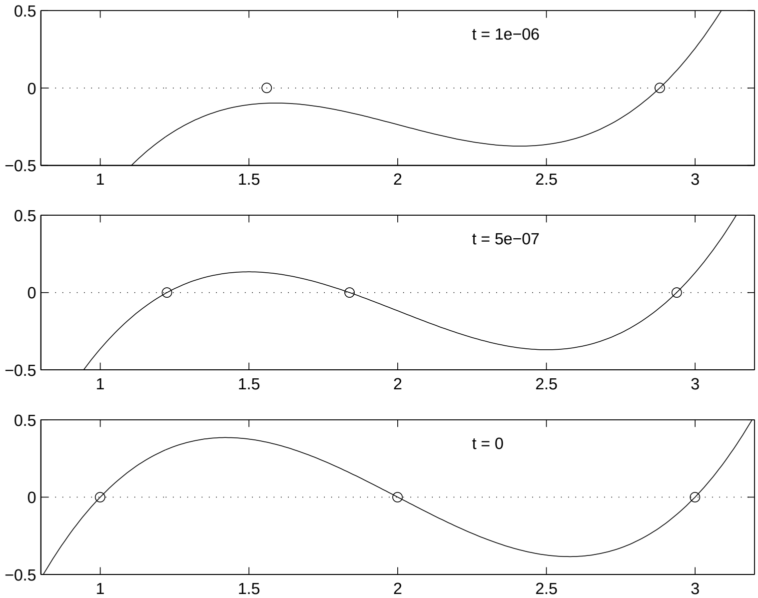
```
pretty(collect(p,x))
```

shows more clearly that `p` is a cubic in `x` whose coefficients vary linearly with `t`

```
    3                 2
[ x   + (– t – 6) x   + (492512 t + 11) x – 6 – 1221271 t ]
```

It turns out that when t is varied over a very small interval, from 0 to 1.0e–6, the desired double root appears. This can best be seen graphically. The first figure shows plots of p, considered as a function of x, for three different values of t: t = 0, t = 0.5e–6, and t = 1.0e–6. For each value, the eigenvalues are computed numerically and also plotted.

```
x = .8:.01:3.2;
for k = 0:2
  c = sym2poly(subs(p,t,k*0.5e-6));
  y = polyval(c,x);
  lambda = eig(double(subs(A,t,k*0.5e-6)));
  subplot(3,1,3-k)
  plot(x,y,'-',x,0*x,':',lambda,0*lambda,'o')
  axis([.8 3.2 -.5 .5])
  text(2.25,.35,['t = ' num2str( k*0.5e-6 )]);
end
```



The bottom subplot shows the unperturbed polynomial, with its three roots at 1, 2, and 3. The middle subplot shows the first two roots approaching each other. In the top subplot, these two roots have become complex and only one real root remains.

The next statements compute and display the actual eigenvalues

```
e = eig(A);
pretty(e)
```

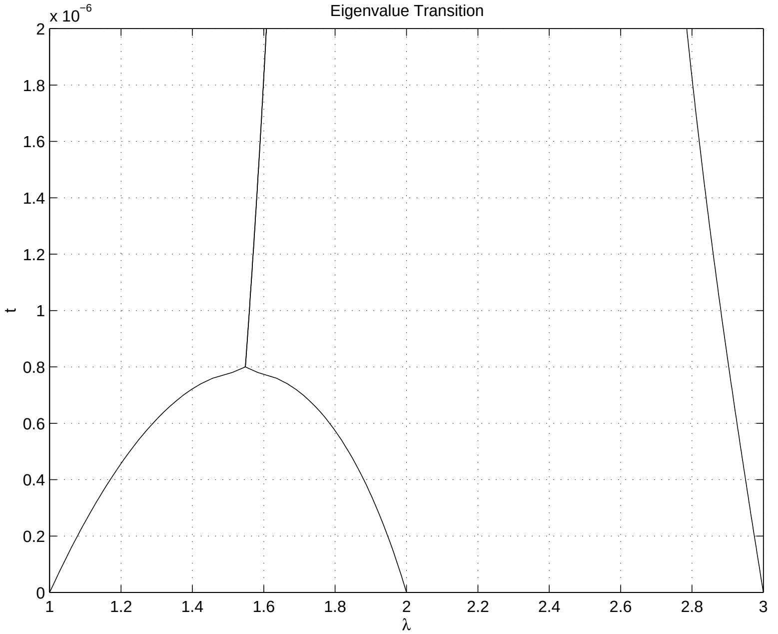
showing that `e(2)` and `e(3)` form a complex conjugate pair.

```
[                           1/3                                    ]
[                    1/3 %1    - 3 %2 + 2 + 1/3 t                  ]
[                                                                  ]
[         1/3                              1/2         1/3         ]
[- 1/6 %1    + 3/2 %2 + 2 + 1/3 t + 1/2 i 3    (1/3 %1    + 3 %2)]
[                                                                  ]
[         1/3                              1/2         1/3         ]
[- 1/6 %1    + 3/2 %2 + 2 + 1/3 t - 1/2 i 3    (1/3 %1    + 3 %2)]

                              2    3
%1 := 3189393 t - 2216286 t  + t  + 3 (-3 + 4432572 t
                         2                      3
     - 1052829647418 t + 358392752910068940 t
                        4 1/2
     - 181922388795 t )

                                 2
      - 1/3 + 492508/3 t - 1/9 t
%2 := ---------------------------
                    1/3
                  %1
```

Next, the symbolic representations of the three eigenvalues are evaluated at many values of t

```
tvals = (2:-.02:0)' * 1.e-6;
r = size(tvals,1);
c = size(e,1);
lambda = zeros(r,c);
for k = 1:c
   lambda(:,k) = double(subs(e(k),t,tvals));
end
plot(lambda,tvals)
xlabel('\lambda'); ylabel('t');
title('Eigenvalue Transition')
```

to produce a plot of their trajectories:



Above t = 0.8e^{-6}, the graphs of two of the eigenvalues intersect, while below t = 0.8e^{-6}, two real roots become a complex conjugate pair. What is the precise value of t that marks this transition? Let τ denote this value of t.

One way to find the *exact* value of τ involves polynomial discriminants. The discriminant of a quadratic polynomial is the familiar quantity under the square root sign in the quadratic formula. When it is negative, the two roots are complex.

There is no `discrim` function in the toolbox, but there is one in Maple and it can be accessed through the toolbox's `maple` function. The statement

```
mhelp discrim
```

provides a brief explanation. Use these commands

```
syms a b c x
maple('discrim', a*x^2+b*x+c, x)
```

to show the generic quadratic's discriminant, $b^2 - 4ac$

```
ans =
-4*a*c+b^2
```

The discriminant for the perturbed cubic characteristic polynomial is obtained, using

```
discrim = maple('discrim',p,x)
```

which produces

```
[discrim =
4-5910096*t+1403772863224*t^2-
477857003880091920*t^3+242563185060*t^4]
```

The quantity τ is one of the four roots of this quartic. Let's find a numeric value for τ

```
digits(24)
s = solve(discrim);
tau = vpa(s)

tau =
[                          1970031.04061804553618913]
[                                   .783792490602e-6]
[ .1076924816049e-5+.318896441018863170083895e-5*i]
[ .1076924816049e-5-.318896441018863170083895e-5*i]
```

Of the four solutions, we know that

```
tau = tau(2)
```

is the transition point

```
tau =
.783792490602e–6
```

because it is closest to our previous estimate.

A more generally applicable method for finding τ is based on the fact that, at a double root, both the function and its derivative must vanish. This results in two polynomial equations to be solved for two unknowns. The statement

```
sol = solve(p,diff(p,x))
```

solves the pair of algebraic equations `p = 0` and `dp/dx = 0` and produces

```
sol =
    t: [4x1 sym]
    x: [4x1 sym]
```

Find τ now by

```
tau = double(sol.t(2))
```

which reveals that the second element of `sol.t` is the desired value of τ

```
format short
tau =
   7.8379e–07
```

Therefore, the second element of `sol.x`

```
sigma = double(sol.x(2))
```

is the double eigenvalue

```
sigma =
   1.5476
```

Let's verify that this value of $\tau = 7.8379e - 07$ does indeed produce a double eigenvalue at $\sigma = 1.5476$. To achieve this, substitute τ for t in the perturbed matrix $A(t) = A + tE$ and find the eigenvalues of $A(t)$. That is,

```
e = eig(double(subs(A,t,tau)))

e =

    1.5476
    1.5476
    2.9047
```

confirms that $\sigma = 1.5476$ is a double eigenvalue of $A(t)$ for $t = 7.8379e - 07$.

Solving Equations

Solving Algebraic Equations

If S is a symbolic expression,

```
solve(S)
```

attempts to find values of the symbolic variable in S (as determined by findsym) for which S is zero. For example

```
syms a b c x
S = a*x^2 + b*x + c;
solve(S)
```

uses the familiar quadratic formula to produce

```
ans =
[1/2/a*(-b+(b^2-4*a*c)^(1/2))]
[1/2/a*(-b-(b^2-4*a*c)^(1/2))]
```

This is a symbolic vector whose elements are the two solutions.

If you want to solve for a specific variable, you must specify that variable as an additional argument. For example, if you want to solve S for b, use the command

```
b = solve(S,b)
```

which returns

```
b =
-(a*x^2+c)/x
```

Note that these examples assume equations of the form $f(x) = 0$. If you need to solve equations of the form $f(x) = q(x)$, you must use quoted strings. In particular, the command

```
s = solve('cos(2*x)+sin(x)=1')
```

returns a vector with four solutions.

```
s =
[      0]
[     pi]
[ 1/6*pi]
[ 5/6*pi]
```

The equation $x^3 - 2x^2 = x - 1$ helps us understand the output of `solve`. Type

```
s = solve('x^3–2*x^2 = x–1')
```

```
s =
[ 1/6*(28+84*i*3^(1/2))^(1/3)+14/3/(28+84*i*3^(1/2))^(1/3)+2/3]
[ –1/12*(28+84*i*3^(1/2))^(1/3)–7/3/(28+84*i*3^(1/2))^(1/3)
        +2/3+1/2*i*3^(1/2)*(1/6*(28+84*i*3^(1/2))^(1/3)
                              –14/3/(28+84*i*3^(1/2))^(1/3))]
[ –1/12*(28+84*i*3^(1/2))^(1/3)–7/3/(28+84*i*3^(1/2))^(1/3)
          +2/3–1/2*i*3^(1/2)*(1/6*(28+84*i*3^(1/2))^(1/3)
                              –14/3/(28+84*i*3^(1/2))^(1/3))]
```

To gain more insight into this solution, let's see the numeric results

```
double(s)
```

```
ans =

  2.24697960371747 + 0.00000000000000i
 –0.80193773580484 + 0.00000000000000i
  0.55495813208737 – 0.00000000000000i
```

It appears that the roots of $x^3 - 2x^2 = x - 1$ are all real. This is misleading. Applying the `vpa` command

```
vpa(s, 10)
```

yields

```
ans =

[  2.246979604+.1e–9*i]
[ –.8019377357+.3e–9*i]
[  .5549581323–.5e–9*i]
```

This shows that the imaginary parts of s are small, but still nonzero.

As another example, consider the commands

```
syms x
s = solve(tan(x)+sin(x)–2);
```

This results in a lengthy 4-by-1 symbolic vector whose qualitative features are, at best, cryptic. As above, use the double command

```
X = double(s)
```

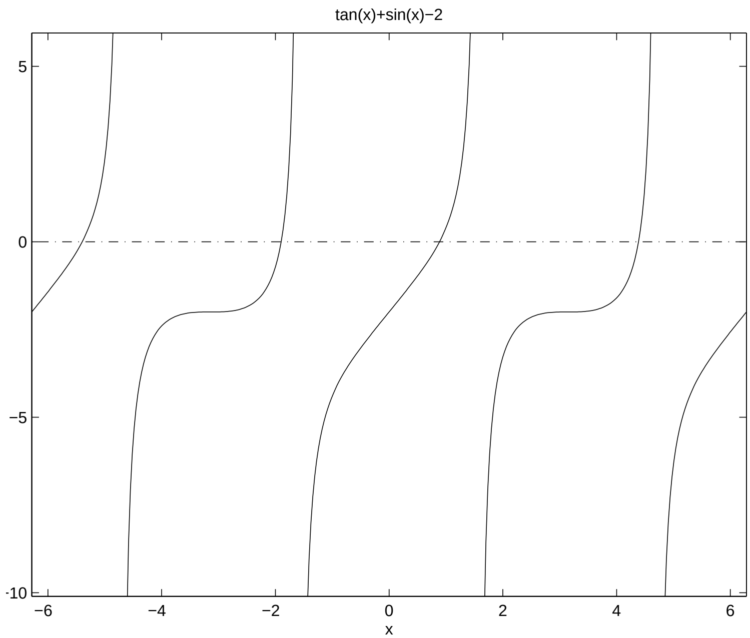
which produces

```
X =
   0.8863
  –1.8979
   2.0766– 1.5151i
   2.0766+ 1.5151i
```

Are these the only two real solutions of $\tan(x) + \sin(x) - 2$ for $x \in [-2\pi, 2\pi]$? To answer this question, plot the function against the real axis

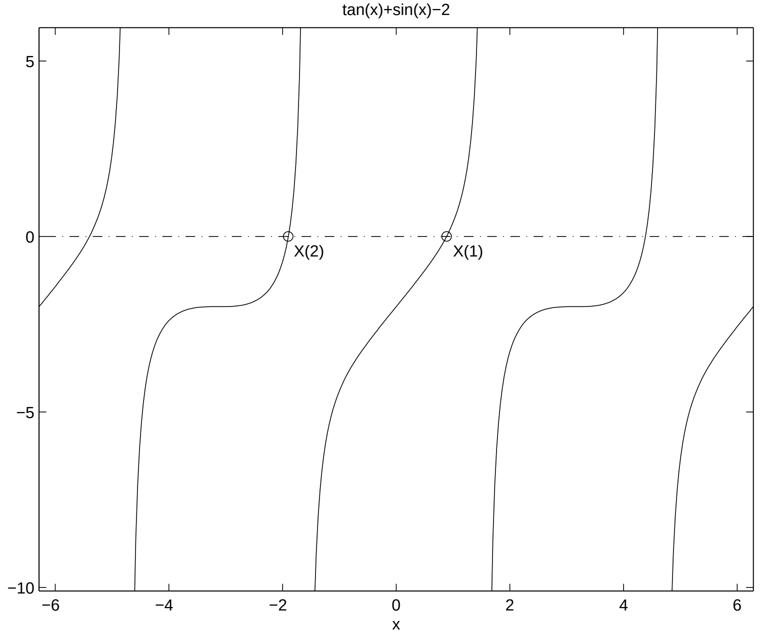
```
ezplot(tan(x)+sin(x)–2)
hold on
w = –2*pi:pi/2:2*pi;
plot(w,0*w,'r-.');
```

The resulting plot



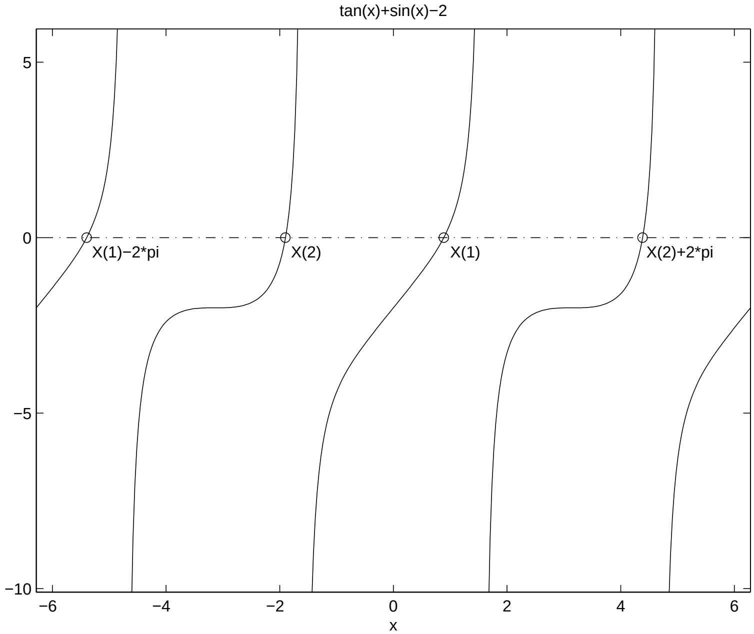
shows that $\tan(x) + \sin(x) - 2$ has four intersections with the real axis for $x \in [-2\pi, 2\pi]$. Since the first two elements of the solution vector X are real, we can plot these on our current figure.

```
RX = [X(1), X(2)]
plot(RX, 0*RX,'gO')
text(-1.8,-0.4,'X(2)')
text(1.0,-0.4,'X(1)')
```



To display the remaining two roots in the interval [–2π, 2π], subtract 2π from X(1) = 0.8863, add 2π to X(2) = –1.8970, and plot the points

```
PX = [X(1)–2*pi,X(2)+2*pi]
plot(PX, 0*PX,'kO')
hold off
```

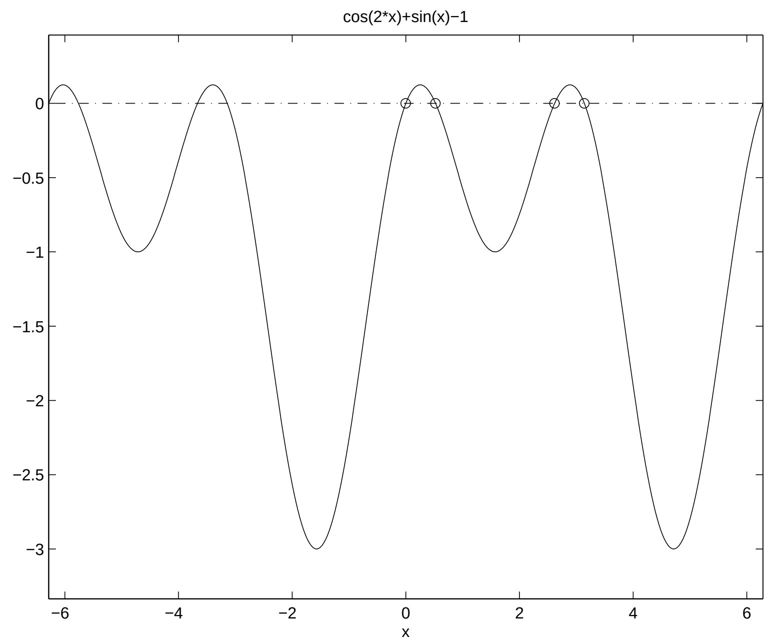


Again for

```
f = cos(2*x) + sin(x) - 1;
s = solve(f);
```

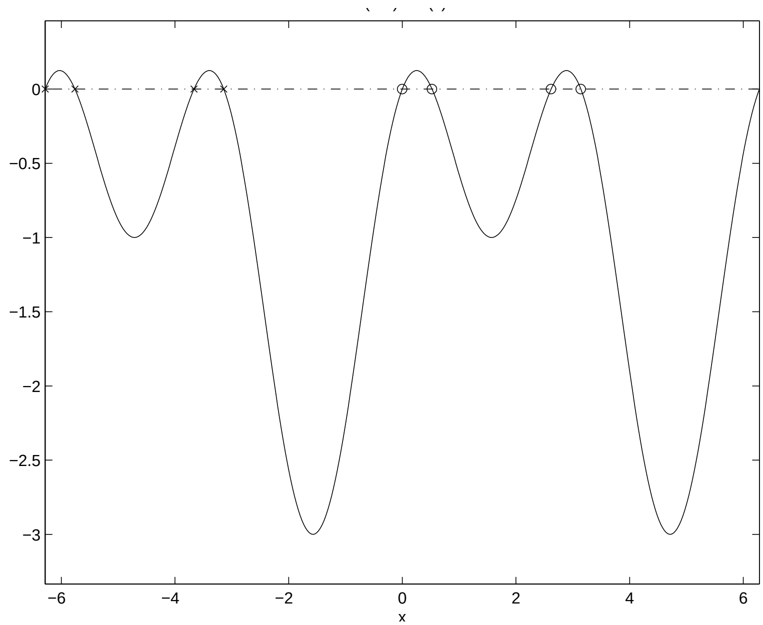
plotting s and f, using

```
ezplot(f)
hold on;
plot(w,0*w,'r-.')
plot(double(s),0*double(s),'gO')
```



makes it clear that solve does not return all solutions. Plotting the points −2*pi, −11/6*pi, −(7/6)*pi, −p along with f shows that these points are also solutions.

```
s2 = [−2*pi, −11/6*pi, −(7/6)*pi, −pi]
plot(s2, 0*s2, 'kX')
```



If we cannot find a symbolic solution, we can compute a variable-precision one. For example

```
digits(6)
syms x
X = solve(log(x)−sin(x)) % Equivalently, solve('log(x)=sin(x)')
```

results in

```
X =
.996957e-1-1.19186*i
```

Several Algebraic Equations

Now let's look at systems of equations. Suppose we have the system

$$x^2 y^2 = 0$$

$$x - \frac{y}{2} = \alpha$$

and we want to solve for x and y. First create the necessary symbolic objects

```
syms x y alpha
```

There are several ways to address the output of `solve`. One is to use a two-output call

```
[x,y] = solve(x^2*y^2; x-(y/2)-alpha)
```

which returns

```
x =
[     0]
[     0]
[ alpha]
[ alpha]

y =
[ -2*alpha]
[ -2*alpha]
[        0]
[        0]
```

Consequently, the solution vector

```
v = [x, y]
```

appears to have redundant components. This is due to the first equation $x^2 y^2 = 0$, which has two solutions in x and y: $x = \pm 0$, $y = \pm 0$. A perturbation to the equations in the form of

```
eqs1 = 'x^2*y^2=1, x-1/2*y-alpha'
```

produces four distinct solutions.

```
x =
[ 1/2*alpha+1/2*(alpha^2+2)^(1/2)]
[ 1/2*alpha–1/2*(alpha^2+2)^(1/2)]
[ 1/2*alpha+1/2*(alpha^2–2)^(1/2)]
[ 1/2*alpha–1/2*(alpha^2–2)^(1/2)]


y =
[ –alpha+(alpha^2+2)^(1/2)]
[ –alpha–(alpha^2+2)^(1/2)]
[ –alpha+(alpha^2–2)^(1/2)]
[ –alpha–(alpha^2–2)^(1/2)]
```

Since we did not specify the dependent variables, `solve` uses `findsym` to determine the variables.

This way of assigning output from `solve` is quite successful for "small" systems. Plainly, if we had, say, a 10-by-10 system of equations, typing

```
[x1,x2,x3,x4,x5,x6,x7,x8,x9,x10] = solve(...)
```

is both awkward and time consuming. To circumvent this difficulty, `solve` can return a structure whose fields are the solutions. In particular, consider the system `u^2–v^2 = a^2`, `u + v = 1`, `a^2–2*a = 3`. The command

```
S = solve('u^2–v^2 = a^2','u + v = 1','a^2–2*a = 3')
```

returns

```
S =
    a: [2x1 sym]
    u: [2x1 sym]
    v: [2x1 sym]
```

The solutions for `a` reside in the "a-field" of `S`. That is,

```
S.a
```

produces

```
ans =
[ –1]
[  3]
```

Similar comments apply to the solutions for u and v. The structure S can now be manipulated by field and index to access a particular portion of the solution. For example, if we want to examine the second solution, we can use the following statement

```
s2 = [S.a(2), S.u(2), S.v(2)]
```

to extract the second component of each field

```
s2 =
[  3,  5, -4]
```

The following statement

```
M = [S.a, S.u, S.v]
```

creates the solution matrix M

```
M =
[ -1,  1,  0]
[  3,  5, -4]
```

whose rows comprise the distinct solutions of the system.

Linear systems of simultaneous equations can also be solved using matrix division. For example,

```
clear u v x y
syms u v x y
S = solve(x+2*y-u, 4*x+5*y-v);
sol = [S.x;S.y]
```

and

```
A = [1 2; 4 5];
b = [u; v];
z = A\b
```

result in

```
sol =

[ –5/3*u+2/3*v]
[  4/3*u–1/3*v]

z =
[ –5/3*u+2/3*v]
[  4/3*u–1/3*v]
```

Thus `s` and `z` produce the same solution, although the results are assigned to different variables.

Single Differential Equation

The function `dsolve` computes symbolic solutions to ordinary differential equations. The equations are specified by symbolic expressions containing the letter `D` to denote differentiation. The symbols `D2`, `D3`, ... `DN`, correspond to the second, third, ..., Nth derivative, respectively. Thus, `D2y` is the Symbolic Math Toolbox equivalent of d^2y/dt^2. The dependent variables are those preceded by `D` and the default independent variable is `t`. Note that names of symbolic variables should not contain `D`. The independent variable can be changed from `t` to some other symbolic variable by including that variable as the last input argument.

Initial conditions can be specified by additional equations. If initial conditions are not specified, the solutions contain constants of integration, `C1`, `C2`, etc.

The output from `dsolve` parallels the output from `solve`. That is, you can call `dsolve` with the number of output variables equal to the number of dependent variables or place the output in a structure whose fields contain the solutions of the differential equations. The table below details the output syntax.

Syntax	Scope
`y = dsolve('Dyt= y0*y')`	One equation, one output

Syntax	Scope
`[u,v] = dsolve('Du = v', 'Dv = u')`	Two equations, two output
`S = dsolve('Df=g','Dg=h','Dh=–f')`	Three equations, output structure
`S.f, S.g, S.h`	Solutions

Example 1

The following call to `dsolve`

```
dsolve('Dy=1+y^2')
```

uses `y` as the dependent variable and `t` as the default independent variable. The output of this command is

```
ans =
tan(t–C1)
```

To specify an initial condition, use

```
y = dsolve('Dy=1+y^2','y(0)=1')
```

This produces

```
y =
tan(t+1/4*pi)
```

Notice that `y` is in the MATLAB workspace, but the independent variable `t` is not. Thus, the command `diff(y,t)` returns an error. To place `t` in the workspace, type `syms t`.

