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C.I.M.E. Director
Pietro ZECCA
Dipartimento di Energetica "S. Stecco"
Università di Firenze
Via S. Marta, 3
50139 Florence
Italy
e-mail: zecca@unifi.it

C.I.M.E. Secretary
Elvira MASCOLO
Dipartimento di Matematica "U. Dini"
Università di Firenze
viale G.B. Morgagni 67/A
50134 Florence
Italy
e-mail: mascolo@math.unifi.it

For more information see CIME's homepage: http://www.cime.unifi.it

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Yves Achdou • Guy Barles • Hitoshi Ishii
Grigory L. Litvinov

Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications

Cetraro, Italy 2011

Editors:
Paola Loreti
Nicoletta Anna Tchou

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Yves Achdou
Laboratoire Jacques-Louis Lions - UPMC
Paris, France

Guy Barles
Laboratoire de Mathématiques et Physique Théorique
Tours, France

Hitoshi Ishii
Department of Mathematics
Waseda University
Shinjuku-ku, Tokyo, Japan

Grigory L. Litvinov
Institution for Information Transmission Problems
Russsian Academy of Sciences
Moscow, Russia

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Foreword

These lecture notes contain the material relative to the courses given at the CIME Summer School held in Cetraro, Italy, from August 29 to September 3, 2011. The topic was *Hamilton–Jacobi Equations: Approximations, Numerical Analysis and Applications*

The courses dealt mostly with the following subjects: first-order and second-order Hamilton–Jacobi–Bellman equations, properties of viscosity solutions, homogenization and asymptotic behaviours, mean field games, approximation and numerical methods, and idempotent analysis. The content of the courses went from an introduction to viscosity solutions to quite advanced topics, at the cutting edge of the research in the field. We believe that they opened perspectives on new and delicate issues.

This volume contains four courses

- Finite Difference Methods for Mean Field Games
  Yves Achdou
- An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton–Jacobi Equations and Applications
  Guy Barles
- A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton–Jacobi Equations
  Hitoshi Ishii
- Idempotent/Tropical Analysis, the Hamilton–Jacobi and Bellman Equations
  Grigory L. Litvinov

A fifth course held at the workshop by Panagiotis E. Souganidis of the University of Chicago (Homogenization and Approximation for Hamilton–Jacobi Equations) is not included in this volume.

The participants came from several countries (ordered decreasingly with the number of participants): Italy, France, the USA, Argentina, Austria, Chile, China, Germany, Japan, Greece, Iran, Rumania, Russia, Sweden and Vietnam.

On September 1st, Paola Loreti, Elvira Mascolo and Nicoletta Tchou organized a session open to the younger researchers. This "CIME-young" session allowed the doctoral students and posdoctoral researchers to present their new results.

Young Speakers

- Moreno Concezzi
  Università Degli Studi Roma Tre, Italy
  Numerical methods and applications-dynamic programming for HCS and fractionary laplacian approximation
- Jean-Paul Daniel
  Laboratoire Jacques-Louis Lions—Université Paris 6, France
  A game interpretation for fully non linear equations with Neumann condition
- Tiziano De Angelis
  Sapienza Università di Roma, Italy
  Optimal stopping of a Hilbert space valued diffusion process
- Joscha Diehl
  University of Berlin, Germany
  Pathwise approach to rough Burger's PDEs
- Benjamin Fehrman
  University of Chicago, USA
  Homogenization of systems of viscous Hamilton–Jacobi equations
- Giulio Galise
  Università degli Studi di Salerno, Italy
  Viscosity solutions of uniformly elliptic equations without boundary and growth conditions at infinity
- Anna Chiara Lai
  Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Italy
  A multi-phalanx self-similar robot finger
- Roberto Mecca
  Dipartimento di Matematica "G. Castelnuovo", Sapienza Università di Roma, Italy
  Shape from shading via photometric stereo technique a new differential approach
- Cristina Pocci
  Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Italy
  Propagation of fronts in nonlinear diffusion equations
- F.J. Silva,
  Dipartimento di Matematica "G. Castelnuovo", Sapienza Università di Roma, Italy
  A semi-Lagrangian scheme for a 1st order-infinite horizon mean field game model

Authors' Short Vitae

- Yves Achdou has been a professor at Université Paris-Diderot since 2000. His research is mainly devoted to the numerical analysis of partial differential equations with several applications ranging from fluid dynamics to mathematical finance.
- Guy Barles has been a professor at Université François Rabelais Tours since 1990. He has worked on viscosity solutions of fully nonlinear elliptic and parabolic equations, including in particular Hamilton–Jacobi equations and various applications like dislocations.
- Hitoshi Ishii was a professor at Chuo University (Tokyo) (April 1989–March 1996) and at Tokyo Metropolitan University (April 1996–March 2001). He has been a professor at Waseda University, Tokyo, since April 2001 and also adjunct professor at King Abdulaziz University, Jeddah, since August 2011. His research interest has been in nonlinear partial differential equations and their applications.
- Grigory L. Litvinov has been a research fellow and lecturer at the Independent University of Moscow since 2002. Since 2006 he is a researcher at the Russian–French Laboratory J.-V. Poncelet. Since 2011 he is a leading scientist at the Institute for Information Transmission Problems of the Russian Academy of Sciences and part-time professor at the National Research University Higher School of Economics, Moscow, Russia. His research interest has been in functional analysis, Lie groups and hypergroups, idempotent mathematics, and computer technology: algorithms and special hardware and software.

Paola Loreti
Dipartimento di Scienze di Base e
Applicate per l'Ingegneria
Sapienza Università di Roma
Via A. Scarpa n.16
00161 Roma
Italy

Nicoletta Tchou
IRMAR
Université de Rennes 1
Campus de Beaulieu
35042 Rennes Cedex
France

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Contents

Contributors

Yves Achdou University of Paris Diderot, Sorbonne Paris Cité, Laboratoire Jacques-Louis Lions, UMR 7598, UPMC, CNRS, F-75205 Paris, France

Guy Barles Laboratoire de Mathématiques et Physique Théorique CNRS UMR 7350 (Tours), Fédération Denis Poisson, Université François Rabelais Tours, Parc de Grandmont 37200 TOURS, France

Hitoshi Ishii Department of Mathematics, Faculty of Education and Integrated Arts and Sciences, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku, Tokyo, 169-8050 Japan

Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, 21589 Saudi Arabia

Grigory L. Litvinov Institute for Information Transmission Problems of Russian Academy of Sciences, B. Karetnyi per. 19/1, Moscow, 127994 Russia

National Research University Higher School of Economics, Moscow, Russia

Finite Difference Methods for Mean Field Games

Yves Achdou

Abstract Mean field type models describing the limiting behavior of stochastic differential game problems as the number of players tends to $+\infty$, have been recently introduced by J-M. Lasry and P-L. Lions. They may lead to systems of evolutive partial differential equations coupling a forward Bellman equation and a backward Fokker–Planck equation. The forward-backward structure is an important feature of this system, which makes it necessary to design new strategies for mathematical analysis and numerical approximation. In this survey, several aspects of a finite difference method used to approximate the previously mentioned system of PDEs are discussed, including: existence and uniqueness properties, a priori bounds on the solutions of the discrete schemes, convergence, and algorithms for solving the resulting nonlinear systems of equations. Some numerical experiments are presented. Finally, the optimal planning problem is considered, i.e. the problem in which the positions of a very large number of identical rational agents, with a common value function, evolve from a given initial spatial density to a desired target density at the final horizon time.

1 Introduction

Mean field type models describing the asymptotic behavior of stochastic differential games (Nash equilibria) as the number of players tends to $+\infty$ have recently been introduced by Lasry and Lions [25–27]. In some cases, they lead to systems of evolutive partial differential equations involving two unknown scalar functions: the density of the agents in a given state x, namely $m = m(t, x)$ and the potential $u = u(t, x)$. Since the present work is devoted to finite difference schemes, we

Y. Achdou (✉)
University of Paris Diderot, Sorbonne Paris Cité, Laboratoire Jacques-Louis Lions, UMR 7598, UPMC, CNRS, F-75205 Paris, France
e-mail: achdou@ljll-univ-paris-diderot.fr

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will assume that the dimension of the state space is $d = 2$ (what follows could be generalized to any dimension d, although in practice, finite difference methods require too many computing resources when $d \geq 4$). In the periodic setting, typical such model comprises the following system of evolution partial differential equations

$$\frac{\partial u}{\partial t}(t,x) - \nu \Delta u(t,x) + H(x, \nabla u(t,x)) = \Phi[m(t,\cdot)](x), \tag{1}$$

$$\frac{\partial m}{\partial t}(t,x) + \nu \Delta m(t,x) + \operatorname{div}\left(m(t,\cdot) \frac{\partial H}{\partial p}(\cdot, \nabla u(t,\cdot)) \right)(x) = 0, \tag{2}$$

in $(0, T) \times \mathbb{T}^2$, with the initial and terminal conditions

$$u(0,x) = u_0(x), \quad m(T,x) = m_T(x), \tag{3}$$

in $\mathbb{T}^2$, given a cost function u_0 and a probability density m_T.

Let us make some comments on the boundary value problem (1)–(3).

First, note that t is the remaining time to the horizon, (the physical time is in fact $T - t$), so u_0 should be seen as a final cost or incitation, whereas m_T is the density of the agents at the beginning of the game.

Here, we denote by $\mathbb{T}^2 = [0,1]^2$ the two-dimensional unit torus, by ν a nonnegative constant and by Δ, ∇ and div, respectively, the Laplace, the gradient and the divergence operator acting on the x variable. By working on the torus $\mathbb{T}^2$, we avoid the discussion of the boundary conditions, but other boundary value problems can be considered, for example, Dirichlet conditions or Neumann conditions if $\nu > 0$.

The system also involves the scalar Hamiltonian $H(x, p)$, which is assumed to be convex with respect to p and $\mathscr{C}^1$ regular w.r.t. x and p. The notation $\frac{\partial H}{\partial p}(x,q)$ is used for the gradient of $p \mapsto H(x, p)$ at $p = q$.

Finally, in the cost term $\Phi[m(t,\cdot)](x)$, Φ may be:

- Either a local operator, i.e. $\Phi[m(t,\cdot)](x) = F(m(t,x))$ where F is a $\mathscr{C}^1$ regular function defined on $\mathbb{R}_+$. In this case, there are existence theorems of either classical (see [15]) or weak solutions (see [26]), under suitable assumptions on the data, H and F.
- Or a non local operator which continuously maps the set of probability measures on $\mathbb{T}^2$ (endowed with the weak * topology) to a bounded subset of $Lip(\mathbb{T}^2)$, the Lipschitz functions on $\mathbb{T}^2$, and for example maps continuously $\mathscr{C}^{k,\alpha}(\mathbb{T}^2)$ to $\mathscr{C}^{k+1,\alpha}(\mathbb{T}^2)$, for all $k \in \mathbb{N}$ and $0 \leq \alpha < 1$. In this case, classical solutions of (1)–(3) are shown to exist under natural assumptions on the data and some technical assumptions on H.

Consider the important special case when the Hamiltonian is of the form

$$H(x, \nabla u) = \sup_{\gamma} \left[\gamma \cdot \nabla u - L(x,\gamma) \right].$$

In this case, if u and m solve the system above, then Dynamic Programming arguments, see Bardi–Capuzzo Dolcetta [8], Fleming–Soner [17], show that the solution u of the forward in time Hamilton–Jacobi–Bellman equation (1) is the value function of an optimal control problem for the controlled dynamics defined on $\mathbb{T}^2$ by

$$dX_s = -\gamma_s \, ds + \sqrt{2\nu} \, dW_s,$$

(here s is the physical time, $t = T - s$ is the time to the horizon, (W_s) is a Brownian motion), and running cost density $L(X_s, \gamma_s) + \Phi[m(s,\cdot)](X_s)$ depending on the position X_s, the control γ_s and the probability density $m(s,\cdot)$. On the other hand, (2) is a backward Fokker–Planck equation with velocity field $\frac{\partial H}{\partial p}(x, \nabla u)$ depending on the value function itself.

We have chosen to focus on the case when the cost $u_{|t=0}$ depends directly on x. In some realistic situations, the final cost may depend on the density of the players, i.e. $u_{|t=0} = \Phi_0[m_{|t=0}](x)$, where Φ_0 is an operator acting on probability densities, which may be local or not. This case can be handled by the methods proposed below, but we will not discuss it in the present work.

System (1)–(2) consists of a forward Bellman equation coupled with a backward Fokker–Planck equation. The forward-backward structure is an important feature of this system, which makes it necessary to design new strategies for its mathematical analysis (see [26, 27]) and for numerical approximation.

The following steady state version of (1)–(3) arises when mean field games with infinite horizon are considered (ergodic problem):

$$-\nu\Delta u(x) + H(x, \nabla u(x)) + \lambda = \Phi[m(\cdot)](x), \quad \text{in } \mathbb{T}^2, \tag{4}$$

$$-\nu\Delta m(x) - \operatorname{div}\left(m \frac{\partial H}{\partial p}(\cdot, \nabla u)\right)(x) = 0, \quad \text{in } \mathbb{T}^2. \tag{5}$$

with the additional normalization of u: $\int_{\mathbb{T}^2} u = 0$. The unknowns in (4)–(5) are the density m, the function u and the scalar λ.

We refer to the mentioned papers of J-M. Lasry and P-L. Lions for analytical results concerning problems (1)–(3) and (4)–(5) as well as for their interpretation in stochastic game theory. Let us only mention here that a very important feature of the mean field model above is that uniqueness and stability may be obtained under reasonable assumptions, see [25–27], in contrast with the Nash system describing the individual behavior of each player, for which uniqueness hardly ever occurs. To be more precise, uniqueness for (1)–(3) is true if Φ is monotonous in the sense that for all probability measures m and $\tilde{m}$ on $\mathbb{T}^2$,

$$\int (\Phi[m](x) - \Phi[\tilde{m}](x))(dm(x) - d\tilde{m}(x)) \le 0 \Rightarrow m = \tilde{m}. \tag{6}$$

Examples of MFG models with applications in economics and social sciences are proposed in [19, 20, 23]. Many important aspects of the mathematical theory developed by J-M. Lasry and P-L. Lions on MFG are not published in journals or books, but can be found in the videos of the lectures of P-L. Lions at Collège de France: see the web site of Collège de France [29]. We also refer to [14] for a nice survey and new results in the deterministic case ($\nu = 0$), and to [7] for an interesting paper on explicit solutions of some linear-quadratic mean field games.

In this survey, we will focus on a finite difference method in order to approximate the solutions of (1)–(3). An important research activity is currently going on about approximation procedures of different types of mean field games models, see [24] for a numerical method based on the reformulation of the model as an optimal control problem for the Fokker–Planck equation with an application in economics and [18] for a work on discrete time, finite state space mean field games. We also refer to [21, 22] for a specific constructive approach when the Hamiltonian is quadratic. Finally, a semi-discrete approximation for a first order mean field games problem has been studied in [13].

The method described below has first been proposed and discussed in [2, 3]. The numerical schemes that we use rely basically on monotone approximations of the Hamiltonian and on a suitable weak formulation of the Fokker–Planck equation. These schemes have several important features:

- Existence and uniqueness for the discretized problems can be obtained by similar arguments as those used in the continuous case.
- They are robust when $\nu \to 0$ (the deterministic limit of the models).
- Bounds on the solutions, which are uniform in the grid step, can be proved under reasonable assumptions on the data.

This survey is organized as follows: Sect. 2 is devoted to the presentation of the finite difference schemes, to existence and uniqueness results under various assumptions, and to a priori estimates on the solutions of the nonlinear system arising from the discretization. An example of a convergence result is given in Sect. 3. Section 4 is devoted to possible algorithms for solving the previously mentioned nonlinear system, with an emphasis on some preconditioned iterative methods for the linearized discrete MFG system. Some numerical simulations are presented in Sect. 5. Finally, Sect. 6 is devoted to the planning problem, in which the initial condition in (3) is replaced with $m(0, x) = m_0(x)$.

2 Finite Difference Schemes

The scheme presented below was originally proposed and studied in [3].

2.1 Description of the Schemes

Let N_T be a positive integer and $\Delta t = T/N_T$, $t_n = n\Delta t$, $n = 0, \dots, N_T$. Let $\mathbb{T}_h^2$ be a uniform grid on the torus with mesh step h, (assuming that $1/h$ is an integer N_h), and x_{ij} denote a generic point in $\mathbb{T}_h^2$. The values of u and m at

$(x_{i,j}, t_n)$ are respectively approximated by $u_{i,j}^n$ and $m_{i,j}^n$. Let u^n (resp. m^n) be the vector containing the values $u_{i,j}^n$ (resp. $m_{i,j}^n$), for $0 \le i, j < N_h$ indexed in the lexicographic order. We may refer to such vectors as *grid functions*. For all grid function z on $\mathbb{T}_h^2$, all i and j, we agree that $z_{i,j} = z_{(i \bmod N_h),(j \bmod N_h)}$.

2.1.1 Elementary Finite Difference Operators

Let us introduce the elementary finite difference operators

$$(D_1^+ u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} \quad \text{and} \quad (D_2^+ u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h}, \tag{7}$$

and define $[D_h u]_{i,j}$ as the collection of the four possible one sided finite differences at $x_{i,j}$:

$$[D_h u]_{i,j} = \Big((D_1^+ u)_{i,j}, (D_1^+ u)_{i-1,j}, (D_2^+ u)_{i,j}, (D_2^+ u)_{i,j-1} \Big) \in \mathbb{R}^4. \tag{8}$$

We will also need the standard five point discrete Laplace operator

$$(\Delta_h u)_{i,j} = -\frac{1}{h^2}(4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}).$$

2.1.2 Discrete Bellman Equation

Numerical Hamiltonian

In order to approximate the term $H(x, \nabla u)$ in (1) or (4), we consider a numerical Hamiltonian $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x, q_1, q_2, q_3, q_4) \mapsto g(x, q_1, q_2, q_3, q_4)$ satisfying the following conditions:

- **($\mathbf{g_1}$)** *Monotonicity*: g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4.
- **($\mathbf{g_2}$)** *Consistency*: $g(x, q_1, q_1, q_2, q_2) = H(x, q), \quad \forall x \in \mathbb{T}^2, \forall q = (q_1, q_2) \in \mathbb{R}^2$.
- **($\mathbf{g_3}$)** *Differentiability*: g is of class $\mathscr{C}^1$.
- **($\mathbf{g_4}$)** *Convexity*: $(q_1, q_2, q_3, q_4) \mapsto g(x, q_1, q_2, q_3, q_4)$ is convex.

We will approximate $H(\cdot, \nabla u)(x_{i,j})$ by $g(x_{i,j}, [D_h u]_{i,j})$.

Standard examples of numerical Hamiltonians fulfilling these requirements are provided by Lax–Friedrichs or Godunov type schemes, see [3]. For example, if the Hamiltonian H is of the form

$$H(x, p) = \mathscr{H}(x) + |p|^\beta, \tag{9}$$

with $\beta > 1$, the conditions above are all fulfilled by the discrete Hamiltonian given by

$$g(x,q) = \mathscr{H}(x) + G(q_1^-, q_2^+, q_3^-, q_4^+), \tag{10}$$

where, for a real number r, $r^+ = \max(r,0)$ and $r^- = \max(-r,0)$ and where $G : (\mathbb{R}_+)^4 \to \mathbb{R}_+$ is given by

$$G(p) = |p|^\beta = (p_1^2 + p_2^2 + p_3^2 + p_4^2)^{\frac{\beta}{2}}. \tag{11}$$

Discrete Version of the Cost Term $\Phi[m(t,\cdot)](x)$

We introduce the compact and convex set

$$\mathscr{K}_h = \{(m_{i,j})_{0\le i,j<N_h} : h^2 \sum_{i,j} m_{i,j} = 1; \quad m_{i,j} \ge 0\} \tag{12}$$

which can be viewed as the set of the discrete probability measures.

We will often make the following assumptions, Φ_h being local or not:

(Φ_{h1}) We assume that Φ_h is continuous on $\mathscr{K}_h$.

(Φ_{h2}) The numerical cost Φ_h is monotone in the following sense:

$$(\Phi_h[m] - \Phi_h[\tilde{m}], m - \tilde{m})_2 \le 0 \Rightarrow \Phi_h[m] = \Phi_h[\tilde{m}], \tag{13}$$

where $(u,v)_2 = \sum_{0\le i,j<N_h} u_{i,j} v_{i,j}$. This assumption and ($g_4$) will be a sufficient condition for the discrete MFG system to have at most a solution, Φ_h being local or not.

If Φ is a local operator, i.e. $\Phi[m](x) = F(m(x))$, F being a continuous function from $\mathbb{R}^+$ to $\mathbb{R}$, then the discrete version of Φ is naturally given by $(\Phi_h[m])_{i,j} = F(m_{i,j})$. In this case, the operator Φ_h is continuous on the set of nonnegative grid functions.

If Φ is a nonlocal operator, then we assume that the discrete operator Φ_h has the following additional properties:

(Φ_{h3}) We assume that there exists a constant C independent of h such that for all grid function $m \in \mathscr{K}_h$,

$$\|\Phi_h[m]\|_\infty \le C \tag{14}$$

and

$$|(\Phi_h[m])_{i,j} - (\Phi_h[m])_{k,\ell}| \le C d_{\mathbb{T}}(x_{i,j}, x_{k,\ell}) \tag{15}$$

where $d_{\mathbb{T}}(x,y)$ is the distance between the two points x and y in the torus $\mathbb{T}^2$.

(Φ_{h4}) Define $\mathscr{K}$ as the set of probability densities, i.e. nonnegative integrable functions m on $\mathbb{T}^2$ such that $\int_{\mathbb{T}^2} m(x)dx = 1$. For a grid function $m_h \in \mathscr{K}_h$, let $\tilde{m}_h$ be the piecewise bilinear interpolation of m_h at the grid nodes: it is clear that $\tilde{m}_h \in \mathscr{K}$. We assume that there exists a continuous and bounded function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$ and for all $m \in \mathscr{K}$, for all sequences $(m_h)_h$, $m_h \in \mathscr{K}_h$,

$$\| \Phi[m] - \Phi_h[m_h] \|_{L^\infty(\mathbb{T}_h^2)} \leq \omega\left(\|m - \tilde{m}_h\|_{L^1(\mathbb{T}^2)}\right). \tag{16}$$

Let $\mathscr{I}_h m$ be the grid function whose value at $x_{i,j}$ is

$$\frac{1}{h^2}\int_{|x-x_{i,j}|_\infty \leq h/2} m(x)dx.$$

It is clear that if $m \in \mathscr{K}$ then $\mathscr{I}_h m \in \mathscr{K}_h$ and that (16) implies that

$$\lim_{h\to 0} \sup_{m\in\mathscr{K}} \| \Phi[m] - \Phi_h[\mathscr{I}_h m] \|_{L^\infty(\mathbb{T}_h^2)} = 0. \tag{17}$$

For example, if $\Phi[m]$ is defined as the solution w of the equation $\Delta^2 w + w = m$ in $\mathbb{T}^2$, (Δ^2 being the bilaplacian), then one can define $\Phi_h[m_h]$ as the solution w_h of $\Delta_h^2 w_h + w_h = m_h$ in $\mathbb{T}_h^2$. It is possible to check that all the above properties are satisfied.

Discrete Bellman Equation

The discrete version of the Bellman equation is obtained by applying a semi-implicit Euler scheme to (1),

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = (\Phi_h[m^n])_{i,j}, \tag{18}$$

for all points in $\mathbb{T}_h^2$ and all n, $0 \leq n < N_T$, where all the discrete operators have been introduced above. Given $(m^n)_{n=0,\dots,N_T-1}$, (18) and the initial condition $u_{i,j}^0 = u_0(x_{i,j})$ for all (i, j) completely characterize $(u^n)_{0\leq n\leq N_T}$.

2.1.3 Discrete Fokker–Planck Equation

In order to approximate equation (2), it is convenient to consider its weak formulation which involves in particular the term

$$\int_{\mathbb{T}^2} \operatorname{div}\left(m\frac{\partial H}{\partial p}(\cdot, \nabla u)\right)(x)w(x)\,dx.$$

By periodicity,

$$\int_{\mathbb{T}^2} \mathrm{div}\left(m\frac{\partial H}{\partial p}(\cdot,\nabla u)\right)(x)w(x)\,dx = -\int_{\mathbb{T}^2} m(x)\frac{\partial H}{\partial p}(x,\nabla u(x))\cdot\nabla w(x)\,dx$$

is valid for any test function w. The right hand side in the identity above will be approximated by

$$-h^2\sum_{i,j} m_{i,j}\nabla_q g(x_{i,j},[D_h u]_{i,j})\cdot[D_h w]_{i,j} = h^2\sum_{i,j}\mathscr{T}_{i,j}(u,m)w_{i,j},$$

where the transport operator $\mathscr{T}$ is defined as follows:

$$\begin{aligned}&\mathscr{T}_{i,j}(u,m)\\ &= \frac{1}{h}\left(\begin{array}{l}\left(\begin{array}{l} m_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j},[D_h u]_{i,j}) - m_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j},[D_h u]_{i-1,j})\\ +m_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j},[D_h u]_{i+1,j}) - m_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j},[D_h u]_{i,j})\end{array}\right)\\ +\left(\begin{array}{l} m_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j},[D_h u]_{i,j}) - m_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1},[D_h u]_{i,j-1})\\ +m_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1},[D_h u]_{i,j+1}) - m_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j},[D_h u]_{i,j})\end{array}\right)\end{array}\right).\end{aligned} \tag{19}$$

The discrete version of (2) is thus chosen as follows:

$$\frac{m_{i,j}^{n+1}-m_{i,j}^{n}}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathscr{T}_{i,j}(u^{n+1},m^n) = 0. \tag{20}$$

This scheme is implicit w.r.t. to m and explicit w.r.t. u because the considered Fokker–Planck equation is backward. Given u, (20) is a system of linear equations for m. It is easy to see that if m^n satisfies (20) for $0 \le n < N_T$ and if $m^{N_T} \in \mathscr{K}_h$, then $m^n \in \mathscr{K}_h$ for all n, $0 \le n < N_T$.

Remark 2.1. It is important to realize that the operator

$$m \mapsto \left(-\nu(\Delta_h m)_{i,j} - \mathscr{T}_{i,j}(u,m)\right)_{i,j}$$

is the adjoint of the linearized version of the operator

$$u \mapsto \left(-\nu(\Delta_h u)_{i,j} + g(x_{i,j},[D_h u]_{i,j})\right)_{i,j}.$$

2.1.4 Summary

The fully discrete scheme for system (1)–(3) is therefore the following: for all $0 \le i, j < N_h$ and $0 \le n < N_T$

$$\begin{cases} \frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = (\Phi_h[m^n])_{i,j}, \\ \frac{m_{i,j}^{n+1}-m_{i,j}^{n}}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathscr{T}_{i,j}(u^{n+1}, m^n) \qquad = 0, \end{cases} \tag{21}$$

with the initial and terminal conditions

$$m_{i,j}^{N_T} = \frac{1}{h^2}\int_{|x-x_{i,j}|_\infty \le h/2} m_T(x)dx, \qquad u_{i,j}^0 = u_0(x_{i,j}), \quad 0 \le i, j < N_h. \tag{22}$$

2.2 *Existence and A priori Bounds*

We recall a useful lemma that can be found in e.g. [16]. We give its proof for completeness.

Lemma 2.1. *Let v be a grid function on $\mathbb{T}_h^2$ and ρ be a positive parameter. Assume that (g_1)–(g_3) hold. There exists a unique grid function u such that*

$$\rho u_{i,j} + g(x_{i,j}, [D_h u]_{i,j}) - \nu(\Delta_h u)_{i,j} = v_{i,j}. \tag{23}$$

Proof. Existence for (23) is proved by using Leray–Schauder fixed point theorem: indeed, we consider the mapping $\mathscr{F} : \mathbb{R}^{N_h^2} \to \mathbb{R}^{N_h^2}$,

$$(\mathscr{F}(u))_{i,j} = \frac{1}{\rho}\left(\nu(\Delta_h u)_{i,j} - g(x_{i,j}, [D_h u]_{i,j}) + v_{i,j}\right),$$

and the real number $r = \max_{(i,j)} \left|H(x_{i,j}, 0) - v_{i,j}\right|/\rho$. From the continuity of g, $\mathscr{F}$ is continuous from $B_r = \{u \in \mathbb{R}^{N_h^2} : \|u\|_\infty \le r\}$ to $\mathbb{R}^{N_h^2}$.

Assuming that $u \in \partial B_r$, there must exist at least one pair of indices (i_0, j_0) such that $u_{i_0,j_0} = \pm r$. Assuming that $u_{i_0,j_0} = r$, we have

$$\nu(\Delta_h u)_{i_0,j_0} - g(x_{i_0,j_0}, [D_h u]_{i_0,j_0}) \le -H(x_{i_0,j_0}, 0),$$

from the monotonicity and the consistency of g. Hence,

$$(\mathscr{F}(u))_{i_0,j_0} \le \frac{1}{\rho}\left(-H(x_{i_0,j_0}, 0) + v_{i_0,j_0}\right) \le r,$$

and $(\mathscr{F}(u))_{i_0,j_0} \neq \lambda u_{i_0,j_0}$ whenever $\lambda > 1$. Similarly, if $u_{i_0,j_0} = -r$, then $(\mathscr{F}(u))_{i_0,j_0} \geq -r$ which implies that $(\mathscr{F}(u))_{i_0,j_0} \neq \lambda u_{i_0,j_0}$. Therefore $\mathscr{F}(u) \neq \lambda u$ for all $\lambda > 1$ and $u \in \partial B_r$. The Leray–Schauder fixed point theorem can be used: there exists a solution of (23) in B_r. Uniqueness for (23) stems from the monotonicity of g. □

We are ready to prove existence for (21)–(22) and a priori bounds if Φ is a nonlocal smoothing operator:

Theorem 2.1. *(a) Assume that* (g_1)*–*(g_3) *and* (Φ_{h1}) *hold, that* $\nu > 0$*, that* u_0 *is a continuous function on* $\mathbb{T}^2$ *and that* $m_T \in \mathscr{K}$*; then, (21)–(22) has a solution such that* $m^n \in \mathscr{K}_h$*,* $\forall n$*.*

(b) Furthermore, under the following conditions:

- Φ_h *satisfies* (Φ_{h3})
- *there exists a constant* C *such that*

$$\left|\frac{\partial g}{\partial x}(x,(q_1,q_2,q_3,q_4))\right| \leq C(1+|q_1|+|q_2|+|q_3|+|q_4|)$$
$$\forall x \in \mathbb{T}^2,\ \forall q_1,q_2,q_3,q_4 \tag{24}$$

- u_0 *is Lipschitz continuous.*

There exists a constant c *independent of* h *and* Δt *such that*

$$\max_{0\leq n\leq N_T}\left(\|u^n\|_\infty + \|D_h u^n\|_\infty\right) \leq c.$$

Proof. We are going to construct a continuous mapping $\chi : \mathscr{K}_h^{N_T} \to \mathscr{K}_h^{N_T}$ and use Brouwer fixed point theorem. Recall that $\mathscr{K}_h$ can be seen as a compact and convex subset of $\mathbb{R}^{N_h^2}$.

We proceed in several steps:

Step 1: A mapping Ψ: $(m^n)_n \mapsto (u^n)_n$

Given u^0, consider the map $\Psi : (m^n)_{0\leq n\leq N_T-1} \in \mathscr{K}_h^{N_T} \mapsto (u^n)_{1\leq n\leq N_T}$, solution of the first equation in (21), i.e.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j},[D_h u^{n+1}]_{i,j}) = (\Phi_h[m^n])_{i,j}, \tag{25}$$

for $n = 0,\dots N_T-1$ and $0 \leq i,j < N_h$. The existence and uniqueness of u^{n+1}, $n = 0,\dots,N_T-1$ is obtained by induction: at each step, we use Lemma 2.1 with $\rho = 1/\Delta t$ and $v_{i,j} = u_{i,j}^n/\Delta t + (\Phi_h[m^n])_{i,j}$.

Step 2: Boundedness and continuity of Ψ

Looking at the proof of Lemma 2.1, we see that

$$\|u^{n+1}\|_\infty \le \max_{(i,j)} \left| \Delta t \left(H(x_{i,j}, 0) - (\Phi_h[m^n])_{i,j} \right) - u^n_{i,j} \right|.$$

From assumption (Φ_{h1}) and the compactness of $\mathcal{K}_h$, we obtain that there exists a constant C depending on h and u^0 but independent of $(m^n)_n$ such that $\|u^n\|_\infty \le C(1+T)$. Therefore, Ψ maps $\mathcal{K}_h^{N_T}$ to a bounded subset of $\mathbb{R}^{N_T \times N_h^2}$. Moreover, by using assumption (Φ_{h1}) and well known results on continuous dependence on the data for monotone schemes (see e.g. [16]), we see that the mapping Ψ is continuous from $\mathcal{K}_h^{N_T}$ to $\mathbb{R}^{N_T \times N_h^2}$.

As a consequence, since all the norms are equivalent in $\mathbb{R}^{N_T \times N_h^2}$, there exists a constant L which depends on $\|D_h u^0\|_\infty$, h and Δt but not on $(m^n)_n$ such that

$$\|D_h u^{n+1}\|_\infty \le L. \tag{26}$$

Moreover, if (Φ_{h3}) holds, then $\|u^n\|_\infty \le C(1+T)$, for a constant which does depend on u^0 but not on h.

Step 3: Discrete Lipschitz continuity estimates on $\Psi((m^n)_{n=0,\dots,N_T-1})$ under additional assumptions

The solution of (25) is noted

$$u^{n+1} = \Theta(u^n, m^{n+1}).$$

Standard arguments on monotone schemes yield that for all $m \in \mathcal{K}_h$, $u, w \in \mathbb{R}^{N_h^2}$,

$$\| \left(\Theta(u,m) - \Theta(w,m)\right)^+ \|_\infty \le \|(u-w)^+\|_\infty, \tag{27}$$

$$\|\Theta(u,m) - \Theta(w,m)\|_\infty \le \|u-w\|_\infty. \tag{28}$$

For $(\ell, m) \in \mathbb{Z}^2$, call $\tau_{\ell,m} u$ the grid function defined by

$$(\tau_{\ell,m} u)_{i,j} = u_{\ell+i, m+j}.$$

It is a simple matter to check that

$$\frac{(\tau_{\ell,m}u)^{n+1}_{i,j}-(\tau_{\ell,m}u)^{n}_{i,j}}{\Delta t}-\nu(\Delta_h(\tau_{\ell,m}u^{n+1}))_{i,j}+g(x_{i,j},[D_h(\tau_{\ell,m}u^{n+1})]_{i,j})$$
$$=(\Phi_h[m^n])_{i,j}+(\Phi_h[m^n])_{i+\ell,j+m}-(\Phi_h[m^n])_{i,j}$$
$$-g(x_{i+\ell,j+m},[D_h(\tau_{\ell,m}u^{n+1})]_{i,j})+g(x_{i,j},[D_h(\tau_{\ell,m}u^{n+1})]_{i,j}),$$

and therefore

$$\tau_{\ell,m}u^{n+1}=\Theta(\tau_{\ell,m}u^n+\Delta t e,m^n),$$
$$e_{i,j}=\begin{pmatrix}(\Phi_h[m^n])_{i+\ell,j+m}-(\Phi_h[m^n])_{i,j}\\ -g(x_{i+\ell,j+m},[D_h(\tau_{\ell,m}u^{n+1})]_{i,j})+g(x_{i,j},[D_h(\tau_{\ell,m}u^{n+1})]_{i,j})\end{pmatrix}.$$

From assumption (Φ_{h3}) and (24), there exists a constant C (independent of n, $(m^n)_n$, h and Δt) such that

$$\|e\|_\infty\le C\left(1+\|D_hu^{n+1}\|_\infty\right)h\sqrt{\ell^2+m^2}.$$

We conclude from (28) that

$$\|\tau_{\ell,m}u^{n+1}-u^{n+1}\|_\infty\le\|\tau_{\ell,m}u^n-u^n\|_\infty+Ch\Delta t\sqrt{\ell^2+m^2}\left(1+\|D_hu^{n+1}\|_\infty\right). \tag{29}$$

Thanks to (29),

$$(1-C\Delta t)\|D_hu^{n+1}\|_\infty\le\|D_hu^n\|_\infty+C\Delta t.$$

A discrete version of Gronwall's lemma yields that there exists a constant L which only depends on C, T and the initial condition $\|D_hu^0\|_\infty$ such that (26) holds for all n, $1\le n\le N_T$; this is a discrete Lipschitz continuity estimate, uniform with respect to $(m^n)_{0\le n\le N_T-1}$, h and Δt.

Step 4: A fixed point problem for $(m^n)_{0\le n\le N_T-1}$

For $(m^n)_{0\le n\le N_T-1}\in\mathcal{K}_h^{N_T}$ and $(u^n)_{1\le n\le N_T}=\Psi((m^n)_{0\le n\le N_T-1})$ and a positive real number μ, consider the following linear problem: find $(\tilde m^n)_{1\le n\le N_T}$ such that

$$\frac{\tilde m^{n+1}_{i,j}-\tilde m^n_{i,j}}{\Delta t}-\mu\tilde m^n_{i,j}+\nu(\Delta_h\tilde m^n)_{i,j}+\mathcal{T}_{i,j}(u^{n+1},\tilde m^n)=-\mu m^n_{i,j}, \tag{30}$$

with the terminal condition $\tilde m^{N_T}=m^{N_T}\in\mathcal{K}_h$.

We are going to prove first that for μ large enough, (30) has a unique solution $(\tilde m^n)_{0\le n\le N_T-1}\in\mathcal{K}_h^{N_T}$, then that the mapping χ: $(m^n)_{1\le n\le N_T}\mapsto(\tilde m^n)_{1\le n\le N_T}$ has a fixed point. Existence for (21)–(22) will then be proved.

Step 5: Existence for (30)

Clearly (30) is the discrete version of a linear backward parabolic equation with a terminal Cauchy condition. It can be written

$$\tilde{m}^n + \Delta t(\mu \tilde{m}^n + A^n \tilde{m}^n) = \tilde{m}^{n+1} + \mu \Delta t m^n, \tag{31}$$

where A^n is a linear operator depending on u^{n+1}.

Assumptions (g_1) and (g_3) imply that the matrix corresponding to $Id + \Delta t A^n$ has positive diagonal entries and nonpositive off-diagonal entries. Furthermore, from assumption (g_3), (26) implies that there exists a constant C which may depend on h, Δt and $\|D^0_{hu}\|$ but not on (m^n), such that for all n, $1 \le n \le N_T$, for all i, j, $0 \le i, j \le N_h$, and for all $\ell = 1, 2, 3, 4$,

$$\left| \frac{\partial g}{\partial q_\ell}(x_{i,j}, [D_h u^n]_{i,j}) \right| \le C. \tag{32}$$

From this and the definition of the discrete transport operator $\mathscr{T}$, we see that for μ large enough but independent of (m^n), the matrix corresponding to $Id + \Delta t(\mu Id + A^n)$ is a M-matrix, and is therefore invertible. The system of linear equations (31) has a unique solution.

Moreover, since $m^n \ge 0$ for all $n = 0, \dots, N_T$ and since $Id + \Delta t(\mu Id + A^n)$ is a M-matrix for all n, $0 \le n \le N_T - 1$, $\tilde{m}^n \ge 0$ for all $n = 0, \dots, N_T - 1$.

We are left with proving that $h^2 \sum_{i,j} \tilde{m}^n_{i,j} = 1$, for all n, $0 \le n < N_T$. We see that for two grid functions w and z, we have

$$\begin{aligned}(A^n w, z)_2 = {}& \nu \sum_{i,j} (D_1^+ w)_{i,j} (D_1^+ z)_{i,j} + \nu \sum_{i,j} (D_2^+ w)_{i,j} (D_2^+ z)_{i,j} \\ &+ \sum_{i,j} w_{i,j} [D_h z]_{i,j} \cdot g_q \left(x_{i,j}, [D_h u^{n+1}]_{i,j} \right).\end{aligned} \tag{33}$$

From (33) and (31), it can be proved by induction that if $h^2(m^{N_T}, 1)_2 = 1$, then the condition $h^2(\tilde{m}^n, 1)_2 = 1$ holds for all n, $0 \le n < N_T$.

Step 6: Existence of a fixed point of χ

From the boundedness and continuity of the mapping Ψ, and from the fact that g is $\mathscr{C}^1$, we obtain that χ is continuous. Therefore, we can apply Brouwer fixed point theorem and obtain that χ has a fixed point.

Conclusion

Assuming that $m^{N_T} \in \mathscr{K}_h$, we have proved that the mapping χ has a fixed point that we call $(m^n)_{0 \le n < N_T}$. Calling $(u^n)_{1 \le n \le N_T} = \Psi((m^n)_{1 \le n \le N_T})$, $(m^n)_{n=0\dots,N_T-1}$

and $(u^n)_{n=1\dots,N_T}$ satisfy (21)–(22). Moreover, under the additional assumptions in the statement of Theorem 2.1, $\max_{0\le n\le N_T}\left(\|u^n\|_\infty + \|D_h u^n\|_\infty\right) \le c$ for a constant c independent of h and Δt. □

2.3 A Fundamental Identity

In this paragraph, we discuss the key identity (38) below, which leads to the stability of the finite difference scheme under additional assumptions.

Let us define the nonlinear functional $\mathscr{G}$ acting on grid functions by

$$\mathscr{G}(m,u,\tilde u) = \sum_{n=1}^{N_T}\sum_{i,j}\mathscr{G}^n_{i,j} \tag{34}$$

where

$$\begin{aligned}\mathscr{G}^n_{i,j} = m^{n-1}_{i,j}\Big(& g(x_{i,j},[D\tilde u^n]_{i,j}) - g(x_{i,j},[Du^n]_{i,j}) - g_q(x_{i,j},[Du^n]_{i,j})\\ &\cdot([D\tilde u^n]_{i,j} - [Du^n]_{i,j})\Big).\end{aligned}$$

Under Assumption (g_4), it is clear that $\mathscr{G}(m,u,\tilde u)\ge 0$ if m is a nonnegative grid function. If g is of the form (10)–(11), we have a more precise estimate.

Consider a perturbed system:

$$\begin{cases}\dfrac{\tilde u^{n+1}_{i,j}-\tilde u^n_{i,j}}{\Delta t} - \nu(\Delta_h\tilde u^{n+1})_{i,j} + g(x_{i,j},[D_h\tilde u^{n+1}]_{i,j}) = (\Phi_h[\tilde m^n])_{i,j} + a^n_{i,j},\\ \dfrac{\tilde m^{n+1}_{i,j}-\tilde m^n_{i,j}}{\Delta t} + \nu(\Delta_h\tilde m^n)_{i,j} + \mathscr{T}_{i,j}(\tilde u^{n+1},\tilde m^n) \qquad = b^n_{i,j}.\end{cases} \tag{35}$$

Multiplying the first equations in (35) and (21) by $m^n_{i,j}-\tilde m^n_{i,j}$ and subtracting, then summing the results for all $n=0,\dots,N_T-1$ and all (i,j), we obtain

$$\begin{aligned}&\sum_{n=0}^{N_T-1}\frac{1}{\Delta t}((u^{n+1}-\tilde u^{n+1})-(u^n-\tilde u^n),(m^n-\tilde m^n))_2\\ &\quad-\nu(\Delta_h(u^{n+1}-\tilde u^{n+1}),m^n-\tilde m^n)_2\\ &\quad+\sum_{n=0}^{N_T-1}\sum_{i,j}(g(x_{i,j},[D_hu^{n+1}]_{i,j})-g(x_{i,j},[D_h\tilde u^{n+1}]_{i,j}))(m^n_{i,j}-\tilde m^n_{i,j})\\ &\quad=\sum_{n=0}^{N_T-1}(\Phi_h[m^n]-\Phi_h[\tilde m^n],m^n-\tilde m^n)_2-\sum_{n=0}^{N_T-1}(a^n,m^n-\tilde m^n)_2,\end{aligned} \tag{36}$$

where $(X, Y)_2 = \sum_{i,j} X_{i,j} Y_{i,j}$. Similarly, subtracting the second equation in (35) from the second equation in (20), multiplying the result by $u_{i,j}^{n+1} - \tilde{u}_{i,j}^{n+1}$ and summing for all $n = 0, \dots, N_T - 1$ and all (i, j) leads to

$$\begin{aligned}
&\sum_{n=0}^{N_T-1} \frac{1}{\Delta} t((m^{n+1} - m^n) - (\tilde{m}^{n+1} - \tilde{m}^n), (u^{n+1} - \tilde{u}^{n+1}))_2 \\
&\quad + \nu((m^n - \tilde{m}^n), \Delta_h(u^{n+1} - \tilde{u}^{n+1}))_2 \\
&\quad - \sum_{n=0}^{N_T-1} \sum_{i,j} m_{i,j}^n [D_h(u^{n+1} - \tilde{u}^{n+1})]_{i,j} \cdot g_q\left(x_{i,j}, [D_h u^{n+1}]_{i,j}\right) \\
&\quad + \sum_{n=0}^{N_T-1} \sum_{i,j} \tilde{m}_{i,j}^n [D_h(u^{n+1} - \tilde{u}^{n+1})]_{i,j} \cdot g_g\left(x_{i,j}, [D_h \tilde{u}^{n+1}]_{i,j}\right) \\
&\quad = -\sum_{n=1}^{N_T} (b^{n-1}, u^n - \tilde{u}^n)_2.
\end{aligned} \tag{37}$$

Adding (36) and (37) leads to the important identity

$$\begin{aligned}
&-\frac{1}{\Delta} t(m^{N_T} - \tilde{m}^{N_T}, u^{N_T} - \tilde{u}^{N_T})_2 + \frac{1}{\Delta} t(m^0 - \tilde{m}^0, u^0 - \tilde{u}^0)_2 \\
&\quad + \mathscr{G}(m, u, \tilde{u}) + \mathscr{G}(\tilde{m}, \tilde{u}, u) + \sum_{n=0}^{N_T-1} (\Phi_h[m^n] - \Phi_h[\tilde{m}^n], m^n - \tilde{m}^n)_2 \\
&\quad = \sum_{n=0}^{N_T-1} (a^n, m^n - \tilde{m}^n)_2 + \sum_{n=1}^{N_T} (b^{n-1}, u^n - \tilde{u}^n)_2.
\end{aligned} \tag{38}$$

It is important to note that under assumptions (g_4) and (Φ_{h2}), the second line of (38) is made of three nonnegative terms. This is the key observation leading to uniqueness for (21)–(22), but it may also lead to a priori estimates or stability estimates under additional assumptions.