Example 2

Nonlinear equations may have multiple solutions, even when initial conditions are given.

```
x = dsolve('(Dx)^2+x^2=1','x(0)=0')
```

results in

```
x =
[–sin(t)]
[ sin(t)]
```

Example 3

Here is a second order differential equation with two initial conditions. The command

```
y = dsolve('D2y=cos(2*x)-y','y(0)=1','Dy(0)=0', 'x')
```

produces

```
y =
-2/3*cos(x)^2+1/3+4/3*cos(x)
```

The key issues in this example are the order of the equation and the initial conditions. To solve the ordinary differential equation

$$\frac{d^3u}{dx^3} = u$$

$$u(0) = 1,\ u'(0) = -1,\ u''(0) = \pi$$

simply type

```
u = dsolve('D3u=u','u(0)=1','Du(0)=-1','D2u(0) = pi','x')
```

Use `D3u` to represent d^3u/dx^3 and `D2u(0)` for $u''(0)$.

Several Differential Equations

The function `dsolve` can also handle several ordinary differential equations in several variables, with or without initial conditions. For example, here is a pair of linear, first order equations.

```
S = dsolve('Df = 3*f+4*g', 'Dg = -4*f+3*g')
```

The computed solutions are returned in the structure `S`. You can determine the values of `f` and `g` by typing

```
f = S.f
f =
C2*exp(3*t)*sin(4*t)-C1*exp(3*t)*cos(4*t)

g = S.g
g =
C1*exp(3*t)*sin(4*t)+C2*exp(3*t)*cos(4*t)
```

If you prefer to recover f and g directly as well as include initial conditions, type

```
[f,g] = dsolve('Df=3*f+4*g, Dg =–4*f+3*g', 'f(0) = 0, g(0) = 1')

f =
exp(3*t)*sin(4*t)

g =
exp(3*t)*cos(4*t)
```

This table details some examples and Symbolic Math Toolbox syntax. Note that the final entry in the table is the Airy differential equation whose solution is referred to as the Airy function.

Differential Equation	MATLAB Command
$\frac{dy}{dt} + 4y(t) = e^{-t}$ $y(0) = 1$	`y = dsolve('Dy+4*y = exp(–t)',` `'y(0) = 1')`
$\frac{d^2y}{dx^2} + 4y(x) = e^{-2x}$ $y(0) = 0, y(\pi) = 0$	`y = dsolve('D2y+4*y = exp(–2*x)',` `'y(0)=0', 'y(pi) = 0', 'x')`

$\frac{d^2y}{dx^2} = xy(x)$
 	
$y(0) = 0, y(3) = \frac{1}{\pi}K_{\frac{1}{3}}(2\sqrt{3})$
 	
(The Airy Equation)
	`y = dsolve('D2y = x*y','y(0) = 0',`
	`'y(3) = besselk(1/3, 2*sqrt(3))/pi',`
	`'x')`

The Airy function plays an important role in the mathematical modeling of the dispersion of water waves. It is a nontrivial exercise to show that the Fourier transform of the Airy function is $\exp(iw^3/3)$. With this in mind, we proceed to the section "Integral Transforms"—the next stop on our tour of the Symbolic Math Toolbox.

Integral Transforms

Integral transforms and their discrete counterparts are powerful and important computational tools in engineering, applied mathematics, and science. The Laplace transform acts on continuous systems (differential equations); its discrete analog, the *z*-transform, operates on discrete systems (difference equations). Similarly, the Fourier transform operates on continuous functions; the discrete Fourier transform (DFT), on finite data samples. The most famous and efficient DFT is the fast Fourier transform (FFT).

The Symbolic Math Toolbox provides commands that enable you to compute the analytic Fourier, Laplace, and *z*-transforms and their inverses. If you want to take the FFT of a data set, use the MATLAB command `fft`. The following sections describe the Symbolic Math Toolbox syntax for these three fundamental transforms and their inverses.

The Fourier and Inverse Fourier Transforms

The Fourier transform of a function $f(x)$ is defined as

$$F[f](w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

and the inverse Fourier transform (IFT) as

$$F^{-1}[f](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)e^{iwx}dw$$

We refer to this formulation as the Fourier transform of f with respect to x as a function of w. Or, more concisely, the Fourier transform of f with respect to x at w. Mathematicians often use the notation $F[f]$ to denote the Fourier transform of f. In this setting, the transform is taken with respect to the independent variable of f (if $f = f(t)$, then t is the independent variable; $f = f(x)$ implies that x is the independent variable, etc.) at the default variable w. We refer to F[f] as the Fourier transform of f at w and $F^{-1}[f]$ is the IFT of f at x. See `fourier` and `ifourier` in the reference pages for tables that show the Symbolic Math Toolbox commands equivalent to various mathematical representations of the Fourier and inverse Fourier transforms.

In the section on differential equations, we mentioned the Airy differential equation

$$\frac{d^2 y}{dx^2} = xy(x)$$

whose solution, $A(x)$, has a Fourier transform of

$$e^{iw^3/3}$$

Let's verify this claim with the Symbolic Math Toolbox. Indeed,

$$F[A] = e^{iw^3/3}$$

if and only if

$$A = F^{-1}[e^{iw^3/3}]$$

The statements

```
syms w x
A = ifourier(exp(i*w^3/3),w,x);
A = simple(A);
pretty(A)
```

return

```
          1/2     1/2                       1/2         1/3   3/2
         3       x     besselk(1/3, 2/9 3       (x 3      )     )
1/3   -----------------------------------------------------------
                                pi
```

$$A(x) = \sqrt{\frac{x}{3}}\,\frac{K_{1/3}\left(\frac{2}{3}x^{3/2}\right)}{\pi}$$

where $K_{1/3}(z)$ is a Bessel function of the second kind of order 1/3. Solving the Airy differential equation

```
Y = dsolve('D2y = x*y','x');
pretty(Y)
```

returns

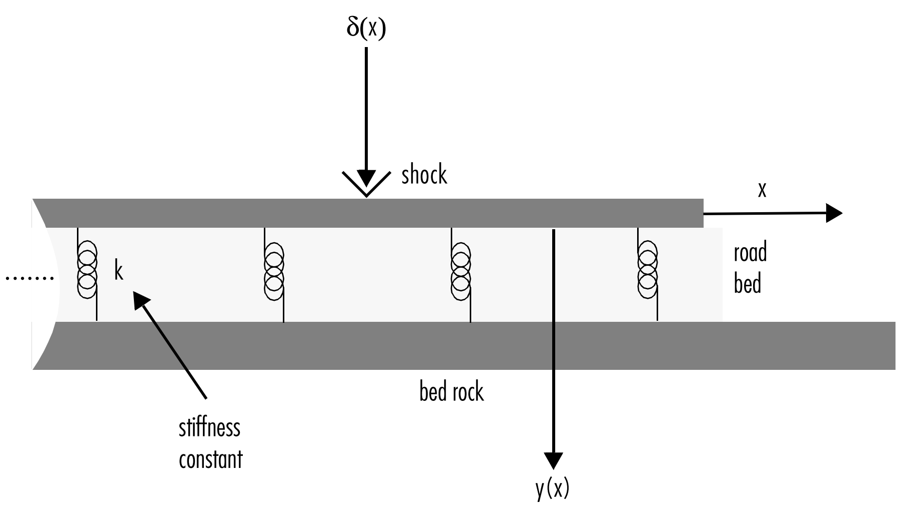
```
     1/2                         3/2         1/2                         3/2
C1 x    besseli(1/3, 2/3 x    ) + C2 x    besselk(1/3, 2/3 x    )
```

and setting

$$C1 = 0, C2 = \frac{1}{\sqrt{3}\pi}$$

yields $A(x)$.

An application of the Fourier transform is the solution of ordinary and partial differential equations over the real line. Consider the deformation of an infinitely long beam resting on an elastic foundation with a shock applied to it at a point. A "real world" analogy to this phenomenon is a set of railroad tracks atop a road bed.



The shock could be induced by a pneumatic hammer blow (or a swing from the mighty John Henry!).

The differential equation idealizing this physical setting is

$$\frac{d^4 y}{dx^4} + \frac{k}{EI} y = \frac{1}{EI}\delta(x), \quad -\infty < x < \infty$$

Here, E represents elasticity of the beam (rail road track), I is the "beam constant," and k is the spring (road bed) stiffness. The shock force on the right hand side of the differential equation is modeled by the Dirac Delta function $\delta(x)$. If you are unfamiliar with $\delta(x)$, you may be surprised to learn that (despite its name), it is not a function at all. Rather, $\delta(x)$ is an example of what mathematicians call a *distribution*. The Dirac Delta function (named after the physicist Paul Dirac) has the following important property

$$\int_{-x}^{\infty} f(x-y)\delta(x)dx = f(x)$$

A definition of the Dirac Delta function is

$$\delta(x) = \lim_{n \to \infty} n\chi_{(-1/2n,\, 1/2n)}(x)$$

where

$$\chi_{(-1/2n,\, 1/2n)}(x) = \begin{cases} 1 & \text{for } -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

You can evaluate the Dirac Delta function at a point (say) $x = 3$, using the commands

```
syms x
del = sym('Dirac(x)');
vpa(subs(del,x,3))
```

which return

```
ans =
   0
```

Returning to the differential equation, let $Y(w) = F[y(x)](w)$ and $\Delta(w) = F[\delta(x)](w)$. Indeed, try the command `fourier(del,x,w)`. The Fourier transform turns differentiation into exponentiation, and, in particular,

$$F\left[\frac{d^4y}{dx^4}\right](w) = w^4 Y(w)$$

To see a demonstration of this property, try this

```
syms w x
fourier(diff(sym('y(x)'),x,4),x,w)
```

which returns

```
ans =
w^4*fourier(y(x),x,w)
```

Note that you can call the `fourier` command with one, two, or three inputs (see the reference pages for `fourier`). With a single input argument, `fourier(f)` returns a function of the default variable `w`. If the input argument is a function of `w`, `fourier(f)` returns a function of `t`. All inputs to `fourier` must be symbolic objects.

We now see that applying the Fourier transform to the differential equation above yields the algebraic equation

$$\left(w^4 + \frac{k}{EI}\right)Y(w) = \Delta(w)$$

or

$$Y(w) = \Delta(w)\, G(w)$$

where

$$G(w) = \frac{1}{w^4 + \frac{k}{EI}} = F[g(x)](w)$$

for some function $g(x)$. That is, g is the inverse Fourier transform of G

$$g(x) = F^{-1}[G(w)](x)$$

The Symbolic Math Toolbox counterpart to the IFT is `ifourier`. This behavior of `ifourier` parallels `fourier` with one, two, or three input arguments (see the reference pages for `ifourier`).

Continuing with the solution of our differential equation, we observe that the ratio

$$\frac{K}{EI}$$

is a relatively "large" number since the road bed has a high stiffness constant k and a rail road track has a low elasticity E and beam constant I. We make the simplifying assumption that

$$\frac{K}{EI} = 1024$$

This is done to ease the computation of $F^{-1}[G(w)](x)$. Proceeding, we type

```
G = 1/(w^4 + 1024);
g = ifourier(G,w,x);
g = simple(expand(g));
pretty(g)
```

and see

```
(-1/1024 + 1/1024i) (exp((4 - 4i)x) Heaviside(x) - exp((4 - 4 i)x)
+ i exp((4 + 4i) x) Heaviside(x) - i Heaviside(x) exp((-4 - 4i)x)
- Heaviside(x) exp((-4 + 4 i) x) - i exp((4 + 4 i) x))
```

Notice that `g` contains the Heaviside distribution

$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \\ \text{singular} & \text{for } x = 0 \end{cases}$$

We can factor out the Heaviside distribution.

```
H = sym('Heaviside(x)');
g = collect(g,H);
pretty(g)
```

returns

```
(– 1/1024 + 1/1024 i) (exp((4 – 4 i) x) + i exp((4 + 4 i) x)
– i exp((–4 – 4 i) x) – exp((–4 + 4 i) x)) Heaviside(x)
+ (– 1/1024 + 1/1024 i) (–exp((4 – 4 i) x) – i exp((4 + 4 i) x))
```

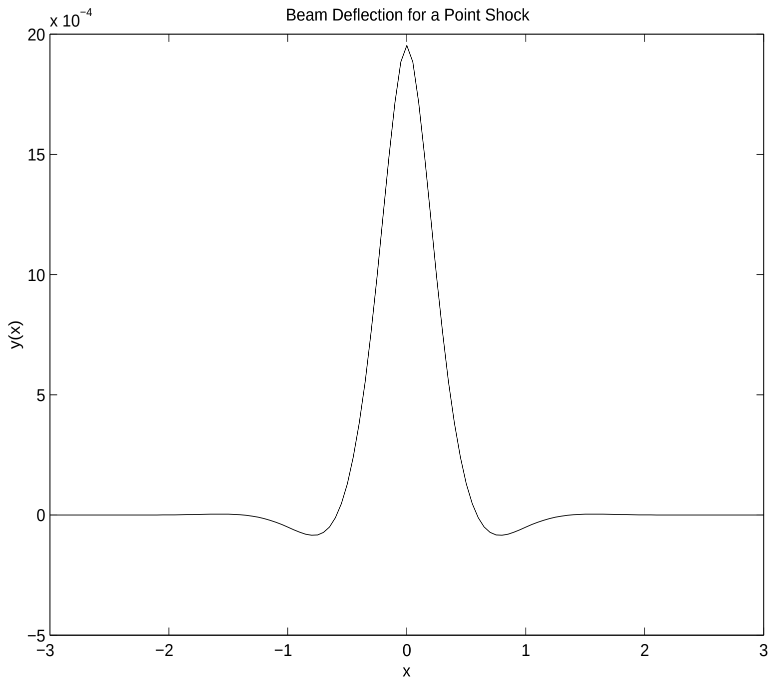
Since Y is the product of Fourier transforms, then y is the convolution of the transformed functions. That is, $F[y] = Y(w) = \Delta(w)\, G(w) = F[\delta]\, F[g]$ implies

$$y(x) = (\delta \cdot g)(x) = \int_{-\infty}^{\infty} g(x-y)\delta(x)dx = g(x)$$

by the special property of the Dirac Delta function. To plot this function, we must substitute the domain of x into $y(x)$, using the `subs` command.

```
XX = –3:0.05:3;
YY = double(subs(g,x,XX));
plot(XX,YY)
title('Beam Deflection for a Point Shock')
xlabel('x'); ylabel('y(x)');
```

The resulting graph



shows that the impact of a blow on a beam is highly localized; the greatest deflection occurs at the point of impact and falls off sharply from there. This is the behavior we expect from experience.

The Laplace and Inverse Laplace Transforms

The Laplace transform of a function $f(t)$ is defined as

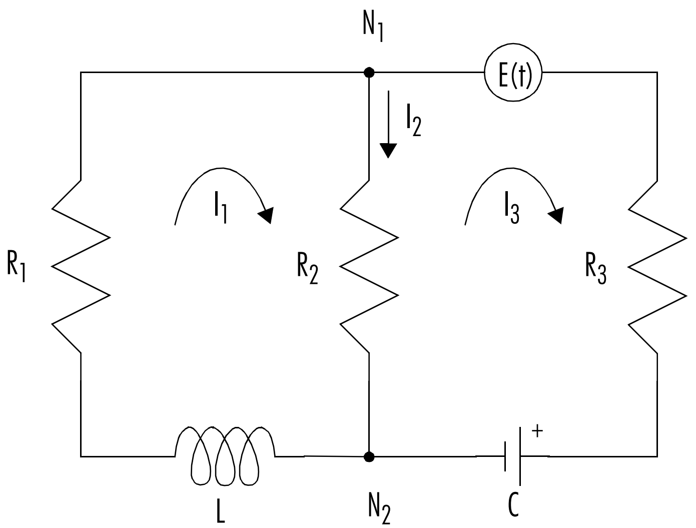
$$L[f](s) = \int_0^\infty f(t)e^{-ts}\,dt$$

while the inverse Laplace transform (ILT) of $f(s)$ is

$$L^{-1}[f](t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}\,ds$$

where c is a real number selected so that all singularities of $f(s)$ are to the left of the line $s = c$. The notation $L[f]$ denotes the Laplace transform of f at s. Similarly, $L^{-1}[f]$ is the ILT of f at t.

The Laplace transform has many applications including the solution of ordinary differential equations/initial value problems. Consider the resistance-inductor-capacitor (RLC) circuit below..



Let Rj and Ij, $j = 1, 2, 3$ be resistances (measured in ohms) and currents (amps), respectively; L be inductance (henrys), and C be capacitance (farads); $E(t)$ be the electromotive force, and $Q(t)$ be the charge

By applying Kirchhoff's voltage and current laws, Ohm's Law, Faraday's Law, and Henry's Law, you can arrive at the following system of simultaneous ordinary differential equations.

$$\frac{dI_1}{dt} + \frac{R_2}{L}\frac{dQ}{dt} = \frac{R_1 - R_2}{L} I_1,\ I_1(0) = I_{10}$$

$$\frac{dQ}{dt} = \frac{1}{R_3 + R_2}\left(E(t) - \frac{1}{C}Q(t)\right) + \frac{R_2}{R_3 + R_2} I_1,\ Q(0) = Q_0$$

Let's solve this system of differential equations using `laplace`. We will first treat the R_j, L, and C as (unknown) real constants and then supply values later on in the computation.

```
syms R1 R2 R3 L C real
dI1 = sym('diff(I1(t),t)'); dQ = sym('diff(Q(t),t)');
I1 = sym('I1(t)'); Q = sym('Q(t)');
syms t s
E = sin(t);  % Voltage
eq1 = dI1 + R2*dQ/L - (R2 - R1)*I1/L;
eq2 = dQ - (E - Q/C)/(R2 + R3) - R2*I1/(R2 + R3);
```

At this point, we have constructed the equations in the MATLAB workspace. An approach to solving the differential equations is to apply the Laplace transform, which we will apply to `eq1` and `eq2`. Transforming `eq1` and `eq2`

```
L1 = laplace(eq1,t,s)
L2 = laplace(eq2,t,s)
```

returns

```
L1 =
[s*laplace(I1(t),t,s) - I1(0) + R2/L*(s*laplace(Q(t),t,s) - Q(0))
-  (R2 - R1)/L*laplace(I1(t),t,s)]

L2 =
[s*laplace(Q(t),t,s) - Q(0) - 1/(R2 + R3)/C*(C/(s^2 + 1) -
laplace(Q(t),t,s)) - R2/(R2 + R3)*laplace(I1(t),t,s)]
```

Now we need to solve the system of equations `L1 = 0`, `L2 = 0` for `laplace(I1(t),t,s)` and `laplace(Q(t),t,s)`, the Laplace transforms of I_1 and Q, respectively. To do this, we need to make a series of substitutions. For the purposes of this example, use the quantities $R_1 = 4\ \Omega$ (ohms), $R_2 = 2\ \Omega$, $R_3 = 3\ \Omega$, $C = 1/4$ farads, $L = 1.6$ H (henrys), $I_1(0) = 15$ amps, and $Q(0) = 2$ amps/sec. Substituting these values in `L1`

```
syms LI1 LQ
NI1 = subs(L1,{R1,R2,R3,L,C,'I1(0)','Q(0)'}, ...
      {4,2,3,1.6,1/4,15,2})
```

returns

```
NI1 =
s*laplace(I1(t),t,s) - 35/2 + 5/4*s*laplace(Q(t),t,s)
+ 5/4*laplace(I1(t),t,s)
```

The substitution

```
NQ = subs(L2,{R1,R2,R3,L,C,'I1(0)','Q(0)'},{4,2,3,1.6,1/4,15,2})
```

returns

```
NQ =
s*laplace(Q(t),t,s) - 2 - 1/5/(s^2+1) + 4/5*laplace(Q(t),t,s)
- 2/5*laplace(I1(t),t,s)
```

To solve for `laplace(I1(t),t,s)` and `laplace(Q(t),t,s)`, we make a final pair of substitutions. First, replace the strings `'laplace(I1(t),t,s)'` and `'laplace(Q(t),t,s)'` by the syms `LI1` and `LQ`, using

```
NI1 =
subs(NI1,{'laplace(I1(t),t,s)','laplace(Q(t),t,s)'},{LI1,LQ})
```

to obtain

```
NI1 =
s*LI1 - 35/2 + 5/4*s*LQ + 5/4*LI1
```

Collecting terms

```
NI1 = collect(NI1,LI1)
```

gives

```
NI1 =
(s + 5/4)*LI1 - 35/2 + 5/4*s*LQ
```

A similar string substitution

```
NQ =
subs(NQ,{'laplace(I1(t),t,s)','laplace(Q(t),t,s)'},{LI1,LQ})
```

yields

```
NQ =
s*LQ - 2 - 1/5/(s^2+1) + 4/5*LQ - 2/5*LI1
```

which, after collecting terms,

```
NQ = collect(NQ,LQ)
```

gives

```
NQ =
(4/5 + s)*LQ - 2/5*LI1 - 2 - 1/5/(s^2+1)
```

Now, solving for `LI1` and `LQ`

```
[LI1,LQ] = solve(NI1,NQ,LI1,LQ)
```

we obtain

```
LI1 =
[5*(59*s + 56 + 56*s^2 + 60*s^3)/(51*s^3 + 40*s^2 + 51*s + 20
+ 20*s^4)]

LQ =
[(44*s + 195 + 190*s^2 + 40*s^3)/(51*s^3 + 40*s^2 + 51*s + 20
+ 20*s^4)]
```

To recover `I1` and `Q` we need to compute the inverse Laplace transform of `LI1` and `LQ`. Inverting `LI1`

```
I1 = ilaplace(LI1,s,t)
```

produces

```
I1 =
-5/51*sin(t)+15*exp(-51/40*t)*cosh(1/40*1001^(1/2)*t)
- 1465/7293*exp(-51/40*t)*1001^(1/2)*sinh(1/40*1001^(1/2)*t)
- 5/51*sin(t)
```

Inverting `LQ`

```
Q = ilaplace(LQ,s,t)
```

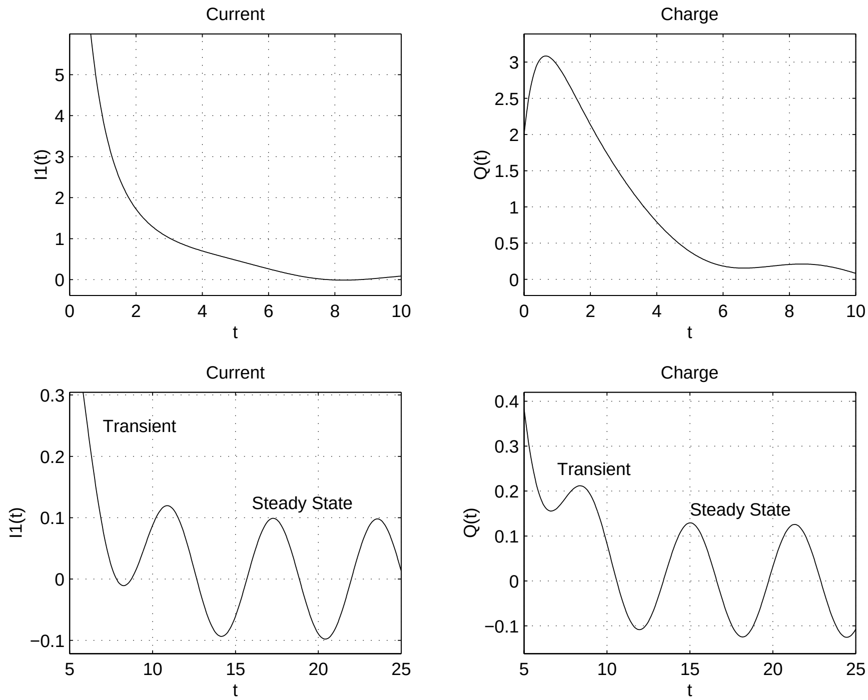
yields

```
Q =
-5/51*cos(t) + 4/51*sin(t)
+ 107/51*exp(-51/40*t)*cosh(1/40*1001^(1/2)*t)
+ 2039/7293*exp(-51/40*t)*1001^(1/2)*sinh(1/40*1001^(1/2)*t)
```

Now let's plot the current `I1(t)` and charge `Q(t)` in two different time domains, $0 \le t \le 10$ and $5 \le t \le 25$. The statements

```
subplot(2,2,1); ezplot(I1,[0,10]);
title('Current'); ylabel('I1(t)'); grid
subplot(2,2,2); ezplot(Q,[0,10]);
title('Charge'); ylabel('Q(t)'); grid
subplot(2,2,3); ezplot(I1,[5,25]);
title('Current'); ylabel('I1(t)'); grid
text(7,0.25,'Transient'); text(16,0.125,'Steady State');
subplot(2,2,4); ezplot(Q,[5,25]);
title('Charge'); ylabel('Q(t)'); grid
text(7,0.25,'Transient'); text(15,0.16,'Steady State');
```

generate the desired plots



Note that the circuit's behavior, which appears to be exponential decay in the short term, turns out to be oscillatory in the long term. The apparent discrepancy arises because the circuit's behavior actually has two components: an exponential part that decays rapidly (the "transient" component) and an oscillatory part that persists (the "steady-state" component).

The Z– and Inverse Z–transforms

The (one-sided) z-transform of a function $f(n)$ is defined as

$$Z[f](z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

The notation $Z[f]$ refers to the z-transform of f at z. Let R be a positive number so that the function $g(z)$ is analytic on and outside the circle $|z| = R$. Then the inverse z-transform (IZT) of g at n is defined as

$$Z^{-1}[g](n) = \frac{1}{2\pi i} \oint_{|z| = R} g(z) z^{n-1} dz, \, n = 1, 2, \ldots$$

The notation $Z^{-1}[f]$ means the IZT of f at n. The Symbolic Math Toolbox commands `ztrans` and `iztrans` apply the z-transform and IZT to symbolic expressions, respectively. See `ztrans` and `iztrans` for tables showing various mathematical representations of the z-transform and inverse z-transform and their Symbolic Math Toolbox counterparts.

The z-transform is often used to solve difference equations. In particular, consider the famous "Rabbit Problem." That is, suppose that rabbits reproduce only on odd birthdays (1, 3, 5, 7, ...). If $p(n)$ is the rabbit population at year n, then p obeys the difference equation

$$p(n+2) = p(n+1) + p(n),\ p(0) = 1,\ p(1) = 2.$$

We can use `ztrans` to find the population each year $p(n)$. First, we apply `ztrans` to the equations

```
pn = sym('p(n)');
pn1 = sym('p(n+1)');
pn2 = sym('p(n+2)');
syms n z
eq = pn2 - pn1 - pn;
Zeq = ztrans(eq, n, z)
```

to obtain

```
Zeq =
z^2*ztrans(p(n),n,z)-p(0)*z^2-p(1)*z-z*ztrans(p(n),n,z)+p(0)*z
-ztrans(p(n),n,z)
```

Next, replace 'ztrans(p(n),n,z)' with Pz and insert the initial conditions for p(0) and p(1).

```
syms Pz
Zeq = subs(Zeq,{'ztrans(p(n),n,z)','p(0)','p(1)'},{Pz,1,2})
```

to obtain

```
Zeq =
z^2*Pz-1*z^2-2*z-z*Pz+1*z-Pz
```

Collecting terms

```
eq = collect(Zeq,Pz)
```

yields

```
eq =
(z^2-z-1)*Pz-z^2-z
```

Now solve for Pz

```
P = solve(eq,Pz)
```

to obtain

```
P =
z*(z + 1)/(z^2 - z - 1)
```

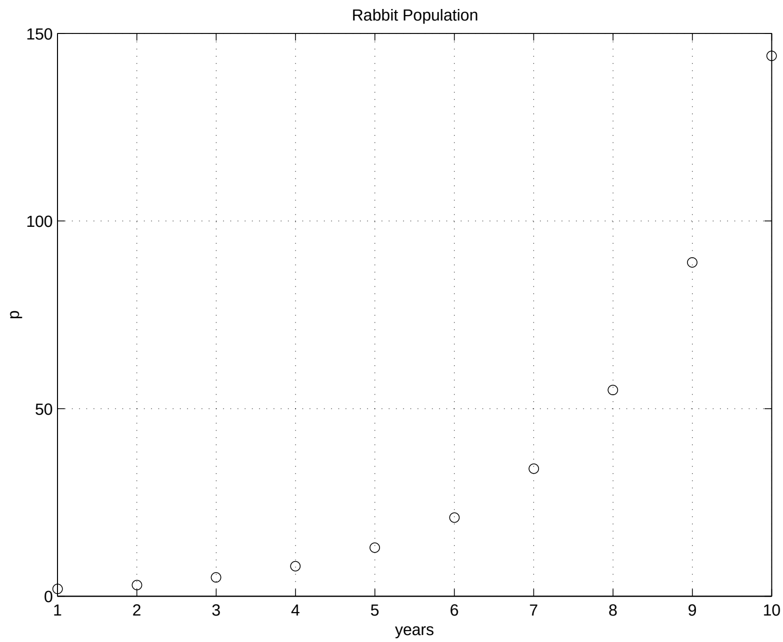
To recover $p(n)$, we take the inverse z-transform of P.

```
p = iztrans(P,z,n);
p = simple(p);
pretty(p)

         n   1/2           n                n   1/2           n
        2   5             2             (-2)   5          (-2)
3/10 --------- + 1/2 -------- - 3/10 ---------- + 1/2 ----------
       1/2    n         1/2   n         1/2    n         1/2    n
     (5   - 1)        (5   - 1)       (5    + 1)       (5    + 1)
```

Finally, let's plot p

```
m = 1:10;
y = double(subs(f,n,m));
plot(m,y,'rO')
title('Rabbit Population');
xlabel('years'); ylabel('f(n)'); grid on
```

to show the growth in rabbit population over time.



References

Andrews, L.C., B.K. Shivamoggi, *Integral Transforms for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1986

Crandall, R.E., *Projects in Scientific Computation*, Springer-Verlag Publishers, New York, 1994

Strang, G., *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, Wellesley, MA, 1986

Special Mathematical Functions

Over fifty of the special functions of classical applied mathematics are available in the toolbox. These functions are accessed with the `mfun` function, which numerically evaluates a special function for the specified parameters. This allows you to evaluate functions that are not available in standard MATLAB, such as the Fresnel cosine integral. In addition, you can evaluate several MATLAB special functions in the complex plane, such as the error function.

For example, suppose you want to evaluate the hyperbolic cosine integral at the points 2+i, 0, and 4.5. First type

```
help mfunlist
```

to see the list of functions available for `mfun`. This list provides a brief mathematical description of each function, its Maple name, and the parameters it needs. From the list, you can see that the hyperbolic cosine integral is called `Chi`, and it takes one complex argument. For additional information, you can access Maple help on the hyperbolic cosine integral using

```
mhelp Chi
```

Now type

```
z = [2+i 0 4.5];
w = mfun('Chi',z)
```

which returns

```
w =
   2.0303 + 1.7227i     NaN       13.9658
```

`mfun` returns `NaN`s where the function has a singularity. The hyperbolic cosine integral has a singularity at $z = 0$.

These special functions can be used with the `mfun` function:

- Airy Functions
- Binomial Coefficients
- Riemann Zeta Functions
- Bernoulli Numbers and Polynomials
- Euler Numbers and Polynomials

- Harmonic Function
- Exponential Integrals
- Logarithmic Integral
- Sine and Cosine Integrals
- Hyperbolic Sine and Cosine Integrals
- Shifted Sine Integral
- Fresnel Sine and Cosine Integral
- Dawson's Integral
- Error Function
- Complementary Error Function and its Iterated Integrals
- Gamma Function
- Logarithm of the Gamma Function
- Incomplete Gamma Function
- Digamma Function
- Polygamma Function
- Generalized Hypergeometric Function
- Bessel Functions
- Incomplete Elliptic Integrals
- Complete Elliptic Integrals
- Complete Elliptic Integrals with Complementary Modulus
- Beta Function
- Dilogarithm Integral
- Lambert's *W* Function
- Dirac Delta Function (distribution)
- Heaviside Function (distribution)

The orthogonal polynomials listed below are available with the Extended Symbolic Math Toolbox:

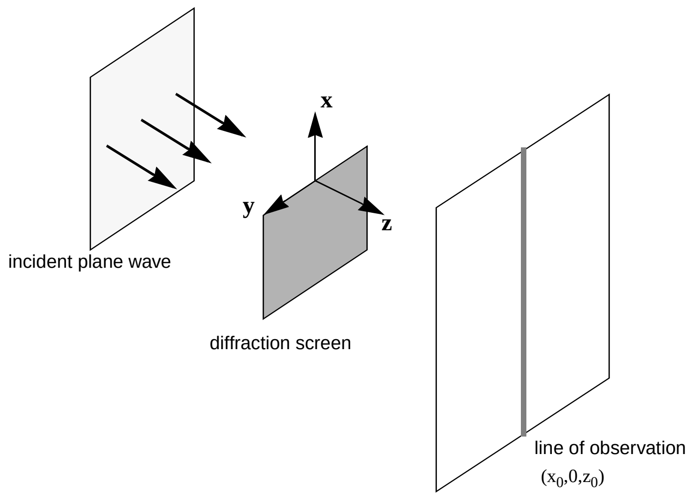
- Gegenbauer
- Hermite

- Laguerre
- Generalized Laguerre
- Legendre
- Jacobi
- Chebyshev of the First and Second Kind

Diffraction

This example is from diffraction theory in classical electrodynamics. (J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, 1962.)

Suppose we have a plane wave of intensity I_0 and wave number k. We assume that the plane wave is parallel to the xy-plane and travels along the z-axis as shown below. This plane wave is called the *incident wave.* A perfectly conducting flat diffraction screen occupies half of the xy-plane, that is $x < 0$. The plane wave strikes the diffraction screen, and we observe the diffracted wave from the line whose coordinates are
$(x, 0, z_0)$, where $z_0 > 0$.



The intensity of the diffracted wave is given by

$$I = \frac{I_0}{2}\left[\left(C(\zeta) + \frac{1}{2}\right)^2 + \left(S(\zeta) + \frac{1}{2}\right)^2\right]$$

where

$$\zeta = \sqrt{\frac{k}{2z_0}} \cdot x$$

and $C(\xi)$ and $S(\xi)$ are the Fresnel cosine and sine integrals:

$$C(\zeta) = \int_0^{\zeta} \cos\left(\frac{\pi}{2} - t^2\right) dt$$

$$S(\zeta) = \int_0^{\zeta} \sin\left(\frac{\pi}{2} - t^2\right) dt$$

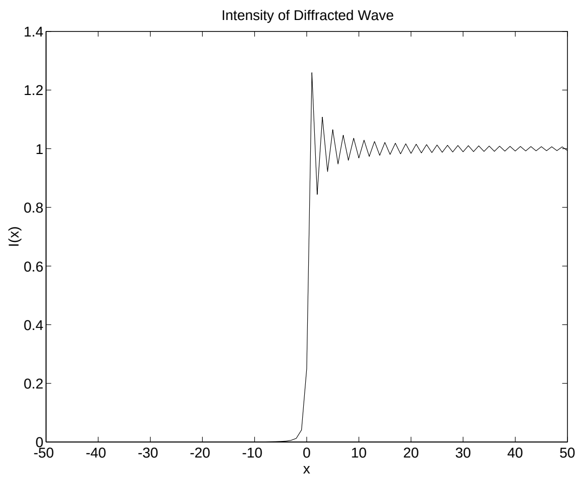
How does the intensity of the diffracted wave behave along the line of observation? Since k and z_0 are constants independent of x, we set

$$\sqrt{\frac{k}{2z_0}} = 1$$

and assume an initial intensity of $I_0 = 1$ for simplicity.

The following code generates a plot of intensity as a function of *x*.

```
x = –50:50;
C = mfun('FresnelC',x);
S = mfun('FresnelS',x);
I0 = 1;
T = (C+1/2).^2 + (S+1/2).^2;
I = (I0/2)*T;
plot(x,I);
xlabel('x');
ylabel('I(x)');
title('Intensity of Diffracted Wave');;
```



We see from the graph that the diffraction effect is most prominent near the edge of the diffraction screen (`x = 0`), as we expect.

Note that values of `x` that are large and positive correspond to observation points far away from the screen. Here, we would expect the screen to have no effect on the incident wave. That is, the intensity of the diffracted wave should be the same as that of the incident wave. Similarly, `x` values that are large and negative correspond to observation points under the screen that are far away from the screen edge. Here, we would expect the diffracted wave to have zero intensity. These results can be verified by setting

```
x = [Inf –Inf]
```

in the code to calculate I.

Using Maple Functions

The maple function lets you access Maple functions directly. This function takes sym objects, strings, and doubles as inputs. It returns a symbolic object, character string, or double corresponding to the class of the input. You can also use the maple function to debug symbolic math programs that you develop.

Simple Example

Suppose we want to write an M-file that takes two polynomials or two integers and returns their greatest common divisor. For example, the greatest common divisor of 14 and 21 is 7. The greatest common divisor of x^2–y^2 and x^3–y^3 is $x - y$.

The first thing we need to know is how to call the greatest common divisor function in Maple. We use the mhelp function to bring up the Maple online help for the greatest common divisor (gcd):

Let's try the gcd function:

```
mhelp gcd
```

which returns

```
Function: gcd - greatest common divisor of polynomials

Function: lcm - least common multiple of polynomials

Calling Sequence:
   gcd(a,b,'cofa','cofb')
   lcm(a,b,...)

Parameters:
   a, b      - multivariate polynomials over the rationals
   cofa,cofb - (optional) unevaluated names
```

```
Description:

- The gcd function computes the greatest common divisor of two
  polynomials a and b with rational coefficients.

- The lcm function computes the least common multiple of an
  arbitrary number of polynomials with rational coefficients.

- The optional third argument cofa is assigned the cofactor
  a/gcd(a,b).

- The optional fourth argument cofb is assigned the cofactor
  b/gcd(a,b).

Examples:
> gcd(x^2-y^2,x^3-y^3);
                                     -x + y

> lcm(x^2-y^2,x^3-y^3);
                              3    4    4      3
                          -y x  + y  - x  + x y

> gcd(6,8,a,b);
                                       2

> a;
                                       3

> b;
                                       4

See Also: gcdex, igcd, ilcm, Gcd, numtheory[GIgcd]
```

Since we now know the Maple calling syntax for gcd, we can write a simple M-file to calculate the greatest common divisor. First, create the M-file gcd in the @sym directory and include the commands below.

```
function g = gcd(a, b)
g = maple('gcd',a, b);
```

If we run this file

```
syms x y
z = gcd(x^2-y^2,x^3-y^3)
w = gcd(6, 24)
```

we get

```
z =
-y+x

w =
6
```

Now let's extend our function so that we can take the gcd of two matrices in a pointwise fashion.

```
function g = gcd(a,b)

if any(size(a) ~= size(b))
  error('Inputs must have the same size.')
end

for k = 1: prod(size(a))
  g(k) = maple('gcd',a(k), b(k));
end

g = reshape(g,size(a));
```

Running this on some test data

```
A = sym([2 4 6; 3 5 6; 3 6 4]);
B = sym([40 30 8; 17 60 20; 6 3 20]);
gcd(A,B)
```

we get the result

```
ans =
[ 2, 2, 2 ]
[ 1, 5, 2 ]
[ 3, 3, 4 ]
```

Vectorized Example

Suppose we want to calculate the sine of a symbolic matrix. One way to do this is:

```
function y = sin1(x)

for k = 1: prod(size(x))
   y(k) = maple('sin',x(k));
end

y = reshape(y,size(x));
```

So the statements

```
syms x y
A = [0 x; y pi/4]
sin1(A)
```

return

```
A =
[    0,    x ]
[    y, pi/4 ]

ans =
[           0,      sin(x) ]
[      sin(y), 1/2*2^(1/2) ]
```

A more efficient way to do this is to call Maple just once, using the Maple `map` function. The `map` function applies a Maple function to each element of an array. In our sine calculation example, this looks like:

```
function y = sin2(x)

if  prod(size(x)) == 1
% scalar case
   y = maple('sin',x);

else
% array case
   y = maple('map','sin',x);

end
```

Note that our `sin2` function treats scalar and array cases differently. It applies the `map` function to arrays but not to scalars. This is because `map` applies a function to each operand of a scalar.

Because our `sin2` function calls Maple only once, it is considerably faster than our `sin1` function, which calls Maple `prod(size(A))` number of times. For example, using `A = x*ones(10,10)` on a Sun-4, `sin1(A)` takes more than six times longer to compute than `sin2(A)`. On a Macintosh 7500/100 PowerPC, the ratio is four-to-one.

The `map` function can also be used for Maple functions that require multiple input arguments. In this case, the syntax is

```
maple('map', Maple function, sym array, arg2, arg3, ..., argn)
```

For example, one way to call the `collect` M-file is `collect(S,x)`. In this case, the `collect` function collects all the coefficients with the same power of `x` for each element in `S`. The core section of the implementation is shown below:

```
r = maple('map','collect',sym(s),sym(x));
```

For additional information on the Maple `map` function, type

```
mhelp map
```

Debugging

The maple command provides two debugging facilities: trace mode and a status output argument.

Trace Mode

The command `maple traceon` causes all subsequent Maple statements and results to be printed to the screen. For example

```
maple traceon
a = sym('a');
exp(2*a)
```

prints all calls made to the Maple kernel for calculating `exp(2*a)`.

```
statement =
(2)*(a);

result =
2*a

statement =
exp(2*a);

result =
exp(2*a)

ans =
exp(2*a)
```

To revert back to suppressed printing, use `maple traceoff`.

Status Output Argument

The `maple` function optionally returns two output arguments, `result` and `status`. If the `maple` call succeeds, Maple returns the result in the `result` argument and zero in the `status` argument. If the call fails, Maple returns an error code (a positive integer) in the `status` argument and a corresponding warning/error message in the `result` argument.

For example, the Maple `discrim` function calculates the discriminant of a polynomial and has the syntax `discrim(p,x)`, where `p` is a polynomial in `x`.

Suppose we forget to supply the second argument when calling the `discrim` function

```
syms a b c x
[result, status] = maple('discrim', a*x^2+b*x+c)
```

Maple returns

```
result =
Error, (in discrim) invalid arguments

status =
     2
```

If we then include `x`

```
[result, status] = maple('discrim', a*x^2+b*x+c, x)
```

we get the following

```
result =
-4*a*c+b^2

status =
     0
```

Extended Symbolic Math Toolbox

The Extended Symbolic Math Toolbox allows you to access all nongraphics Maple packages, Maple programming features, and Maple procedures. The Extended Toolbox thus provides access to a large body of mathematical software written in the Maple language.

Maple programming features include looping (`for ... do ... od`, `while ... do ... od`) and conditionals (`if ... elif ... else ... fi`). Please see *The Maple Handbook* for information on how to use these and other features.

This section explains how to load Maple packages and how to use Maple procedures. For additional information, please consult these references.

Char, B.W., K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, and S.M. Watt, *First Leaves: A Tutorial Introduction to Maple V*, Springer-Verlag, NY, 1991.

Char, B.W., K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, and S.M. Watt, *Maple V Language Reference Manual*, Springer-Verlag, NY, 1991.

Char, B.W., K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, and S.M. Watt, *Maple V Library Reference Manual*, Springer-Verlag, NY, 1991.

Heck, A., *Introduction to Maple*, Springer-Verlag, NY, 1996.

Nicolaides, R. and N. Walkington, *Maple: A Comprehensive Introduction*, Cambridge University Press, Cambridge, 1996.

Packages of Library Functions

Specialized libraries, or “packages,” can be used through the Extended Toolbox. These packages include:

- Combinatorial Functions
- Differential Equation Tools
- Differential Forms
- Domains of Computation
- Euclidean Geometry
- Gaussian Integers
- Gröbner Bases

- Permutation and Finitely Presented Groups
- Lie Symmetries
- Boolean Logic
- Graph Networks
- Newman-Penrose Formalism
- Number Theory
- Numerical Approximation
- Orthogonal Polynomials
- p-adic Numbers
- Formal Power Series
- Projective Geometry
- Simplex Linear Optimization
- Statistics
- Total Orders on Names
- Galois Fields
- Linear Recurrence Relation Tools
- Financial Mathematics
- Rational Generating Functions
- Tensor Computations

You can use the Maple `with` command to load these packages. Say, for example, that you want to use the orthogonal polynomials package. First get the Maple name of this package, using the statement

```
mhelp index[packages]
```

which returns

```
HELP FOR: Index of descriptions for packages of library functions

SYNOPSIS:
- The following packages are available:
                ...
                orthopoly:   orthogonal polynomials
                ...
```

You can then can access information about the package

```
mhelp orthopoly
```

To load the package, type

```
maple('with(orthopoly);')
```

This returns

```
ans =
[G, H, L, P, T, U]
```

which is a listing of function names in the `orthopoly` package. These functions are now loaded in the Maple workspace, and you can use them as you would any regular Maple function.

Procedure Example

The following example shows how you can access a Maple procedure through the Extended Symbolic Math Toolbox. The example computes either symbolic or variable-precision numeric approximations to π, using a method derived by

Richard Brent based from the arithmetic-geometric mean algorithm of Gauss. Here is the Maple source code.

```
pie := proc(n)
  # pie(n) takes n steps of an arithmetic–geometric mean
  # algorithm for computing pi. The result is a symbolic
  # expression whose length roughly doubles with each step.
  # The number of correct digits in the evaluated string also
  # roughly doubles with each step.

  # Example: pie(5) is a symbolic expression with 1167
  # characters which, when evaluated, agrees with pi to 84
  # decimal digits.

  local a,b,c,d,k,t;

  a := 1:
  b := sqrt(1/2):
  c := 1/4:
  t := 1:

  for k from 1 to n do
     d := (b–a)/2:
     b := sqrt(a*b):
     a := a+d:
     c := c–t*d^2:
     t := 2*t:
  od;

  (a+b)^2/(4*c):

end;
```

Assume the source code for this Maple procedure is stored in the file `pie.src`. Using the Extended Symbolic Math Toolbox, the MATLAB statement

```
procread('pie.src')
```

reads the specified file, deletes comments and newline characters, and sends the resulting string to Maple. (The MATLAB `ans` variable then contains a string representation of the `pie.src` file.)

You can use the pie function, using the maple function. The statement

```
p = maple('pie',5)
```

returns a symbolic object p that begins and ends with

```
p =
1/4*(1/32+1/64*2^(1/2)+1/32*2^(3/4)+ ...
     ... *2^(1/2))*2^(3/4))^(1/2))^(1/2))^2)
```

It is interesting to change the computation from symbolic to numeric. The assignment to the variable b in the second executable line is key. If the assignment statement is simply

```
b := sqrt(1/2)
```

the entire computation is done symbolically. But if the assignment statement is modified to include decimal points

```
b := sqrt(1./2.)
```

the entire computation uses variable-precision arithmetic at the current setting of digits. If this change is made, then

```
digits(100)
procread('pie.src')
p = maple('pie',5)
```

produces a 100-digit result

```
p =
3.14159265358979323 ... 5628703211672038
```

The last 16 digits differ from those of π because, with five iterations, the algorithm gives only 84 digits.

Note that you can define your own MATLAB M-file that accesses a Maple procedure

```
function p = pie1(n)
p = maple('pie',n)
```

Precompiled Maple Procedures

When Maple loads a source (ASCII text) procedure into its workspace, it compiles (translates) the procedure into an internal format. You can subsequently use the maple function to save the procedures in the internal format. The advantage is you avoid recompiling the procedure the next time you load it, thereby speeding up the process.

For example, you can convert the pie.src procedure developed in the preceding example to a precompiled Maple procedure, using the commands

```
clear maplemex
procread('pie.src')
maple('save(`pi.m`)');
```

The clear maplemex command resets the Maple workspace to its initial state. Since the Maple save command saves all variables in the current session, we want to remove extraneous variables. Note that you must use back quotes around the function name.

To read the precompiled procedure into a subsequent MATLAB session, type

```
maple('read','`pie.m`');
```

Again, as with the ASCII text form, you can access the function using maple

```
p = maple('pie',5)
```

Note that precompiled Maple procedures have .m extensions. Hence, you must take care to avoid confusing them with MATLAB M-files, which also have .m extensions.

2

Reference

This chapter provides detailed descriptions of all Symbolic Math Toolbox functions. It begins with tables of these functions and continues with the reference entries in alphabetical order.

Calculus	
`diff`	Differentiate.
`int`	Integrate.
`jacobian`	Jacobian matrix.
`limit`	Limit of an expression.
`symsum`	Summation of series.
`taylor`	Taylor series expansion.

Linear Algebra	
`colspace`	Basis for column space.
`det`	Determinant.
`diag`	Create or extract diagonals.
`eig`	Eigenvalues and eigenvectors.
`expm`	Matrix exponential.
`inv`	Matrix inverse.
`jordan`	Jordan canonical form.
`null`	Basis for null space.
`poly`	Characteristic polynomial.
`rank`	Matrix rank.
`rref`	Reduced row echelon form.

Linear Algebra	
`svd`	Singular value decomposition.
`tril`	Lower triangle.
`triu`	Upper triangle.

Simplification	
`collect`	Collect common terms.
`expand`	Expand polynomials and elementary functions.
`factor`	Factor.
`horner`	Nested polynomial representation.
`numden`	Numerator and denominator.
`simple`	Search for shortest form.
`simplify`	Simplification.
`subexpr`	Rewrite in terms of subexpressions.

Solution of Equations	
`compose`	Functional composition.
`dsolve`	Solution of differential equations.
`finverse`	Functional inverse.
`solve`	Solution of algebraic equations.

Variable Precision Arithmetic	
digits	Set variable precision accuracy.
vpa	Variable precision arithmetic.

Arithmetic Operations	
+	Addition.
–	Subtraction.
*	Multiplication.
.*	Array multiplication.
/	Right division.
./	Array right division.
\	Left division.
.\	Array left division.
^	Matrix or scalar raised to a power.
.^	Array raised to a power.
'	Complex conjugate transpose.
.'	Real transpose.

Special Functions	
`cosint`	Cosine integral, *Ci(x)*.
`hypergeom`	Generalized hypergeometric function.
`lambertw`	Solution of $\lambda(x)e^{\lambda(x)} = x$.
`sinint`	Sine integral, *Si(x)*.
`zeta`	Riemann zeta function.

Access To Maple	
`maple`	Access Maple kernel.
`mapleinit`	Initialize Maple.
`mfun`	Numeric evaluation of Maple functions.
`mhelp`	Maple help.
`mfunlist`	List of functions for `mfun`.
`procread`	Install a Maple procedure.

Pedagogical and Graphical Applications	
`ezcontour`	Contour plotter.
`ezcontourf`	Filled contour plotter.
`ezmesh`	Mesh plotter.
`ezmeshc`	Combined mesh and contour plotter.

Pedagogical and Graphical Applications	
`ezplot`	Function plotter.
`ezplot3`	3-D curve plotter.
`ezpolar`	Polar coordinate plotter.
`ezsurf`	Surface plotter.
`ezsurfc`	Combined surface and contour plotter.
`funtool`	Function calculator.
`rsums`	Riemann sums.
`taylortool`	Taylor series calculator.