Remark 2.2. It has been proved in [1] that if:

1. g is of the form (10)–(11) with $\beta \geq 2$ and if m is a nonnegative grid function bounded from below by $\underline{m}$, then

$$\mathscr{G}(m, u, \tilde{u}) \geq \frac{\underline{m}}{2^{2\beta-3}(\beta - 1)} \sum_{n=1}^{N_T} \sum_{i,j} \left| [D\tilde{u}^n]_{i,j} - [Du^n]_{i,j} \right|^\beta. \tag{39}$$

2. if g is of the form (10)–(11) with $1 < \beta < 2$, and $m_{i,j}^n \geq \underline{m}$, then for all $(\tilde{u}_{i,j}^n)_{i,j,n}$ such that for all n, $\tilde{u}_{i,j}^n$ does not depend on i and j,

$$\mathscr{G}(m, \tilde{u}, u) = \mathscr{G}(m, 0, u) \geq 2^{2\beta-6}\beta(\beta-1)\underline{m} \sum_{n=1}^{N_T} \sum_{i,j} |[Du^n]_{i,j}|^\beta.$$

Hence, in these cases, (38) leads to stability estimates for (21)–(22).

2.4 Uniqueness

Theorem 2.2. *If (g_1)–(g_4) and (Φ_{h1})–(Φ_{h2}) hold and if $\nu > 0$, then (21)–(22) has a unique solution.*

Proof. Take two solutions of (21)–(22), (u, m) and $(\tilde{u}, \tilde{m})$ and use (38): the terms in the first line and in the right hand side of (38) are zero. The three terms in the second line are nonnegative, so each of them is actually zero. From (Φ_{h2}), $\Phi_h[m^n] = \Phi_h[\tilde{m}^n]$, for all n. From this and Assumption (g_1), which yields the uniqueness for the discrete Cauchy problem with the Bellman equation (18), we deduce that $u^n = \tilde{u}^n$ for each n. Injecting this piece of information into the discrete Fokker–Planck equations for m and $\tilde{m}$ finally implies that $m = \tilde{m}$. □

2.5 A priori Estimates for (21)–(22) with Local Operators Φ

Below, we give a result similar to Theorem 2.7 in [27]; this result which was originally proposed in [1]:

Proposition 2.1. *Assume that $0 \leq m_T(x) \leq \bar{m}_T$, that u_0 is a continuous function, that g is of the form (10)–(11) with $\beta > 1$. If $(\Phi_h[m])_{i,j} = F(m_{i,j})$, where*

(F_1) F is a $\mathscr{C}^0$ function defined on $[0, \infty)$.
(F_2) There exist three constants $\delta > 0$ and $\gamma > 1$ and $C_1 \geq 0$ such that

$$mF(m) \geq \delta |F(m)|^\gamma - C_1, \quad \forall m \geq 0,$$

then there exist two constants $c \in \mathbb{R}$ and $C > 0$ such that the solution of (21)–(22) satisfies:

- *$u_{i,j}^n \geq c$, for all n, i and j*
-

$$h^2 \Delta t \mathscr{G}(\bar{m}_T, 0, u) + h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} \left| F(m_{i,j}^n) \right|^\gamma \leq C \qquad (40)$$

- $$\max_{0\le n\le N_T} h^2 \sum_{i,j} |u^n_{i,j}| \le C \tag{41}$$

- *Finally, let us call U^n the sum $h^2 \sum_{i,j} u^n_{i,j}$ and U_h the piecewise linear function obtained by interpolating the values U^n at the points (t_n): the family of functions (U_h) is bounded in $W^{1,1}(0,T)$ by a constant independent of h and Δt.*

Proof. From the two assumptions on F, we deduce that $\underline{F} \equiv \inf_{m\in\mathbb{R}_+} F(m)$ is a real number and that $\underline{F} = \min_{m\ge 0} F(m)$. Note that $\underline{F} = F(0)$ if F is nondecreasing.

A standard comparison argument shows that

$$u^n_{i,j} \ge \min_{x\in\mathbb{T}^2} u_0(x) + \left(\underline{F} - \max_{x\in\mathbb{T}^2} \mathscr{H}(x)\right) t_n \ge \min_{x\in\mathbb{T}^2} u_0(x) - T\left(\underline{F} - \max_{x\in\mathbb{T}^2} \mathscr{H}(x)\right)^-,$$

so $u^n_{i,j}$ is bounded from below by a constant independent of h and Δt.

Consider $\tilde{u}^n_{i,j} = n\Delta t F(\bar{m}_T)$ and $\tilde{m}^n_{i,j} = \bar{m}_T$ for all i, j, n. We have

$$\begin{cases} \dfrac{\tilde{u}^{n+1}_{i,j} - \tilde{u}^n_{i,j}}{\Delta t} - \nu(\Delta_h \tilde{u}^{n+1})_{i,j} + g(x_{i,j}, [D_h \tilde{u}^{n+1}]_{i,j}) = F(\bar{m}_T) + \mathscr{H}(x_{i,j}), \\ \dfrac{\tilde{m}^{n+1}_{i,j} - \tilde{m}^n_{i,j}}{\Delta t} + \nu(\Delta_h \tilde{m}^n)_{i,j} + \mathscr{T}_{i,j}(\tilde{u}^{n+1}, \tilde{m}^n) \qquad = 0. \end{cases}$$

Since $D_h \tilde{u}^n = 0$ for all n, identity (38) becomes

$$\begin{aligned} &h^2\Delta t \mathscr{G}(m,u,0) + h^2 \Delta t \mathscr{G}(\tilde{m},0,u) + h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} (F(m^n_{i,j}) \\ &\quad - F(\bar{m}_T)(m^n_{i,j} - \bar{m}_T) \\ &\qquad = h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} \mathscr{H}(x_{i,j})(m^n_{i,j} - \tilde{m}^n_{i,j}) \\ &\qquad\quad + h^2(m^{N_T} - \bar{m}_T, u^{N_T} - T\, F(\bar{m}_T))_2 - h^2(m^0 - \bar{m}_T, u^0)_2. \end{aligned} \tag{42}$$

On the other hand:

1. Since the function $x \to \mathscr{H}(x)$ is bounded, and m^n is a discrete probability density, there exists a constant C such that

$$\left| h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} \mathscr{H}(x_{i,j})(m^n_{i,j} - \tilde{m}^n_{i,j}) \right| \le C.$$

2. Since $m^{N_T} - \bar{m}_T$ is nonpositive with a bounded mass, and since u^n is bounded from below by a constant, there exists a constant C such that

$$h^2(m^{N_T} - \bar{m}_T, u^{N_T} - TF(\bar{m}_T))_2 \leq C.$$

3. Since u_0 is continuous on $\mathbb{T}^2$ and m_0 is a discrete probability density, there exists a constant C such that

$$-h^2(m^0 - \bar{m}_T, u^0)_2 \leq C.$$

4. Finally, we know that

$$\begin{aligned}(F(m^n_{i,j}) - F(\bar{m}_T))(m^n_{i,j} - \bar{m}_T)\\ \geq \delta \left| F(m^n_{i,j} \right|^\gamma - C_1 - \bar{m}_T F(m^n_{i,j}) - m^n_{i,j} F(\bar{m}_T) + \bar{m}_T F(\bar{m}_T).\end{aligned}$$

Moreover, since $\gamma > 1$, there exist two constants $c = \frac{\delta}{2}$ and C such that $\delta \left| F(m^n_{i,j}) \right|^\gamma - \bar{m}_T F(m^n_{i,j}) \geq c \left| F(m^n_{i,j}) \right|^\gamma - C$. Since $m^n \in \mathscr{K}_h$, summing yields that for a possibly different constant C,

$$h^2 \sum_{i,j} (F(m^n_{i,j}) - F(\bar{m}_T))(m^n_{i,j} - \bar{m}_T) \geq ch^2 \sum_{i,j} \left| F(m^n_{i,j}) \right|^\gamma - C.$$

We get (40) from (42) and from the four points above. Note that (40) implies that there exists a constant C such that

$$h^2 \Delta t \sum_{\ell=0}^{N_T-1} \sum_{i,j} g(x_{i,j}, [D_h u^{\ell+1}]_{i,j}) \leq C, \tag{43}$$

because, from the special form of g,

$$\mathscr{G}(\bar{m}_T, u, 0) = \bar{m}_T \sum_{\ell=0}^{N_T-1} \sum_{i,j} \left(g(x_{i,j}, [D_h u^{\ell+1}]_{i,j}) - \mathscr{H}(x_{i,j}) \right).$$

Again from the special form of g and the boundedness of $\mathscr{H}$, we deduce that

$$h^2 \Delta t \sum_{\ell=0}^{N_T-1} \sum_{i,j} \left| g(x_{i,j}, [D_h u^{\ell+1}]_{i,j}) \right| \leq C. \tag{44}$$

Finally, summing the first equation in (21) for all i, j, $0 \leq \ell < n$ one gets that

$$h^2 \sum_{i,j} u^n_{i,j} + h^2 \Delta t \sum_{\ell=0}^{q-1} \sum_{i,j} g(x_{i,j}, [D_h u^{\ell+1}]_{i,j}) = h^2 \Delta t \sum_{\ell=0}^{n-1} \sum_{i,j} F(m^\ell_{i,j}) + h^2 \sum_{i,j} u^0_{i,j}.$$

Using (40) and (44), we get that there exists a constant C such that

$$h^2 \sum_{i,j} u_{i,j}^n \le C,$$

and since $u_{i,j}^n$ is bounded from below by a constant, we get (41).

Finally, remember that U^n is the sum $h^2 \sum_{i,j} u_{i,j}^n$; summing the first equations in (21) for all i, j, we obtain that

$$\frac{U^{n+1} - U^n}{\Delta} t = G^{n+1} \equiv -h^2 \sum_{i,j} \left(g(x_{i,j}, [D_h u^{n+1}]_{i,j}) + F(m_{i,j}^n) \right).$$

The bounds (40) and (44) imply that the piecewise linear function U_h obtained by interpolating the values U^n at the points (t_n) is bounded in $W^{1,1}(0,T)$ by a constant independent of h and Δt. □

3 Examples of Convergence Results

It is possible to obtain various convergence results depending on the assumptions on g and Φ_h. In the case when Φ is a nonlocal smoothing operator and assumptions (Φ_{h1}), (Φ_{h2}), (Φ_{h3}), and (Φ_{h4}) hold for Φ_h, things are easier because of the uniform Lipschitz bound given in Theorem 2.1, and it is possible to prove convergence in various norms, in particular a uniform convergence for the potential u:

Theorem 3.1. *Let us make the following assumptions on the data: $\nu > 0$, $\beta > 1$; the function $x \to \mathscr{H}(x)$ is $\mathscr{C}^1$ on $\mathbb{T}^2$, the functions u_0 and m_T are smooth, and $m_T \in \mathscr{K}$ is bounded from below by a positive number. We assume that Φ is monotone in the sense of (6), nonlocal and smoothing, so that there is a unique classical solution (u, m) of (1)–(3) such that $m > 0$.*

Consider a numerical Hamiltonian given by (10)–(11) and a numerical cost function Φ_h such that (Φ_{h1}), (Φ_{h2}), (Φ_{h3}), and (Φ_{h4}) hold. Let u_h (resp. m_h) be the piecewise trilinear function in $\mathscr{C}([0,T] \times \mathbb{T}^2)$ obtained by interpolating the values $u_{i,j}^n$ (resp $m_{i,j}^n$) at the nodes of the space-time grid. The functions u_h converge uniformly and in $L^{\max(\beta,2)}(0,T; W^{1,\max(\beta,2)}(\mathbb{T}^2))$ to u as h and Δt tend to 0. If $\beta \ge 2$ the functions m_h converge to m in $C^0([0,T]; L^2(\mathbb{T}^2)) \cap L^2(0,T; H^1(\mathbb{T}^2))$ as h and Δt tend to 0. If $1 < \beta < 2$, the functions m_h converge to m in $L^2((0,T) \times \mathbb{T}^2)$.

Proof. See [1]. □

In the case when Φ is a local operator, it is still possible to state convergence results. Here, for simplicity, we are going to focus on the case when H is given by (9) with $\beta > 1$ and we make the assumption that the continuous problem has a classical solution: existence of a classical solution can be true for local operators

Φ: for example, it has been proved in [15] that if $\beta = 2$, and F is $\mathscr{C}^1$ and bounded from below, and if the functions u_0 and m_T are $\mathscr{C}^2$ then there is a classical solution.

Remark 3.1. For the stationary problem (4), it can be proved that, if (F_2) holds with $\gamma > 2$ (2 is the space dimension) and F is nondecreasing, then (4) has a classical solution for any $\beta > 1$, by using the weak Bernstein method studied in [28].

Theorem 3.2. *Let us make the following assumptions on the data: $\nu > 0$, H is given by (9) with $\beta > 1$, $x \to \mathscr{H}(x)$ is $\mathscr{C}^1$ on $\mathbb{T}^2$ and g is given by (10)–(11), the functions u_0 and m_T are $\mathscr{C}^2$, and $m_T \in \mathscr{K}$ is bounded from below by a positive number. We assume that (F_1) and (F_2) hold, and that there exist three positive constants $\underline{\delta}$, $\eta_1 > 0$ and $0 < \eta_2 < 1$ such that $F'(m) \geq \underline{\delta} \min(m^{\eta_1}, m^{-\eta_2})$.*

We assume that there is unique classical solution (u, m) of (1)–(3) such that $m > 0$.

Consider a numerical Hamiltonian given by (10)–(11). Let u_h (resp. m_h) be the piecewise trilinear function in $\mathscr{C}([0,T] \times \mathbb{T}^2)$ obtained by interpolating the values $u^n_{i,j}$ (resp $m^n_{i,j}$) at the nodes of the space-time grid. The functions u_h converge in $L^\beta(0,T; W^{1,\beta}(\mathbb{T}^2))$ to u as h and Δt tend to 0. The functions m_h converge to m in $L^{2-\eta_2}((0,T) \times \mathbb{T}^2)$ as h and Δt tend to 0.

Proof. For simplicity, we give the proof in the case $\beta \geq 2$ only. For the case $1 < \beta < 2$, the proof is a bit more complicated, and we refer to [1]. Call $\bar{m} = \max m(t,x)$ and $0 < \underline{m} = \min m(t,x)$.

Note that $m_h(t,\cdot) \in \mathscr{K}$ for any $t \in [0,T]$.

We call $\tilde{u}^n$ and $\tilde{m}^n$ the grid functions such that $\tilde{u}^n_{i,j} = u(n\Delta t, x_{i,j})$ and $\tilde{m}^n = \mathscr{I}_h(m(t_n,\cdot))$. The functions $\tilde{u}$ and $\tilde{m}^n$ are solutions of (35) where a and b are consistency errors. From the fact that (u,m) is a classical solution of (1)–(3), we infer from the consistency of the scheme (in particular from (17)) that $\max_{0\leq n<N_T}(\|a^n\|_{L^\infty(\mathbb{T}^2_h)} + \|b^n\|_{L^\infty(\mathbb{T}^2_h)})$ tends to zero as h and Δt tend to zero.

Step 1

As a consequence of the previous observations, the fundamental identity (38) holds, and from (22), can be written as follows:

$$h^2\Delta t \mathscr{G}(m,u,\tilde{u}) + h^2 \Delta t \mathscr{G}(\tilde{m},\tilde{u},u) + h^2\Delta t \sum_{n=0}^{N_T-1} (\Phi_h[m^n] - \Phi_h[\tilde{m}^n], m^n - \tilde{m}^n)_2$$
$$= h^2 \Delta t \sum_{n=0}^{N_T-1} (a^n, m^n - \tilde{m}^n)_2 + h^2 \Delta t \sum_{n=1}^{N_T} (b^{n-1}, u^n - \tilde{u}^n)_2. \tag{45}$$

From Proposition 2.1, the a priori bound (41) holds for u_h. This implies that

$$\lim_{h,\Delta t \to 0} h^2 \max_n |(b^{n-1}, u^n - \tilde{u}^n)_2| = 0.$$

From the fact that $m^n \in \mathscr{K}_h$, we also get that $\lim_{h,\Delta t\to 0} h^2 \max_n |(a^n, m^n - \tilde{m}^n)_2| = 0$.

Therefore, since $\beta \geq 2$, and since $\tilde{m}^n_{i,j} \geq \underline{m}$, we deduce from (45) and (39) that

$$h^2 \Delta t \sum_{n=1}^{N_T} \sum_{i,j} \left| [D_h \tilde{u}^n]_{i,j} - [D_h u^n]_{i,j} \right|^\beta = o(1). \tag{46}$$

Step 2

We also obtain from (45) that

$$h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} (F(m^n_{i,j}) - F(\tilde{m}^n_{i,j})(m^n_{i,j} - \tilde{m}^n_{i,j}) = o(1). \tag{47}$$

We split the sum w.r.t. (i,j) in the left hand side of (47) into

$$S_1^n = \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^-)^2 \int_0^1 F'(\tilde{m}^n_{i,j} + t(m^n_{i,j} - \tilde{m}^n_{i,j}))dt,$$

$$S_2^n = \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^+)^2 \int_0^1 F'(\tilde{m}^n_{i,j} + t(m^n_{i,j} - \tilde{m}^n_{i,j}))dt.$$

Call $\bar{m} = \max m(t,x)$ and $\underline{m} = \min m(t,x) > 0$; there exists a positive number c depending on $\underline{m}$ and $\bar{m}$ but independent of h and Δt, and (i,j,n) such that

$$\begin{aligned} S_1^n &\geq c \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^-)^2 \int_0^1 (\tilde{m}^n_{i,j} + t(m^n_{i,j} - \tilde{m}^n_{i,j}))^{\eta_1} dt \\ &= \frac{c}{\eta_1 + 1} \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^-)((\tilde{m}^n_{i,j})^{\eta_1+1} - (m^n_{i,j})^{\eta_1+1}) \\ &\geq \frac{c}{\eta_1 + 1} \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^-)^2 (\tilde{m}^n_{i,j})^{\eta_1}. \end{aligned}$$

The latter inequality comes from the nondecreasing character of the function $\chi : [0,y] \to \mathbb{R}$, $\chi(z) = \frac{y^{\eta_1+1} - z^{\eta_1+1}}{y-z}$. Thus, $\chi(z) \geq \chi(0) = y^{\eta_1}$. Hence, there exists a constant c depending on the bounds on the density m solution of (1)–(3) but not on h and Δt, and (i,j,n) such that

$$S_1^n \geq c \sum_{i,j} ((m^n_{i,j} - \tilde{m}^n_{i,j})^-)^2.$$

On the other hand

$$\begin{aligned} S_2^n &\geq c \sum_{i,j} ((m_{i,j}^n - \tilde{m}_{i,j}^n)^+)^2 \int_0^1 (\tilde{m}_{i,j}^n + t(m_{i,j}^n - \tilde{m}_{i,j}^n))^{-\eta_2} dt \\ &= \frac{c}{1-\eta_2} \sum_{i,j} ((m_{i,j}^n - \tilde{m}_{i,j}^n)^+)((m_{i,j}^n)^{1-\eta_2} - (\tilde{m}_{i,j}^n)^{1-\eta_2}) \\ &\geq c \sum_{i,j} ((m_{i,j}^n - \tilde{m}_{i,j}^n)^+)^2 (m_{i,j}^n)^{-\eta_2}. \end{aligned}$$

But there exists a constant c such that for all $y \in [\underline{m}, \bar{m}]$: if $z \geq y+1$

$$(z-y)^2 z^{-\eta_2} \geq (z-y)^{2-\eta_2} \inf_{z \geq y+1} \frac{(z-y)^{\eta_2}}{z^{\eta_2}} \geq c(z-y)^{2-\eta_2},$$

and if $y \leq z \leq y+1$,

$$(z-y)^2 z^{-\eta_2} \geq c(z-y)^2.$$

Therefore there exists a constant c such that

$$\begin{aligned} S_1^n + S_2^n \geq c \Bigg(& \sum_{i,j} (m_{i,j}^n - \tilde{m}_{i,j}^n)^2 1_{\{m_{i,j}^n \leq \tilde{m}_{i,j}^n + 1\}} \\ & + \sum_{i,j} (m_{i,j}^n - \tilde{m}_{i,j}^n)^{2-\eta_2} 1_{\{m_{i,j}^n \geq \tilde{m}_{i,j}^n + 1\}} \Bigg). \end{aligned}$$

Then (47) implies that

$$\lim_{h,\Delta t \to 0} h^2 \Delta t \sum_{n=0}^{N_T-1} \left(\begin{array}{l} \sum_{i,j} (m_{i,j}^n - \tilde{m}_{i,j}^n)^2 1_{\{m_{i,j}^n \leq \tilde{m}_{i,j}^n + 1\}} + \\ + \sum_{i,j} (m_{i,j}^n - \tilde{m}_{i,j}^n)^{2-\eta_2} 1_{\{m_{i,j}^n \geq \tilde{m}_{i,j}^n + 1\}} \end{array} \right) = 0.$$

A Hölder inequality leads to

$$\lim_{h,\Delta t \to 0} h^2 \Delta t \sum_{n=0}^{N_T-1} \sum_{i,j} |m_{i,j}^n - \tilde{m}_{i,j}^n|^{2-\eta_2} = 0. \tag{48}$$

Step 3

From the previous two steps, up to an extraction of a sequence, $m_h \to m$ in $L^{2-\eta_2}((0,T) \times \mathbb{T}^2)$ and almost everywhere in $(0,T) \times \mathbb{T}^2$, ∇u_h converges to ∇u

strongly in $L^\beta((0,T)\times\mathbb{T}^2)$. Moreover, from the last point in Proposition 2.1, the sequence of piecewise linear functions (U_h) on $[0,T]$ obtained by interpolating the values $U^n = h^2\sum_{i,j} u^n_{i,j}$ at the points (t_n) is bounded in $W^{1,1}(0,T)$, so up to a further extraction of a subsequence, it converges to some function U in $L^\beta(0,T)$. As a result, there exists a function ψ of the variable t such that $u_h \to u+\psi$ in $L^\beta(0,T;W^{1,\beta}(\mathbb{T}^2))$.