Conversions	
`char`	Convert `sym` object to string.
`double`	Convert symbolic matrix to `double`.
`poly2sym`	Function calculator.
`sym2poly`	Symbolic polynomial to coefficient vector.

Basic Operations	
`ccode`	C code representation of a symbolic expression.
`conj`	Complex conjugate.
`findsym`	Determine symbolic variables.
`fortran`	Fortran representation of a symbolic expression.
`imag`	Imaginary part of a complex number.

Basic Operations	
`latex`	LaTeX representation of a symbolic expression.
`pretty`	Pretty print a symbolic expression.
`real`	Real part of an imaginary number.
`sym`	Create symbolic object.
`syms`	Shortcut for creating multiple symbolic objects.

Integral Transforms	
`fourier`	Fourier transform.
`ifourier`	Inverse Fourier transform.
`ilaplace`	Inverse Laplace transform.
`iztrans`	Inverse *z*-transform.
`laplace`	Laplace transform.
`ztrans`	*z*-transform.

Arithmetic Operations

Purpose

Perform arithmetic operations on symbols.

Syntax

```
A+B
A–B
A*B  A.*B
A\B  A.\B
A/B  A./B
A^B  A.^B
A'  A.'
```

Description

`+` Matrix addition. `A + B` adds `A` and `B`. `A` and `B` must have the same dimensions, unless one is scalar.

`–` Subtraction. `A – B` subtracts `B` from `A`. `A` and `B` must have the same dimensions, unless one is scalar.

`*` Matrix multiplication. `A*B` is the linear algebraic product of `A` and `B`. The number of columns of `A` must equal the number of rows of `B`, unless one is a scalar.

`.*` Array multiplication. `A.*B` is the entry-by-entry product of `A` and `B`. `A` and `B` must have the same dimensions, unless one is scalar.

`\` Matrix left division. `X = A\B` solves the symbolic linear equations `A*X=B`. Note that `A\B` is roughly equivalent to `inv(A)*B`. Warning messages are produced if `X` does not exist or is not unique. Rectangular matrices `A` are allowed, but the equations must be consistent; a least squares solution is *not* computed.

`.\` Array left division. `A.\B` is the matrix with entries `B(i,j)/A(i,j)`. `A` and `B` must have the same dimensions, unless one is scalar.

`/` Matrix right division. `X=B/A` solves the symbolic linear equation `X*A=B`. Note that `B/A` is the same as `(A.'\B.')`. Warning messages are produced if `X` does not exist or is not unique. Rectangular matrices `A` are allowed, but the equations must be consistent; a least squares solution is not computed.

`./` Array right division. `A./B` is the matrix with entries `A(i,j)/B(i,j)`. `A` and `B` must have the same dimensions, unless one is scalar.

^ Matrix power. X^P raises the square matrix X to the integer power P. If X is a scalar and P is a square matrix, X^P raises X to the matrix power P, using eigenvalues and eigenvectors. X^P, where X and P are both matrices, is an error.

.^ Array power. A.^B is the matrix with entries A(i,j)^B(i,j). A and B must have the same dimensions, unless one is scalar.

' Matrix Hermition transpose. If A is complex, A' is the complex conjugate transpose.

.' Array transpose. A.' is the real transpose of A. A.' does not conjugate complex entries.

Examples

The following statements

```
syms a b c d;
A = [a b; c d];
A*A/A
A*A–A^2
```

return

```
[ a, b]
[ c, d]

[ 0, 0]
[ 0, 0]
```

The following statements

```
syms a11 a12 a21 a22 b1 b2;
A = [a11 a12; a21 a22];
B = [b1 b2];
X = B/A;
x1 = X(1)
x2 = X(2)
```

return

```
x1 =
(a21*b2-b1*a22)/(-a11*a22+a12*a21)

x2 =
(-a11*b2+a12*b1)/(-a11*a22+a12*a21)
```

See Also

`null`, `solve`

Purpose C code representation of a symbolic expression.

Syntax

```
ccode(s)
```

Description `ccode(s)` returns a fragment of C that evaluates the symbolic expression `s`.

Examples The statements

```
syms x
f = taylor(log(1+x));
ccode(f)
```

return

```
t0 = x-x*x/2+x*x*x/3-pow(x,4.0)/4+pow(x,5.0)/5;
```

The statements

```
H = sym(hilb(3));
ccode(H)
```

return

```
H[0][0] = 1.0;        H[0][1] = 1.0/2.0;    H[0][2] = 1.0/3.0;
H[1][0] = 1.0/2.0;    H[1][1] = 1.0/3.0;    H[1][2] = 1.0/4.0;
H[2][0] = 1.0/3.0;    H[2][1] = 1.0/4.0;    H[2][2] = 1.0/5.0;
```

See Also `fortran`, `latex`, `pretty`

collect

Purpose

Collect coefficients.

Syntax

```
R = collect(S)
R = collect(S,v)
```

Description

For each polynomial in the array `S` of polynomials, `collect(S)` collects terms containing the variable `v` (or `x`, if `v` is not specified). The result is an array containing the collected polynomials.

Examples

The following statements

```
syms x y;
R1 = collect((exp(x)+x)*(x+2))
R2 = collect((x+y)*(x^2+y^2+1), y)
R3 = collect([(x+1)*(y+1),x+y])
```

return

```
R1 =
x^2+(exp(x)+2)*x+2*exp(x)

R2 =
y^3+x*y^2+(x^2+1)*y+x*(x^2+1)

R3 =
[(y+1)*x+y+1, x+y]
```

See Also

`expand`, `factor`, `simple`, `simplify`, `syms`

Purpose Basis for column space.

Syntax

```
B = colspace(A)
```

Description `colspace(A)` returns a matrix whose columns form a basis for the column space of `A`. `A` is a symbolic or numeric matrix. Note that `size(colspace(A),2)` returns the rank of `A`.

Examples The statements

```
A = sym([2,0;3,4;0,5])
B = colspace(A)
```

return

```
A =
[2,0]
[3,4]
[0,5]

B =
[    1,   0]
[    0,   1]
[-15/8, 5/4]
```

See Also `null`

`orth` in the online MATLAB Function Reference

compose

Purpose Functional composition.

Syntax

```
compose(f,g)
compose(f,g,z)
compose(f,g,x,z)
compose(f,g,x,y,z)
```

Description

`compose(f,g)` returns `f(g(y))` where `f = f(x)` and `g = g(y)`. Here `x` is the symbolic variable of `f` as defined by `findsym` and `y` is the symbolic variable of `g` as defined by `findsym`.

`compose(f,g,z)` returns `f(g(z))` where `f = f(x)`, `g = g(y)`, and `x` and `y` are the symbolic variables of `f` and `g` as defined by `findsym`.

`compose(f,g,x,z)` returns `f(g(z))` and makes `x` the independent variable for `f`. That is, if `f = cos(x/t)`, then `compose(f,g,x,z)` returns `cos(g(z)/t)` whereas `compose(f,g,t,z)` returns `cos(x/g(z))`.

`compose(f,g,x,y,z)` returns `f(g(z))` and makes `x` the independent variable for `f` and `y` the independent variable for `g`. For `f = cos(x/t)` and `g = sin(y/u)`, `compose(f,g,x,y,z)` returns `cos(sin(z/u)/t)` whereas `compose(f,g,x,u,z)` returns `cos(sin(y/z)/t)`.

Examples

Suppose

```
syms x y z t u;
f = 1/(1 + x^2); g = sin(y); h = x^t; p = exp(-y/u);
```

Then

```
compose(f,g)   ->  1/(1+sin(x)^2)
compose(f,g,t) ->  1/(1+sin(t)^2)
compose(h,g,x,z)   ->  sin(z)^t
compose(h,g,t,z)   ->  x^sin(z)
compose(h,p,x,y,z) ->  exp(-z/u)^t
compose(h,p,t,u,z) ->  x^exp(-y/z)
```

See Also `finverse`, `subs`, `syms`

Purpose Symbolic conjugate.

Syntax `conj(X)`

Description `conj(X)` is the complex conjugate of `X`.

For a complex `X`, `conj(X) = real(X) - i*imag(X)`.

See Also `real`, `imag`

cosint

Purpose

Cosine integral function.

Syntax

```
Y = cosint(X)
```

Description

`cosint(X)` evaluates the cosine integral function at the elements of `X`, a numeric matrix, or a symbolic matrix. The cosine integral function is defined by:

$$Ci(x) = \gamma + \ln(x) + \int_0^x \frac{\cos t - 1}{t}\,dt$$

where γ is Euler's constant 0.577215664...

Examples

`cosint(7.2)` returns `0.0960`.

`cosint([0:0.1:1])` returns

```
Columns 1 through 7

    Inf   –1.7279   –1.0422   –0.6492   –0.3788   –0.1778   –0.0223

Columns 8 through 11

    0.1005    0.1983    0.2761    0.3374
```

The statements

```
syms x;
f = cosint(x);
diff(x)
```

return

```
cos(x)/x
```

See Also

`sinint`

Purpose Matrix determinant.

Syntax

```
r = det(A)
```

Description `det(A)` computes the determinant of `A`, where `A` is a symbolic or numeric matrix. `det(A)` returns a symbolic expression, if `A` is symbolic; a numeric value, if `A` is numeric.

Examples The statements

```
syms a b c d;
det([a, b; c, d])
```

return

```
a*d - b*c
```

The statements

```
A = sym([2/3 1/3;1 1])
r = det(A)
```

return

```
A =
[ 2/3, 1/3]
[   1,   1]

r = 1/3
```

diag

Purpose

Create or extract symbolic diagonals.

Syntax

```
diag(A,k)
diag(A)
```

Description

`diag(A,k)`, where `A` is a row or column vector with `n` components, returns a square symbolic matrix of order `n+abs(k)`, with the elements of `A` on the `k`-th diagonal. `k = 0` signifies the main diagonal; `k > 0`, the `k`-th diagonal above the main diagonal; `k < 0`, the `k`-th diagonal below the main diagonal.

`diag(A,k)`, where `A` is a square symbolic matrix, returns a column vector formed from the elements of the `k`-th diagonal of `A`.

`diag(A)`, where `A` is a vector with `n` components, returns an `n`-by-`n` diagonal matrix having `A` as its main diagonal.

`diag(A)`, where `A` is a square symbolic matrix, returns the main diagonal of `A`.

Examples

Suppose

```
v = [a b c]
```

Then both `diag(v)` and `diag(v,0)` return

```
[ a, 0, 0 ]
[ 0, b, 0 ]
[ 0, 0, c ]
```

`diag(v,–2)` returns

```
[ 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0]
[ a, 0, 0, 0, 0]
[ 0, b, 0, 0, 0]
[ 0, 0, c, 0, 0]
```

Suppose

```
A =
    [ a, b, c ]
    [ 1, 2, 3 ]
    [ x, y, z ]
```

`diag(A)` returns

```
[ a ]
[ 2 ]
[ z ]
```

`diag(A,1)` returns

```
[ b ]
[ 3 ]
```

See Also

`tril`, `triu`

diff

Purpose

Differentiate.

Syntax

```
diff(S,'v')
diff(S,n)
diff(S,'v',n)
```

Description

`diff(S)` differentiates a symbolic expression `S` with respect to its free variable as determined by `findsym`.

`diff(S,'v')` or `diff(S,sym('v'))` differentiates `S` with respect to `v`.

`diff(S,n)`, for a positive integer `n`, differentiates *S* `n` times.

`diff(S,'v',n)` and `diff(S,n,'v')` are also acceptable.

Examples

Assume

```
syms x t
```

Then

```
diff(sin(x^2))
```

returns

```
2*cos(x^2)*x
```

and

```
diff(t^6,6)
```

returns

```
720
```

See also

`int`, `jacobian`, `findsym`

Purpose Set variable precision accuracy.

Syntax

```
digits(d)
d = digits
digits
```

Description `digits` specifies the number of significant decimal digits that Maple uses to do variable precision arithmetic (VPA). The default value is 32 digits.

`digits(d)` sets the current VPA accuracy to `d` digits.

`digits` returns the current VPA accuracy.

Examples If

```
z = 1.0e–16
x = 1.0e+2
digits(14)
```

then

```
y = vpa(x*z+1)
```

uses 14-digit decimal arithmetic and returns

```
y =
1.0000000000000
```

Whereas

```
digits(15)
y = vpa(x*z+1)
```

used 15-digit decimal arithmetic and returns

```
y =
1.00000000000001
```

See Also `double`, `vpa`

double

Purpose Convert symbolic matrix to MATLAB numeric form.

Syntax

```
R = double(S)
```

Description `double(S)` converts the symbolic object `S` to a numeric object. If `S` is a symbolic constant or constant expression, `double` returns a double-precision floating-point number representing the value of `S`. If `S` is a symbolic matrix whose entries are constants or constant expressions, `double` returns a matrix of double precision floating-point numbers representing the values of `S`'s entries.

Examples `double(sym('(1+sqrt(5))/2'))` returns `1.6180`.

The following statements

```
a = sym(2*sqrt(2));
b = sym((1–sqrt(3))^2);
T = [a, b]
double(T)
```

return

```
ans =
    2.8284    0.5359
```

See Also `sym`, `vpa`

Purpose

Symbolic solution of ordinary differential equations.

Syntax

```
r = dsolve('eq1,eq2,...', 'cond1,cond2,...', 'v')
r = dsolve('eq1','eq2',...,'cond1','cond2',...,'v')
```

Description

`dsolve('eq1,eq2,...', 'cond1,cond2,...', 'v')` symbolically solves the ordinary differential equation(s) specified by `eq1, eq2,...` using `v` as the independent variable and the boundary and/or initial condition(s) specified by `cond1,cond2,...`.

The default independent variable is `t`.

The letter `D` denotes differentiation with respect to the independent variable; with the primary default, this is `d/dx`. A `D` followed by a digit denotes repeated differentiation. For example, `D2` is `d2/dx2`. Any character immediately following a differentiation operator is a dependent variable. For example, `D3y` denotes the third derivative of `y(x)` or `y(t)`.

Initial/boundary conditions are specified with equations like `y(a) = b` or `Dy(a) = b`, where `y` is a dependent variable and `a` and `b` are constants. If the number of initial conditions specified is less than the number of dependent variables, the resulting solutions will contain the arbitrary constants `C1`, `C2,...`.

You can also input each equation and/or initial condition as a separate symbolic equation. `dsolve` accepts up to 12 input arguments.

With no output arguments, `dsolve` returns a list of solutions.

`dsolve` returns a warning message, if it cannot find an analytic solution for an equation. In such a case, you can find a numeric solution, using MATLAB's `ode23` or `ode45` function.

Examples

`dsolve('Dy = a*y')` returns

```
exp(a*t)*C1
```

`dsolve('Df = f + sin(t)')` returns

```
–1/2*cos(t)–1/2*sin(t)+exp(t)*C1
```

`dsolve('(Dy)^2 + y^2 = 1','s')` returns

```
—sin(—s+C1)
```

`dsolve('Dy = a*y', 'y(0) = b')` returns

```
exp(a*t)*b
```

`dsolve('D2y = —a^2*y', 'y(0) = 1', 'Dy(pi/a) = 0')` returns

```
cos(a*t)
```

`dsolve('Dx = y', 'Dy = —x')` returns

```
x=
cos(t)*C1+sin(t)*C2

y =
—sin(t)*C1+cos(t)*C2
```

Diagnostics

If `dsolve` cannot find an analytic solution for an equation, it prints the warning

```
Warning: explicit solution could not be found
```

and return an empty `sym` object.

See Also

`syms`

Purpose

Symbolic matrix eigenvalues and eigenvectors.

Syntax

```
lambda = eig(A)
[V,D] = eig(A)
[V,D,P] = eig(A)
lambda = eig(vpa(A))
[V,D] = eig(vpa(A))
```

Description

`lambda=eig(A)` returns a symbolic vector containing the eigenvalues of the square symbolic matrix `A`.

`[V,D] = eig(A)` returns a matrix `V` whose columns are eigenvectors and a diagonal matrix `D` containing eigenvalues. If the resulting `V` is the same size as `A`, then `A` has a full set of linearly independent eigenvectors that satisfy `A*V = V*D`.

`[V,D,P]=eig(A)` also returns `P`, a vector of indices whose length is the total number of linearly independent eigenvectors, so that `A*V = V*D(P,P)`.

`lambda = eig(VPA(A))` and `[V,D] = eig(VPA(A))` compute numeric eigenvalues and eigenvectors, respectively, using variable precision arithmetic. If `A` does not have a full set of eigenvectors, the columns of `V` will not be linearly independent.

Examples

The statements

```
R = sym(gallery('rosser'));
eig(R)
```

return

```
ans =
[                  0]
[               1020]
[  510+100*26^(1/2)]
[  510–100*26^(1/2)]
[     10*10405^(1/2)]
[    –10*10405^(1/2)]
[               1000]
[               1000]
```

`eig(vpa(R))` returns

```
ans =


[    -1020.049018429968238463137913055]
[ .56512999999999999999999999999800e-28]
[  .98048640721516997177589097485157e-1]
[     1000.000000000000000000000000002]
[     1000.000000000000000000000000003]
[     1019.901951359278483002822410902]
[     1020.000000000000000000000000003]
[     1020.049018429968238463137913055]
```

The statements

```
A = sym(gallery(5));
[v,lambda] = eig(A)
```

return

```
v =
[       0]
[  21/256]
[ -71/128]
[ 973/256]
[       1]

lambda =

[ 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0]
```

See Also

`jordan`, `poly`, `svd`, `vpa`

Purpose Symbolic matrix exponential.

Syntax

```
expm(A)
```

Description `expm(A)` is the matrix exponential of the symbolic matrix `A`.

Examples The statements

```
syms t;
A = [0 1; –1 0];
expm(t*A)
```

return

```
[  cos(t),  sin(t)]
[ –sin(t),  cos(t)]
```

expand

Purpose Symbolic expansion.

Syntax

```
R = expand(S)
```

Description `expand(S)` writes each element of a symbolic expression `S` as a product of its factors. `expand` is most often used only with polynomials, but also expands trigonometric, exponential, and logarithmic functions.

Examples `expand((x–2)*(x–4))` returns

```
x^2–6*x+8
```

`expand(cos(x+y))` returns

```
cos(x)*cos(y)–sin(x)*sin(y)
```

`expand(exp((a+b)^2))` returns

```
exp(a^2)*exp(a*b)^2*exp(b^2)
```

`expand(log(a*b/sqrt(c)))` returns

```
log(a)+log(b)–1/2*log(c)
```

`expand([sin(2*t), cos(2*t)])` returns

```
[2*sin(t)*cos(t), 2*cos(t)^2–1]
```

See Also `collect`, `factor`, `horner`, `simple`, `simplify`, `syms`

Purpose Contour plotter.

Syntax

```
ezcontour(f)
ezcontour(f,domain)
ezcontour(...,n)
```

Description

`ezcontour(f)` plots the contour lines of *f(x,y)*, where `f` is a symbolic expression that represents a mathematical function of two variables, such as *x* and *y*.

The function *f* is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function *f* is not defined (singular) for points on the grid, then these points are not plotted.

`ezcontour(f,domain)` plots *f(x,y)* over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If *f* is a function of the variables *u* and *v* (rather than *x* and *y*), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezcontour(u^2 - v^3,[0,1],[3,6])` plots the contour lines for $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezcontour(...,n)` plots *f* over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezcontour` automatically adds a title and axis labels.

Examples

The following mathematical expression defines a function of two variables, *x* and *y*:

$$f(x, y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$

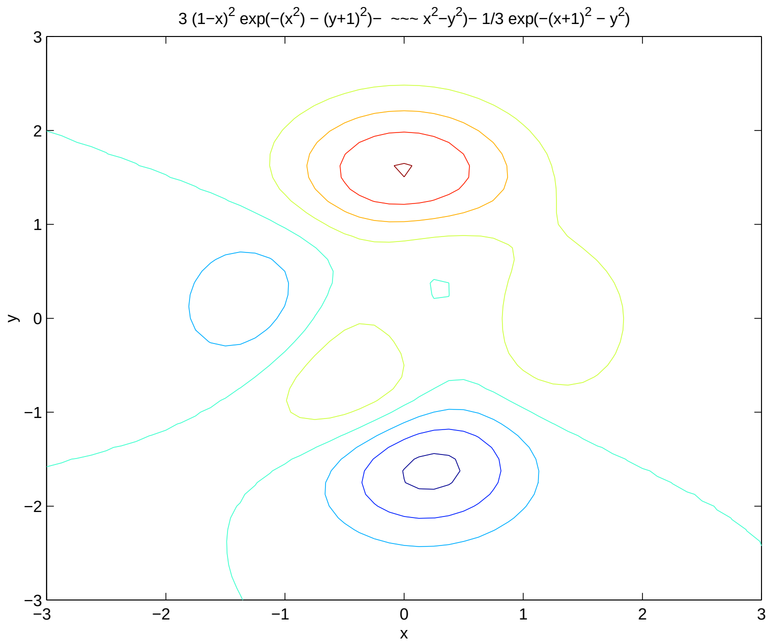
`ezcontour` requires a `sym` argument that expresses this function using MATLAB syntax to represent exponents, natural logs, etc. This function is represented by the symbolic expression:

```
syms x y
f = 3*(1-x)^2*exp(-(x^2)-(y+1)^2) ...
   - 10*(x/5 - x^3 - y^5)*exp(-x^2-y^2) ...
   - 1/3*exp(-(x+1)^2 - y^2);
```

For convenience, this expression is written on three lines.

Pass the sym `f` to `ezcontour` along with a domain ranging from -3 to 3 and specify a computational grid of 49-by-49:

```
ezcontour(f,[-3,3],49)
```



In this particular case, the title is too long to fit at the top of the graph so MATLAB abbreviates the string.

See Also

`contour`, `ezcontourf`, `ezmesh`, `ezmeshc`, `ezplot`, `ezplot3`, `ezpolar`, `ezsurf`, `ezsurfc`

Purpose

Filled contour plotter.

Syntax

```
ezcontourf(f)
ezcontourf(f,domain)
ezcontourf(...,n)
```

Description

`ezcontour(f)` plots the contour lines of *f(x,y)*, where `f` is a `sym` that represents a mathematical function of two variables, such as x and y.

The function f is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function f is not defined (singular) for points on the grid, then these points are not plotted.

`ezcontour(f,domain)` plots *f(x,y)* over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If f is a function of the variables u and v (rather than x and y), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezcontourf(u^2 - v^3,[0,1],[3,6])` plots the contour lines for $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezcontourf(...,n)` plots f over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezcontourf` automatically adds a title and axis labels.

Examples

The following mathematical expression defines a function of two variables, x and y:

$$f(x, y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$$

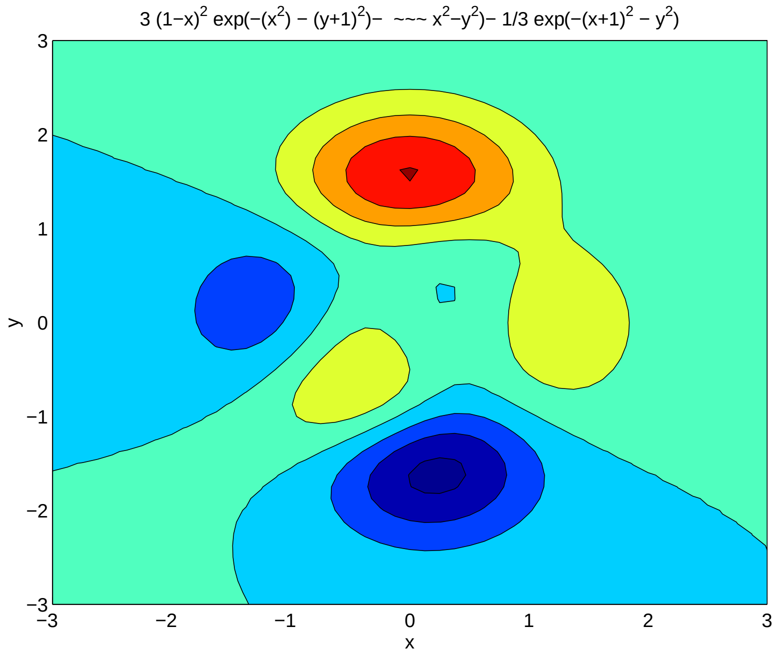
`ezcontourf` requires a `sym` argument that expresses this function using MATLAB syntax to represent exponents, natural logs, etc. This function is represented by the symbolic expression:

```
syms x y
f = 3*(1-x)^2*exp(-(x^2)-(y+1)^2) ...
    - 10*(x/5 - x^3 - y^5)*exp(-x^2-y^2) ...
    - 1/3*exp(-(x+1)^2 - y^2);
```

For convenience, this expression is written on three lines.

Pass the sym `f` to `ezcontourf` along with a domain ranging from `-3` to `3` and specify a grid of 49-by-49:

```
ezcontourf(f,[-3,3],49)
```



In this particular case, the title is too long to fit at the top of the graph so MATLAB abbreviates the string.

See Also

`contourf`, `ezcontour`, `ezmesh`, `ezmeshc`, `ezplot`, `ezplot3`, `ezpolar`, `ezsurf`, `ezsurfc`

Purpose 3-D mesh plotter.

Syntax

```
ezmesh(f)
ezmesh(f,domain)
ezmesh(x,y,z)
ezmesh(x,y,z,[smin,smax,tmin,tmax]) or ezmesh(x,y,z,[min,max])
ezmesh(...,n)
ezmesh(...,'circ')
```

Description

`ezmesh(f)` creates a graph of $f(x,y)$, where `f` is a symbolic expression that represents a mathematical function of two variables, such as x and y.

The function f is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function f is not defined (singular) for points on the grid, then these points are not plotted.

`ezmesh(f,domain)` plots f over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If f is a function of the variables u and v (rather than x and y), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezmesh(u^2 - v^3,[0,1],[3,6])` plots $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezmesh(x,y,z)` plots the parametric surface $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ over the square: $-2\pi < s < 2\pi$, $-2\pi < t < 2\pi$.

`ezmesh(x,y,z,[smin,smax,tmin,tmax])` or `ezmesh(x,y,z,[min,max])` plots the parametric surface using the specified domain.

`ezmesh(...,n)` plots f over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezmesh(...,'circ')` plots f over a disk centered on the domain

Remarks

`rotate3d` is always on. To rotate the graph, click and drag with the mouse.

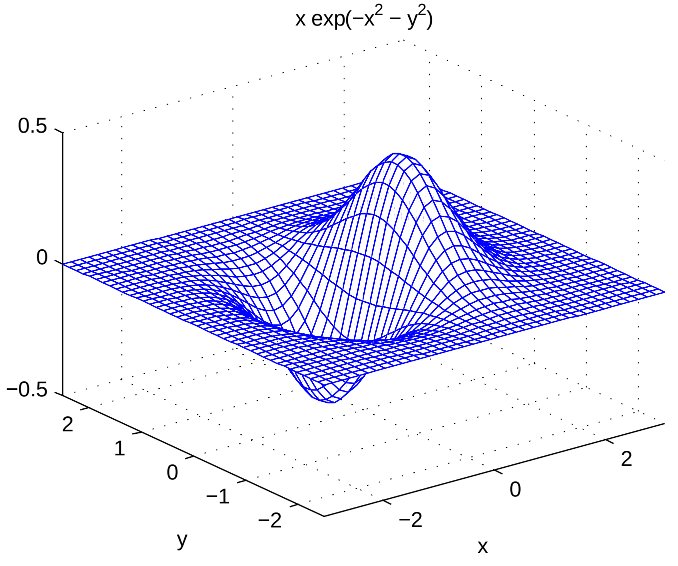
Examples

This example visualizes the function,

$$f(x, y) = xe^{-x^2 - y^2}$$

with a mesh plot drawn on a 40-by-40 grid. The mesh lines are set to a uniform blue color by setting the colormap to a single color:

```
syms x y
ezmesh(x*exp(-x^2-y^2),40)
colormap [0 0 1]
```



See Also

`ezcontour`, `ezcontourf`, `ezmeshc`, `ezplot`, `ezplot3`, `ezpolar`, `ezsurf`, `ezsurfc`, `mesh`

Purpose

Combined mesh/contour plotter.

Syntax

```
ezmeshc(f)
ezmeshc(f,domain)
ezmeshc(x,y,z)
ezmeshc(x,y,z,[smin,smax,tmin,tmax]) or ezmeshc(x,y,z,[min,max])
ezmeshc(...,n)
ezmeshc(...,'circ')
```

Description

`ezmeshc(f)` creates a graph of *f(x,y)*, where *f* is a symbolic expression that represents a mathematical function of two variables, such as *x* and y.

The function *f* is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function *f* is not defined (singular) for points on the grid, then these points are not plotted.

`ezmeshc(f,domain)` plots *f* over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If *f* is a function of the variables *u* and *v* (rather than *x* and *y*), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezmeshc(u^2 - v^3,[0,1],[3,6])` plots $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezmeshc(x,y,z)` plots the parametric surface $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ over the square: $-2\pi < s < 2\pi$, $-2\pi < t < 2\pi$.

`ezmeshc(x,y,z,[smin,smax,tmin,tmax])` or `ezmeshc(x,y,z,[min,max])` plots the parametric surface using the specified domain.

`ezmeshc(...,n)` plots *f* over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezmeshc(...,'circ')` plots *f* over a disk centered on the domain

Remarks

`rotate3d` is always on. To rotate the graph, click and drag with the mouse.

Examples

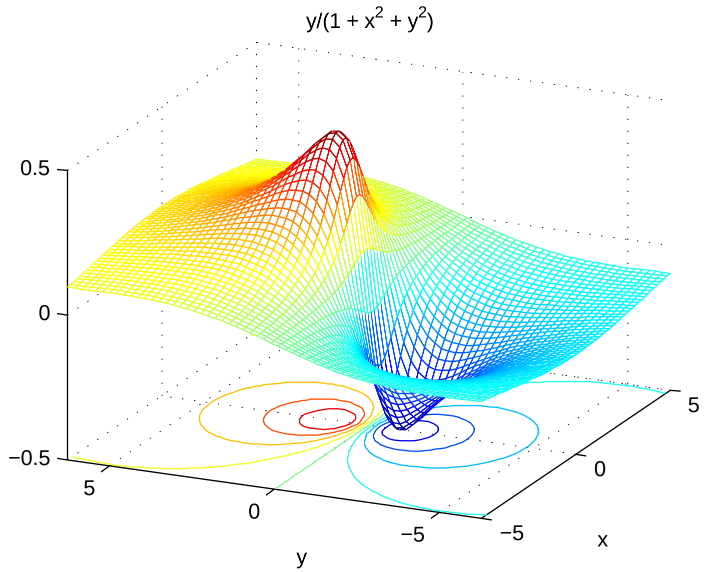
Create a mesh/contour graph of the expression,

$$f(x, y) = \frac{y}{1 + x^2 + y^2}$$

over the domain $-5 < x < 5$, $-2*\text{pi} < y < 2*\text{pi}$:

```
syms x y
ezmeshc(y/(1 + x^2 + y^2),[-5,5,-2*pi,2*pi])
```

Use the mouse to rotate the axes to better observe the contour lines (this picture uses a view of azimuth = –65.5 and elevation = 26).



See Also

`ezcontour`, `ezcontourf`, `ezmesh`, `ezplot`, `ezplot3`, `ezpolar`, `ezsurf`, `ezsurfc`, `meshc`

Purpose

Function plotter.

Syntax

```
ezplot(f)
ezplot(f,[min,max])
ezplot(f,[xmin,xmax,ymin,ymax])
ezplot(x,y)
ezplot(x,y,[tmin,tmax])
ezplot(...,figure)
```

Description

`ezplot(f)` plots the expression $f = f(x)$ over the default domain $-2\pi < x < 2\pi$.

`ezplot(f,[xmin xmax])` plots $f = f(x)$ over the specified domain. It opens and displays the result in a window labeled **Figure No. 1**. If any plot windows are already open, `ezplot` displays the result in the highest numbered window.

`ezplot(f,[xmin xmax],fign)` opens (if necessary) and displays the plot in the window labeled `fign`.

For implicitly defined functions, $f = f(x,y)$:

`ezplot(f)` plots $f(x,y) = 0$ over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$.

`ezplot(f,[xmin,xmax,ymin,ymax])` plots $f(x,y) = 0$ over `xmin` < x < `xmax` and `ymin` < y < `ymax`.

`ezplot(f,[min,max])`plots $f(x,y) = 0$ over `min` < x < `max` and `min` < y < `max`.

If f is a function of the variables u and v (rather than x and y), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezplot(u^2 – v^2 – 1,[–3,2,–2,3])` plots $u^2 - v^2 - 1 = 0$ over $-3 < u < 2$, $-2 < v < 3$.

`ezplot(x,y)` plots the parametrically defined planar curve $x = x(t)$ and $y = y(t)$ over the default domain $0 < t < 2\pi$.

`ezplot(x,y,[tmin,tmax])` plots $x = x(t)$ and $y = y(t)$ over `tmin` < t < `tmax`.

`ezplot(...,figure)` plots the given function over the specified domain in the Figure window identified by the handle `figure`.

Algorithm

If you do not specify a plot range, `ezplot` samples the function between `–2*pi` and `2*pi` and selects a subinterval where the variation is significant as the plot domain. For the range, ezplot omits extreme values associated with singularities.

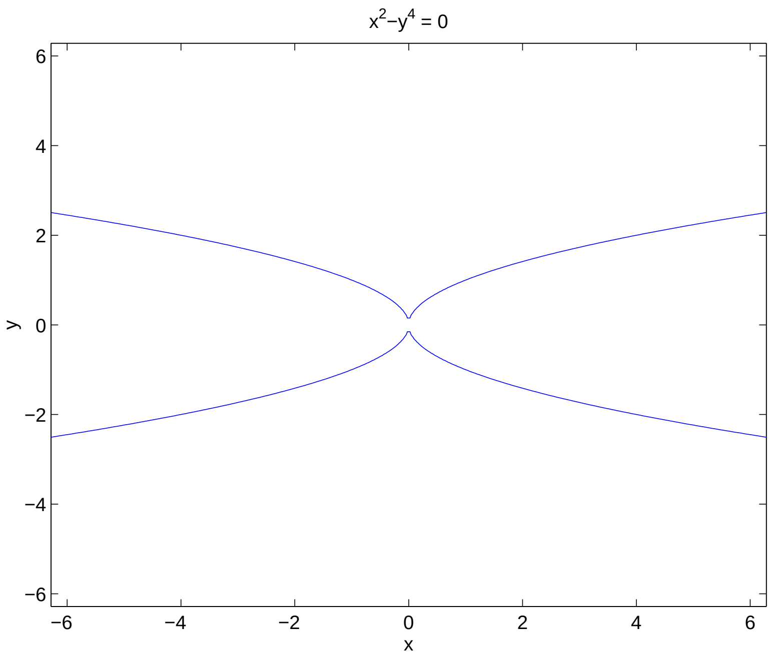
Examples

This example plots the implicitly defined function,

$$x^2 - y^4 = 0$$

over the domain $[-2\pi, 2\pi]$:

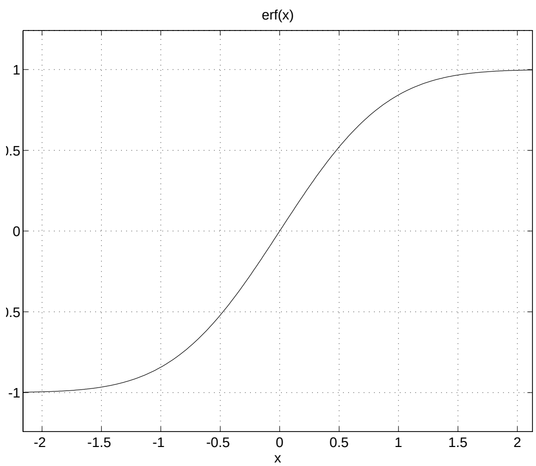
```
syms x y
ezplot(x^2-y^4)
```



The following statements

```
syms x
ezplot(erf(x))
grid
```

plot a graph of the error function:



See Also

`ezcontour`, `ezcontourf`, `ezmesh`, `ezmeshc`, `ezplot3`, `ezpolar`, `ezsurf`, `ezsurfc`, `plot`

ezplot3

Purpose 3-D parametric curve plotter.

Syntax

```
ezplot3(x,y,z)
ezplot3(x,y,z,[tmin,tmax])
ezplot3(...,'animate')
```

Description

`ezplot3(x,y,z)` plots the spatial curve $x = x(t)$, $y = y(t)$, and $z = z(t)$ over the default domain $0 < t < 2\pi$.

`ezplot3(x,y,z,[tmin,tmax])` plots the curve $x = x(t)$, $y = y(t)$, and $z = z(t)$ over the domain `tmin` < t < `tmax`.

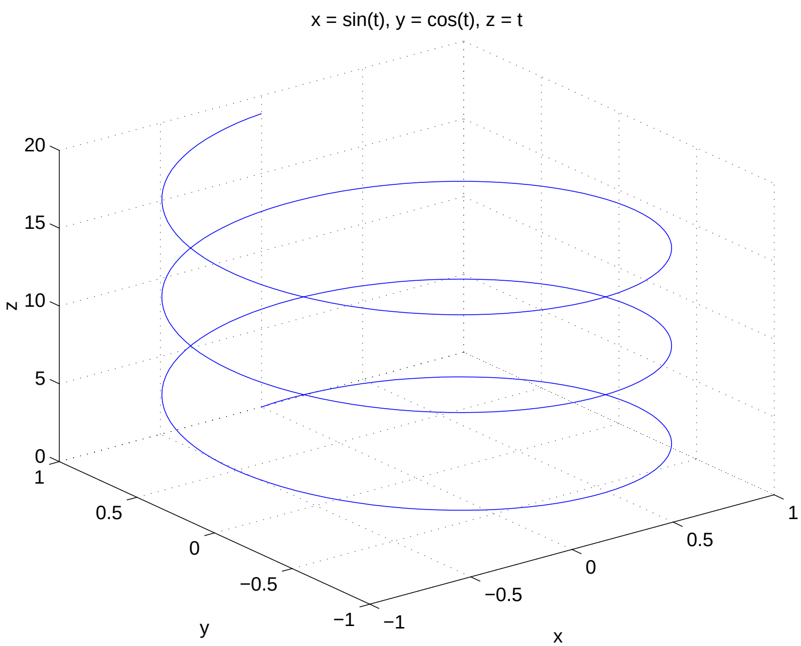
`ezplot3(...,'animate')` produces an animated trace of the spatial curve.

Examples

This example plots the parametric curve,

over the domain $[0,6\pi]$:

```
syms t; ezplot3(sin(t), cos(t), t,[0,6*pi])
```



See Also

`ezcontour`, `ezcontourf`, `ezmesh`, `ezmeshc`, `ezplot`, `ezpolar`, `ezsurf`, `ezsurfc`, `plot3`

ezpolar

Purpose Polar coordinate plotter.

Syntax

```
ezpolar(f)
ezpolar(f,[a,b])
```

Description

`ezpolar(f)` plots the polar curve *rho* = *f(theta)* over the default domain $0 < \text{theta} < 2\pi$.

`ezpolar(f,[a,b])` plots *f* for `a` < *theta* < `b`.

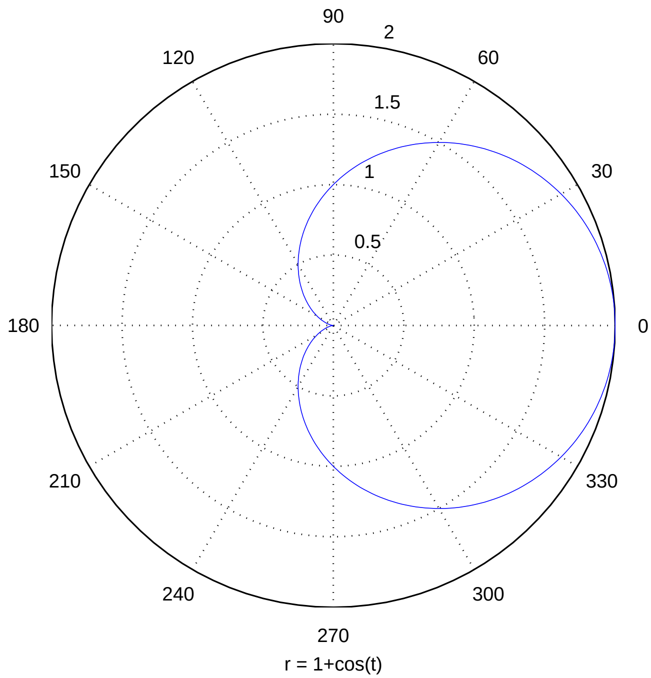
Example

This example creates a polar plot of the function,

1 + cos(t)

over the domain $[0, 2\pi]$:

```
syms t
ezpolar(1+cos(t))
```



Purpose

3-D colored surface plotter.

Syntax

```
ezsurf(f)
ezsurf(f,domain)
ezsurf(x,y,z)
ezsurf(x,y,z,[smin,smax,tmin,tmax]) or ezsurf(x,y,z,[min,max])
ezsurf(...,n)
ezsurf(...,'circ')
```

Purpose

`ezsurf(f)` creates a graph of $f(x,y)$, where `f` is a symbolic expression that represents a mathematical function of two variables, such as x and y.

The function f is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function f is not defined (singular) for points on the grid, then these points are not plotted.

`ezsurf(f,domain)` plots f over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If f is a function of the variables u and v (rather than x and y), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezsurf(u^2 - v^3,[0,1],[3,6])` plots $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezsurf(x,y,z)` plots the parametric surface $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ over the square: $-2\pi < s < 2\pi$, $-2\pi < t < 2\pi$.

`ezsurf(x,y,z,[smin,smax,tmin,tmax])` or `ezsurf(x,y,z,[min,max])` plots the parametric surface using the specified domain.

`ezsurf(...,n)` plots f over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezsurf(...,'circ')` plots f over a disk centered on the domain

Remarks

`rotate3d` is always on. To rotate the graph, click and drag with the mouse.

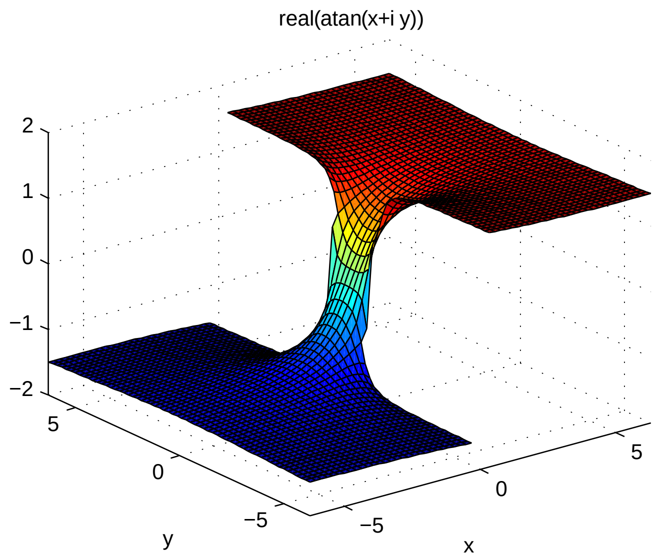
Examples

`ezsurf` does not graph points where the mathematical function is not defined (these data points are set to `NaN`s, which MATLAB does not plot). This example illustrates this filtering of singularities/discontinuous points by graphing the function,

$$f(x, y) = real(atan(x + iy))$$

over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$:

```
syms x y
ezsurf(real(atan(x+i*y)))
```



Note also that `ezsurf` creates graphs that have axis labels, a title, and extend to the axis limits.

See Also

`ezcontour`, `ezcontourf`, `ezmesh`, `ezmeshc`, `ezplot`, `ezpolar`, `ezsurfc`, `surf`

Purpose Combined surface/contour plotter.

Syntax

```
ezsurfc(f)
ezsurfc(f,domain)
ezsurfc(x,y,z)
ezsurfc(x,y,z,[smin,smax,tmin,tmax]) or ezsurfc(x,y,z,[min,max])
ezsurfc(...,n)
ezsurfc(...,'circ')
```

Description

`ezsurfc(f)` creates a graph of $f(x,y)$, where `f` is a symbolic expression that represents a mathematical function of two variables, such as x and y.

The function f is plotted over the default domain $-2\pi < x < 2\pi$, $-2\pi < y < 2\pi$. MATLAB chooses the computational grid according to the amount of variation that occurs; if the function f is not defined (singular) for points on the grid, then these points are not plotted.

`ezsurfc(f,domain)` plots f over the specified `domain`. `domain` can be either a 4-by-1 vector [xmin, xmax, ymin, ymax] or a 2-by-1 vector [min, max] (where, min < x < max, min < y < max).

If f is a function of the variables u and v (rather than x and y), then the domain endpoints umin, umax, vmin, and vmax are sorted alphabetically. Thus, `ezsurfc(u^2 - v^3,[0,1],[3,6])` plots $u^2 - v^3$ over $0 < u < 1$, $3 < v < 6$.

`ezsurfc(x,y,z)` plots the parametric surface $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$ over the square: $-2\pi < s < 2\pi$, $-2\pi < t < 2\pi$.

`ezsurfc(x,y,z,[smin,smax,tmin,tmax])` or `ezsurfc(x,y,z,[min,max])` plots the parametric surface using the specified domain.

`ezsurfc(...,n)` plots f over the default domain using an `n`-by-`n` grid. The default value for `n` is 60.

`ezsurfc(...,'circ')` plots f over a disk centered on the domain.

Remarks

`rotate3d` is always on. To rotate the graph, click and drag with the mouse.

ezsurfc

Examples

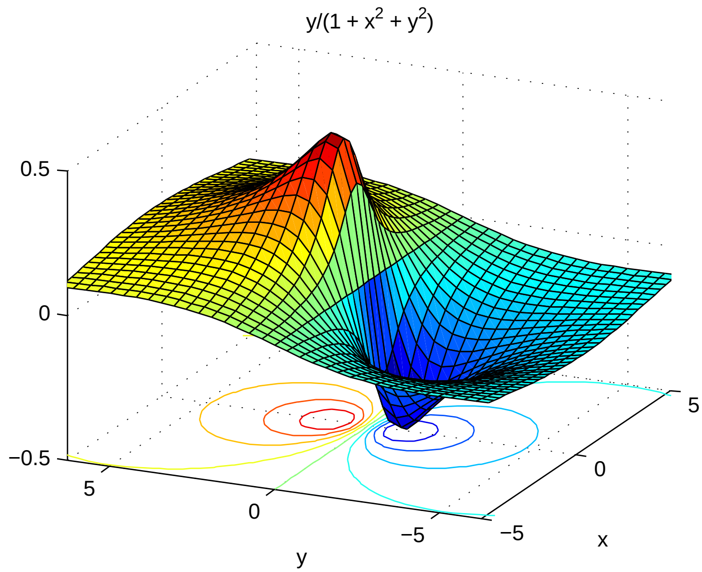
Create a surface/contour plot of the expression,

$$f(x, y) = \frac{y}{1 + x^2 + y^2}$$

over the domain $-5 < x < 5$, $-2*\text{pi} < y < 2*\text{pi}$, with a computational grid of size 35-by-35:

```
syms x y
ezsurfc(y/(1 + x^2 + y^2),[-5,5,-2*pi,2*pi],35)
```

Use the mouse to rotate the axes to better observe the contour lines (this picture uses a view of azimuth = –65.5 and elevation = 26)



See Also

`ezcontour`, `ezcontourf`, `ezmesh`, `ezmeshc`, `ezplot`, `ezpolar`, `ezsurf`, `surfc`

Purpose Factorization.

Syntax

```
factor(X)
```

Description `factor` can take a positive integer, an array of symbolic expressions, or an array of symbolic integers as an argument. If `N` is a positive integer, `factor(N)` returns the prime factorization of `N`.

If `S` is a matrix of polynomials or integers, `factor(S)` factors each element. If any element of an integer array has more than 16 digits, you must use `sym` to create that element, for example, `sym('N')`.

Examples `factor(x^3–y^3)` returns

```
(x–y)*(x^2+x*y+y^2)
```

`factor([a^2–b^2, a^3+b^3])` returns

```
[(a–b)*(a+b), (a+b)*(a^2–a*b+b^2)]
```

`factor(sym('12345678901234567890'))` returns

```
(2)*(3)^2*(5)*(101)*(3803)*(3607)*(27961)*(3541)
```

See Also `collect`, `expand`, `horner`, `simplify`, `simple`

findsym

Purpose

Finds the variables in a symbolic expression or matrix.

Syntax

```
r = findsym(S)
r = findsym(S,n)
```

Description

`findsym(S)` returns all symbolic variables in `S` in alphabetical order, separated by commas. If `S` does not contain any variables, `findsym` returns an empty string.

`findsym(S,n)` returns the `n` variables alphabetically closest to `x`.

Note A symbolic variable is an alphanumeric name, other than `i` or `j`, that begins with an alphabetic character.

Examples

```
syms a x y z t
```

`findsym(sin(pi*t))` returns `pi, t`.

`findsym(x+i*y–j*z)` returns `x, y, z`.

`findsym(a+y,1)` returns `y`.

See Also

`compose`, `diff`, `int`, `limit`, `taylor`

Purpose

Functional inverse.

Syntax

```
g = finverse(f)
g = finverse(f,u)
```

Description

`g = finverse(f)` returns the functional inverse of `f`. `f` is a scalar `sym` representing a function of one symbolic variable, say `x`. Then `g` is a scalar `sym` that satisfies `g(f(x)) = x`. That is, `finverse(f)` returns f^{-1}, provided f^{-1} exists.

`g = finverse(f,v)` uses the symbolic variable `v`, where `v` is a `sym`, as the independent variable. Then `g` is a scalar `sym` that satisfies `g(f(v)) = v`. Use this form when `f` contains more than one symbolic variable.

Examples

`finverse(1/tan(x))` returns

```
atan(1/x)
```

`finverse(exp(u–2*v),u)` returns

```
2*v+log(u)
```

See Also

`compose`, `syms`

fortran

Purpose Fortran representation of a symbolic expression.

Syntax `fortran(S)`

Description `fortran(S)` returns the Fortran code equivalent to the expression `S`.

Examples The statements

```
syms x
f = taylor(log(1+x));
fortran(f)
```

return

```
t0 = x–x**2/2+x**3/3–x**4/4+x**5/5
```

The statements

```
H = sym(hilb(3));
fortran(H)
```

return

```
H(1,1) = 1           H(1,2) = 1.E0/2.E0    H(1,3) = 1.E0/3.E0
H(2,1) = 1.E0/2.E0   H(2,2) = 1.E0/3.E0    H(2,3) = 1.E0/4.E0
H(3,1) = 1.E0/3.E0   H(3,2) = 1.E0/4.E0    H(3,3) = 1.E0/5.E0
```

See Also `ccode`, `latex`, `pretty`

Purpose

Fourier integral transform.

Syntax

```
F = fourier(f)
F = fourier(f,v)
F = fourier(f,u,v)
```

Description

`F = fourier(f)` is the Fourier transform of the symbolic scalar `f` with default independent variable `x`. The default return is a function of `w`. The Fourier transform is applied to a function of `x` and returns a function of `w`.

$$f = f(x) \Rightarrow F = F(w)$$

If `f = f(w)`, `fourier` returns a function of `t`.

$$F = F(t)$$

By definition

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

where `x` is the symbolic variable in `f` as determined by `findsym`.