From the a priori estimate (40), the sequence $(F(m_h))$ is bounded in $L^\gamma((0,T)\times\mathbb{T}^2)$ for some $\gamma>1$, which implies that it is uniformly integrable on $(0,T)\times\mathbb{T}^2$. On the other hand, $F(m_h)$ converges almost everywhere to $F(m)$. Therefore, from Vitali's theorem, $F(m_h)$ converges to $F(m)$ in $L^1((0,T)\times\mathbb{T}^2)$, (in fact, it is also possible to show that $F(m_h)$ converges to $F(m)$ in $L^q((0,T)\times\mathbb{T}^2)$ for all $q\in[1,\gamma)$).

It is then possible to pass to the limit in the discrete Bellman equation, which yields that $\frac{\partial\psi}{\partial t}=0$ in the sense of distributions in $(0,T)$. Hence ψ is a constant.

We are left with proving that ψ is indeed 0. For that, we split $\frac{\partial u_h}{\partial t}$ into the sum $\mu_h+\eta_h$, where

- $\mu_h|_{t\in(t_n,t_{n+1}]}$ is constant w.r.t. t and piecewise linear w.r.t. x, and takes the value $\eta(\Delta_h u^{n+1})_{i,j}$ at the node $\xi_{i,j}$.
- η_h is the remainder, see (18).

From the observations above, (η_h) converges in $L^1((0,T)\times\mathbb{T}^2)$. On the other hand, from (46), it is not difficult to see that (μ_h) is a Cauchy sequence in $L^\beta(0,T;(W^{s,\beta/(\beta-1)}(\mathbb{T}^2))')$ for s large enough, (here $(W^{s,\beta/(\beta-1)}(\mathbb{T}^2))'$ is the topological dual of $W^{s,\beta/(\beta-1)}(\mathbb{T}^2)$).

Hence, $(\frac{\partial u_h}{\partial t})$ converges in $L^1(0,T;(W^{s,\beta/(\beta-1)}(\mathbb{T}^2))')$. Therefore, u_h converges in $\mathscr{C}^0([0,T];(W^{s,\beta/(\beta-1)}(\mathbb{T}^2))')$; since $(u_h(t=0))$ converges to u_0, we see that $\psi=0$.

This implies that the extracted sequence u_h converges to u in $L^\beta(0,T;W^{1,\beta}(\mathbb{T}^2))$. Since the limit is unique, the whole family (u_h) converges to u in $L^\beta(0,T;W^{1,\beta}(\mathbb{T}^2))$ as h and Δt tend to 0. □

We give the corresponding theorem in the ergodic case, without proof, because it is quite similar to that of Theorem 3.2.

Theorem 3.3. *Let us make the following assumptions on the data:* $\nu>0$, $\beta>1$*; the function* $x\to\mathscr{H}(x)$ *is* $\mathscr{C}^1$ *on* $\mathbb{T}^2$*. We assume that* (F_1) *and* (F_2) *hold, and that there exist three positive constants* $\underline{\delta}$*,* $\eta_1>0$ *and* $0<\eta_2<1$ *such that* $F'(m)\ge\underline{\delta}\min(m^{\eta_1},m^{-\eta_2})$.

We assume that there is unique classical solution (u,m,λ) *of (4)–(5) such that* $m>0$ *and* $\int_{\mathbb{T}^2}u(x)dx=0$.

Consider a numerical Hamiltonian given by (10)–(11). Let u_h *(resp.* m_h*) be the piecewise bilinear function in* $\mathscr{C}(\mathbb{T}^2)$ *obtained by interpolating the values* $u_{i,j}$ *(resp* $m_{i,j}$*) at the nodes of* $\mathbb{T}^2_h$*, where* $\big((u_{i,j}),(m_{i,j}),\lambda_h\big)$ *is the unique solution of the following system:*

for all $0\le i,j<N_h$, $m_{i,j}\ge 0$,

$$\begin{cases} -\nu(\Delta_h u)_{i,j} + g(x_{i,j}, [D_h u]_{i,j}) + \lambda_h = F(m_{i,j}), \\ -\nu(\Delta_h m)_{i,j} - \mathscr{T}_{i,j}(u, m) \qquad\qquad\qquad = 0, \\ h^2 \sum_{i,j} u_{i,j} = 0, \quad h^2 \sum_{i,j} m_{i,j} \qquad\quad = 1. \end{cases} \tag{49}$$

As h tends to 0, the functions u_h converge in $W^{1,\beta}(\mathbb{T}^2)$ to u, the functions m_h converge to m in $L^{2-\eta_2}(\mathbb{T}^2)$, and λ_h tends to λ.

4 Algorithms for Solving the Discrete Linear Systems

The algorithms described below were originally proposed in [4], among other methods.

4.1 Newton Methods for Solving (21)–(22)

In this paragraph, we assume that $p \mapsto H(x, p)$ and $q \mapsto g(x, q)$ are $\mathscr{C}^2$ regular. This will allow us to use Newton like algorithms for (21)–(22). We also assume that assumptions (g_1)–(g_3) hold and that Φ is a local operator, i.e. $\Phi[m](x) = F(m(x))$, where F is a $\mathscr{C}^1$ and strictly increasing function.

System (21)–(22) can be seen as a forward discrete Bellman equation for u with a Cauchy condition at $t = 0$ coupled with a backward discrete Fokker–Planck equation for m with a Cauchy condition at final time. This structure prohibits the use of a straightforward time-marching solution procedure.

Call $\mathscr{U}$ and $\mathscr{M}$ the vectors of $\mathbb{R}^{N^T N^2}$ such that $\mathscr{U}_{kN^2+iN+j} = u^k_{i,j}$ and $\mathscr{M}_{kN^2+iN+j} = m^{k-1}_{i,j}$. (recall that u_0 and m^{N_T} are given). The system of nonlinear equations can be written

$$\mathscr{F}_U(\mathscr{U}, \mathscr{M}) = 0, \quad \text{and} \quad \mathscr{F}_M(\mathscr{U}, \mathscr{M}) = 0, \tag{50}$$

with

- $\mathscr{F}_U(\mathscr{U}, \mathscr{M}) = 0 \Leftrightarrow$ (18) $\quad \forall n, 0 \le n < N_T, \forall i, j.$
- $\mathscr{F}_M(\mathscr{U}, \mathscr{M}) = 0 \Leftrightarrow$ (20) $\quad \forall n, 0 \le n < N_T, \forall i, j.$

In order to discuss the Newton method for solving (50), we use the following notation

$$\begin{aligned} A_{U,U}(\mathscr{U}, \mathscr{M}) &= D_{\mathscr{U}} \mathscr{F}_{\mathscr{U}}(\mathscr{U}, \mathscr{M}), \; A_{U,M}(\mathscr{U}, \mathscr{M}) = D_{\mathscr{M}} \mathscr{F}_{\mathscr{U}}(\mathscr{U}, \mathscr{M}), \\ A_{M,U}(\mathscr{U}, \mathscr{M}) &= D_{\mathscr{U}} \mathscr{F}_{\mathscr{M}}(\mathscr{U}, \mathscr{M}), \; A_{M,M}(\mathscr{U}, \mathscr{M}) = D_{\mathscr{M}} \mathscr{F}_{\mathscr{M}}(\mathscr{U}, \mathscr{M}). \end{aligned} \tag{51}$$

The matrices $A_{UU}(\mathscr{U}, \mathscr{M})$ and $A_{UM}(\mathscr{U}, \mathscr{M})$ have the form

$$A_{UU} = \begin{pmatrix} D_1 & 0 & \dots & \dots & 0 \\ -\frac{1}{\Delta}tI & D_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\frac{1}{\Delta}tI & D_{N_T} \end{pmatrix}, A_{UM} = \begin{pmatrix} E_1 & 0 & \dots & \dots & 0 \\ 0 & E_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & & 0 & E_{N_T} \end{pmatrix}. \quad (52)$$

The blocks of $A_{UU}(\mathscr{U}, \mathscr{M})$ are sparse. The block D_n corresponds to the discrete operator $(z_{i,j}) \mapsto \left(\frac{1}{\Delta t} z_{i,j} - \nu(\Delta_h z)_{i,j} + [D_h z]_{i,j} \cdot g_q(x_{i,j}, [D_h u^n]_{i,j})\right)$ coming from the linearization of the discrete Bellman equation. From the assumptions (g_1) and (g_3), D_n is a M-matrix, thus A_{UU} is invertible.

The blocks of $A_{UM}(\mathscr{U}, \mathscr{M})$ are diagonal matrices, (note that they would be dense matrices if Φ was a nonlocal operator). We have assumed that $F' > 0$ so the diagonal entries of $A_{UM}(\mathscr{U}, \mathscr{M})$ are negative.

From Remark 2.1, the matrices $A_{MM}(\mathscr{U}, \mathscr{M})$ and $A_{MU}(\mathscr{U}, \mathscr{M})$ have the form

$$A_{MM} = A_{UU}^T, \qquad A_{MU} = \begin{pmatrix} \tilde{E}_1 & 0 & \dots & \dots & 0 \\ 0 & \tilde{E}_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \tilde{E}_{N_T-1} & 0 \\ 0 & \dots\dots & & 0 & \tilde{E}_{N_T} \end{pmatrix}. \quad (53)$$

The block A_{MM} corresponds to a discrete linear transport equation. Note that

$$\mathscr{V}^T \tilde{E}_n \mathscr{W} = \sum_{i,j} m_{i,j}^{n-1} [D_h v]_{i,j} \cdot g_{q,q}(x_{i,j}, [D_h u^n]_{i,j}) [D_h w]_{i,j}.$$

From the convexity of g, we see that the block $\tilde{E}_n$ is symmetric and positive semi-definite if m^{n-1} is a nonnegative grid function.

In [2], it is proved that under Assumptions (g_1)–(g_4) and if F is strictly increasing, and if the iterate produced by the Newton method satisfies $\mathscr{M} \geq 0$, then the Jacobian matrix

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix}$$

is invertible. The proof is similar to that used for the uniqueness of the solution of (21)–(22). The positivity of $\mathscr{M}$ is not guaranteed though, but if the initial guess is close enough to a solution $(\hat{\mathscr{U}}, \hat{\mathscr{M}})$ with $\hat{\mathscr{M}} > 0$, then the iterates $\mathscr{M}$ will stay positive.

Assuming the invertibility of the matrix, the most time consuming part of the procedure lies in solving the system of linear equations

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \mathscr{U} \\ \mathscr{M} \end{pmatrix} = \begin{pmatrix} G_U \\ G_M \end{pmatrix}. \tag{54}$$

As explained above, the Newton method described above may break down if in the Newton loop, the approximation of m_h takes negative values. A similar phenomenon was observed by Benamou et al. [10, 11] when they studied a somewhat similar but simpler penalty method (using conjugate gradient iterations instead of Newton) for computing a mixed L^2-Wasserstein distance between two probability densities. This is of course a drawback of the method. However, breakdown does not happen if the initial guess is close enough to a solution. Therefore, it is important to find good initial guesses for the Newton method.

A possible way of avoiding breakdowns is to start solving (21)–(22)with a rather high value of the parameter ν (of the order of 1), then gradually decrease ν down to the desired value, the solution of (13) found by the Newton method for a given value of ν being used as an initial guess for the next and smaller value of ν. Doing so in our tests, we have avoided breakdowns of the Newton method. For values of ν between 1 and 0.1, the number of iterations of the Newton method to achieve that the ℓ_2 norm of the residual be smaller than 10^{-5} was found to be less than 10 and to increase as ν decreases.

In the sequel, we propose possible iterative strategies for solving (54), which are based on eliminating $\mathscr{U}$ from the Bellman equation. Other iterative strategies based on eliminating $\mathscr{M}$ from the Bellman equation can be designed, see [4], but we will not present them here, since they are more involved and require efficient multigrid preconditioners.

4.2 *Iterative Strategies for Solving (54) Based on Eliminating $\mathscr{U}$*

4.2.1 A Basic Iterative Method

The principle of the method is as follows:

1. First solve

$$A_{U,U}\tilde{\mathscr{U}} = G_U. \tag{55}$$

This is done by sequentially solving

$$D_1\tilde{U}^1 = G_U^1, \tag{56}$$

then

$$D_k\tilde{U}^k = \frac{1}{\Delta t}\tilde{U}^{k-1} + G_U^k, \quad \text{for } k > 1, \tag{57}$$

i.e. marching in time in the forward direction. We know that (56) and (57) have a unique solution if (g_1) and (g_3) hold.

2. Introducing $\overline{\mathscr{U}} = \mathscr{U} - \tilde{\mathscr{U}}$, the vector $(\overline{\mathscr{U}}, \mathscr{M})^T$ satisfies

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \overline{\mathscr{U}} \\ \mathscr{M} \end{pmatrix} = \begin{pmatrix} 0 \\ G_M - A_{M,U}\tilde{\mathscr{U}} \end{pmatrix}, \tag{58}$$

   which implies

$$\left(A_{M,M} - A_{M,U} A_{U,U}^{-1} A_{U,M}\right) \mathscr{M} = G_M - A_{M,U}\tilde{\mathscr{U}}. \tag{59}$$

3. Once (59) is solved, $\overline{\mathscr{U}}$ is obtained by solving the discrete forward linearized Bellman equation

$$A_{U,U}\overline{\mathscr{U}} = -A_{U,M}\mathscr{M} \tag{60}$$

   by the same method as for (56), (55).

The system (59) is solved by means of an iterative method, for example, the BiCGstab algorithm [31] in all what follows; it only requires an implementation of the matrix-vector product with the matrix $A_{M,M} - A_{M,U} A_{U,U}^{-1} A_{U,M}$. Of course, this matrix is not assembled: the matrix-vector product involves matrix-vector products with the matrices $A_{M,M}$, $A_{M,U}$ and $A_{U,M}$ and solving a linear system of the form (55), similar to that appearing in the first step.

Numerical tests not reported here show that, with the previously described iterative method, the number of iterations to reduce the error by a fixed factor increases as the size of the mesh grows; this can also be foreseen by using arguments similar to those in Sect. 4.2.2 below: hence it is desirable to modify this basic method by using a suitable preconditioner.

4.2.2 Preconditioned Iterative Methods

We propose to use A_{MM} as a preconditioner for (59): it amounts to applying an iterative algorithm (i.e. the BiCGstab algorithm) to

$$\left(I - A_{M,M}^{-1} A_{M,U} A_{U,U}^{-1} A_{U,M}\right) \mathscr{M} = A_{M,M}^{-1}(G_M - A_{M,U}\tilde{\mathscr{U}}) \tag{61}$$

rather than to (59).

A Heuristic Interpretation in Terms of Partial Differential Operators

A heuristic explanation for this preconditioner choice is as follows: calling v and n two functions on $\mathbb{T}^2$,

- A_{UU} is the matrix counterpart of the linearized Bellman operator (advection-diffusion operator):

$$v \mapsto \text{Lin-HJB}(v) := \frac{\partial v}{\partial t} - \nu \Delta v + \frac{\partial H}{\partial p}(x, \nabla u) \cdot \nabla v$$

- $A_{U,M}$ is the matrix counterpart of the operator: $n \mapsto -F'(m)n$
- A_{MM} is the matrix counterpart of the Fokker–Planck operator (transport-diffusion operator):
$$n \mapsto \ \mathrm{FP}(n) := -\frac{\partial n}{\partial t} - \nu \Delta n - \mathrm{div}(n\frac{\partial H}{\partial p}(x, \nabla u))$$
- $A_{M,U}$ is the matrix counterpart of the operator: $v \mapsto -\mathrm{div}\left(mH_{pp}(x,\nabla u)\nabla v\right)$, where $H_{pp}(x,q)$ stands for the Hessian of $p \mapsto H(x,p)$ at $p = q$

Let us define $\text{Lin-HJB}^{-1}(w)$ as the unique solution v of the Cauchy problem involving the linearized Bellman equation:

$$\begin{aligned} \frac{\partial v}{\partial t} - \nu \Delta v + \frac{\partial H}{\partial p}(x, \nabla u) \cdot \nabla v = w \quad & \text{in } (0,T] \times \mathbb{T}^2, \\ v|_{t=0} = 0 \quad & \text{in } \mathbb{T}^2, \end{aligned} \tag{62}$$

and $\mathrm{FP}^{-1}(r)$ as the unique solution n of the backward Cauchy problem involving the Fokker–Planck equation:

$$\begin{aligned} \frac{\partial n}{\partial t} + \nu \Delta n + \mathrm{div}(n\frac{\partial H}{\partial p}(x, \nabla u)) = -r \quad & \text{in } (0,T] \times \mathbb{T}^2, \\ n|_{t=T} = 0 \quad & \text{in } \mathbb{T}^2, \end{aligned} \tag{63}$$

The matrix $-A_{M,M}^{-1} A_{M,U} A_{U,U}^{-1} A_{U,M}$ is the counterpart of the nonlocal operator:

$$n \mapsto \Big(\ \mathrm{FP}^{-1} \circ \big(v \mapsto -\mathrm{div}\left(mH_{pp}(x,\nabla u)\nabla v\right)\big) \circ \ \text{Lin-HJB}^{-1}\Big)\big(F'(m)n\big).$$

Now, assuming that u and m belong to $\mathscr{C}^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{T}^2)$, it can be shown that the latter operator maps continuously $\mathscr{C}^{\alpha/2,\alpha}([0,T]\times\mathbb{T}^2)$ to $\mathscr{C}^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{T}^2)$, so it is a compact operator on $\mathscr{C}^{\alpha/2,\alpha}([0,T]\times\mathbb{T}^2)$. Compactness in $L^2((0,T)\times\mathbb{T}^2)$ is also true. Hence $I - A_{M,M}^{-1} A_{M,U} A_{U,U}^{-1} A_{U,M}$ is the discrete version of the perturbation of the identity by a compact operator. Therefore, the convergence of the BiCGstab algorithm should not depend on the size the grid. This will be confirmed by the numerical experiments below.

The PDE interpretation of the preconditioner also leads us to predict that the number of iterations needed by the iterative solver should increase as ν decreases to zero, which will indeed appear clearly in the tests.

Algorithm

The matrix $A_{M,M}^{-1} A_{M,U} A_{U,U}^{-1} A_{U,M}$ is not assembled. The proposed method only requires an implementation of the matrix-vector product with the matrix $A_{M,M} - A_{M,U} A_{U,U}^{-1} A_{U,M}$ as discussed above (it does not need the matrix $A_{U,U}^{-1}$), and solving systems of linear equations of the form

$$A_{M,M}\tilde{\mathscr{M}} = G_M. \tag{64}$$

This is done by sequentially solving

$$D_{N_T}^T \tilde{M}^{N_T-1} = G_M^{N_T}, \tag{65}$$

then

$$D_k^T \tilde{M}^{k-1} = -\frac{1}{\Delta} t \tilde{M}^k + G_M^k, \quad \text{for } 1 \le k < N_T, \tag{66}$$

i.e. marching in time in the backward direction. It has already been seen that the blocks D_k are invertible, and so are the blocks D_k^T.

Note that an iteration of the preconditioned BiCGstab method involves two solves of systems of the type (55) and two solves of systems of the type (64).

Solving the systems (56), (57), (65), (66) (two-dimensional problems) can be done with fast direct solvers: in our implementation, we have used the open source library UMFPACK [30] which contains an Unsymmetric MultiFrontal method for solving linear systems.

Numerical Tests

We consider the following case:

$$T = 1, \tag{67}$$

$$H(x, p) = \sin(2\pi x_1) + \sin(2\pi x_2) + \cos(4\pi x_1) + |p|^3, \tag{68}$$

$$\Phi(m) = m, \tag{69}$$

$$u_0(x) = 0, \tag{70}$$

$$m_T(x) = 1, \tag{71}$$

and g corresponds to a classical Godunov scheme (10)–(11).

For what follows it interesting to plot the contours of m at time $t = T/2 = 0.5$: it was observed in [2] that up to the addition of a constant to u, the solution at $t = T/2$ is close to the solution of the ergodic problem (4)–(6); on Fig. 1, we display the contours of m for $\nu = 0.6$ and $\nu = 0.08$. Note that for $\nu = 0.08$, i.e. rather close to the deterministic case $\nu = 0$, $m(x)$ is small (smaller than 0.01) in a large region.

In Table 1, we show the number of iterations needed to decrease the residual norm by a factor 10^{-3} or 10^{-7} with the preconditioned BiCGstab method. In our tests, choosing an error reduction of 10^{-3} instead of 10^{-7} had no effect on the convergence of the inexact Newton method.

We see that, as expected, the number of BiCGstab iterations is small and does not depend on the size of the grid, and that it increases as ν decreases.

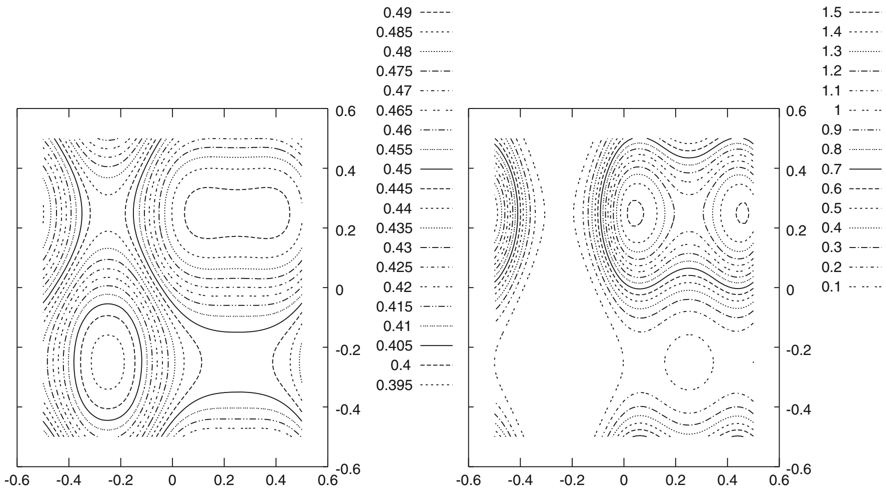

Fig. 1 Contours of m for $\nu = 0.6$ (*left*) and $\nu = 0.08$ (*right*) at time $t = 0.5$

Table 1 Average (on the Newton loop) number of iterations of BiCGstab to decrease the residual by a factor 10^{-3} or 10^{-7}

Grid	$32 \times 32 \times 32$		$64 \times 64 \times 64$		$128 \times 128 \times 64$	
Rel. accur.	10^{-3}	10^{-7}	10^{-3}	10^{-7}	10^{-3}	10^{-7}
$\nu = 0.6$	1	2	1	2	1	2
$\nu = 0.36$	1.75	2	1.75	2	1.8	2
$\nu = 0.2$	2	3.5	2	3.5	2	4
$\nu = 0.12$	3	6	3	6	3	6.1
$\nu = 0.046$	4.9	10	5.1	10	5.1	10

Table 2 Average computing time for solving the linearized problem

$\nu \setminus$ grid	$32 \times 32 \times 32$	$64 \times 64 \times 64$	$128 \times 128 \times 128$
0.6	2.06	19.9	234.7
0.12	5.02	50.03	577.25

Table 2 contains the average computing times for solving the linear systems of the form (54) for different grids, for $\nu = 0.6$ and $\nu = 0.12$, on a Dell server with Six-core 2.93 GHz Intel(R) Xeon(R) X5670 processors. It can be seen that the computing times are not far from scaling linearly with the total number of unknowns. In order to achieve almost optimal complexity, it is possible to use multigrid methods for solving (57) and (66), see [4].