`F = fourier(f,v)` makes `F` a function of the symbol `v` instead of the default `w`.

$$F(v) = \int_{-\infty}^{\infty} f(x)e^{-ivx}dx$$

`F = fourier(f,u,v)` makes `f` a function of `u` and `F` a function of `v` instead of the default variables `x and w`, respectively.

$$F(v) = \int_{-\infty}^{\infty} f(u)e^{-ivu}du$$

fourier

Examples

Fourier Transform	MATLAB Command
$f(x) = e^{-x^2}$	`f = exp(-x^2)`
$F[f](w) = \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$	`fourier(f)`
	returns
$= \sqrt{\pi}\, e^{-w^2/4}$	`pi^(1/2)*exp(-1/4*w^2)`
$g(w) = e^{-\lvert w \rvert}$	`g = exp(-abs(w))`
$F[g](t) = \int_{-\infty}^{\infty} g(w) e^{-itw} dw$	`fourier(g)`
	returns
$= \dfrac{2}{1 + t^2}$	`2/(1+t^2)`
$f(x) = x e^{-\lvert x \rvert}$	`f = x*exp(-abs(x))`
$F[f](u) = \int_{-\infty}^{\infty} f(x) e^{-ixu} dx$	`fourier(f,u)`
	returns
$= -\dfrac{4i}{(1 + u^2)^2} u$	`-4*i/(1+u^2)^2*u`

Fourier Transform	MATLAB Command
$f(x, v) = e^{-x^2\|v\|}\frac{\sin v}{v}$, x real	`syms x real` `f = exp(-x^2*abs(v))*sin(v)/v`
$F[f(v)](u) = \int_{-\infty}^{\infty} f(x, v)e^{-ivu}dv$	`fourier(f,v,u)`
	returns
$= -\operatorname{atan}\frac{u-1}{x^2} + \operatorname{atan}\frac{u+1}{x^2}$	`-atan((u-1)/x^2)+atan((u+1)/x^2)`

See Also

`ifourier`, `laplace`, `ztrans`

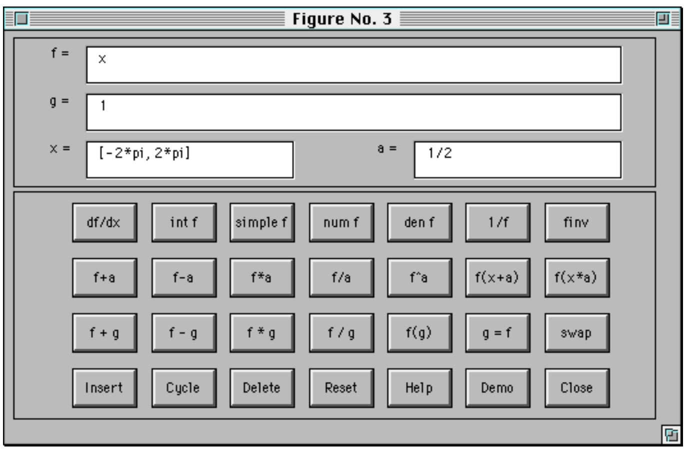
funtool

Purpose Function calculator.

Syntax `funtool`

Description `funtool` is a visual function calculator that manipulates and displays functions of one variable. At the click of a button, for example, `funtool` draws a graph representing the sum, product, difference, or ratio of two functions that you specify. `funtool` includes a function memory that allows you to store functions for later retrieval.

At startup, `funtool` displays graphs of a pair of functions, `f(x) = x` and `g(x) = 1`. The graphs plot the functions over the domain `[–2*pi, 2*pi]`. `funtool` also displays a control panel that lets you save, retrieve, redefine, combine, and transform `f` and `g`.



Text Fields. The top of the control panel contains a group of editable text fields.

f= Displays a symbolic expression representing `f`. Edit this field to redefine `f`.

g= Displays a symbolic expression representing `g`. Edit this field to redefine `g`.

x= Displays the domain used to plot `f` and `g`. Edit this field to specify a different domain.

a= Displays a constant factor used to modify `f` (see button descriptions in the next section). Edit this field to change the value of the constant factor.

`funtool` redraws `f` and `g` to reflect any changes you make to the contents of the control panel's text fields.

Control Buttons. The bottom part of the control panel contains an array of buttons that transform `f` and perform other operations.

The first row of control buttons replaces `f` with various transformations of `f`.

df/dx	Derivative of `f`
int f	Integral of `f`
simple f	Simplified form of `f`, if possible
num f	Numerator of `f`
den f	Denominator of `f`
1/f	Reciprocal of `f`
finv	Inverse of `f`

The operators **intf** and **finv** may fail if the corresponding symbolic expressions do not exist in closed form.

The second row of buttons translates and scales `f` and the domain of `f` by a constant factor. To specify the factor, enter its value in the field labeled **a=** on the calculator control panel. The operations are

f+a	Replaces `f(x)` by `f(x) + a`.
f–a	Replaces `f(x)` by `f(x) – a`.
f*a	Replaces `f(x)` by `f(x) * a`.
f/a	Replaces `f(x)` by `f(x) / a`.
f^a	Replaces `f(x)` by `f(x) ^ a`.
f(x+a)	Replaces `f(x)` by `f(x + a )`.
f(x*a)	Replaces `f(x)` by `f(x * a)`.

The first four buttons of the third row replace `f` with a combination of `f` and `g`.

f+g Replaces `f(x)` by `f(x) + g(x)`.

f–g Replaces `f(x)` by `f(x)–g(x)`.

f*g Replaces `f(x)` by `f(x) * g(x)`.

f/g Replaces `f(x)` by `f(x) / g(x)`.

The remaining buttons on the third row interchange `f` and `g`.

g=f Replaces `g` with `f`.

swap Replaces `f` with `g` and `g` with `f`.

The first three buttons in the fourth row allow you to store and retrieve functions from the calculator's function memory.

Insert Adds `f` to the end of the list of stored functions.

Cycle Replaces `f` with the next item on the function list.

Delete Deletes `f` from the list of stored functions.

The other four buttons on the fourth row perform miscellaneous functions:

Reset Resets the calculator to its initial state.

Help Displays the online help for the calculator.

Demo Runs a short demo of the calculator.

Close Closes the calculator's windows.

See Also

`ezplot`, `syms`

Purpose Horner polynomial representation.

Syntax

```
R = horner(P)
```

Description Suppose `P` is a matrix of symbolic polynomials. `horner(P)` transforms each element of `P` into its Horner, or nested, representation.

Examples

`horner(x^3-6*x^2+11*x-6)` returns

```
-6+(11+(-6+x)*x)*x
```

`horner([x^2+x;y^3-2*y])` returns

```
[    (1+x)*x]
[(-2+y^2)*y]
```

See Also `expand`, `factor`, `simple`, `simplify`, `syms`

hypergeom

Purpose

Generalized hypergeometric function.

Syntax

```
hypergeom(n, d, z)
```

Description

`hypergeom(n, d, z)` is the generalized hypergeometric function *F(n, d, z)*, also known as the Barnes extended hypergeometric function and denoted by ${}_jF_k$ where `j = length(n)` and `k = length(d)`. For scalar `a`, `b`, and `c`, `hypergeom([a,b],c,z)` is the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$.

The definition by a formal power series is

$$F(n, d, z) = \sum_{k=0}^{\infty} \frac{C_{n,k}}{C_{d,k}} \cdot \frac{z^k}{k!}$$

where

$$C_{\vec{v},k} = \prod_{j=1}^{|\vec{v}|} \frac{\Gamma(v_j + k)}{\Gamma(v_j)}$$

Either of the first two arguments may be a vector providing the coefficient parameters for a single function evaluation. If the third argument is a vector, the function is evaluated pointwise. The result is numeric if all the arguments are numeric and symbolic if any of the arguments is symbolic.

See Abramowitz and Stegun, *Handbook of Mathematical Functions*, chapter 15.

Examples

`syms a z`

`hypergeom([],[],z)` returns `exp(z)`

`hypergeom(1,[],z)` returns `-1/(-1+z)`

`hypergeom(1,2,'z')` returns `(exp(z)-1)/z`

`hypergeom([1,2],[2,3],'z')` returns `-2*(-exp(z)+1+z)/z^2`

`hypergeom(a,[],z)` returns `(1-z)^(-a)`

`hypergeom([],1,-z^2/4)` returns `besselj(0,z)` `hypergeom([-n, n],1/2,(1-z)/2)` returns

```
expand(cos(n*acos(z)))
```

which is *T(n, z)*, the `n`-th Chebyshev polynomial.

ifourier

Purpose

Inverse Fourier integral transform.

Syntax

```
f = ifourier(F)
f = ifourier(F,u)
f = ifourier(F,v,u)
```

Description

`f = ifourier(F)` is the inverse Fourier transform of the scalar symbolic object `F` with default independent variable `w`. The default return is a function of `x`. The inverse Fourier transform is applied to a function of `w` and returns a function of `x`.

$$F = F(w) \Rightarrow f = f(x)$$

If `F = F(x)`, `ifourier` returns a function of `t`.

$$f = f(t)$$

By definition

$$f(x) = 1/(2\pi) \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

`f = ifourier(F,u)` makes `f` a function of `u` instead of the default `x`.

$$f(u) = 1/(2\pi) \int_{-\infty}^{\infty} F(w) e^{iwu} dw$$

Here `u` is a scalar symbolic object.

`f = ifourier(F,v,u)` takes `F` to be a function of `v` and `f` to be a function of `u` instead of the default `w` and `x`, respectively.

$$f(u) = 1/(2\pi) \int_{-\infty}^{\infty} F(v) e^{ivu} dv$$

Examples

Inverse Fourier Transform	MATLAB Command

$f(w) = e^{w^2/(4a^2)}$
 	
$F^{-1}[f](x) = \int_{-\infty}^{\infty} f(w)e^{ixw}dw$
 	
$= \frac{a}{\sqrt{\pi}}e^{-(ax)^2}$
	`syms a real`
	`f = exp(-w^2/(4*a^2))`
	
	`F = ifourier(f)`
	`F = simple(F)`
	
	returns
	
	`a*exp(-x^2*a^2)/pi^(1/2)`

$g(x) = e^{-\|x\|}$
 	
$F^{-1}[g](t) = \int_{-\infty}^{\infty} g(x)e^{itx}dx$
 	
$= \frac{\pi}{1+t^2}$
	`g = exp(-abs(x))`
	
	`ifourier(g)`
	
	returns
	
	`1/(1+t^2)/pi`

$f(w) = 2e^{-\|w\|} - 1$
 	
$F^{-1}[f](t) = \int_{-\infty}^{\infty} f(w)e^{itw}dw$
 	
$= \frac{2 - \pi\delta(t)(1-t^2)}{\pi(1+t^2)}$
	`f = 2*exp(-abs(w)) - 1`
	
	`simple(ifourier(f,t))`
	
	returns
	
	`(2-pi*Dirac(t)-pi*Dirac(t)*t^2)/`
	`(pi+pi*t^2)`

Inverse Fourier Transform	MATLAB Command
$f(w, v) = e^{-w^2\|v\|}\frac{\sin v}{v}$, w real	`syms w real` `f = exp(-w^2*abs(v))*sin(v)/v`
$F^{-1}[f(v)](t) = \int_{-\infty}^{\infty} f(w, v)e^{ivt}dv$	`ifourier(f,v,t)` returns
$= \frac{1}{2\pi}\left(\operatorname{atan}\frac{t+1}{w^2} - \operatorname{atan}\frac{t-1}{w^2}\right)$	`1/2*(atan((t+1)/w^2)-` `atan((-1+t)/w^2))/pi`

See Also

`fourier`, `ilaplace`, `iztrans`

Purpose

Inverse Laplace transform.

Syntax

```
F = ilaplace(L)
F = ilaplace(L,y)
F = ilaplace(L,y,x)
```

Description

`F = ilaplace(L)` is the inverse Laplace transform of the scalar symbolic object `L` with default independent variable `s`. The default return is a function of `t`. The inverse Laplace transform is applied to a function of `s` and returns a function of `t`.

$$L = L(s) \Rightarrow F = F(t)$$

If `L = L(t)`, `ilaplace` returns a function of `x`.

$$F = F(x)$$

By definition

$$F(t) = \int_{c - i\infty}^{c + i\infty} L(s) e^{st} ds$$

where `c` is a real number selected so that all singularities of `L(s)` are to the left of the line `s = c, i`.

`F = ilaplace(L,y)` makes `F` a function of `y` instead of the default `t`.

$$F(y) = \int_{c - i\infty}^{c + i\infty} L(y) e^{sy} ds$$

Here `y` is a scalar symbolic object.

`F = ilaplace(L,y,x)` takes `F` to be a function of `x` and `L` a function of `y` instead of the default variables `t` and `s`, respectively.

$$F(x) = \int_{c - i\infty}^{c + i\infty} L(y) e^{xy} dy$$

Examples

Inverse Laplace Transform	MATLAB Command

$f(s) = \frac{1}{s^2}$
$L^{-1}[f] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds$
$= t$
	`f = 1/s^2`
	`ilaplace(f)`
	returns
	`t`

$g(t) = \frac{1}{(t-a)^2}$
$L^{-1}[g] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(t)e^{xt}dt$
$= xe^{ax}$
	`g = 1/(t-a)^2`
	`ilaplace(g)`
	returns
	`x*exp(a*x)`

$f(u) = \frac{1}{u^2 - a^2}$
$L^{-1}[f] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(u)e^{xu}du$
$= \frac{1}{2ae^{ax}} - \frac{1}{2ae^{-ax}}$
	`f = 1/(u^2-a^2)`
	`ilaplace(f,x)`
	returns
	`1/2/a*exp(a*x)-1/2/a*exp(-a*x)`

Inverse Laplace Transform	MATLAB Command
$f(s, v) = \dfrac{s^3 v}{s^2 + v^2}$	`f = s^3*v/(s^2+v^2)`
$L^{-1}[f] = \dfrac{1}{2\pi i} \displaystyle\int_{c - i\infty}^{c + i\infty} f(s, v) e^{xv} dv$	`laplace(f,v,x)`
$= s^3 \cos sx$	returns `s^3*cos(s*x)`

See Also

`ifourier`, `iztrans`, `laplace`

imag

Purpose Symbolic imaginary part.

Syntax `imag(Z)`

Description `imag(Z)` is the imaginary part of a symbolic `Z`.

See Also conj, real

Purpose

Integrate.

Syntax

```
R = int(S)
R = int(S,v)
R = int(S,a,b)
R = int(S,v,a,b)
```

Description

`int(S)` returns the indefinite integral of `S` with respect to its symbolic variable as defined by `findsym`.

`int(S,v)` returns the indefinite integral of `S` with respect to the symbolic scalar variable `v`.

`int(S,a,b)` returns the definite integral from `a` to `b` of each element of `S` with respect to each element's default symbolic variable. `a` and `b` are symbolic or double scalars.

`int(S,v,a,b)` returns the definite integral of `S` with respect to `v` from `a` to `b`.

Examples

`int(-2*x/(1+x^2)^2)` returns

```
1/(1+x^2)
```

`int(x/(1+z^2),z)` returns

```
atan(z)*x
```

`int(x*log(1+x),0,1)` returns

```
1/4
```

`int(2*x, sin(t), 1)` returns

```
1-sin(t)^2
```

`int([exp(t),exp(alpha*t)])` returns

```
[exp(t), 1/alpha*exp(alpha*t)]
```

See Also

`diff`, `symsum`

Purpose Matrix inverse.

Syntax

```
R = inv(A)
```

Description `inv(A)` returns inverse of the symbolic matrix `A`.

Examples The statements

```
A = sym([2,–1,0;–1,2,–1;0,–1,2]);
inv(A)
```

return

```
[ 3/4, 1/2, 1/4]
[ 1/2,   1, 1/2]
[ 1/4, 1/2, 3/4]
```

The statements

```
syms a b c d
A = [a b; c d]
inv(A)
```

return

```
[  d/(a*d–b*c), –b/(a*d–b*c)]
[ –c/(a*d–b*c),  a/(a*d–b*c)]
```

Suppose you have created the following M-file:

```
%% Generate a symbolic N-by-N Hilbert matrix.
function A = genhilb(N)
syms t;
for i = 1:N
        for j = 1:N
        A(i,j) = 1/(i + j – t);
        end
end
```

Then, the following statement

```
inv(genhilb(2))
```

returns

```
[      -(-3+t)^2*(-2+t), (-3+t)*(-2+t)*(-4+t)]
[(-3+t)*(-2+t)*(-4+t),       -(-3+t)^2*(-4+t)]
```

the symbolic inverse of the 2-by-2 Hilbert matrix.

See Also

vpa

Arithmetic Operations page

iztrans

Purpose

Inverse z-transform.

Syntax

```
f = iztrans(F)
f = iztrans(F,k)
f = iztrans(F,w,k)
```

Description

`f = iztrans(F)` is the inverse z-transform of the scalar symbolic object `F` with default independent variable `z`. The default return is a function of `n`:

$$f(n) = \frac{1}{2\pi i} \oint_{|z| = R} F(z) z^{n-1} dz, \; n = 1, 2, \ldots$$

where R is a positive number chosen so that the function $F(z)$ is analytic on and outside the circle $|z| = R$

If `F = F(n)`, `iztrans` returns a function of `k`:

$$f = f(k)$$

`f = iztrans(F,k)` makes `f` a function of `k` instead of the default `n`. Here `k` is a scalar symbolic object.

`f = iztrans(F,w,k)` takes `F` to be a function of `w` instead of the default `findsym(F)` and returns a function of `k`:

$$F = F(w) \Rightarrow f = f(k)$$

Examples

Inverse Z-Transform	MATLAB Operation
$f(z) = \frac{2z}{(z-2)^2}$	`f = 2*z/(z–2)^2`
$Z^{-1}[f] = \frac{1}{2\pi i} \oint_{\lvert z \rvert = R} f(s) z^{n-1} dz$	`iztrans(f)` returns
$= n2^n$	2^n*n

Inverse Z-Transform	MATLAB Operation

$g(n) = \frac{n(n+1)}{n^2+2n+1}$
$Z^{-1}[g] = \frac{1}{2\pi i} \oint_{\lvert n\rvert = R} g(n) n^{k-1} dn$
$= -1^k$
	`g = n*(n+1)/(n^2+2*n+1)`
	`iztrans(g)`
	returns
	`(-1)^k`

$f(z) = \frac{z}{z-a}$
$Z^{-1}[f] = \frac{1}{2\pi i} \oint_{\lvert z\rvert = R} f(z) z^{k-1} dz$
$= a^k$
	`f = z/(z-a)`
	`iztrans(f,k)`
	returns
	`a^k`

$f(x,z) = \frac{x(x-e^z)}{x^2-2xe^z+e^{2z}}$
$Z^{-1}[f] = \frac{1}{2\pi i} \oint_{\lvert x\rvert = R} f(x,z) x^{k-1} dx$
$= e^{kz}$
	`f = x*(x-exp(z))/`
	`(x^2-2*x*exp(z)+exp(2*z))`
	`iztrans(f,x,k)`
	returns
	`exp(z)^k`

See Also

`ifourier`, `ilaplace`, `ztrans`

jacobian

Purpose

Jacobian matrix.

Syntax

```
R = jacobian(w,v)
```

Description

`jacobian(w,v)` computes the Jacobian of `w` with respect to `v`. `w` is a symbolic scalar expression or a symbolic column vector. `v` is a symbolic row vector. The `(i,j)`-th entry of the result is ∂`w(i)`/∂`v(j)`.

Examples

The statements

```
w = [x*y*z; y; x+z];
v = [x,y,z];
R = jacobian(w,v)
b = jacobian(x+z, v)
```

return

```
R =
[y*z, x*z, x*y]
[  0,   1,   0]
[  1,   0,   1]

b =
[1, 0, 1]
```

See Also

`diff`

Purpose Jordan canonical form.

Syntax

```
J = jordan(A)
[V,J] = jordan(A)
```

Description

jordan(A) computes the Jordan canonical (normal) form of A, where A is a symbolic or numeric matrix. The matrix must be known exactly. Thus, its elements must be integers or ratios of small integers. Any errors in the input matrix may completely change the Jordan canonical form.

[V,J] = jordan(A) computes both J, the Jordan canonical form, and the similarity transform, V, whose columns are the generalized eigenvectors. Moreover, V\A*V=J.

Examples

The statements

```
A = [1 –3 –2; –1  1 –1; 2 4 5]
[V,J] = jordan(A)
```

return

```
A =
     1     –3     –2
    –1      1     –1
     2      4      5

V =
[–1, –2, 2]
[ 0, –2, 0]
[ 0,  4, 0]

J =
[3, 0, 0]
[0, 2, 1]
[0, 0, 2]
```

Then the statements

```
V = double(V)
V\A*V
```

`return`

```
V =
    -1    -2     2
     0    -2     0
     1     4     0

ans =
     3     0     0
     0     2     1
     0     0     2
```

See Also `eig`, `poly`

Purpose

Lambert's *W* function.

Syntax

```
Y = lambertw(X)
```

Description

`lambertw(X)` evaluates Lambert's *W* function at the elements of `X`, a numeric matrix or a symbolic matrix. Lambert's *W* solves the equation

$$we^{w} = x$$

for `w` as a function of `x`.

Examples

`lambertw([0 –exp(–1); pi 1])` returns

```
          0    –1.0000
     1.0737     0.5671
```

The statements

```
syms x y
lambertw([0 x;1 y])
```

return

```
[            0, lambertw(x)]
[  lambertw(1), lambertw(y)]
```

References

[1] Corless, R.M, Gonnet, G.H., Hare, D.E.G., and Jeffrey, D.J., *Lambert's W Function in Maple*, Technical Report, Dept. of Applied Math., Univ. of Western Ontario, London, Ontario, Canada.

[2] Corless, R.M, Gonnet, G.H., Hare, D.E.G., and Jeffrey, D.J., *On Lambert's W Function*, Technical Report, Dept. of Applied Math., Univ. of Western Ontario, London, Ontario, Canada.

Both papers are available by anonymous FTP from

```
cs–archive.uwaterloo.ca
```

laplace

Purpose

Laplace transform.

Syntax

```
laplace(F)
laplace(F,t)
fourier(F,w,z)
```

Description

`L = laplace(F)` is the Laplace transform of the scalar symbol `F` with default independent variable `t`. The default return is a function of `s`. The Laplace transform is applied to a function of `t` and returns a function of `s`.

$$F = F(t) \Rightarrow L = L(s)$$

If `F = F(s)`, `laplace` returns a function of `t`.

$$L = L(t)$$

By definition

$$L(s) = \int_0^\infty F(t)e^{-st}dt$$

where `t` is the symbolic variable in `F` as determined by `findsym`.

`L = laplace(F,t)` makes `L` a function of `t` instead of the default `s`.

$$L(t) = \int_0^\infty F(x)e^{-tx}dx$$

Here `L` is returned as a scalar symbol.

`L = laplace(F,w,z)` makes `L` a function of `z` and `F` a function of `w` instead of the default variables `s` and `t`, respectively.

$$L(z) = \int_0^\infty F(w)e^{-zw}dw$$

Examples

Laplace Transform	MATLAB Command

$f(t) = t^4$
$L[f] = \int_0^\infty f(t)e^{-ts}dt$
$= \frac{24}{s^5}$
	`f = t^4`
	`laplace(f)`
	returns
	`24/s^5`

$g(s) = \frac{1}{\sqrt{s}}$
$L[g](t) = \int_0^\infty g(s)e^{-st}ds$
$= \sqrt{\frac{\pi}{s}}$
	`g = 1/sqrt(s)`
	`laplace(g)`
	returns
	`1/s^(1/2)*pi^(1/2)`

$f(t) = e^{-at}$
$L[f](x) = \int_0^\infty f(t)e^{-tx}dt$
$= \frac{1}{x+a}$
	`f = exp(-a*t)`
	`laplace(f,x)`
	returns
	`1/(x + a)`

laplace

Laplace Transform	MATLAB Command

$f(t, v) = 1 - \cos tv$
$L[f](x) = \int_0^{\infty} f(t, v) e^{-vx} dv$
$= \frac{1}{x} - \frac{x}{x^2 + t^2} = \frac{t^2}{x^3 + xt^2}$
	`f = 1 - cos(t*v)`
	`laplace(f,v,x)`
	returns
	`1/x-x/(x^2+t^2)`

See Also

`fourier`, `ilaplace`, `ztrans`

Purpose LaTeX representation of a symbolic expression.

Syntax

```
latex(S)
```

Description `latex(S)` returns the LaTeX representation of the symbolic expression `S`.

Examples The statements

```
syms x
f = taylor(log(1+x));
latex(f)
```

return

```
x-1/2\,{x}^{2}+1/3\,{x}^{3}-1/4\,{x}^{4}+1/5\,{x}^{5}
```

The statements

```
H = sym(hilb(3));
latex(H)
```

return

```
\left [\begin {array}{ccc} 1&1/2&1/3\\\noalign{\medskip}1/2&1/
3&1/4
\\\noalign{\medskip}1/3&1/4&1/5\end {array}\right ]
```

The statements

```
syms alpha t
A = [alpha t alpha*t];
latex(A)
```

return

```
\left [\begin {array}{ccc} \alpha&t&\alpha\,t\end {array}\right ]
```

See Also `pretty`, `ccode`, `fortran`

limit

Purpose

Limit of a symbolic expression.

Syntax

```
limit(F,x,a)
limit(F,a)
limit(F)
limit(F,x,a,'right')
limit(F,x,a,'left')
```

Description

`limit(F,x,a)` takes the limit of the symbolic expression `F` as `x -> a`.

`limit(F,a)` uses `findsym(F)` as the independent variable.