Remark 4.1. It is possible to use this family of algorithms when the cost operator Φ is non local, at least if (Φ_{h2}) holds.

Remark 4.2. It is possible to use this family of algorithms when the monotonicity assumption on F is not fulfilled, for example $F(m) = -\log(m)$.

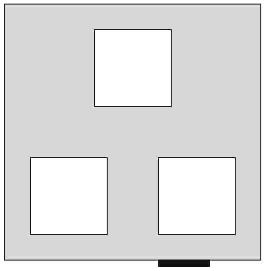

Fig. 2 The geometry of the problem

5 Some Simulations

We use the mean field games theory to model a situation where a population is driven to leave a given closed room with obstacles: one can imagine for example a situation of panic in a closed building, in which the population tries to reach the exit door. The chosen geometry is represented on Fig. 2. The domain Ω is the unit square $(0,1)^2$ perforated with three square holes whose side is 0.3. The exit door is taken to be the line segment $\Gamma_D = [0.6, 0.8] \times \{0\}$. Let Γ_N be given by $\Gamma_N = \partial\Omega \backslash \Gamma_D$.

Realistic models take into account congestion, i.e. the fact that it is more difficult for an individual to move if the density is locally high; this translates into the fact that the running cost density is of the more complex form $L(X_s, m(s, X_s), \gamma_s) + \Phi(m(s,(X_s))$ and that the Hamiltonian becomes

$$H(x, m(x), \nabla u) = \sup_\gamma \left[\gamma \cdot \nabla u - L(x, m(x), \gamma)\right].$$

The agents minimize the expectation of

$$c + \int_0^{t^*} \left(L(X_s, m(s, X_s), \gamma_s) + \Phi(m(s,(X_s))\right) ds$$

where t^* is the first time when the exit door Γ_D is reached. Here c is the exit cost.

The MFG system of PDEs becomes

$$\frac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + H(x, m(t,x), \nabla u(t,x)) = \Phi(m(t,x),$$
$$\text{in } (0,T)\times\Omega,$$
$$\frac{\partial m}{\partial t}(t,x) + \nu\Delta m(t,x) + \operatorname{div}\left(m(t,\cdot)\frac{\partial H}{\partial p}(\cdot, m(\cdot), \nabla u(t,\cdot))\right)(x) = 0,$$
$$\text{in } (0,T)\times\Omega,$$

We choose H as follows:

$$H(x, m, p) = \mathscr{H}(x) + \frac{|p|^\beta}{(c_0 + c_1 m)^\gamma}. \tag{72}$$

with $c_0 > 0$, $c_1 \geq 0$, $\beta > 1$ and $0 \leq \gamma < 4(\beta - 1)/\beta$. For such coefficients, existence and uniqueness was proven by Lions [29]. The function $\mathscr{H}(x)$ models the panic in the room and we have chosen $\mathscr{H}(x) = -k$, where $k \geq 0$ is called the panic coefficient.

The boundary Γ_N corresponds to the walls of the room, so the natural boundary condition on Γ_N is a homogeneous Neumann boundary condition on u: $\frac{\partial u}{\partial n} = 0$ which says that the velocity of the agents is tangential to the walls. The same condition holds for m, namely $\frac{\partial m}{\partial n} = 0$, which says that nobody escapes or enters the room through Γ_N. To summarize, the boundary conditions on Γ_N are

$$\frac{\partial u}{\partial n}(t, x) = \frac{\partial m}{\partial n}(t, x) = 0, \qquad \text{on } \Gamma_N. \tag{73}$$

For the numerical scheme, we add a layer of virtual nodes outside Ω and we apply a first order scheme at the nodes on Γ_N to discretize the Neumann condition: this implies that the values of u (resp. m) at the virtual nodes is the value of u (resp. m) at their neighbor nodes in Γ_N, and we use these values to apply the scheme (21) at the nodes on Γ_N.

The boundary conditions at the exit door are chosen as follows: there is a Dirichlet condition for u at the door: $u = c$ where c is a small enough number; in our simulations, we have chosen $c = 0$. For m, we may assume that $m = 0$ outside the domain, so we also get a Dirichlet condition for m at Γ_D. Hence

$$u(t, x) = m(t, x) = 0, \qquad \text{on } \Gamma_D. \tag{74}$$

In the numerical method, we add a layer of nodes outside Ω and we apply the scheme (21) at the boundary nodes on Γ_D having fixed the value of u and m to zero on this additional layer of node outside Ω.

Note that it is also possible and arguably more realistic to replace the Dirichlet condition on m by a Robin condition.

In the simulation, we have chosen

$$\begin{aligned}
&\nu = 0.0375, \quad T = 6, \\
&\Phi(m) = m, \quad H(x, m, p) = -0.1 + \frac{|p|^2}{(1 + 4m)^{3/2}}, \\
&m_T(x, y) = 4(1_{\{|x-1/4|<1/10\}} + 1_{\{|x-3/4|<1/10\}})1_{\{y>4/5\}}, \\
&u_0 = 0.
\end{aligned}$$

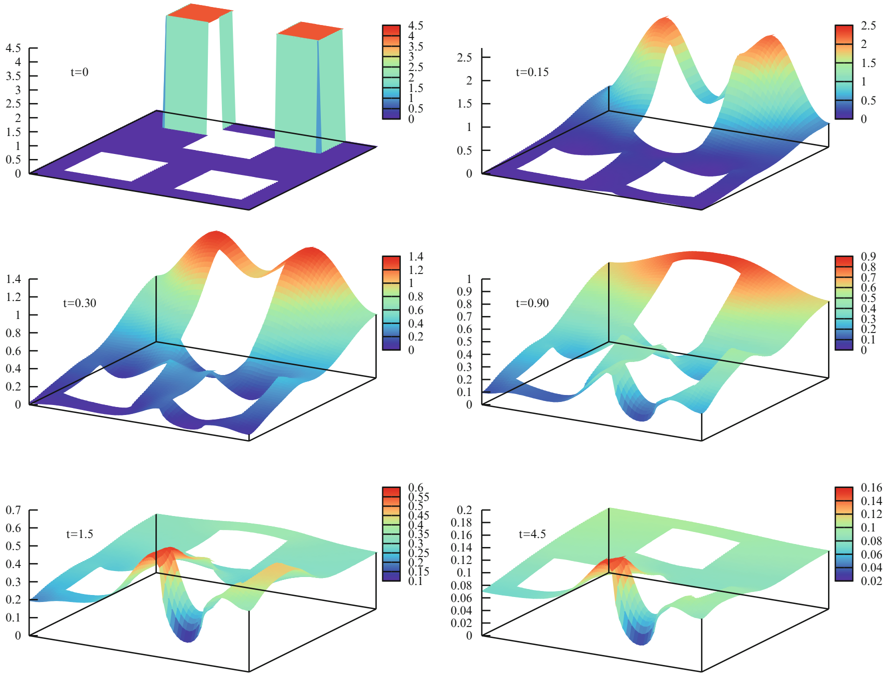

Fig. 3 The density at different times: the scales are adapted and differ from one time another

The grid parameters are $h = 1/64$ and $\Delta t = 0.015$.

In Figs. 3 and 4, we have plotted the graph of the densities at different physical times. We can give the following interpretation of these plots: the population is initially confined in small regions near the top part of the domain, on the left and right sides of the top obstacle. We see that the population tends first to occupy as much space as possible instead of aiming directly at the exit door: for example, at $t = 0.9$ the maximum of the density is behind the top obstacle, i.e. far from the exit door; this is caused by the cost term $\Phi(m(x))$, which models the fact that people do not like to be confined in regions of high density. As a result, in a first phase, the population gets distributed close to symmetrically with respect to the axis $x = 0.5$. We also see that it takes a rather long time for the population to leave the top part of the domain: this is caused by the congestion factor: the agents move slowlier if the density is high. Later, people take the direction of the exit door; most of the population goes round the right obstacle, because the exit time is smaller on these trajectories: the densities on the right and middle corridors are of the same orders, and higher than the density in the left corridor. Finally, there is a higher density of people in the right side of the middle corridor which is the locus of the shortest path to the exit. In Fig. 5, we have plotted the velocities given by $v = -\frac{\partial H}{\partial q}(x, m, \nabla u)$ at time $t = 1.5$.

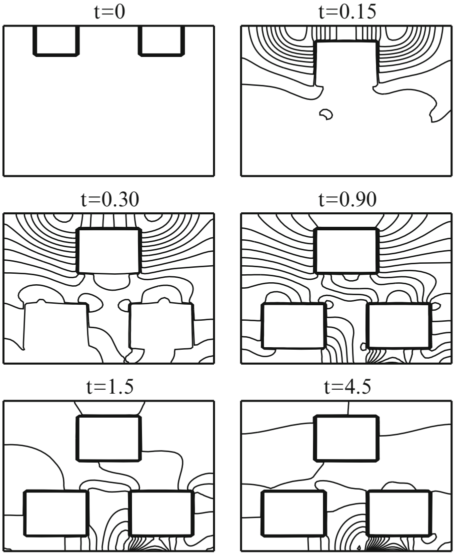

Fig. 4 The density at different times: *contour lines*

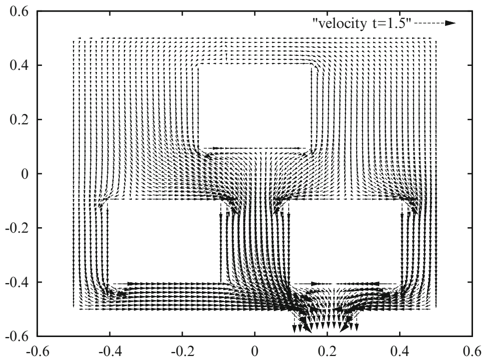

Fig. 5 The velocity $v = -H_p(x, m, \nabla u)$ at time $t = 1.5$

Finally, we keep all the parameters unchanged in the simulation, except the Hamiltonian, which becomes

$$H(x, m, p) = -0.1 + \frac{|p|^2}{(1 + 8m)^{1.8}}. \tag{75}$$

The evolution of the density is plotted on Fig. 6: congestion is stronger, which leads to a slower exit and a more symmetric distribution of the agents.

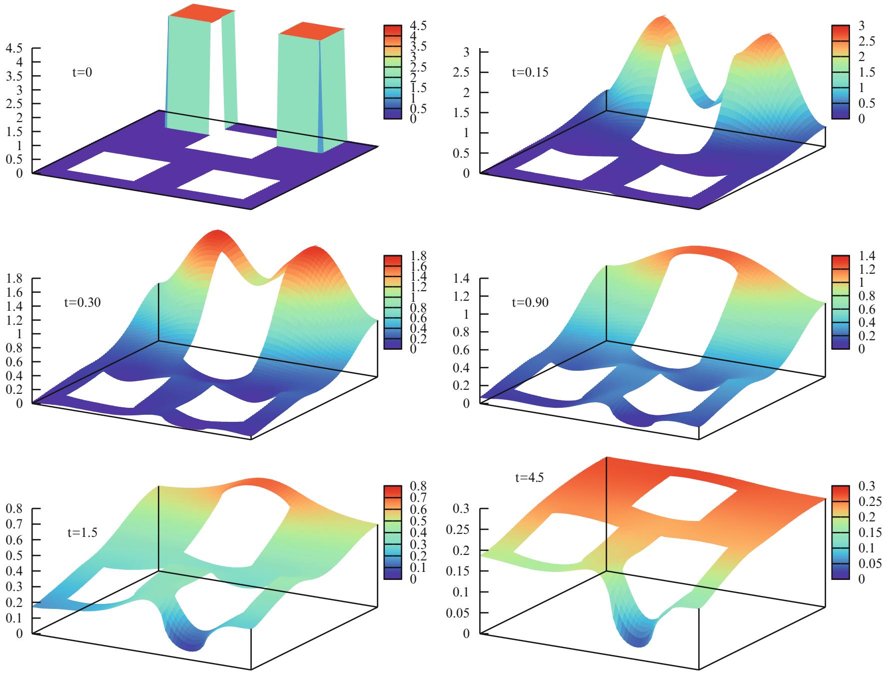

Fig. 6 Everything is kept unchanged but the Hamiltonian, which is given by (75): the density at different times: the scales are adapted and differ from one time another

6 The Planning Problem

6.1 Description of the Planning Problem

In the planning problem, the system of PDEs (1)–(2) is kept unchanged, but the final and terminal conditions become

$$m(0,x) = m_0(x), \quad m(T,x) = m_T(x), \quad \text{in } \mathbb{T}^2. \tag{76}$$

The difference with (3) is that the initial condition lies on m and no longer on u. If the Hamiltonian is of the form $H(x, \nabla u) = \sup_\gamma \left[\gamma \cdot \nabla u - L(x,\gamma)\right]$, conditions (76) represent the requirement that the positions of a very large number of identical rational agents whose dynamics is given by $dX_s = -\frac{\partial H}{\partial p}(X_s, \nabla u(s, X_s))\, ds + \sqrt{2\nu}\, dW_s$ and running cost density is given by $L(X_s, \gamma_s) + \Phi[m_s](X_s)$, evolve from a given spatial density m_T at $s = 0 \Leftrightarrow t = T$ to a desired target density m_0 at $s = T \Leftrightarrow t = 0$.

Whereas existence (and uniqueness) results for (1)–(3) are available under fairly general assumptions, see [26, 27], much less is known concerning (1), (2), (76).

Indeed, as far as we know, P-L. Lions has proved existence for (1), (2), (76) in mainly two cases:

1. $\nu = 0$ (deterministic case), H is a smooth and strictly convex Hamiltonian such that $\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = +\infty$, $\Phi[m](x) = F(m(x))$ where F is a smooth and strictly increasing function, m_0 and m_T are smooth functions bounded away from 0.
2. $\nu > 0$, $H(p) = c|p|^2$ or $H(p)$ is close to $c|p|^2$, $\Phi[m](x) = F(m(x))$ where F is a smooth, bounded and nondecreasing function, m_0 and m_T are smooth functions bounded away from 0, but existence is still an open question when $\nu > 0$ and the Hamiltonian is more general. P-L. Lions has also proved that if H is sublinear with respect to p and if $m_0 \neq m_T$, then there are no solutions if T is small enough. Therefore, existence may only result from combined nonlinear effects. It is also worth to observe that the planning problem described above can be seen as a generalization of the simpler system, (with in particular $F = 0$, $\nu = 0$),

$$\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = 0\,, \qquad \frac{\partial m}{\partial t} + \operatorname{div}(m\nabla u) = 0\,, \tag{77}$$

$$m(0,x) = m_0(x), \quad m(T,x) = m_T(x) \tag{78}$$

which was introduced by Benamou and Brenier [9], see also [32], as a fluid mechanics formulation of the Monge–Kantorovich mass transfer problem. In [9], a numerical method for the solution of (77), (78) is proposed on the basis of a reformulation of the problem as the system of optimality conditions for a suitably constructed primal-dual pair of convex optimal control problems for the transport equation

$$\frac{\partial m}{\partial t} + \operatorname{div}(m\,\gamma) = 0\,,$$

the velocity field $\gamma(x,t)$ playing here the role of a distributed control. Similarly, the mean field games models can also be reformulated as an optimal control problems for a density driven by a Fokker–Planck equation, see [26, 27].

In what follows, we are going to give an existence result for the discrete version of (1), (2), (76) in the particular case when Φ is a local operator; the main idea is to use the optimal control formulation of the discrete schemes, following ideas in [9, 27, 32].

6.2 *The Finite Difference Scheme and an Optimal Control Formulation*

The arguments below were originally published in [2].

The finite difference scheme for the planning problem is obviously given by (21) with the initial and terminal conditions: for $0 \le i, j < N_h$,

$$m_{i,j}^{N_T} = \frac{1}{h^2} \int_{|x-x_{i,j}|_\infty \le h/2} m_T(x)dx, \qquad m_{i,j}^{0} = \frac{1}{h^2} \int_{|x-x_{i,j}|_\infty \le h/2} m_0(x)dx. \quad (79)$$

We assume that Φ is a local operator, i.e. $\Phi[m](x) = F(m(x))$, and that $F = W'$ where $W : \mathbb{R} \to \mathbb{R}$ is a strictly convex, coercive $\mathscr{C}^2$ function. It follows that the image of the interval $(0, +\infty)$ by F is some interval $\mathscr{J}_F = (\underline{F}, +\infty)$.

We also assume that (g_1)–(g_4) hold and that the numerical Hamiltonian has the further coercivity property

($\mathbf{g_5}$) *Coercivity*:

$$\lim_{q_1 \to -\infty} \frac{g(x,q_1,q_2,q_3,q_4)}{|q_1|} = +\infty \text{ uniformly w.r.t. } x, q_2, q_3, q_4,$$
$$\lim_{q_2 \to +\infty} \frac{g(x,q_1,q_2,q_3,q_4)}{q_2} = +\infty \text{ uniformly w.r.t. } x, q_1, q_3, q_4,$$
$$\lim_{q_3 \to -\infty} \frac{g(x,q_1,q_2,q_3,q_4)}{|q_3|} = +\infty \text{ uniformly w.r.t. } x, q_1, q_2, q_4,$$
$$\lim_{q_4 \to +\infty} \frac{g(x,q_1,q_2,q_3,q_4)}{|q_4|} = +\infty \text{ uniformly w.r.t. } x, q_1, q_2, q_3.$$

This coercivity property implies that

$$\lim_{\| [D_h U] \|_\infty \to \infty} \frac{\max_{i,j} g(x_{i,j}, [D_h U]_{i,j})}{\| [D_h U] \|_\infty} = +\infty. \quad (80)$$

We are going to introduce an optimal control problem whose optimality conditions are interpreted as the semi-implicit scheme (21), (79). In this way, using a convex duality argument based on the Fenchel–Rockafellar Theorem, we are going to prove the existence of a solution of (21), (79), see Theorem 6.1 below.

If χ denotes the indicator function of the set $\{m \ge 0\}$, the Legendre–Fenchel transform of $W + \chi$ is defined by

$$\big(W + \chi\big)^*(\alpha) = \sup_m \big[\alpha m - W(m) - \chi(m)\big].$$

It is clear that $\big(W + \chi\big)^*$ is convex, continuous and non decreasing. If $\alpha \in \mathscr{J}_F$ then $\big(W + \chi\big)^*(\alpha) = \alpha F^{-1}(\alpha) - W(F^{-1}(\alpha))$. If $\alpha \notin \mathscr{J}_F$ then $\big(W + \chi\big)^*(\alpha) = -W(0)$.

Consider now the convex functional Θ^* on $\mathbb{R}^{N_T \times N_h^2} \times \mathbb{R}^{4N_T \times N_h^2}$:

$$\Theta^*(\alpha, \beta) = \sum_{n=1}^{N_T} \sum_{i,j} (W + \chi)^* \big(\alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j})\big) .$$

where $\alpha = (\alpha_{i,j}^n)$, $\beta = ([\beta^n]_{i,j})$ and $[\beta^n]_{i,j} = (\beta_{i,j}^{1,n}, \beta_{i,j}^{2,n}, \beta_{i,j}^{3,n}, \beta_{i,j}^{4,n})$, $1 \le n \le N_T$, $1 \le i, j \le N_h$. The Legendre–Fenchel transform of Θ^* is defined by

$$\Theta(m,z) = \sup_{\alpha,\beta}\left(\sum_{n=1}^{N_T}\sum_{i,j} m_{i,j}^{n-1}\alpha_{i,j}^{n} + \langle [z^{n-1}]_{i,j}, [\beta^n]_{i,j}\rangle \right.$$
$$\left. -(W+\chi)^*\big(\alpha_{i,j}^{n} + g(x_{i,j},[\beta^n]_{i,j})\big)\right) \tag{81}$$

where $m = (m_{i,j}^n)$, $z = ([z^n]_{i,j})$ and $[z^n]_{i,j} = (z_{i,j}^{1,n}, z_{i,j}^{2,n}, z_{i,j}^{3,n}, z_{i,j}^{4,n})$, $0 \le n < N_T$, $1 \le i,j \le N_h$ and $\langle [z],[\beta]\rangle = \sum_{k=1}^4 \beta^k z^k$.

Remark 6.1. Note that in our definition, for $n = 1,\dots,N_T$, the dual variable of $\alpha_{i,j}^n$ is $m_{i,j}^{n-1}$, and the dual variable of $[\beta^n]_{i,j}$ is $[z^{n-1}]_{i,j}$. This lag in the time index n will prove convenient for our purpose.

Let us introduce the minimization problem

$$\left\{\begin{array}{r}\text{Minimize } \Theta(m,z) \text{ subject to the constraint}\\[2ex] \dfrac{m_{i,j}^n - m_{i,j}^{n-1}}{\Delta t} + \nu(\Delta_h m^{n-1})_{i,j} + \mathrm{div_h}(z^{n-1})_{i,j} = 0, \quad 1 \le n \le N_T,\\ m_{i,j}^{N_T} = (m_T)_{i,j},\\ m_{i,j}^0 = (m_0)_{i,j},\end{array}\right. \tag{82}$$

where $(m_T)_{i,j}$ and $(m_0)_{i,j}$ are the right hand sides in (79) and

$$\mathrm{div_h}(z^{n-1})_{i,j} = (D_1^+ z^{1,n-1})_{i-1,j} + (D_1^+ z^{2,n-1})_{i,j} + (D_2^+ z^{3,n-1})_{i,j-1} + (D_2^+ z^{4,n-1})_{i,j}$$

The above minimization problem is an optimal control problem for a discrete density driven by a discrete Fokker–Planck equation. The data $(m_0)_{i,j}, (m_T)_{i,j} \in \mathcal{K}_h$ are discrete probability densities.

We are going to prove next that if the initial datum satisfies $(m_0)_{i,j} > 0$ for all i,j, then the optimal control problem above has at least a solution (m,z), that there exists a solution (α,β) of the dual problem and that the optimality conditions at the saddle point coincide with the discrete scheme (21), (79). The argument is based on convex duality and the Fenchel–Rockafellar theorem.

Let us introduce for this purpose the functionals $\mathcal{L}, \Lambda, \Sigma^*$ by setting

$$\mathcal{L}(\psi) = \frac{1}{\Delta t}\left(\sum_{i,j}(m_0)_{i,j}\psi_{i,j}^0 - \sum_{i,j} m_{T,i,j}\psi_{i,j}^{N_T}\right) \tag{83}$$

$$(\alpha,\beta) = \Lambda(\psi) \quad \Leftrightarrow \quad \left\{\begin{array}{rl} \alpha_{i,j}^{n+1} &= \dfrac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{\Delta t} - \nu(\Delta_h\psi^{n+1})_{i,j},\\ {[\beta^{n+1}]_{i,j}} &= [D_h\psi^{n+1}]_{i,j}, \quad 0 \le n < N_T,\end{array}\right. \tag{84}$$

and, finally,

$$\Sigma^*(\alpha,\beta) = \begin{cases} \mathscr{L}(\psi) \text{ if } \exists\psi \quad \text{s.t.} \quad (\alpha,\beta) = \Lambda(\psi) \text{ and } \sum_{i,j} \psi^0_{i,j} = 0, \\ +\infty \quad \text{otherwise.} \end{cases} \tag{85}$$

Lemma 6.1. *The functional Θ^* is convex and continuous. The functional Σ^* is convex and lower semicontinuous. Moreover, the following constraints qualification property holds: there exists (α,β) such that $\Sigma^*(\alpha,\beta) < +\infty$ (and of course $\Theta^*(\alpha,\beta) < +\infty$).*

Proof. Convexity and continuity/semicontinuity are straightforward to check. For the constraint qualification it is enough to solve

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = r^{n+1}_{i,j},$$

where $r^{n+1}_{i,j} \in \mathscr{J}_F$ for all i, j, n, with an initial datum $u^0_{i,j}$ such that $\sum_{i,j} u^0_{i,j} = 0$. Then, take (α,β) be such that

$$\begin{cases} \alpha^{n+1}_{i,j} = \dfrac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j}, \\ [\beta^{n+1}]_{i,j} = [D_h u^{n+1}]_{i,j}, \quad 0 \le n < N_T - 1, \end{cases}$$

Thus

$$\Sigma^*(\alpha,\beta) = -\frac{1}{\Delta} t \sum_{i,j} (m_T)_{i,j} u^{N_T}_{i,j} + \frac{1}{\Delta} t \sum_{i,j} (m_0)_{i,j} u^0_{i,j} < +\infty.$$

□

Lemma 6.2. *The functionals Θ and Σ are convex and lower semicontinuous. Moreover,*

$$\Theta(m,z) = \sum_{n=1}^{N_T} \sum_{i,j} (W + \chi)(m^{n-1}_{i,j})$$
$$+ \sup_\beta \left\{ \sum_{n=1}^{N_T} \sum_{i,j} \langle [z^{n-1}_{i,j}], [\beta^n]_{i,j} \rangle - m^{n-1}_{i,j} g(x_{i,j}, [\beta^n]_{i,j}) \right\}$$

and

$$\Sigma(m,z) = \sup_{\psi} \left\{ \begin{array}{l} \frac{1}{\Delta}t \sum_{i,j} ((m_T)_{i,j} + m_{i,j}^{N_T})\psi_{i,j}^{N_T} - \frac{1}{\Delta}t \sum_{i,j} ((m_0)_{i,j} + m_{i,j}^0)\psi_{i,j}^0 \\ + \sum_{n=0}^{N_T-1} \sum_{i,j} \psi_{i,j}^{n+1} \left(\frac{m_{i,j}^n - m_{i,j}^{n+1}}{\Delta t} - \nu(\Delta_h m^n)_{i,j} - \mathrm{div_h}(z^n)_{i,j} \right) \end{array} \right\}.$$

Proof. Convexity and semi-continuity are a direct consequence of the previous lemma and the properties of the Legendre–Fenchel transform. Adding and subtracting a same term in (81), we get

$$\Theta(m,z) = \sup_{\alpha,\beta} \left\{ \begin{array}{l} \sum_{n=1}^{N_T} \sum_{i,j} -m_{i,j}^{n-1} g(x_{i,j}, [\beta^n]_{i,j}) + \langle [\beta^n]_{i,j}, [z^{n-1}]_{i,j} \rangle \\ + \sum_{n=1}^{N_T} \sum_{i,j} m_{i,j}^{n-1} \left(\alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j}) \right) \\ -(W+\chi)^* \left(\alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j}) \right) \end{array} \right\}.$$

A simple computation shows that this can be written as

$$\sup_{\gamma,\beta} \left\{ \sum_{n=1}^{N_T} \sum_{i,j} -m_{i,j}^{n-1} g(x_{i,j}, [\beta^n]_{i,j}) + \langle [\beta^n]_{i,j}, [z^{n-1}]_{i,j} \rangle + m_{i,j}^{n-1} \gamma_{i,j}^n - (W+\chi)^* \left(\gamma_{i,j}^n \right) \right\}$$

and the formula for Θ in the statement follows. As for Σ, observe that

$$\Sigma(m,z) = \sup_{\alpha,\beta} \left(\sum_{n=0}^{N_T-1} \sum_{i,j} m_{i,j}^n \alpha_{i,j}^{n+1} + \langle [z^n]_{i,j}, [\beta^{n+1}]_{i,j} \rangle - \Sigma^*(\alpha,\beta) \right).$$

Thus, taking the definition of Σ^* and Λ into account,

$$\Sigma(m,z) = \sup_{\psi} \left\{ \begin{array}{l} \frac{1}{\Delta}t \sum_{i,j} (m_T)_{i,j} \psi_{i,j}^{N_T} - \frac{1}{\Delta}t \sum_{i,j} (m_0)_{i,j} \psi_{i,j}^0 \\ + \sum_{n=0}^{N_T-1} \sum_{i,j} m_{i,j}^n \left(\frac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{\Delta t} - \nu(\Delta_h \psi^{n+1})_{i,j} \right) \\ + \langle [z^n]_{i,j}, [D_h \psi^{n+1}]_{i,j} \rangle \end{array} \right\}$$

and the claimed formula for Σ easily follows by a discrete integration by part. □

Using Lemma 6.2, it easy to realize that the optimal control problem (82) can be equivalently formulated as the unconstrained minimization problem

$$\min_{m,z} \quad \Theta(m,z) + \Sigma(-m,-z). \tag{86}$$

The qualification condition is fulfilled for this problem also:

Lemma 6.3. *Assume that $(m_0)_{i,j} > 0$ for all i, j. Then there exists (m, z) such that*

$$\begin{cases} \Theta(m, z) < +\infty, \\ \Sigma(-m, -z) < +\infty, \\ \Theta \text{ is continuous in a neighborhood of } m, z. \end{cases}$$

Proof. Take $m^n_{i,j} = \frac{n}{N_T}(m_T)_{i,j} + (1 - \frac{n}{N_T})(m_0)_{i,j}$, and choose ϕ^n such that

$$\Delta_h \phi^n = \frac{1}{\Delta} t(m^{n+1} - m^n) + \nu \Delta_h m^n, \quad n = 0, \dots, N_T - 1.$$

Since ϕ^n is unique up to the addition of a constant, one can always choose the constant in such a way that $\phi^n < \underline{\phi} < 0$, where $\underline{\phi}$ is a fixed negative number.

Set then

$$z^{1,n}_{i,j} = \frac{\phi^n_{i,j}}{h}, \quad z^{2,n}_{i,j} = -\frac{\phi^n_{i,j}}{h}, \quad z^{3,n}_{i,j} = \frac{\phi^n_{i,j}}{h}, \quad z^{4,n}_{i,j} = -\frac{\phi^n_{i,j}}{h}.$$

We have

$$\mathrm{div_h}(z^n)_{i,j} = -(\Delta_h \phi^n)_{i,j}.$$

Therefore

$$\begin{cases} \dfrac{m^{n+1}_{i,j} - m^n_{i,j}}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathrm{div_h}(z^n)_{i,j} = 0, \quad 0 \le n < N_T, \\ m^{N_T}_{i,j} = (m_T)_{i,j}, \\ m^0_{i,j} = (m_0)_{i,j}, \\ m_{i,j} \ge 0. \end{cases}$$

Observe that the assumption $(m_0)_{i,j} > 0$ implies $m^n \ge \underline{m} > 0$ for all $n < N_T$.

Using Lemma 6.2, this implies $\Sigma(-m, -z) = 0$. Also, taking the definition of z into account,

$$\Theta(m, z) = \sum_{n=1}^{N_T} \sum_{i,j} W(m^{n-1}_{i,j}) + \sup_{\beta} \Big\{ \sum_{n=1}^{N_T} \sum_{i,j} \Big(\frac{\phi^{n-1}_{i,j}}{h} (\beta^{1,n}_{i,j} - \beta^{2,n}_{i,j} + \beta^{3,n}_{i,j} - \beta^{4,n}_{i,j}) \\ - m^{n-1}_{i,j} g(x_{i,j}, [\beta^n]_{i,j}) \Big) \Big\}.$$

Since $\phi < 0$ and $m^n > \underline{m} > 0$, $n = 0, \dots, N_T - 1$, from the coercivity ($\mathbf{g}_5$) of g we deduce that $\Theta(m, z)$ is finite and Θ is continuous in a neighborhood of (m, z). □

The next result gives sufficient conditions for the existence of a solution of the discrete system (15).

Theorem 6.1. *Assume that*

(i) ($\mathbf{g_1}$)–($\mathbf{g_5}$) *hold.*
(ii) $\Phi[m](x) = F(m(x))$, *and* $F = W'$ *where* $W : \mathbb{R} \to \mathbb{R}$ *is a strictly convex, coercive* $\mathscr{C}^2$ *function.*
(iii) $(m_0)_{i,j}, (m_T)_{i,j} \in \mathscr{K}_h$ *with* $(m_0)_{i,j} > 0$, $\forall i, j$.
(iv) *either* $\nu > 0$ *or* $\left(\nu = 0 \text{ and } (m_T)_{i,j} > 0, \forall i, j\right)$.

Then the saddle point problem:

$$\min_{m,z} \quad \Theta(m,z) + \Sigma(-m,-z) = -\min_{\alpha,\beta} \left(\Theta^*(\alpha,\beta) + \Sigma^*(\alpha,\beta)\right) \tag{87}$$

has a solution $(m,z), (\alpha,\beta)$ *and there exists* u *such that* $(\alpha,\beta) = \Lambda(u)$. *Moreover,* (m,z) *and* u *satisfy the optimality conditions of (87)*

$$-\Lambda^*(m,z) \in \partial \mathscr{L}(u), \tag{88}$$

$$\Lambda(u) \in \partial \Theta(m,z), \tag{89}$$

which are equivalent to the discrete system (21)–(79).

Proof. By applying the Fenchel–Rockafellar Duality Theorem to Θ^* and Σ^* (see for example [5,6,12,32]) and using Lemma 6.1, there exists a solution (m,z) of the problem

$$\begin{aligned} &\Theta(m,z) + \Sigma(-m,-z) \\ &\quad = \inf_{m,z} \left(\Theta(m,z) + \Sigma(-m,-z)\right) = -\inf_{\alpha,\beta} \left(\Theta^*(\alpha,\beta) + \Sigma^*(\alpha,\beta)\right). \end{aligned} \tag{90}$$

By applying the Fenchel–Rockafellar Duality Theorem to $(m,z) \mapsto \Theta(m,z)$ and $(m,z) \mapsto \Sigma(-m,-z)$, and using Lemmas 6.2 and 6.3, we deduce that there exist (α,β) such that

$$\begin{aligned} \Theta^*(\alpha,\beta) + \Sigma^*(\alpha,\beta) &= \inf_{\alpha,\beta} \left(\Theta(\alpha,\beta) + \Sigma(\alpha,\beta)\right) \\ &= -\inf_{m,z}\left(\Theta(m,z) + \Sigma(-m,-z)\right). \end{aligned} \tag{91}$$

We have thus proved the existence of a solution of the saddle point problem (87).

By the optimality conditions, see [6, Theorem 2.4 page 205], we get

$$-\Lambda^*(m,z) \in \partial \mathscr{L}(u), \tag{92}$$

$$(\alpha,\beta) = \Lambda(u) \in \partial \Theta(m,z). \tag{93}$$

Recalling the definition of $\mathscr{L}$, (92) is seen to be in fact equivalent to

$$\begin{cases} \dfrac{m_{i,j}^{n+1} - m_{i,j}^n}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathrm{div_h}(z^n)_{i,j} = 0, & 0 \le n < N_T, \\ m_{i,j}^{N_T} = (m_T)_{i,j}, \\ m_{i,j}^0 = (m_0)_{i,j}. \end{cases} \tag{94}$$

On the other hand, it is easy to see that (93) is equivalent to

$$\Theta(m,z) = \sum_{n=1}^{N_T} \sum_{i,j} \Big(m_{i,j}^{n-1} \alpha_{i,j}^n + \langle [z^{n-1}]_{i,j}, [\beta^n]_{i,j} \rangle \\ - (W + \chi)^* \big(\alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j}) \big) \Big).$$

Introducing $\gamma_{i,j}^n = \alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j})$, $n = 1, \ldots, N_T$, the latter equation is equivalent to

$$z_{i,j}^{k,n} = m_{i,j}^n \frac{\partial g}{\partial q_k}(x_{i,j}, [\beta^{n+1}]_{i,j}), \qquad k = 1, \ldots, 4, \tag{95}$$

$$0 = \sum_{n=1}^{N_T} \sum_{i,j} \Big(m_{i,j}^{n-1} \gamma_{i,j}^n - (W + \chi)^*(\gamma_{i,j}^n) - (W + \chi)(m_{i,j}^{n-1}) \Big). \tag{96}$$

Equation (96) is equivalent to

$$\begin{aligned} & m_{i,j}^n \ge 0, \\ & \gamma_{i,j}^{n+1} = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = W'(m_{i,j}^n) \\ & \qquad \text{if } m_{i,j}^n > 0, \\ & \gamma_{i,j}^{n+1} = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) \le W'(m_{i,j}^n) \\ & \qquad \text{if } m_{i,j}^n = 0, \end{aligned} \tag{97}$$

for $0 \le n < N_T$.

From (94) and (95), we deduce

$$\frac{m_{i,j}^{n+1} - m_{i,j}^n}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathcal{T}_{i,j}(u^{n+1}, m^n) = 0, \quad 0 \le i, j < N_h, \ 0 \le n < N_T. \tag{98}$$

The fact that $m^{N_T} \in \mathcal{K}_h$ and (98) imply that $h^2 \sum_{i,j} m_{i,j}^n = 1$ for all n, $0 \le n < N_T$. Finally $m^n \in \mathcal{K}_h$ because of (98).

Finally, let us prove that $m^n > 0$ for all $0 \le n < N_T$. Indeed, assume that the minimum of $m_{i,j}^n$ is 0 and is reached at $n_0 < N_T$, i_0, j_0. Equation (98) for $n = n_0$, $i = i_0$ and $j = j_0$ can be written

$$0 = \frac{1}{\Delta} t m^{n_0+1}_{i_0,j_0} + \frac{\nu}{h^2}(m^{n_0}_{i_0+1,j_0} + m^{n_0}_{i_0-1,j_0} + m^{n_0}_{i_0,j_0+1} + m^{n_0}_{i_0,j_0-1})$$
$$-\frac{1}{h}\left\{\begin{array}{l} m^{n_0}_{i_0-1,j_0}\dfrac{\partial g}{\partial q_1}(x_{i_0-1,j_0}, [D_h u^{n_0+1}]_{i_0-1,j_0}) \\ -m^{n_0}_{i_0+1,j_0}\dfrac{\partial g}{\partial q_2}(x_{i_0+1,j_0}, [D_h u^{n_0+1}]_{i_0+1,j_0}) \end{array}\right\}$$
$$-\frac{1}{h}\left\{\begin{array}{l} m^{n_0}_{i_0,j_0-1}\dfrac{\partial g}{\partial q_3}(x_{i_0,j_0-1}, [D_h u^{n_0+1}]_{i_0,j_0-1}) \\ -m^{n_0}_{i_0,j_0+1}\dfrac{\partial g}{\partial q_4}(x_{i_0,j_0+1}, [D_h u^{n_0+1}]_{i_0,j_0+1}) \end{array}\right\}.$$

If $\nu > 0$, then the nonnegativity of m and the monotonicity of g imply that $m^{n_0}_{i_0\pm1,j_0} = m^{n_0}_{i_0,j_0\pm1} = 0$. We can therefore repeat the argument for the triplets of indices $(n_0, i_0 \pm 1, j_0)$ and $(n_0, i_0, j_0 \pm 1)$. Repeating the argument as many times as necessary, we finally obtain that $m^{n_0} = 0$, which is impossible since $m^{n_0} \in \mathscr{K}_h$.

If $\nu = 0$ and $m^{N_T} > 0$, a similar argument gives that $m^{n_0+1}_{i_0,j_0} = 0$. After a finite number of steps, we get that $m^{N_T}_{i_0,j_0} = 0$, which is in contradiction with the hypothesis.

As a consequence, (94), (95) and (97) can be written:

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = F(m^n_{i,j}), \tag{99}$$

$$\frac{m^{n+1}_{i,j} - m^n_{i,j}}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathscr{T}_{i,j}(u^{n+1}, m^n) = 0, \tag{100}$$

for $n = 0, \dots N_T - 1$ and $0 \le i, j < N_h$, with

$$m^{N_T}_{i,j} = (m_T)_{i,j}, \qquad m^0_{i,j} = (m_0)_{i,j}, \quad 0 \le i, j < N_h, \tag{101}$$

and

$$m^n \in \mathscr{K}_h, \quad 0 \le n \le N_T. \tag{102}$$

Recognizing that (99) to (102) comprise indeed the semi-implicit finite difference scheme (21)–(79), the proof is complete. □

6.3 *Uniqueness*

System (21)–(79) also enjoys some uniqueness property, see [2] for the proof:

Proposition 6.1. *Under the same assumptions as in Theorem 6.1, if* $(u^n_{i,j}, m^n_{i,j})_{n,i,j}$ *and* $(\tilde{u}^n_{i,j}, \tilde{m}^n_{i,j})_{n,i,j}$ *are solutions of system (21)–(79), then*

$$m^n_{i,j} = \tilde{m}^n_{i,j} \ \text{ for all } n = 0, \dots, N_T, \text{ and for all } (i, j).$$

Moreover if the numerical Hamiltonian g is strictly convex, there exists a constant c_u such that

$$u^n_{i,j} - \tilde{u}^n_{i,j} = c_u \ \text{for all } n = 0, \dots, N_T, \ \text{and for all } (i,j)\,.$$

6.4 A Penalty Method

We consider the penalized version of (21)–(79), namely

$$\frac{u^{\epsilon,n+1}_{i,j} - u^{\epsilon,n}_{i,j}}{\Delta t} - \nu(\Delta_h u^{\epsilon,n+1})_{i,j} + g(x_{i,j}, [D_h u^{\epsilon,n+1}]_{i,j}) = F(m^n_{i,j}), \quad (103)$$

$$\frac{m^{\epsilon,n+1}_{i,j} - m^{\epsilon,n}_{i,j}}{\Delta t} + \nu(\Delta_h m^{\epsilon,n})_{i,j} + \mathscr{T}_{i,j}(u^{\epsilon,n+1}, m^{\epsilon,n}) = 0, \quad (104)$$

for $n = 0, \dots N_T - 1$ and $0 \le i,j < N_h$, with the terminal and initial conditions

$$u^{\epsilon,0}_{i,j} = \frac{1}{\epsilon}(m^{\epsilon,0}_{i,j} - (m_0)_{i,j}), \qquad m^{\epsilon,N_T}_{i,j} = (m_T)_{i,j}, \quad \forall\, 0 \le i,j < N_h. \quad (105)$$

With this penalized version, algorithms close to those described in Sect. 4 have been used in [2]. Note that the small parameter ϵ makes the convergence slower.

The following result was proved in [2]:

Proposition 6.2. *We make the same assumptions as in Theorem 6.1. For a subsequence still called ϵ, let $(u^{\epsilon,n}, m^{\epsilon,n})$ be a solution of (103)–(105) and (m^n) be a family of grid functions in $\mathscr{K}_h$ such that $\lim_{\epsilon\to 0} \max_n \|m^{\epsilon,n} - m^n\|_\infty = 0$. There exists a family of grid functions (u^n) such that up to a further extraction of a subsequence, $\lim_{\epsilon\to 0} \max_n \|u^{\epsilon,n} - u^n\|_\infty = 0$ and $(u^n, m^n)_n$ is a solution of (21)–(79).*

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An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton–Jacobi Equations and Applications

Guy Barles

Abstract In this course, we first present an elementary introduction to the concept of viscosity solutions for first-order Hamilton–Jacobi Equations: definition, stability and comparison results (in the continuous and discontinuous frameworks), boundary conditions in the viscosity sense, Perron's method, Barron–Jensen solutions ... etc. We use a running example on exit time control problems to illustrate the different notions and results. In a second part, we consider the large time behavior of periodic solutions of Hamilton–Jacobi Equations: we describe recents results obtained by using partial differential equations type arguments. This part is complementary of the course of H. Ishii which presents the dynamical system approach ("weak KAM approach").

1 Introduction

This text contains two main parts: in the first one, we present an elementary introduction of the notion of viscosity solutions in which we restrict ourselves to the case of first-order Hamilton–Jacobi Equations (we do not present the uniqueness arguments for second-order equations). We recall that this notion of solutions was introduced in the 1980s by Crandall and Lions [22] (see also Crandall et al. [21]). In the second part, we describe recent results on the large time behavior of solutions of Hamilton–Jacobi Equations which are obtained by using partial differential equations type arguments: this part is complementary of the course of H. Ishii which presents the dynamical system approach ("weak KAM approach").

Despite the main focus of this article will be on first-order equations, we point out that the natural framework for presenting viscosity solutions' theory is to

G. Barles (✉)
Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 7350), Fédération Denis Poisson (FR CNRS 2964), Université de Tours, Parc de Grandmont, 37200 Tours, France
e-mail: barles@lmpt.univ-tours.fr

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consider fully nonlinear degenerate elliptic equations (and even equations with integro-differential operators under suitable assumptions); we will use this natural framework when there will be no additional difficulty.

We refer the reader to the book of Bardi and Capuzzo Dolcetta [2] for a more complete presentation of this notion of solutions including applications to deterministic optimal control problems and differential games, to the "Users guide" of Crandall et al. [23] for extensions to second-order equations and to the book of Fleming and Soner [26] where the applications to deterministic and stochastic optimal control are also described. An introduction to the notion of viscosity solutions as well as applications in various directions can also be found in the 1995 CIME course [3].