`limit(F)` uses `a = 0` as the limit point.

`limit(F,x,a,'right')` or `limit(F,x,a,'left')` specify the direction of a one-sided limit.

Examples

Assume

```
syms x a t h;
```

Then

```
limit(sin(x)/x)                 => 1
limit(1/x,x,0,'right')          => inf
limit(1/x,x,0,'left')           => –inf
limit((sin(x+h)–sin(x))/h,h,0)  => cos(x)
v = [(1 + a/x)^x, exp(–x)];
limit(v,x,inf,'left')           => [exp(a),  0]
```

See Also

`pretty`, `ccode`, `fortran`

Purpose

Access Maple kernel.

Syntax

`r = maple('statement')`
`r = maple('function',arg1,arg2,...)`
`[r, status] = maple(...)`
`maple('traceon')` or `maple trace on`
`maple('traceoff')` or `maple trace off`

Description

`maple('statement')` sends `statement` to the Maple kernel and returns the result. A semicolon for the Maple syntax is appended to `statement` if necessary.

`maple('function',arg1,arg2,...)` accepts the quoted name of any Maple function and associated input arguments. The arguments are converted to symbolic expressions if necessary, and `function` is then called with the given arguments. If the input arguments are `sym`s, then `maple` returns a `sym`. Otherwise, it returns a result of class `char`.

`[r,status] = maple(...)` is an option that returns the warning/error status. When the statement execution is successful, `r` is the result and `status` is `0`. If the execution fails, `r` is the corresponding warning/error message, and `status` is a positive integer.

`maple('traceon')` (or `maple trace on`) causes all subsequent Maple statements and results to be printed. `maple('traceoff')` (or `maple trace off`) turns this feature off.

Examples

Each of the following statements evaluate π to 100 digits:

```
maple('evalf(Pi,100)')
maple evalf Pi 100
maple('evalf','Pi',100)
```

The statement

```
[result,status] = maple('BesselK',4.3)
```

returns the following output because Maple's `BesselK` function needs two input arguments:

```
result =
Error, (in BesselK) invalid arguments
status =
2
```

The `traceon` command shows how Symbolic Math Toolbox commands interact with Maple. For example, the statements

```
syms x
v = [x^2-1;x^2-4]
maple traceon % or maple trace on
w = factor(v)
```

return

```
v =
[x^2-1]
[x^2-4]

statement =
map(ifactor,array([[x^2-1],[x^2-4]]));

result =
Error, (in ifactor) invalid arguments

statement =
map(factor,array([[x^2-1],[x^2-4]]));

result =
MATRIX([[(x-1)*(x+1)], [(x-2)*(x+2)]])

w =
[(x-1)*(x+1)]
[(x-2)*(x+2)]
```

This example reveals that the `factor` statement first invokes Maple's integer factor (`ifactor`) statement to determine whether the argument is a factorable integer. If Maple's integer factor statement returns an error, the Symbolic

Math Toolbox `factor` statement then invokes Maple's expression factoring statement.

See Also

`mhelp`, `procread`

mapleinit

Purpose Initialize the Maple kernel.

Syntax

```
mapleinit
```

Description

`mapleinit` determines the path to the directory containing the Maple Library, loads the Maple linear algebra and integral transform packages, initializes `digits`, and establishes several aliases. `mapleinit` is called by the MEX-file interface to Maple.

You can edit the `mapleinit` M-file to change the pathname to the Maple library. You do this by changing the `initstring` variable in `mapleinit.m` to the full pathname of the Maple library, as described below.

UNIX. Suppose you already have a copy of the Library for Maple V, Release 4 in the UNIX directory `/usr/local/Maple/lib`. You can edit `mapleinit.m` to contain

```
maplelib = '/usr/local/Maple/lib'
```

and then delete the copy of the Maple Library that is distributed with MATLAB.

MS-Windows. Suppose you already have a copy of the Library for Maple V, Release 4 in the directory `C:\MAPLE\LIB`. You can edit `mapleinit.m` to contain

```
maplelib = 'C:\MAPLE\LIB'
```

and then delete the copy of the Maple Library that is distributed with MATLAB.

Macintosh. Suppose you already have a copy of the Library for Maple V, Release 4 in the directory `MyDisk:Maple:Lib`. You can edit `mapleinit.m` to contain

```
maplelib = 'MyDisk:Maple:Lib'
```

and then delete the copy of the Maple Library that is distributed with MATLAB.

Purpose Numeric evaluation of Maple mathematical function.

Syntax

```
Y = mfun('function',par1,par2,par3,par4)
```

Description `mfun('function',par1,par2,par3,par4)` numerically evaluates one of the special mathematical functions known to Maple. Each `par` argument is a numeric quantity corresponding to a Maple parameter for `function`. You can use up to four parameters. The last parameter specified can be a matrix, usually corresponding to `X`. The dimensions of all other parameters depend on the Maple specifications for `function`. You can access parameter information for Maple functions using one of the following commands:

```
help mfunlist
mhelp function
```

Maple evaluates `function` using 16 digit accuracy. Each element of the result is a MATLAB numeric quantity. Any singularity in `function` is returned as `NaN`.

Examples `mfun('FresnelC',0:5)` returns

```
0    0.7799    0.4883    0.6057    0.4984    0.5636
```

`mfun('Chi',[3*i 0])` returns

```
0.1196 + 1.5708i   NaN
```

See Also `mfunlist`, `mhelp`

mfunlist

Purpose

List special functions for use with `mfun`.

Syntax

```
mfunlist % help mfunlist on the Macintosh
```

Description

`mfunlist` lists the special mathematical functions for use with the `mfun` function. The following tables describe these special functions.

You can access more detailed descriptions by typing

```
mhelp function
```

Limitations

In general, the accuracy of a function will be lower near its roots and when its arguments are relatively large.

Runtime depends on the specific function and its parameters. In general, calculations are slower than standard MATLAB calculations.

See Also

`mfun`, `mhelp`

References

[1] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, Dover Publications, 1965.

Table Conventions

The following conventions are used in Table 2-1: MFUN Special Functions, unless otherwise indicated in the **Arguments** column:

`x, y`	real argument
`z, z1, z2`	complex argument
`m, n`	integer argument

Table 2-1: MFUN Special Functions

Function Name	Definition	mfun Name	Arguments
Bernoulli Numbers and Polynomials	Generating functions: $$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \cdot \frac{t^{n-1}}{n!}$$	`bernoulli(n)` `bernoulli(n,t)`	`n ≥ 0` $0 < \lvert t\rvert < 2\pi$

Bessel Functions	`BesselI`, `BesselJ` – Bessel functions of the first kind.
	`BesselK`, `BesselY` – Bessel functions of the second kind.
		`BesselJ(v,x)`
		`BesselY(v,x)`
		`BesselI(v,x)`
		`BesselK(v,x)`
			`v` is real.

Beta Function	$$B(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$	`Beta(x,y)`	
Binomial Coefficients	$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$ $$= \frac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-n+1)}$$	`binomial(m,n)`	

Complete Elliptic Integrals	Legendre's complete elliptic integrals of the first, second, and third kind.	`LegendreKc(k)`
		`LegendreEc(k)`
		`LegendrePic(a,k)`
			`a` is real
			`–Inf < a < Inf`
			
			`k` is real
			`0 < k < 1`

Table 2-1: MFUN Special Functions (Continued)

Function Name	Definition	mfun Name	Arguments

Complete Elliptic Integrals with Complementary Modulus	Associated complete elliptic integrals of the first, second, and third kind using complementary modulus.	`LegendreKc1(k)`
		`LegendreEc1(k)`
		`LegendrePic1(a,k)`
			`a` is real
			`–Inf < a < Inf`
			
			`k` is real
			`0 < k < 1`

Complementary Error Function and Its Iterated Integrals	$erfc(z) = \frac{2}{\sqrt{\pi}} \cdot \int_z^{\infty} e^{-t^2} dt = 1 - erf(z)$ $erfc(-1, z) = \frac{2}{\sqrt{\pi}} \cdot e^{-z^2}$ $erfc(n, z) = \int_z^{\infty} erfc(n-1, z)\ dt$	`erfc(z)` `erfc(n,z)`	`n > 0`
Dawson's Integral	$F(x) = e^{-x^2} \cdot \int_0^x e^{-t^2} dt$	`dawson(x)`	
Digamma Function	$\psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$	`Psi(x)`	

Table 2-1: MFUN Special Functions (Continued)

Function Name	Definition	mfun Name	Arguments
Dilogarithm Integral	$f(x) = \int_1^x \frac{\ln(t)}{1-t} dt$	`dilog(x)`	`x > 1`
Error Function	$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$	`erf(z)`	
Euler Numbers and Polynomials	Generating function for Euler numbers: $\frac{1}{ch(t)} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$	`euler(n)` `euler(n,z)`	$n \geq 0$ $\lvert t\rvert < \frac{\pi}{2}$
Exponential Integrals	$Ei(n, z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt$ $Ei(x) = PV - \int_{-\infty}^{x} \frac{e^t}{t}$	`Ei(n,z)` `Ei(x)`	`n ≥ 0` `Real(z) > 0`

Table 2-1: MFUN Special Functions (Continued)

Function Name	Definition	mfun Name	Arguments
Fresnel Sine and Cosine Integrals	$C(x) = \int_0^x \cos\left(\frac{\pi}{2} \cdot t^2\right) dt$ $S(x) = \int_0^x \sin\left(\frac{\pi}{2} \cdot t^2\right) dt$	`FresnelC(x)` `FresnelS(x)`	
Gamma Function	$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$	`GAMMA(z)`	
Harmonic Function	$h(n) = \sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma$	`harmonic(n)`	`n > 0`
Hyperbolic Sine and Cosine Integrals	$Shi(z) = \int_0^z \frac{\sinh(t)}{t} dt$ $Chi(z) = \gamma + \ln(z) + \int_0^z \frac{\cosh(t) - 1}{t} dt$	`Shi(z)` `Chi(z)`	

Table 2-1: MFUN Special Functions (Continued)

Function Name	Definition	mfun Name	Arguments

(Generalized) Hypergeometric Function	$$F(n,d,z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{j} \frac{\Gamma(n_i+k)}{\Gamma(n_i)} \cdot z^k}{\prod_{i=1}^{m} \frac{\Gamma(d_i+k)}{\Gamma(d_i)} \cdot k!}$$ where `j` and `m` are the number of terms in `n` and `d`, respectively.	`hypergeom(n,d,x)`
		
		where
		`n = [n1,n2,...]`
		`d = [d1,d2,...]`
			`n1,n2,...` are real.
			`d1,d2,...` are real and non-negative.

Incomplete Elliptic Integrals	Legendre's incomplete elliptic integrals of the first, second, and third kind.	`LegendreF(x,k)`
		`LegendreE(x,k)`
		`LegendrePi(x,a,k)`
			`0 < x ≤ Inf`
			
			`a` is real
			`–Inf < a < Inf`
			
			`k` is real
			`0 < k < 1`

Incomplete Gamma Function	$$\Gamma(a,z) = \int_{z}^{\infty} e^{-t} \cdot t^{a-1} dt$$	`GAMMA(z1,z2)`	
Logarithm of the Gamma Function	$$\ln\Gamma(z) = \ln(\Gamma(z))$$	`lnGAMMA(z)`	

Table 2-1: MFUN Special Functions (Continued)

Function Name	Definition	mfun Name	Arguments
Logarithmic Integral	$Li(x) = PV\left\{\int_0^x \frac{dt}{\ln t}\right\} = Ei(\ln x)$	`Li(x)`	`x > 1`
Polygamma Function	$\psi^{(n)}(z) = \frac{d^n}{dz}\psi(z)$ where $\psi(z)$ is the Digamma function.	`Psi(n,z)`	n ≥ 0
Shifted Sine Integral	$Ssi(z) = Si(z) - \frac{\pi}{2}$	`Ssi(z)`	

Orthogonal Polynomials

The following functions require the Maple Orthogonal Polynomial Package. They are available only with the Extended Symbolic Math Toolbox. Before using these functions, you must first initialize the Orthogonal Polynomial Package by typing

```
maple('with','orthopoly')
```

Note that in all cases, `n` is a non-negative integer and `x` is real.

Table 2-2: Orthogonal Polynomials

Polynomial	Maple Name	Arguments
Gegenbauer	`G(n,a,x)`	`a` is a nonrational algebraic expression or a rational number greater than –1/2.
Hermite	`H(n,x)`	
Laguerre	`L(n,x)`	
Generalized Laguerre	`L(n,a,x)`	`a` is a nonrational algebraic expression or a rational number greater than –1.
Legendre	`P(n,x)`	
Jacobi	`P(n,a,b,x)`	`a`, `b` are nonrational algebraic expressions or rational numbers greater than –1.
Chebyshev of the First and Second Kind	`T(n,x)` `U(n,x)`	

mhelp

Purpose Maple help.

Syntax

```
mhelp topic
mhelp('topic')
```

Description `mhelp topic` and `mhelp('topic')` both return Maple's online documentation for the specified Maple `topic`.

Examples `mhelp BesselI` and `mhelp('BesselI')` both return Maple's online documentation for the Maple `BesselI` function.

See Also `maple`

Purpose Basis for null space.

Syntax

```
Z = null(A)
```

Description The columns of `Z = null(A)` form a basis for the null space of `A`.

`size(Z,2)` is the nullity of `A`.

`A*Z` is zero.

If `A` has full rank, `Z` is empty.

Examples The statements

```
A = sym(magic(4));
Z = null(A)
A*Z
```

return

```
[ –1]
[ –3]
[  3]
[  1]

[ 0]
[ 0]
[ 0]
[ 0]
```

See Also arithmetic operators, `colspace`, `rank`, `rref`, `svd`
`null` in the online MATLAB Function Reference.

numden

Purpose

Numerator and denominator.

Syntax

```
[N,D] = numden(A)
```

Description

`[N,D] = numden(A)` converts each element of `A` to a rational form where the numerator and denominator are relatively prime polynomials with integer coefficients. `A` is a symbolic or a numeric matrix. `N` is the symbolic matrix of numerators, and `D` is the symbolic matrix of denominators.

Examples

`[n,d] = numden(4/5)` returns `n = 4` and `d = 5`.

`[n,d] = numden(x/y + y/x)` returns

```
n =
x^2+y^2

d =
y*x
```

The statements

```
A = [a, 1/b]
[n,d] = numden(A)
```

return

```
A =
[a, 1/b]

n =
[a, 1]

d =
[1, b]
```

Purpose Characteristic polynomial of a matrix.

Syntax

```
p = poly(A)
p = poly(A, v)
```

Description If `A` is a numeric array, `poly(A)` returns the coefficients of the characteristic polynomial of `A`. If `A` is symbolic, `poly(A)` returns the characteristic polynomial of `A` in terms of the default variable `x`. The variable `v` can be specified in the second input argument.

Note that if `A` is numeric, `poly(sym(A))` approximately equals `poly2sym(poly(A))`. The approximation is due to roundoff error.

Examples The statements

```
A = gallery(3)
p = poly(A)
q = poly(sym(A))
s = poly(sym(A),z)
```

return

```
A =
  –149   –50  –154
   537   180   546
   –27    –9   –25

p =
1.0000   –6.0000   11.0000   –6.0000

q=
x^3–6*x^2+11*x–6

s =

z^3–6*z^2+11*z–6
```

See Also `poly2sym`, `jordan`, `eig`, `solve`

poly2sym

Purpose Polynomial coefficient vector to symbolic polynomial.

Syntax

```
r = poly2sym(c)
r = poly2sym(c, v)
```

Description `r = poly2sym(c)` returns a symbolic representation of the polynomial whose coefficients are in the numeric vector `c`. The default symbolic variable is `x`. The variable `v` can be specified as a second input argument. If `c = [c1 c2 ... cn]`, `r=poly2sym(c)` has the form:

$$c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n$$

`poly2sym` uses `sym`'s default (rational) conversion mode to convert the numeric coefficients to symbolic constants. This mode expresses the symbolic coefficient approximately as a ratio of integers, if `sym` can find a simple ratio that approximates the numeric value, otherwise as an integer multiplied by a power of 2.

If `x` has a numeric value and `sym` expresses the elements of `c` exactly, `eval(poly2sym(c))` returns the same value as `polyval(c,x)`.

Examples `poly2sym([1 3 2])` returns

```
x^2 + 3*x + 2
```

`poly2sym([.694228, .333, 6.2832])` returns

```
6253049924220329/9007199254740992*x^2+333/1000*x+3927/625
```

`poly2sym([1 0 1 –1 2], y)` returns

```
y^4+y^2–y+2
```

See Also `sym`, `sym2poly`
`polyval` in the online MATLAB Function Reference

Purpose Prettyprint symbolic expressions.

Syntax

```
pretty(S)
pretty(S,n)
```

Description The `pretty` function prints symbolic output in a format that resembles typeset mathematics.

`pretty(S)` prettyprints the symbolic matrix `S` using the default line width of 79.

`pretty(S,n)` prettyprints `S` using line width `n` instead of 79.

Examples The following statements:

```
A = sym(pascal(2))
B = eig(A)
pretty(B)
```

return

```
A =
[1, 1]
[1, 2]

B =
[3/2+1/2*5^(1/2)]
[3/2–1/2*5^(1/2)]

[             1/2 ]
[ 3/2 + 1/2 5     ]
[                 ]
[             1/2 ]
[ 3/2 – 1/2 5     ]
```

procread

Purpose

Install a Maple procedure.

Syntax

procread('*filename*')

Description

procread('*filename*') reads the specified file, which should contain the source text for a Maple procedure. It deletes any comments and newline characters, then sends the resulting string to Maple.

The Extended Symbolic Math Toolbox is required.

Examples

Suppose the file ident.src contains the following source text for a Maple procedure.

```
ident := proc(A)
#  ident(A) computes A*inverse(A)
   local X;
   X := inverse(A);
   evalm(A &* X);
end;
```

Then the statement

```
procread('ident.src')
```

installs the procedure. It can be accessed with

```
maple('ident',magic(3))
```

or

```
maple('ident',vpa(magic(3)))
```

See Also

maple

Purpose Symbolic matrix rank.

Syntax `rank(A)`

Description `rank(A)` is the rank of the symbolic matrix `A`.

Examples `rank([a b;c d])` is 2.

`rank(sym(magic(4)))` is 3.

real

Purpose Symbolic real part.

Syntax `real(Z)`

Description `real(Z)` is the real part of a symbolic `Z`.

See Also `conj`, `imag`

Purpose Reduced row echelon form.

Syntax

```
rref(A)
```

Description `rref(A)` is the reduced row echelon form of the symbolic matrix `A`.

Examples `rref(sym(magic(4)))` returns

```
[  1,  0,  0,  1]
[  0,  1,  0,  3]
[  0,  0,  1, –3]
[  0,  0,  0,  0]
```

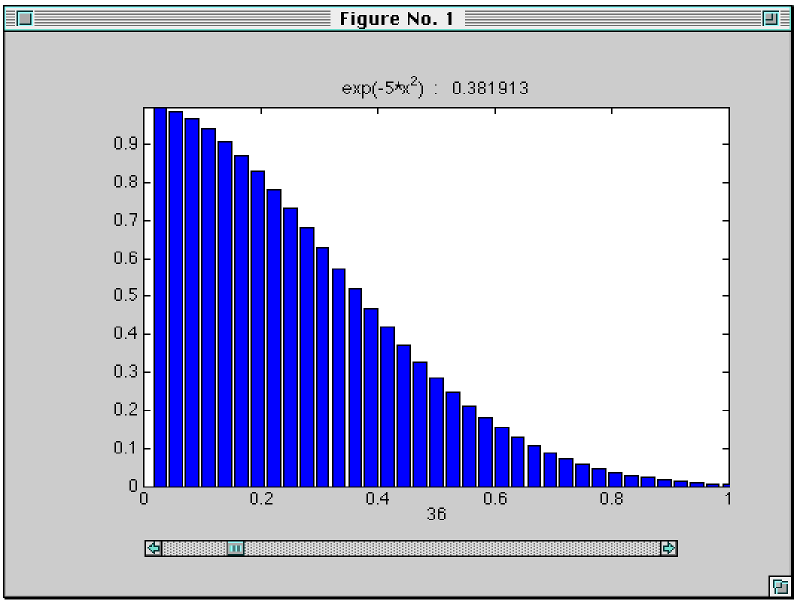
rsums

Purpose Interactive evaluation of Riemann sums.

Syntax `rsums(f)`

Description `rsums(f)` interactively approximates the integral of *f(x)* by Riemann sums. `rsums(f)` displays a graph of *f(x)*. You can then adjust the number of terms taken in the Riemann sum by using the slider below the graph. The number of terms available ranges from 2 to 256.

Examples `rsums exp(–5*x^2)` creates the following plot



Purpose

Search for a symbolic expression's simplest form.

Syntax

```
r = simple(S)
[r,how] = simple(S)
```

Description

`simple(S)` tries several different algebraic simplifications of the symbolic expression `S`, displays any that shorten the length of `S`'s representation, and returns the shortest. `S` is a `sym`. If `S` is a matrix, the result represents the shortest representation of the entire matrix, which is not necessarily the shortest representation of each individual element. If no return output is given, `simple(S)` displays all possible representations and returns the shortest.

`[r,how] = simple(S)` does not display intermediate simplifications, but returns the shortest found, as well as a string describing the particular simplification. `r` is a `sym`. `how` is a string.

Examples

Expression	Simplification	Simplification Method
`cos(x)^2+sin(x)^2`	`1`	`simplify`
`2*cos(x)^2–sin(x)^2`	`3*cos(x)^2–1`	`simplify`
`cos(x)^2–sin(x)^2`	`cos(2*x)`	`combine(trig)`
`cos(x)+ (–sin(x)^2)^(1/2)`	`cos(x)+i*sin(x)`	`radsimp`
`cos(x)+i*sin(x)`	`exp(i*x)`	`convert(exp)`
`(x+1)*x*(x–1)`	`x^3–x`	`collect(x)`
`x^3+3*x^2+3*x+1`	`(x+1)^3`	`factor`
`cos(3*acos(x))`	`4*x^3–3*x`	`expand`

See Also

`collect`, `expand`, `factor`, `horner`, `simplify`

simplify

Purpose

Symbolic simplification.

Syntax

```
R = simplify(S)
```

Description

`simplify(S)` simplifies each element of the symbolic matrix `S` using Maple simplification rules.

Examples

`simplify(sin(x)^2 + cos(x)^2)` returns

```
1
```

`simplify(exp(c*log(sqrt(a+b))))` returns

```
(a+b)^(1/2*c)
```

The statements

```
S = [(x^2+5*x+6)/(x+2),sqrt(16)];
R = simplify(S)
```

return

```
R = [x+3,4]
```

See Also

`collect`, `expand`, `factor`, `horner`, `simple`

Purpose Sine integral function.

Syntax

```
Y = sinint(X)
```

Description `sinint(X)` evaluates the sine integral function at the elements of `X`, a numeric matrix, or a symbolic matrix. The result is a numeric matrix. The sine integral function is defined by

$$Si(x) = \int_0^x \frac{\sin t}{t} dt$$

Examples `sinint([pi 0;–2.2 exp(3)])` returns

```
  1.8519         0
 –1.6876    1.5522
```

`sinint(1.2)` returns `1.1080`.

`diff(sinint(x))` returns `sin(x)/x`.

See Also `cosint`

size

Purpose

Symbolic matrix dimensions.

Syntax

```
d = size(A)
[m,n] = size(A)
d= size(A, n)
```

Description

Suppose `A` is an `m`-by-`n` symbolic or numeric matrix. The statement `d = size(A)` returns a numeric vector with two integer components, `d = [m,n]`.

The multiple assignment statement `[m,n] = size(A)` returns the two integers in two separate variables.

The statement `d = size(A,n)` returns the length of the dimension specified by the scalar n. For example, `size(A,1)` is the number of rows of `A` and `size(A,2)` is the number of columns of `A`.

Examples

The statements

```
syms a b c d
A = [a b c ; a b d; d c b; c b a];
d = size(A)
r = size(A, 2)
```

return

```
d =
     4     3

r =
     3
```

See Also

`length`, `ndims` in the online MATLAB Function Reference

Purpose

Symbolic solution of algebraic equations.

Syntax

```
g = solve(eq)
g = solve(eq,var)
g = solve(eq1,eq2,...,eqn)
g = solve(eq1,eq2,...,eqn,var1,var2,...,varn)
```

Description

Single Equation/Expression. The input to `solve` can be either symbolic expressions or strings. If `eq` is a symbolic expression (`x^2–2*x+1`) or a string that does not contain an equal sign (`'x^2–2*x+1'`), then `solve(eq)` solves the equation `eq=0` for its default variable (as determined by `findsym`).

`solve(eq,var)` solves the equation `eq` (or `eq=0` in the two cases cited above) for the variable `var`.

System of Equations. The inputs are either symbolic expressions or strings specifying equations. `solve(eq1,eq2,...,eqn)` solves the system of equations implied by `eq1,eq2,...,eqn` in the `n` variables determined by applying `findsym` to the system.

Three different types of output are possible. For one equation and one output, the resulting solution is returned with multiple solutions for a nonlinear equation. For a system of equations and an equal number of outputs, the results are sorted alphabetically and assigned to the outputs. For a system of equations and a single output, a structure containing the solutions is returned.

For both a single equation and a system of equations, numeric solutions are returned if symbolic solutions cannot be determined.