By *"fully nonlinear degenerate elliptic equations"*, we mean equations which can be written as

$$F(y, u, Du, D^2u) = 0 \quad \text{in } \mathscr{O} \ , \tag{1}$$

where $\mathscr{O}$ is a domain in $\mathbb{R}^N$ and F is, say, a continuous, real-valued function defined on $\mathscr{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N$, $\mathscr{S}^N$ being the space of $N \times N$ symmetric matrices, and which satisfies the *(degenerate) ellipticity condition*

$$F(y, r, p, M_1) \le F(y, r, p, M_2) \quad \text{if } M_1 \ge M_2 \ , \tag{2}$$

for any $y \in \mathscr{O}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, $M_1, M_2 \in \mathscr{S}^N$. The solution u is a scalar function and Du, D^2u denote respectively its gradient and Hessian matrix.

Of course, first-order equations obviously enter in this framework since, in that case, F does not depend on D^2u and is therefore elliptic. We also point out that parabolic/first-order evolution equations like

$$u_t + H(x, t, u, D_x u) - \varepsilon \Delta^2_{xx} u = 0 \quad \text{in } \Omega \times (0, T) \ ,$$

are also degenerate elliptic equations if $\varepsilon \ge 0$ (including $\varepsilon = 0$) with the domain $\mathscr{O} = \Omega \times (0, T)$ and the variable $y = (x, t)$; in other words, a classical (possibly degenerate) parabolic equation is a degenerate elliptic equation.

The ellipticity property is a key property for defining the notion of viscosity solutions: this fact will become clear in Sect. 3. From now on, we will always assume it is satisfied by the equations we consider.

In fact, the notion of viscosity solutions applies naturally to (a priori) any type of equations modelling *monotone phenomenas*. A famous result in this direction is given by Alvarez et al. [1] for image analysis (see also Biton [19]): a multiscale analysis which satisfies some locality, regularity, causality and *monotonicity* properties is given by a fully nonlinear parabolic pde, and even by the viscosity solution of this pde. Furthermore, one has a geometrical counterpart of this result in [14] for front propagation problems, where monotonicity has to be understood in the inclusion sense. We will emphasize this monotonicity feature, starting, in Sect. 2, with a running example on exit time control problems.

The article is organized as follows: in Sect. 3, we provide the definition of *continuous* viscosity sub and supersolutions and their first properties (different formulations, connections with classical properties, changes of variables, … etc); we also provide a first stability result for continuous solutions (Sect. 4). Section 5 describes what is called (improperly) "uniqueness results": in fact, these are "comparison results" of Maximum Principle type which (roughly speaking) implies that subsolutions are below supersolutions. After describing the basic arguments (doubling of variables and basic estimates), we show how to obtain such comparison results in various situations (in particular for problems set in $\mathbb{R}^N \times (0,T)$ with or without "finite speed of propagation" type properties). In Sect. 6, we describe the notion of viscosity solutions for discontinuous solutions and equations: the main motivation comes from the discontinuous stability result ("half relaxed limit method") which allows passage to the limit with only a uniform (L^∞) bound on the solutions. This last result leads us to the existence properties for viscosity obtained by the Perron's method (Sect. 7). In Sect. 8, we show how to prove regularity results: Lipschitz continuity, semi-concavity, … etc and we conclude by the Barron–Jensen's approach for first-order equations with convex Hamiltonians (Sect. 9).

In a second part, in Sect. 10, we provide an application of the presented tools to the study (by pde methods) of the large time behavior of solutions of Hamilton–Jacobi Equations: we present the various difficulties and key results for these problems (basic estimates, ergodic problem, … etc.) and we describe the two main convergence results, namely the Namah–Roquejoffre framework [42] and what we name as the "strictly convex" framework, even if the Hamiltonians do not really need to be strictly convex, related to the result by Souganidis and the author [15]; while the Namah–Roquejoffre result relies on rather classical viscosity solutions' methods, the "strictly convex" one uses a more surprising asymptotic monotone property of the solutions in t.

2 Preliminaries: A Running Example

In this section, we present an example which is used in the sequel to illustrate several concepts or results related to viscosity solutions. This example concerns deterministic control problems and, more precisely, exit time control problems. We describe it now.

We consider a controlled system whose state is described by the solution y_x of the ordinary differential equation (the "dynamic")

$$\begin{cases} \dot{y}_x(s) = b(y_x(s), \alpha(s)) & \text{for } s > 0, \\ y_x(0) = x \in \Omega\ . \end{cases} \tag{3}$$

where Ω is a bounded domain of $\mathbb{R}^N$ (Ω or its closure $\overline{\Omega}$ represents the possible "states of the system"), $\alpha(\cdot)$, the control, is a measurable function which takes its value in a compact metric space $\mathscr{A}$ and $b : \mathbb{R}^N \times \mathscr{A} \to \mathbb{R}^N$ is a function satisfying, for some constant $C > 0$ and for any $x, y \in \overline{\Omega}$, $\alpha \in \mathscr{A}$

$$\begin{cases} b \text{ is a continuous function from } \mathbb{R}^N \times \mathscr{A} \text{ into } \mathbb{R}^N . \\ |b(x,\alpha) - b(y,\alpha)| \leq C|y-x| \ , \quad |b(x,\alpha)| \leq C \ , \end{cases} \tag{4}$$

Because of this assumption, the ordinary differential equation (3) has a unique solution which is defined for all $s > 0$.

The trajectories y_x depend both on the starting point x but also on the choice of the control $\alpha(\cdot)$. We omit this second dependence for the sake of simplicity of notations.

The "value function" is then defined, for $x \in \Omega$ (or $\overline{\Omega}$) and $t \in [0, T]$, by

$$\mathbf{U}(x,t) = \inf_{\alpha(.)} \left\{ \int_0^\tau f(y_x(s), \alpha(s))ds + \varphi(y_x(\tau))\mathbf{1}_{\{\tau \leq t\}} + u_0(y_x(t))\mathbf{1}_{\{\tau > t\}} \right\} , \tag{5}$$

where f, φ, u_0 are continuous functions defined respectively on $\overline{\Omega} \times \mathscr{A}$, $\partial\Omega$ and $\overline{\Omega}$ which takes values in $\mathbb{R}$. We denote by τ the first exit time of the trajectory y_x from Ω, i.e.

$$\tau = \inf\{t \geq 0 \ ; \ y_x(t) \notin \Omega \ \} \ .$$

Of course, τ depends on x and $\alpha(\cdot)$ but we drop this dependence, again for the sake of simplicity of notations. Finally, for any set A, $\mathbf{1}_A$ denotes the indicator function of the set A. For reasons which will be clear later on, we assume the compatibility condition

$$u_0 = \varphi \text{ on } \overline{\Omega} \ . \tag{6}$$

In the sequel, we will say that the "control assumptions", and we will write **(CA)**, are satisfied if (4) holds, if f, φ, u_0 are continuous functions and if we have (6).

The first remark that we can make on this example concerns the monotonicity: keeping the same dynamic, if we consider different costs f_1, φ_1, u_0^1 and f_2, φ_2, u_0^2 with

$$f_1 \leq f_2 \quad \text{on } \overline{\Omega} \times \mathscr{A}, \quad \varphi_1 \leq \varphi_2 \quad \text{on } \partial\Omega, \quad u_0^1 \leq u_0^2 \quad \text{on } \overline{\Omega} \ ,$$

then the associated value functions satisfy $\mathbf{U}_1 \leq \mathbf{U}_2$ on $\overline{\Omega} \times [0, T]$. In other words, the value functions depends in a monotone way of the data.

We will see that the value function $\mathbf{U}$ is a solution of

$$\mathbf{U}_t + H(x, D\mathbf{U}) = 0 \quad \text{ in } \Omega \times (0, T) \ , \tag{7}$$

where $H(x, p) := \sup_{\alpha \in \mathscr{A}} \{-b(x,\alpha) \cdot p - f(x,\alpha)\}$, with the Dirichlet boundary condition

$$\mathbf{U}(x,t) = \varphi(x,t) \quad \text{ on } \partial\Omega \times (0, T) \ , \tag{8}$$

and the initial condition

$$\mathbf{U}(x,0) = u_0(x) \quad \text{on } \overline{\Omega}\ . \tag{9}$$

We have to answer to several questions in the sequel:

- A priori, the value function $\mathbf{U}$ is not regular: in which sense can it be a solution of (7)–(9)?
- How is the boundary data achieved? In which sense?
- Is the value function the unique solution of (7)–(9)?
- Are we able to prove directly that a solution of (7)–(9) satisfies the monotonicity property?

We conclude this section by (very) few some references on exit time control problems. The work of Soner [44] on state constraints problems is the first article which studies this kind of problems in connections with viscosity solutions, uses boundary conditions in the viscosity solutions' sense and provides a general argument to prove uniqueness results. Boundary conditions in the viscosity solutions' sense have been considered previously for Neumann/reflection problems by Lions [37]. Pushing their ideas a little bit further, Perthame and the author [8–10] (see also [5]) systematically study Dirichlet/exit time control problems (including state constraints problems). For stochastic control, we refer the reader to [12] and references therein.

3 The Notion of Continuous Viscosity Solutions: Definition(s) and First Properties

3.1 Why a "Good" Notion of Weak Solution is Needed?

We give now few concrete examples of equations where there will be a unique viscosity solution but either no smooth solutions or with several *generalized* solutions (i.e. solutions which are locally Lipschitz continuous and satisfy the equation almost everywhere). We refer to Sect. 5 for the proof of the uniqueness results we are going to use.

The first example is

$$\frac{\partial u}{\partial t} + \left|\frac{\partial u}{\partial x}\right| = 0 \quad \text{in } \mathbb{R}\times(0,+\infty)\ . \tag{10}$$

We first remark that (10) enters into our framework with $\mathscr{O} = \mathbb{R}\times(0,+\infty)$, the variable is $y=(x,t)$, $Du = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t})$[1] and

$$F(y,u,p,M) = p_t + |p_x|\ ,$$

[1] Here we use the notation Du for the full gradient of u in space and time but, in general, we will use it for the gradient in space of u.

with $p = (p_x, p_t)$.

It can be shown that the function u defined in $\mathbb{R} \times (0, +\infty)$ by

$$u(x,t) = -(|x| + t)^2 \ ,$$

is the unique viscosity solution of (10) in $C(\mathbb{R}\times(0,+\infty))$ (see Sect. 5.3). It is worth remarking in this example that u is only locally Lipschitz continuous for $t > 0$ despite the initial data

$$u(x,0) = -x^2 \quad \text{in } \mathbb{R} \ ,$$

is in $C^\infty(\mathbb{R})$. In particular, this problem has no smooth solution as it is generally the case for such nonlinear hyperbolic equations.

Moreover, if we consider (10) together with the initial data

$$u(x,0) = |x| \quad \text{in } \mathbb{R} \ , \tag{11}$$

then the functions $u_1(x,t) = |x|-t$ and $u_2(x,t) = (|x|-t)^+$ are two "generalized" solutions in the sense that they satisfy the equation almost everywhere (at each of their points of differentiability). This problem of nonuniqueness is solved by the notion of viscosity solutions since it can be shown that u_2 is the unique continuous viscosity solution of (10)–(11) (see again Sect. 5.3). In that case, the notion of viscosity solutions selects the "good" solution which is here the value-function of the associated deterministic control problem (cf. Bardi and Capuzzo Dolcetta [2] and Fleming and Soner [26]). An other remark (or interpretation) is that the notion of viscosity solutions selects the solution which satisfies the right monotonicity property: indeed the initial data is positive and therefore the solution has to be positive since 0 is a (natural) solution.

For second-order equations, non-smooth solutions appear generally as a consequence of the degeneracy of the equation. We refer to [23] for details in this direction.

3.2 Continuous Viscosity Solutions

As we already mention it in the introduction, we are going to present the different definitions of viscosity solutions in the framework of *fully nonlinear degenerate elliptic equations* i.e. equations like (1) which satisfies the *ellipticity condition* (2).

In order to introduce the notion of viscosity solutions and to show the importance of the ellipticity condition, we first give an equivalent definition of the notion of classical solution which only uses the Maximum Principle.

Theorem 3.1 (Classical Solutions and Maximum Principle). *$u \in C^2(\mathcal{O})$ is a classical solution of* (1) ***if and only if***
for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local maximum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \leq 0,$$

***and**, for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local minimum point of $u - \varphi$, one has*

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0\ .$$

Proof. The proof of this result is very simple: the first part of the equivalence just comes from the classical properties $Du(y_0) = D\varphi(y_0)$, $D^2u(y_0) \leq D^2\varphi(y_0)$, at a maximum point y_0 of $u - \varphi$ (recall that u and φ are smooth) or $Du(y_0) = D\varphi(y_0)$, $D^2u(y_0) \geq D^2\varphi(y_0)$, at a minimum point y_0 of $u - \varphi$. One has just to use these properties together with the ellipticity property (2) of F to obtain the inequalities of the theorem.

The second part is a consequence of the fact that we can take $\varphi = u$ as test-function and therefore $F(y_0, u(y_0), Du(y_0), D^2u(y_0))$ is both positive and negative at any point y_0 of $\mathcal{O}$ since any $y_0 \in \mathcal{O}$ is both a local maximum and minimum point of $u - u$.

Now we simply remark that the equivalent definition of classical solutions which is given here in terms of test-functions φ does not require the existence of first and second derivatives of u. For example, the continuity of u is sufficient to give a sense to this equivalent definition; therefore we use this formulation to define viscosity solutions.

Definition 3.1 (Continuous Viscosity Solutions). The function $u \in C(\mathcal{O})$ is a viscosity solution of (1) **if and only if**
for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local maximum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \leq 0,$$

and, for any $\varphi \in C^2(\mathcal{O})$, if $y_0 \in \mathcal{O}$ is a local minimum point of $u - \varphi$, one has

$$F(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0\ .$$

If u only satisfies the first property of Definition 3.1 (with maximum points), we will say that u is a viscosity subsolution of the equation, while it is called a viscosity supersolution if it only satisfies the second one. From now on, we will talk only of subsolution, supersolution and solution considering that they will be anytime taken in the viscosity sense. This notion of solution was called "viscosity solution" because for first-order equations, as we will see it below, viscosity solutions were first obtained as limits in the "vanishing viscosity method", i.e. by an approximation procedure involving a $-\varepsilon\Delta$ term.

For first-order equations (otherwise this remark makes no sense), it is worth pointing out that a solution of $F = 0$ is not necessarily a solution of $-F = 0$: the sign of the nonlinearity plays a role. This phenomena can be understood in the following way: the viscosity solution of the equation $F = 0$ when unique can be thought as being obtained through the vanishing viscosity approximation $-\varepsilon\Delta + F = 0$ and there is no reason why the other vanishing approximation

$\varepsilon\Delta + F = 0$ (which leads in fact to a solution of $-F = 0$) converges to the same solution.

Finally we remark that parabolic equations are just a particular case of (degenerate) elliptic equations: the y—variable is just the (x,t)—variable and, of course, Du, D^2u have to be understood as the gradient and Hessian matrix of u with respect to the variable (x,t).

3.3 Back to the Running Example (I): The Value Function U *is a Viscosity Solution of (7)*

The key result is the *Dynamic Programming Principle*

Theorem 3.2. *Under the hypothesis* **(CA)***, if $x \in \Omega$, $0 < t \leq T$, the value function satisfies, for $S > 0$ small enough*

$$\mathbf{U}(x,t) = \inf_{\alpha(.)} \left[\int_0^S f(y_x(s),\alpha(s))ds + \mathbf{U}(y_x(S), t-S) \right]. \tag{12}$$

We leave the proof of this result to the reader and show how it implies that **U** is a viscosity solution of (7). To do so, we assume that **U** is continuous (an assumption which will be removed later on). We only prove that it is a supersolution, the subsolution property being easier to obtain.

Let $\phi \in C^1(\Omega \times (0,T))$ and assume that $(x,t) \in \Omega \times (0,T)$ is a local minimum point of $\mathbf{U} - \phi$. There exists $r > 0$ such that, if $|x' - x| \leq r$ and $|t' - t| \leq r$, then $x' \in \Omega$, $t' > 0$ and

$$\mathbf{U}(x',t') - \phi(x',t') \geq \mathbf{U}(x,t) - \phi(x,t)\ .$$

Using the Dynamic Programming Principle with S small enough in order to have $S \leq r$ and $|y_x(S) - x| \leq r$ (recall that b is uniformly bounded), we obtain

$$\phi(x,t) \geq \inf_{\alpha(.)} \left[\int_0^S f(y_x(s),\alpha(s))ds + \phi(y_x(S), t-S) \right]\ .$$

But, by standard calculus

$$\begin{aligned}&\phi(y_x(S), t-S)\\ &\quad = \phi(x,t) + \int_0^S \left(D\phi(y_x(s), t-s)\cdot b(y_x(s),\alpha(s)) - \phi_t(y_x(s), t-s)\right) ds\ .\end{aligned}$$

And therefore

$$0 \geq \inf_{\alpha(.)} \left[\int_0^S (D\phi(y_x(s), t-s) \cdot b(y_x(s), \alpha(s)) - \phi_t(y_x(s), t-s) + f(y_x(s), \alpha(s)))\, ds \right] ,$$

or

$$\sup_{\alpha(.)} \left[\int_0^S (-D\phi(y_x(s), t-s) \cdot b(y_x(s), \alpha(s)) + \phi_t(y_x(s), t-s) - f(y_x(s), \alpha(s)))\, ds \right] \geq 0 .$$

Next, we remark that the integrand can be replaced by (the larger quantity)

$$\phi_t(y_x(s), t-s) + H(y_x(s), D\phi(y_x(s), t-s))$$

and then, because of the regularity of ϕ and the continuity property of H, by $\phi_t(x,t) + H(x, D\phi(x,t)) + o(1)$ where $o(1)$ denotes a quantity which tends to 0 as $S \to 0$, uniformly with respect to the control. Finally

$$\sup_{\alpha(.)} \left[\int_0^S (\phi_t(x,t) + H(x, D\phi(x,t)) + o(1))\, ds \right] \geq 0 ,$$

and the conclusion follows by dividing by S and letting S tends to 0, noticing that the sup can be dropped.

Remark 3.1. The above argument is a key one and it is worth pointing out that it just uses the fact that

$$u(x,t) = G(S, x, t, u(\cdot)) ,$$

where G is *monotone in* $u(\cdot)$ and *consistent* with the equation, in the sense that

$$\frac{\phi(x,t) - G(S, x, t, \phi(\cdot))}{S} \to \phi_t(x,t) + H(x, D\phi(x,t) \text{ as } S \to 0 ,$$

for any smooth function ϕ.[2] Therefore it is a rather general argument which connects "monotonicity" and "viscosity solutions": it appears in various situations such as the convergence of numerical scheme (see in particular [13]), the connection of monotone semi-group with viscosity solutions (see, for instance, [1,19,36]), ... etc.

[2]Here we have also used a less important (but simplifying) property, namely the commutation with constants: for any $c \in \mathbb{R}$, S, x, t and for any function $u(\cdot)$, $G(S, x, t, u(\cdot) + c) = G(S, x, t, u(\cdot)) + c$.

3.4 An Equivalent Definition and Its Consequences

We continue by giving some equivalent definitions which may be useful.

Proposition 3.1. *An equivalent definition of subsolution, supersolution and solution is obtained by replacing in Definition 3.1:*

1. *"$\phi \in C^2(\mathscr{O})$" by "$\phi \in C^k(\mathscr{O})$" ($2 < k < +\infty$) or by "$\phi \in C^\infty(\mathscr{O})$"*
2. *"$\phi \in C^2(\mathscr{O})$" by "$\phi \in C^1(\mathscr{O})$" in the case of first-order equations*
3. *"local maximum" or "local minimum" by "strict local maximum" or "strict local minimum" or by "global maximum" or "global minimum" or by "strict global maximum" or "strict global minimum".*

This proposition is useful since, in general, the proofs are simplified by a right choice of the definition. In particular the definition with "global maximum points" or "global minimum points" in order to avoid heavy localisation arguments.

The proof of this proposition is left as an exercise (despite it is not obvious at all): it is based on classical Analysis type arguments, some of them being rather delicate.

We give now a more "pointwise" definition using generalized derivatives ("sub and super-differential" or "semi-jets") which plays a central role for second-order equations.

Definition 3.2 (Second-order sub and super-differential of a continuous function). The second-order superdifferential of $u \in C(\mathscr{O})$ at $y \in \mathscr{O}$ is the, possibly empty, convex subset of $\mathbb{R}^N \times \mathscr{S}^N$, denoted by $D^{2,+}u(y)$, of all couples $(p, M) \in \mathbb{R}^N \times \mathscr{S}^N$ satisfying

$$u(y+h) - u(y) - (p,h) - \frac{1}{2}(Mh,h) \leq o(|h|^2) ,$$

for $h \in \mathbb{R}^N$ small enough.

The second-order subdifferential of $u \in C(\mathscr{O})$ at $y \in \mathscr{O}$ is the, possibly empty, convex subset of $\mathbb{R}^N \times \mathscr{S}^N$, denoted by $D^{2,-}u(y)$, of all couples $(p, M) \in \mathbb{R}^N \times \mathscr{S}^N$ satisfying

$$u(y+h) - u(y) - (p,h) - \frac{1}{2}(Mh,h) \geq o(|h|^2) ,$$

for $h \in \mathbb{R}^N$ small enough.

As indicated in the definition, these subsets can be empty, even both as it is the case, at the point $y = 0$, for the function $y \mapsto \sqrt{|y|}\sin(\dfrac{1}{y^2})$ extended at 0 by 0.

If u is twice differentiable at y then

$$D^{2,+}u(y) = \{(Du(y), M);\ M \geq D^2u(y)\}\ ,$$

$$D^{2,-}u(y) = \{(Du(y), M);\ M \leq D^2u(y)\}\ ,$$

Now we turn to the connections between sub and super-differentials with viscosity solutions.

Theorem 3.3. *(i) $u \in C(\mathcal{O})$ is a subsolution of (1) iff, for any $y \in \mathcal{O}$ and for any $(p, M) \in D^{2,+}u(y)$*

$$F(y, u(y), p, M) \leq 0. \tag{13}$$

(ii) $u \in C(\mathcal{O})$ is a supersolution of (1) iff, for any $y \in \mathcal{O}$ and for any $(p, M) \in D^{2,-}u(y)$

$$F(y, u(y), p, M) \geq 0\ . \tag{14}$$

Before giving some elements of the proof of Theorem 3.3, we provide some easy (but useful) consequences.

Corollary 3.1. *(i) If $u \in C^2(\mathcal{O})$ satisfies $F(y, u(y), Du(y), D^2u(y)) = 0$ in $\mathcal{O}$ then u is a viscosity solution of (1).*

(ii) If $u \in C(\mathcal{O})$ is a viscosity solution of (1) and if u is twice differentiable at $y_0 \in \mathcal{O}$ then

$$F(y_0, u(y_0), Du(y_0), D^2u(y_0)) = 0\ .$$

(iii) If $u \in C(\mathcal{O})$ is a viscosity solution of (1) and if $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2—function such that $\varphi' > 0$ on $\mathbb{R}$ then the function v defined by $v = \varphi(u)$ is a viscosity solution of

$$K(y, v, Dv, D^2v) = 0 \quad in\ \mathcal{O}\ ,$$

where $K(y, z, p, M) = F(y, \psi(z), \psi'(z)p, \psi'(z)M + \psi''(z)p \otimes p)$ and $\psi = \varphi^{-1}$.