Examples

`solve('a*x^2 + b*x + c')` returns

```
[ 1/2/a*(–b+(b^2–4*a*c)^(1/2)),
1/2/a*(–b–(b^2–4*a*c)^(1/2))]
```

`solve('a*x^2 + b*x + c','b')` returns

```
–(a*x^2+c)/x
```

`solve('x + y  = 1','x – 11*y = 5')` returns

```
y = –1/3, x = 4/3
```

```
A = solve('a*u^2 + v^2', 'u – v = 1', 'a^2 – 5*a + 6')
```

returns

```
A = 

    a: [1x4 sym]
    u: [1x4 sym]
    v: [1x4 sym]
```

where

```
A.a =
   [ 2, 2, 3, 3]

A.u =
   [ 1/3+1/3*i*2^(1/2), 1/3–1/3*i*2^(1/2),
   1/4+1/4*i*3^(1/2), 1/4–1/4*i*3^(1/2)]

A.v =
   [ –2/3+1/3*i*2^(1/2), –2/3–1/3*i*2^(1/2),
   –3/4+1/4*i*3^(1/2), –3/4–1/4*i*3^(1/2)]
```

See Also

arithmetic operators, dsolve, findsym

Purpose Rewrite a symbolic expression in terms of common subexpressions.

Syntax

```
[Y,SIGMA] = subexpr(X,SIGMA)
[Y,SIGMA] = subexpr(X,'SIGMA')
```

Description `[Y,SIGMA] = subexpr(X,SIGMA)` or `[Y,SIGMA] = subexpr(X,'SIGMA')` rewrites the symbolic expression `X` in terms of its common subexpressions. These are the subexpressions that are written as `%1`, `%2`, etc. by `pretty(S)`.

Examples The statements

```
t = solve('a*x^3+b*x^2+c*x+d = 0');
[r,s] = subexpr(t,'s');
```

return the rewritten expression for `t` in `r` in terms of a common subexpression, which is returned in `s`.

See Also `pretty`, `simple`, `subs`

subs

Purpose

Symbolic substitution in a symbolic expression or matrix.

Syntax

```
R = subs(S)
R = subs(S,old,new)
```

Description

`subs(S)` replaces all occurrences of variables in the symbolic expression `S` with values obtained from the calling function, or the MATLAB workspace.

`subs(S,old,new)` replaces `old` with `new` in the symbolic expression `S`. `old` is a symbolic variable or a string representing a variable name. `new` is a symbolic or numeric variable or expression.

If `old` and `new` are cell arrays of the same size, each element of `old` is replaced by the corresponding element of `new`. If `S` and `old` are scalars and `new` is an array or cell array, the scalars are expanded to produce an array result. If `new` is a cell array of numeric matrices, the substitutions are performed elementwise (i.e., `subs(x*y,{x,y},{A,B})` returns `A.*B` when `A` and `B` are numeric).

If `subs(s,old,new)` does not change `s`, `subs(s,new,old)` is tried. This provides backwards compatibility with previous versions and eliminates the need to remember the order of the arguments. `subs(s,old,new)` does not switch the arguments if `s` does not change.

Examples

Single input:

Suppose a = 980 and C1 = 3 exist in the workspace.

The statement

```
y = dsolve('Dy = -a*y')
```

produces

```
y = exp(-a*t)*C1
```

Then the statement

```
subs(y)
```

produces

```
ans = 3*exp(-980*t)
```

Single Substitution:

`subs(a+b,a,4)` returns `4+b`.

Multiple Substitutions:

`subs(cos(a)+sin(b),{a,b},{sym('alpha'),2})` returns

```
cos(alpha)+sin(2)
```

Scalar Expansion Case:

`subs(exp(a*t),'a',-magic(2))` returns

```
[   exp(-t), exp(-3*t)]
[ exp(-4*t), exp(-2*t)]
```

Multiple Scalar Expansion:

`subs(x*y,{x,y},{[0 1;-1 0],[1 -1;-2 1]})` returns

```
[  0, -1]
[  2,  0]
```

See Also

`simplify`, `subexpr`

svd

Purpose Symbolic singular value decomposition.

Syntax

```
sigma = svd(A)
sigma = svd(vpa(A))
[U,S,V] = svd(A)
[U,S,V] = svd(vpa(A))
```

Description `sigma = svd(A)` is a symbolic vector containing the singular values of a symbolic matrix `A`.

`sigma = svd(vpa(A))` computes numeric singular values, using variable precision arithmetic.

`[U,S,V] = svd(A)` and `[U,S,V] = svd(vpa(A))` return numeric unitary matrices `U` and `V` whose columns are the singular vectors and a diagonal matrix `S` containing the singular values. Together, they satisfy `A = U*S*V'`.

Symbolic singular vectors are not available.

Examples The statements

```
digits(3)
A = sym(magic(4));
svd(A)
svd(vpa(A))
[U,S,V] = svd(A)
```

return

```
[         0]
[        34]
[ 2*5^(1/2)]
[ 8*5^(1/2)]

[ .311e-6*i]
[      4.47]
[      17.9]
[      34.1]
```

```
U =

[ -.500,  .671,  .500, -.224]
[ -.500, -.224, -.500, -.671]
[ -.500,  .224, -.500,  .671]
[ -.500, -.671,  .500,  .224]


S =

[     34.0,         0,         0,         0]
[        0,      17.9,         0,         0]
[        0,         0,      4.47,         0]
[        0,         0,         0, .835e-15]


V =

[ -.500,  .500,  .671, -.224]
[ -.500, -.500, -.224, -.671]
[ -.500, -.500,  .224,  .671]
[ -.500,  .500, -.671,  .224]
```

See Also

`digits`, `eig`, `vpa`

sym

Purpose

Construct symbolic numbers, variables and objects.

Syntax

```
S = sym(A)
x = sym('x')
x = sym('x','real')
x = sym('x','unreal')
S = sym(A,flag) where flag is one of 'r', 'd', 'e', or 'f'.
```

Description

`S = sym(A)` constructs an object `S`, of class 'sym', from `A`. If the input argument is a string, the result is a symbolic number or variable. If the input argument is a numeric scalar or matrix, the result is a symbolic representation of the given numeric values.

`x = sym('x')` creates the symbolic variable with name `'x'` and stores the result in `x`. `x = sym('x','real')` also assumes that `x` is real, so that `conj(x)` is equal to `x`. `alpha = sym('alpha')` and `r = sym('Rho','real')` are other examples. `x = sym('x','unreal')` makes `x` a purely formal variable with no additional properties (i.e., ensures that `x` is *not* real). See also the reference pages on `syms`.

Statements like `pi = sym('pi')` and `delta = sym('1/10')` create symbolic numbers that avoid the floating-point approximations inherent in the values of `pi` and `1/10`. The `pi` created in this way temporarily replaces the built-in numeric function with the same name.

`S = sym(A,flag)` converts a numeric scalar or matrix to symbolic form. The technique for converting floating-point numbers is specified by the optional second argument, which can be `'f'`, `'r'`, `'e'` or `'d'`. The default is `'r'`.

`'f'` stands for "floating-point." All values are represented in the form `'1.F'*2^(e)` or `'–1.F'*2^(e)` where `F` is a string of 13 hexadecimal digits and `e` is an integer. This captures the floating-point values exactly, but may not be convenient for subsequent manipulation. For example, `sym(1/10,'f')` is `'1.999999999999a'*2^(–4)` because `1/10` cannot be represented exactly in floating-point.

`'r'` stands for "rational." Floating-point numbers obtained by evaluating expressions of the form `p/q`, `p*pi/q`, `sqrt(p)`, `2^q`, and `10^q` for modest sized integers `p` and `q` are converted to the corresponding symbolic form. This effectively compensates for the roundoff error involved in the original evaluation, but may not represent the floating-point value precisely. If no simple rational approximation can be found, an expression of the form p*2^q with large integers p and q reproduces the floating-point value exactly. For example, sym(4/3,'r') is '4/3', but sym(1+sqrt(5),'r') is 7286977268806824*2^(–51)

'e' stands for "estimate error." The 'r' form is supplemented by a term involving the variable 'eps', which estimates the difference between the theoretical rational expression and its actual floating-point value. For example, sym(3*pi/4) is 3*pi/4–103*eps/249.

'd' stands for "decimal." The number of digits is taken from the current setting of digits used by vpa. Fewer than 16 digits loses some accuracy, while more than 16 digits may not be warranted. For example, with digits(10), sym(4/3,'d') is 1.333333333, while with digits digits(20), sym(4/3,'d') is 1.3333333333333332593, which does not end in a string of 3's, but is an accurate decimal representation of the floating-point number nearest to 4/3.

See Also

digits, double, syms

eps in the online MATLAB Function Reference

syms

Purpose

Short-cut for constructing symbolic objects.

Syntax

```
syms arg1 arg2 ...
syms arg1 arg2 ... real
syms arg1 arg2 ... unreal
```

Description

`syms arg1 arg2 ...` is short-hand notation for

```
arg1 = sym('arg1');
arg2 = sym('arg2'); ...
```

`syms arg1 arg2 ... real` is short-hand notation for

```
arg1 = sym('arg1','real');
arg2 = sym('arg2','real'); ...
```

`syms arg1 arg2 ... unreal` is short-hand notation for

```
arg1 = sym('arg1','unreal');
arg2 = sym('arg2','unreal'); ...
```

Each input argument must begin with a letter and can contain only alphanumeric characters.

Examples

`syms x beta real` is equivalent to:

```
x = sym('x','real');
beta = sym('beta','real');
```

To clear the symbolic objects `x` and `beta` of `'real'` status, type

```
syms x beta unreal
```

Note `clear x` will *not* clear the symbolic object `x` of its status `'real'`. You can achieve this, using the commands `syms x unreal` or `clear mex` or `clear all`. In the latter two cases, the Maple kernel will have to be reloaded in the MATLAB workspace. (This is inefficient and time consuming).

See Also

`sym`

Purpose

Symbolic-to-numeric polynomial conversion.

Syntax

```
c = sym2poly(s)
```

Description

`sym2poly` returns a row vector containing the numeric coefficients of a symbolic polynomial. The coefficients are ordered in descending powers of the polynomial's independent variable. In other words, the vector's first entry contains the coefficient of the polynomial's highest term; the second entry, the coefficient of the second highest term; and so on.

Examples

The commands

```
syms x u v;
sym2poly(x^3 - 2*x - 5)
```

return

```
1     0    -2    -5
```

while `sym2poly(u^4 - 3 + 5*u^2)` returns

```
1     0     5     0    -3
```

and `sym2poly(sin(pi/6)*v + exp(1)*v^2)` returns

```
2.7183    0.5000         0
```

See Also

`poly2sym`
`polyval` in the online MATLAB Function Reference

symsum

Purpose

Symbolic summation.

Syntax

```
r = symsum(s)
r = symsum(s,v)
r = symsum(s,a,b)
r = symsum(s,v,a,b)
```

Description

`symsum(s)` is the summation of the symbolic expression `s` with respect to its symbolic variable `k` as determined by `findsym` from `0` to `k–1`.

`symsum(s,v)` is the summation of the symbolic expression `s` with respect to the symbolic variable `v` from `0` to `v–1`.

`symsum(s,a,b)` and `symsum(s,v,a,b)` are the definite summations of the symbolic expression from `v=a` to `v=b`.

Examples

The commands

```
syms k n x
symsum(k^z)
```

return

```
1/3*k^3–1/2*k^2+1/6*k
```

`symsum(k)` returns

```
1/2*k^2–1/2*k
```

`symsum(sin(k*pi)/k,0,n)` returns

```
–1/2*sin(k*(n+1))/k+1/2*sin(k)/k/(cos(k)–1)*cos(k*(n+1))–
1/2*sin(k)/k/(cos(k)–1)
```

`symsum(k^2,0,10)` returns

```
385
```

`symsum(x^k/sym('k!'), k, 0,inf)` returns

```
exp(x)
```

Note The preceding example uses `sym` to create the symbolic expression `k!` in order to bypass MATLAB's expression parser, which does not recognize `!` as a factorial operator.

See Also

`findsym`, `int`, `syms`

taylor

Purpose

Taylor series expansion.

Syntax

```
r = taylor(f)
r = taylor(f,n,v)
r = taylor(f,n,v,a)
```

Description

`taylor(f,n,v)` returns the (n–1)-order Maclaurin polynomial approximation to `f`, where `f` is a symbolic expression representing a function and `v` specifies the independent variable in the expression. `v` can be a string or symbolic variable.

`taylor(f,n,v,a)` returns the Taylor series approximation to `f` about `a`. The argument `a` can be a numeric value, a symbol, or a string representing a numeric value or an unknown.

You can supply the arguments `n`, `v`, and `a` in any order. `taylor` determines the purpose of the arguments from their position and type.

You can also omit any of the arguments `n`, `v`, and `a`. If you do not specify `v`, `taylor` uses `findsym` to determine the function's independent variable. `n` defaults to `6`.

The Taylor series for an analytic function $f(x)$ about the basepoint $x=a$ is given below.

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n \cdot \frac{f^{(n)}(a)}{n!}$$

Examples

This table describes the various uses of the `taylor` command and its relation to Taylor and MacLaurin series.

Mathematical Operation	MATLAB
$\sum_{n=0}^{5} x^n \cdot \frac{f^{(n)}(0)}{n!}$	`syms x` `taylor(f)`

Mathematical Operation	MATLAB
$\sum_{n=0}^{m} x^n \cdot \frac{f^{(n)}(0)}{n!}$, m is a positive integer	`taylor(f,m)` m is a positive integer
$\sum_{n=0}^{5} (x-a)^n \cdot \frac{f^{(n)}(a)}{n!}$, a is a real number	`taylor(f,a)` a is a real number
$\sum_{n=0}^{m_1} (x-m_2)^n \cdot \frac{f^{(n)}(m_2)}{n!}$ m_1, m_2 are positive integers	`taylor(f,m1,m2)` m_1, m_2 are positive integers
$\sum_{n=0}^{m} (x-a)^n \cdot \frac{f^{(n)}(a)}{n!}$ a is real and m is a positive integer	`taylor(f,m,a)` a is real and m is a positive integer

In the case where `f` is a function of two or more variables (`f=f(x,y,...)`), there is a fourth parameter that allows you to select the variable for the Taylor expansion. Look at this table for illustrations of this feature.

Mathematical Operation	MATLAB
$\sum_{n=0}^{5} \frac{y^n}{n!} \cdot \frac{\partial^n}{\partial y^n} f(x, y = 0)$	`taylor(f,y)`

Mathematical Operation	MATLAB
$\sum_{n=0}^{m} \frac{y^n}{n!} \cdot \frac{\partial^n}{\partial y^n} f(x, y = 0)$ m is a positive integer	`taylor(f,y,m)` or `taylor(f,m,y)` m is a positive integer
$\sum_{n=0}^{m} \frac{(y-a)^n}{n!} \cdot \frac{\partial^n}{\partial y^n} f(x, y = a)$ a is real and m is a positive integer	`taylor(f,m,y,a)` a is real and m is a positive integer
$\sum_{n=0}^{5} \frac{(y-a)^n}{n!} \cdot \frac{\partial^n}{\partial y^n} f(x, y = a)$ a is real	`taylor(f,y,a)` a is real

See Also

`findsym`

Purpose Taylor series calculator.

Syntax

```
taylortool
taylortool('f')
```

Description `taylortool` initiates a GUI that graphs a function against the Nth partial sum of its Taylor series about a basepoint `x = a`. The default function, value of N, basepoint, and interval of computation for `taylortool` are `f = x*cos(x)`, `N = 7`, `a = 0`, and `[-2*pi,2*pi]`, respectively.

`taylortool('f')` initiates the GUI for the given expression `f`.

Examples

```
taylortool('exp(x*sin(x))')
```

```
taylortool('sin(tan(x)) — tan(sin(x))')
```

See Also `funtools`, `rsums`

tril

Purpose

Symbolic lower triangle.

Syntax

```
tril(X)
tril(X,K)
```

Description

`tril(X)` is the lower triangular part of `X`.

`tril(X,K)` returns a lower triangular matrix that retains the elements of `X` on and below the `k`-th diagonal and sets the remaining elements to 0. The values `k=0`, `k>0`, and `k<0` correspond to the main, superdiagonals, and subdiagonals, respectively.

Examples

Suppose

```
A =
[    a,    b,    c ]
[    1,    2,    3 ]
[  a+1,  b+2,  c+3 ]
```

Then `tril(A)` returns

```
[    a,    0,    0 ]
[    1,    2,    0 ]
[  a+1,  b+2,  c+3 ]
```

`tril(A,1)` returns

```
[    a,    b,    0 ]
[    1,    2,    3 ]
[  a+1,  b+2,  c+3 ]
```

`triu(A,–1)` returns

```
[    0,    0,    0 ]
[    1,    0,    0 ]
[  a+1,  b+2,    0 ]
```

See Also

`diag`, `triu`

Purpose Symbolic upper triangle.

Syntax

```
triu(X)
triu(X, K)
```

Description `triu(X)` is the upper triangular part of `X`.

`triu(X, K)` returns an upper triangular matrix that retains the elements of `X` on and above the `k`-th diagonal and sets the remaining elements to 0. The values `k=0`, `k>0`, and `k<0` correspond to the main, superdiagonals, and subdiagonals, respectively.

Examples Suppose

```
A =
[    a,    b,    c ]
[    1,    2,    3 ]
[  a+1,  b+2,  c+3 ]
```

Then `triu(A)` returns

```
[    a,    b,    c ]
[    0,    2,    3 ]
[    0,    0,  c+3 ]
```

`triu(A,1)` returns

```
[ 0, b, c ]
[ 0, 0, 3 ]
[ 0, 0, 0 ]
```

`triu(A,–1)` returns

```
[    a,    b,    c ]
[    1,    2,    3 ]
[    0,  b+2,  c+3 ]
```

See Also `diag`, `tril`

Purpose Variable precision arithmetic.

Syntax

```
R = vpa(A)
R = vpa(A,d)
```

Description

`vpa(A)` uses variable-precision arithmetic (VPA) to compute each element of `A` to `d` decimal digits of accuracy, where `d` is the current setting of `digits`. Each element of the result is a symbolic expression.

`vpa(A,d)` uses `d` digits, instead of the current setting of `digits`.

Examples

The statements

```
digits(25)
q = vpa(sym(sin(pi/6)))
p = vpa(pi)
w = vpa('(1+sqrt(5))/2')
```

return

```
q =
.5000000000000000000000000

p = 3.141592653589793238462643

w =
1.618033988749894848204587
```

`vpa pi 75` computes π to 75 digits.

`The statements`

```
A = vpa(hilb(2),25)
B = vpa(hilb(2),5)
```

return

```
A =
[                        1., .50000000000000000000000000]
[.50000000000000000000000000, .33333333333333333333333333]

B =
[    1., .50000]
[.50000, .33333]
```

See Also

digits, double

Purpose Riemann Zeta function.

Syntax

```
Y = zeta(X)
Y = zeta(n, X)
```

Description `zeta(X)` evaluates the Zeta function at the elements of `X`, a numeric matrix, or a symbolic matrix. The Zeta function is defined by

$$\zeta(w) = \sum_{k=1}^{\infty} \frac{1}{k^w}$$

`zeta(n, X)` returns the `n`-th derivative of `zeta(X)`.

Examples `zeta(1.5)` returns `2.6124.`

`zeta(1.2:0.1:2.1)` returns

```
  Columns 1 through 7

     5.5916    3.9319    3.1055    2.6124    2.2858    2.0543    1.8822

  Columns 8 through 10

      1.7497    1.6449    1.5602
```

`zeta([x 2;4 x+y])` returns

```
  [    zeta(x),  1/6*pi^2]
  [ 1/90*pi^4, zeta(x+y)]
```

`diff(zeta(x),x,3)` returns `zeta(3,x)`.

Purpose z-transform.

Syntax

```
F = ztrans(f)
F = ztrans(f,w)
F = ztrans(f,k,w)
```

Description `F = ztrans(f)` is the z-transform of the scalar symbol `f` with default independent variable `n`. The default return is a function of `z`.

$$f = f(n) \Rightarrow F = F(z)$$

The z-transform of `f` is defined as:

$$F(z) = \sum_{0}^{\infty} \frac{f(n)}{z^n}$$

where `n` is `f`'s symbolic variable as determined by `findsym`. If

`f = f(z)`, then `ztrans(f)` returns a function of `w`.

$$F = F(w)$$

`F = ztrans(f,w)` makes `F` a function of the symbol `w` instead of the default `z`.

$$F(w) = \sum_{0}^{\infty} \frac{f(n)}{w^n}$$

`F = ztrans(f,k,w)` takes `f` to be a function of the symbolic variable `k`.

$$F(w) = \sum_{0}^{\infty} \frac{f(k)}{w^k}$$

Examples

Z-Transform	MATLAB Operation

$f(n) = n^4$
 $Z[f] = \sum_{n=0}^{\infty} f(n) z^{-n}$
 $= \frac{z(z^3 + 11z^2 + 11z + 1)}{(z-1)^5}$
	`f = n^4`
	 `ztrans(f)`
	 returns
	 `z*(z^3+11*z^2+11*z+1)/(z-1)^5`

$g(z) = a^z$
 $Z[g] = \sum_{z=0}^{\infty} g(z) w^{-z}$
 $= \frac{w}{a-w}$
	`g = a^z`
	 `ztrans(g)`
	 returns
	 `-w/(-w+a)`

$f(n) = \sin an$
 $Z[f] = \sum_{n=0}^{\infty} f(n) w^{-n}$
 $= \frac{w \sin a}{1 - 2w\cos a + w^2}$
	`f = sin(a*n)`
	 `ztrans(f,w)`
	 returns
	 `sin(a)*w/(1-w*cos(a)+w^2)`

Z-Transform	MATLAB Operation
$f(n,k) = e^{-kn^2}\cos kn$	`f = exp(k*n^2)*cos(k*n)`
$Z[f] = \sum_{k=0}^{\infty} f(n,k)x^{-k}$	`ztrans(f,k,x)`

$= \dfrac{x^2-(e^{-n^2}\cos n)x}{x^2-(2e^{-n^2}\cos n)x+e^{-2n^2}}$	returns
	`x*(-exp(-n*2)*cos(n)+x)/`
	`(-2*x*exp(-n^2)*cos(n) + x^2+`
	`exp(-2*n^2))`

See Also

`fourier`, `iztrans`, `laplace`

A

Compatibility Guide

Compatibility with Earlier Versions

Earlier versions of the Symbolic Math Toolboxes work with version 4.0 or 4.1 of MATLAB and version V, release 2 of Maple. The goal was to provide access to Maple with a language syntax that is familiar to MATLAB users. This was been done without modifying either of the two underlying systems.

However, it is not possible to provide completely seamless integration without modifying MATLAB. For example, if `f` and `g` are strings representing symbolic expressions, we would prefer to use the notation `f+g` for their sum, instead of `symadd(f,g)`. But `f+g` attempts to add the individual characters in the two strings, rather than concatenate them with a plus sign in between. Similarly, if `A` is a matrix whose elements are symbolic expressions, we would prefer to use `A(i,j)` to access a individual expression, instead of `sym(A,i,j)`. But if `A` is a matrix of strings, then `A(i,j)` is a single character, not a complete expression.

This version of the Symbolic Math Toolboxes makes extensive use of the new MATLAB object capabilities and works with Maple V, Release 4. For this reason, it is not fully compatible with version 1 of the Symbolic Math Toolbox.

Obsolete Functions

This version maintains some compatibility with version 1. For example, the following obsolete functions continue to be available in version 2, though you should avoid using them as future releases may not include them.

Function	Description
`determ`	Symbolic matrix determinant
`linsolve`	Solve simultaneous linear equations
`eigensys`	Symbolic eigenvalues and eigenvectors
`singvals`	Symbolic singular values and singular vectors
`numeric`	Convert symbolic matrix to numeric form
`symop`	Symbolic operations
`symadd`	Add symbolic expressions
`symsub`	Subtract symbolic expressions
`symmul`	Multiply symbolic expressions
`symdiv`	Divide symbolic expressions
`sympow`	Power of symbolic expression
`eval`	Evaluate a symbolic expression

In version 1, these functions accepted strings as arguments and returned strings as results. In version 2, they accept either strings or symbolic objects as input arguments and produce symbolic objects as results. version 2 provides overloaded MATLAB operators or new functions that you can use to replace most of these functions in your existing code.

For example, the version 1 statements

```
f = '1/(5+4*cos(x))'
g = int(int(diff(f,2)))
e = symsub(f,g)
simple(e)
```

continue to work in version 2. However, with version 2, the preferred approach is

```
syms x
f = 1/(5+4*cos(x))
g = int(int(diff(f,2)))
e = f - g
simple(e)
```

The version 1 statements

```
H = sym(hilb(3))
I = sym(eye(3))
X = linsolve(H,I)
t = sym(0)
for j = 1:3
   t = symadd(t,sym(X,j,j))
end
t
```

continue to work in version 2. However, the preferred approach is

```
H = sym(hilb(3))
I = eye(3)
X = H\I
t = sum(diag(X))
```

You can no longer use the `sym` function in this way.

```
M = sym(3,3,'1/(i+j–t)')
```

Instead, you must change the code to something like this

```
syms t
[J,I] = meshgrid(1:3)
M = 1./(I+J–t)
```

As in version 1, you can supply `diff`, `int`, `solve`, and `dsolve` with string arguments in version 2. In version 2, however, these functions return symbolic objects instead of strings.

For some computations, the new release of Maple produces results in a different format.

For example, with version 1, the statement

```
[x,y] = solve('x^2 + 2*x*y + y^2 = 4', 'x^3 + 4*y^3 = 1')
```

produces

```
x =
      [    -RootOf(_Z^3-2*_Z^2-4*_Z-3)-2]
      [-RootOf(3*_Z^3+6*_Z^2-12*_Z+7)+2]

y =
      [    RootOf(_Z^3-2*_Z^2-4*_Z-3)]
      [RootOf(3*_Z^3+6*_Z^2-12*_Z+7)]
```

The same statement works in version 2, but produces results with the `RootOf` expressions expanded to exhibit the multiple solutions.

Index

G

H

I

J

L

M

S