The proof of this Corollary is based on the classical technics of calculus and is left as an exercise.

This corollary is formulated in terms of "solution" but, of course analogous results hold for subsolutions and supersolutions.

A lot of different changes can be considered instead of the one in the result (iii): as long as signs are preserved in order to keep the inequalities satisfied by the sub or superdifferentials or, if the minima are not transformed in maxima and vice-versa, such result remains true. Let us mention, for example, the transformations: $v = u + \psi$, with ψ being of classe C^2 or $v = \chi u + \psi$, χ, ψ being of classe C^2 and $\chi \geq \alpha > 0 \ldots$ etc.

In the case when "signs are changed", we have the following proposition.

Proposition 3.2. *$u \in C(\mathcal{O})$ is a subsolution (resp. supersolution) of (1) iff $v = -u$ is a supersolution (resp. subsolution) of*

$$-F(y, -v, -Dv, -D^2v) = 0 \quad in\ \mathcal{O}\ .$$

The *proof of Theorem 3.3* (that Proposition 3.2 allows us to do only in the subsolution case) relies only on two arguments; the first is elementary: if ϕ a C^2 test-function and if y_0 a local maximum point of $u - \phi$ then, by combining the regularity of ϕ and the property of local maximum, we get

$$\begin{aligned} u(y) &\leq \phi(y) + u(y_0) - \phi(y_0) \\ &\leq u(y_0) + (D\phi(y_0), y - y_0) + \frac{1}{2} D^2\phi(y_0)(y - y_0)\cdot(y - y_0) + o(|y - y_0|^2) \ . \end{aligned}$$

Therefore $(D\phi(y_0), D^2\phi(y_0))$ is in $D^{2,+}u(y_0)$.

The second one is not as simple as the first one and is described in the following lemma.

Lemma 3.1. *If $(p, M) \in D^{2,+}u(y_0)$, there exists a C^2—function $\phi : \mathcal{O} \to \mathbb{R}$ such that $D\phi(y_0) = p$, $D^2\phi(y_0) = M$ and such that y_0 is a local maximum point of $u - \phi$.*

The proof of this lemma uses classical but rather tricky Analysis tools, in particular regularization arguments. We skip it since it is rather long and not in the central scope of this course. We refer to Crandall et al. [21] or Lions [36] for a complete proof.

4 The First Stability Result for Viscosity Solutions

There is no need to recall here that problems involving passage to the limit in nonlinear equations when we have only a weak convergence is one of the fundamental problem of nonlinear Analysis. We call "stability result" a result showing under which conditions a limit of a sequence of sub or supersolutions is still a sub or a supersolution.

We present in theses notes two types of stability results which are of different natures: the first one looks rather classical since it requires compactness (or convergence) properties on the considered sequences. It may be a priori of a rather difficult use since the needed estimates on the solutions are not so easy to obtain in concrete situations. The second one, on the contrary, will be far less classical and requires only easy estimates but rather strong uniqueness properties for the limiting equation: we present this second stability result in Sect. 6 since it requires the notion of discontinuous viscosity solutions. We state both results in the framework of second-order equations since there are no additional difficulties.

The first result is the

Theorem 4.1. *Assume that, for $\varepsilon > 0$, $u_\varepsilon \in C(\mathcal{O})$ is a subsolution (resp. a supersolution) of the equation*

$$F_\varepsilon(y, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0 \quad in\ \mathcal{O}\ , \tag{15}$$

where $(F_\varepsilon)_\varepsilon$ is a sequence of continuous functions satisfying the ellipticity condition. If $u_\varepsilon \to u$ in $C(\mathscr{O})$ and if $F_\varepsilon \to F$ in $C(\mathscr{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathscr{S}^N)$ then u is a subsolution (resp. a supersolution) of the equation

$$F(y,u,Du,D^2u) = 0 \quad \text{in } \mathscr{O}\ .$$

We first recall that the convergence in the spaces of continuous functions $C(\mathscr{O})$ or $C(\mathscr{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathscr{S}^N)$ is the uniform convergence on compact subsets.

This result allows to pass to the limit in a nonlinear equation (and in particular with a nonlinearity on the gradient and the Hessian matrix of the solutions) with only the local uniform convergence of the sequence $(u_\varepsilon)_\varepsilon$, which, of course, does not imply any strong convergence (for example, a convergence in the almost everywhere sense) neither on the gradient nor a fortiori on the Hessian matrix of the solutions.

An unusual characteristic of this result is to consider separately the convergence of the equation—or more precisely of the nonlinearities F_ε—and of the solutions u_ε. Classical arguments would lead to a question like: is the convergence of u_ε strong enough in order to pass to the limit in the equality $F_\varepsilon(y,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon) = 0$?". In this case, the necessary convergence on u_ε would have depended strongly on the equations through the properties of the F_ε. Here this is not at all the case: the required convergences for F_ε and u_ε are fixed a priori.

The most classical example of application of this result is the vanishing viscosity method

$$-\varepsilon\Delta u_\varepsilon + H(y,u_\varepsilon,Du_\varepsilon) = 0 \quad \text{in } \mathscr{O}.$$

This explains why we present the above result in the second-order framework. In this case, the nonlinearity F_ε is given by

$$F_\varepsilon(y,u,p,M) = -\varepsilon \mathrm{Tr}(M) + H(y,u,p)\ ,$$

and its convergence in $C(\mathscr{O}\times\mathbb{R}\times\mathbb{R}^N\times\mathscr{S}^N)$ to $H(y,u,p)$ is obvious. If u_ε converges uniformly to u, then Theorem 4.1 implies that u is a solution of

$$H(y,u,Du) = 0 \quad \text{in } \mathscr{O}\ .$$

The above example shows that the solutions of Hamilton–Jacobi Equations—and more generally of nonlinear elliptic equations—obtained by the vanishing viscosity method are viscosity solutions of these equations, and this justifies the terminology.

In practical use, most of the time, Theorem 4.1 is applied to a subsequence of $(u_\varepsilon)_\varepsilon$ instead of the sequence itself. When one wants to pass to the limit in an equation of the type (15), one proceeds, in general, as follows:

1. One proves that u_ε is locally bounded in L^∞, uniformly w.r.t $\varepsilon > 0$.
2. One shows that u_ε is locally bounded in some Hölder space $C^{0,\alpha}$ for some $0 \leq \alpha < 1$ or in $W^{1,\infty}$, uniformly w.r.t $\varepsilon > 0$.
3. Because of the two first steps, by Ascoli's Theorem, the sequence $(u_\varepsilon)_\varepsilon$ is in a compact subset of $C(K)$ for any $K \subset\subset \mathscr{O}$.

4. One applies the stability result to a converging subsequence of $(u_\varepsilon)_\varepsilon$ which is obtained by a diagonal extraction procedure.

This method will be really complete only when we will have a uniqueness result: indeed, the above argument shows that all converging subsequence of the sequence $(u_\varepsilon)_\varepsilon$ converges to A viscosity solution of the limiting equation. If there exists only one solution of this equation then all the converging subsequences converge to THE viscosity solution of the limiting equation that we denote by u. A classical compactness and separation argument then implies that all the sequence $(u_\varepsilon)_\varepsilon$ converge to u (exercise!).

But, in order to have uniqueness and to justify this argument, one has to impose boundary conditions and also to be able to pass to the limit in these boundary conditions ... (to be continued!).

We now give an example of application of this method.

Example. This example is unavoidably a little bit formal since our aim is to show a mechanism of passage to the limit by viscosity solutions' methods and we do not intend to obtain the estimates we need in full details. In particular, we are to use the Maximum Principle in $\mathbb{R}^N$ without justification.

For $\varepsilon > 0$, let $u_\varepsilon \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be the unique solution of the equation

$$-\varepsilon \Delta u_\varepsilon + H(Du_\varepsilon) + u_\varepsilon = f(x) \quad \text{in } \mathbb{R}^N,$$

where H is a locally Lipschitz continuous function on $\mathbb{R}^N$, $H(0) = 0$ and $f \in W^{1,\infty}(\mathbb{R}^N)$. By the Maximum Principle, we have

$$-||f||_\infty \le u_\varepsilon \le ||f||_\infty \quad \text{in } \mathbb{R}^N,$$

because $-||f||_\infty$ and $||f||_\infty$ are respectively sub- and supersolution of the equation. Moreover, if $h \in \mathbb{R}^N$, since $u_\varepsilon(. + h)$ is a solution of an analogous equation where $f(.)$ is replaced by $f(. + h)$ in the right-hand side, the Maximum Principle also implies

$$||u_\varepsilon(. + h) - u_\varepsilon(.)||_\infty \le ||f(. + h) - f(.)||_\infty \quad \text{in } \mathbb{R}^N,$$

and, since f is Lipschitz continuous, the right-hand side is estimated by $C|h|$ where C is the Lipschitz constant of f. This yields

$$||u_\varepsilon(. + h) - u_\varepsilon(.)||_\infty \le C|h| \quad \text{in } \mathbb{R}^N .$$

Since this inequality is true for any h, it implies that u_ε est Lipschitz continuous with Lipschitz constant C.

Using the Ascoli's Theorem and a diagonal extraction procedure, we can extract a subsequence still denoted by $(u_\varepsilon)_\varepsilon$ which converges to a continuous function u which is, by Theorem 4.1, a solution of the equation

$$H(Du) + u = f(x) \quad \text{in } \mathbb{R}^N \ .$$

In this example, we perform a passage to the limit in a singular perturbation problem without facing much difficulties; again this example will be complete only when we will know that u is the unique solution of the limiting equation since it will imply that the whole sequence $(u_\varepsilon)_\varepsilon$ converges to u by a classical compactness and separation argument.

Now we turn to the *Proof of Theorem 4.1*. We prove the result only in the subsolution case, the other case being shown in an analogous way.

We consider $\phi \in C^2(\mathcal{O})$ and $y_0 \in \mathcal{O}$ a local maximum point of $u-\phi$. Subtracting if necessary a term like $\chi(y) = |y - y_0|^4$ to $u-\phi$, one can always assume that y_0 is a *strict* local maximum point. We then use the following lemma (left as an exercise).

Lemma 4.1. *Let $(v_\varepsilon)_\varepsilon$ be a sequence of continuous functions on an open subset $\mathcal{O}$ which converge in $C(\mathcal{O})$ to v. If $y_0 \in \mathcal{O}$ is a strict local maximum point of v, there exists a sequence of local maximum points of v_ε, denoted by $(y_\varepsilon)_\varepsilon$, which converges to y_0.*

One uses Lemma 4.1 with $v_\varepsilon = u_\varepsilon - (\phi + \chi)$ and $v = u - (\phi + \chi)$. Since u_ε is a subsolution of (15) and since y^ε is a local maximum of $u_\varepsilon - (\phi + \chi)$, we have, by definition

$$F_\varepsilon\Big(y^\varepsilon, u_\varepsilon(y^\varepsilon), D\phi(y^\varepsilon) + D\chi(y^\varepsilon), D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon)\Big) \leq 0 \ .$$

Now we have just to pass to the limit in this inequality: since $y^\varepsilon \to y_0$, we use the regularity of the test-functions ϕ and χ which implies

$$D\phi(y^\varepsilon) + D\chi(y^\varepsilon) \to D\phi(y_0) + D\chi(y_0) = D\phi(y_0) \ ,$$

and

$$D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon) \to D^2\phi(y_0) + D^2\chi(y_0) = D^2\phi(y_0) \ .$$

Moreover, because of the local uniform convergence of u_ε, we have $u_\varepsilon(y^\varepsilon) \to u(y_0)$, and the convergence of F_ε finally yields

$$F_\varepsilon\Big(y^\varepsilon, u_\varepsilon(y^\varepsilon), D\phi(y^\varepsilon) + D\chi(y^\varepsilon), D^2\phi(y^\varepsilon) + D^2\chi(y^\varepsilon)\Big)$$
$$\to F\Big(y_0, u(y_0), D\phi(y_0), D^2\phi(y_0)\Big) \ .$$

Therefore

$$F\Big(y_0, u(y_0), D\phi(y_0), D^2\phi(y_0)\Big) \leq 0.$$

And the proof is complete.

5 Uniqueness: The Basic Arguments and Additional Recipes

5.1 A First Basic Result

In this section, we present the basic arguments to obtain "comparison results" for viscosity solutions. In order to simplify the presentation, we begin with a simple result and then we show (few) additional arguments which are needed in order to extend it to different situations.

We consider the equation

$$u_t + H(x, t, Du) = 0 \quad \text{in } \Omega \times (0, T) \,, \tag{16}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $T > 0$ and, here, Du denotes the gradient of u in the space variable x and H is a continuous function. We use the (standard) notations

$$Q = \Omega \times (0, T) \;\text{ and }\; \partial_p Q = \partial\Omega \times [0, T] \cup \overline{\Omega} \times \{0\} \,.$$

$\partial_p Q$ is called the parabolic boundary of Q.

By "comparison result", we mean the following

If $u, v \in C(\overline{Q})$ are respectively subsolution and supersolution of (16) *and if $u \leq v$ on $\partial_p Q$ then*

$$u \leq v \quad \text{on } \overline{Q}.$$

To state and prove the main result, we use the following assumption

(H1) There exists a modulus $m : [0, +\infty) \to [0, +\infty)$ such that, for any $x, y \in \overline{\Omega}$, $t \in (0, T]$ and $p \in \mathbb{R}^N$

$$|H(x, t, p) - H(y, t, p)| \leq m\big(|x - y|(1 + |p|)\big) \,.$$

We recall that a modulus m is an increasing, positive function, defined on $[0, +\infty)$ such that $m(r) \to 0$ when $r \downarrow 0$.

The result is the following.

Theorem 5.1. *If (H1) holds, we have a comparison result for* (16)*. Moreover, the result remains true if we replace the hypothesis* (H1) *by either "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.*

This result means that the Maximum Principle, which is classical for elliptic and parabolic equations, extends to viscosity solutions of first-order Hamilton–Jacobi Equations.

At first glance, assumption (H1) does not seem to be a very natural assumption. We first remark that, if H is a locally Lipschitz continuous function in x for any

$t \in (0,T]$ and for any $p \in \mathbb{R}^N$, (H1) is satisfied if there exists a constant $C > 0$, such that, for any $t \in (0,T]$ and $p \in \mathbb{R}^N$

$$|\frac{\partial H}{\partial x}(x,t,p)| \le C(1+|p|) \quad \text{a.e. in } \mathbb{R}^N .$$

This version of (H1) is perhaps easier to understand.

In order to justify (H1), let us consider the case of the transport equation

$$u_t - b(x)\cdot Du = f(x) \quad \text{in } Q \ . \tag{17}$$

It is clear that the hypothesis (H1) is satisfied if b is a Lipschitz continuous vector field on $\overline{\Omega}$ and the function f has to be continuous on $\overline{\Omega}$.

In this example, the Lipschitz assumption on b is the most restrictive and important in order to have (H1): we will see in the proof of Theorem 5.1 the central role of the term $|x-y|.|p|$ in (H1) which comes from this hypothesis. But it is well-known that the properties of (17) are connected to those of the dynamical system

$$\dot{x}(t) = b(x(t)) \ . \tag{18}$$

Indeed, one can compute the solutions of (17) by solving this ode through the *Method of Characteristics*. Therefore the Lipschitz assumption on b appears as being rather natural since it is also the standard assumption to have existence and uniqueness for (18) by the Cauchy–Lipschitz Theorem.[3]

Remark 5.1. It is worth pointing out that, in Theorem 5.1, no assumption is made on the behavior of H en p (except indirectly with the restrictions coming from (H1)). For example, one has a uniqueness result for the equation

$$u_t + H(Du) = f(x,t) \quad \text{in } Q \ ,$$

if f is continuous on Q, for *any* continuous function H, without any growth condition.

There are a lot of variations for Theorem 5.1: for example, one can play with (H1) and the regularity of the solutions (as it is already the case in the statement of Theorem 5.1).

A classical and useful corollary of Theorem 5.1 is the one when we do not assume anything on the sub and supersolution on the parabolic boundary of Q

[3] In Biton [19], a non-trivial counterexample to the uniqueness for (17) is given in a situation where the Cauchy–Lipschitz Theorem cannot be applied to (18).

Corollary 5.1. *Under the assumptions of Theorem 5.1, if $u, v \in C(\overline{Q})$ are respectively sub and supersolutions of (16) then*

$$\max_{\overline{Q}}\,(u-v)^+ \le \max_{\partial_p Q}\,(u-v)^+ \ .$$

Moreover, the result remains true if we replace (H1) by "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.

The proof of Corollary 5.1 is immediate by remarking that, if we set $C = \max_{\partial_p Q}\,(u-v)^+$, $v + C$ is still a supersolution of (16) and $u \le v + C$ on $\partial_p Q$. Theorem 5.1 implies then $u \le v + C$ on $\overline{Q}$, which is the desired result.

Remark 5.2. As the above proof shows it, this type of corollary is an immediate consequence of all comparison results with a suitable change on the sub or supersolution which may be more complicated depending on the dependence of H in u. We can have also more precise results by applying the comparison property on sub-intervals.

Now we turn to the *Proof of Theorem 5.1*. The aim of is to show that $M = \max\limits_{\overline{Q}}\big(u-v\big)$ is less or equal to 0. We argue by contradiction assuming that $M > 0$.

In order to simplify the proof, we are going to make some reductions and to give preliminary results.

First, changing u in $u_\eta(x,t) := u(x,t) - \eta t$ for some $\eta > 0$ (small), we may assume without loss of generality that u a strict subsolution of (16) since u_η is a subsolution of

$$(u_\eta)_t + H(x,t,Du_\eta) \le -\eta < 0 \quad \text{in } \Omega \times (0,T) \tag{19}$$

To complete the proof, it suffices to show that $u_\eta \le v$ on $\overline{Q}$ for any η and then to let η tends to 0. Notice also that we still have $u_\eta \le v$ on $\partial_p Q$. To simplify the notations and since the proof is clearly reduced to compare u_η and v, we drop the η and use the notation u instead of u_η.

Next, we consider the difficulty with $\Omega \times \{T\}$: a priori, we do not know if $u \le v$ on this part of the boundary and a maximum point of $u - v$ (or related functions) can be located there. It is solved by the

Lemma 5.1. *If $u, v \in C(\overline{Q})$ are respectively sub and supersolutions of (16) in Q, they are also sub and supersolutions in $\Omega \times (0,T]$. More precisely the viscosity inequalities hold if the maximum or minimum points are on $\Omega \times \{T\}$.*

We leave the simple checking of this result to the reader: if (x_0,T) is a strict maximum point of $u - \varphi$, where φ is a smooth function, we consider the function $u(x,t) - \varphi(x,t) - \frac{\eta}{T-t}$ for $\eta > 0$ small enough. By Lemma 4.1, this function has a maximum point at a nearby point (x_η, t_η) $(t_\eta < T)$ and $(x_\eta,t_\eta) \to (x_0,T)$; in order to conclude, it suffices to pass to the limit in the viscosity inequality at the point (x_η,t_η), remarking that the term $\frac{\eta}{T-t}$ has a positive derivative which can be dropped.

Next, since u and v are not smooth, we need an argument in order to be able to use the definition of viscosity solutions. This argument is the "doubling of variables". For $0 < \varepsilon, \alpha \ll 1$, we introduce the "test-function"

$$\psi_{\varepsilon,\alpha}(x,t,y,s) = u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} .$$

The function $\psi_{\varepsilon,\alpha}$ being continuous on $\overline{Q} \times \overline{Q}$, it achieves its maximum at a point which we denote by $(\overline{x},\overline{t},\overline{y},\overline{s})$ and we set $\overline{M} := \psi_{\varepsilon,\alpha}(\overline{x},\overline{t},\overline{y},\overline{s})$; we have dropped the dependences of $\overline{x},\overline{t},\overline{y},\overline{s}$ and $\overline{M}$ in all the parameters in order to avoid heavy notations.

Because of the "penalisation" terms $\left(\frac{|x-y|^2}{\varepsilon^2}\right.$ and $\left.\frac{|t-s|^2}{\alpha^2}\right)$ which imposes to the maximum points $(\overline{x},\overline{t},\overline{y},\overline{s})$ of $\psi_{\varepsilon,\alpha}$ to verify $(\overline{x},\overline{t}) \sim (\overline{y},\overline{s})$ if ε, α are small enough, one can think that the maximum of $\psi_{\varepsilon,\alpha}$ looks like the maximum of $u - v$. This idea is justified by the following lemma which plays a key role in the proof.

Lemma 5.2. *The following properties hold*

1. *When* $\varepsilon, \alpha \to 0$, $\overline{M} \to M$.
2. $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) \to M$ *when* $\varepsilon, \alpha \to 0$.
3. *We have*

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}, \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to 0 \quad \text{when } \varepsilon, \alpha \to 0 .$$

Moreover, if u or v is Lipschitz continuous in x, then $\overline{p} := \dfrac{2(\overline{x}-\overline{y})}{\varepsilon^2}$ *is bounded by twice the (uniform in t) Lipschitz constant of u or v.*

4. $(\overline{x},\overline{t})$, $(\overline{y},\overline{s}) \in \Omega \times (0,T]$ *if* ε, α *are sufficiently small.*

We conclude the proof of the theorem by using the lemma. We assume that ε, α are sufficiently small in order that the last point of the lemma holds true. Since $(\overline{x},\overline{t},\overline{y},\overline{s})$ is a maximum point of $\psi_{\varepsilon,\alpha}$, $(\overline{x},\overline{t})$ is a maximum point of the function

$$(x,t) \mapsto u(x,t) - \varphi^1(x,t) ,$$

where

$$\varphi^1(x,t) = v(\overline{y},\overline{s}) + \frac{|x-\overline{y}|^2}{\varepsilon^2} + \frac{|t-\overline{s}|^2}{\alpha^2} ;$$

but u is viscosity subsolution of (19) and $(\overline{x},\overline{t}) \in \Omega \times (0,T]$, therefore

$$\frac{\partial \varphi^1}{\partial t}(\overline{x},\overline{t}) + H\left(\overline{x},\overline{t}, D\varphi^1_\varepsilon(\overline{x},\overline{t})\right) = \frac{2(\overline{t}-\overline{s})}{\alpha^2} + H\left(\overline{x},\overline{t}, \frac{2(\overline{x}-\overline{y})}{\varepsilon^2}\right) \leq -\eta .$$

In the same way, $(\overline{y},\overline{s})$ is a maximum point of the function

$$(y,s) \mapsto -v(y,s) + \varphi^2(y,s) ,$$

where

$$\varphi^2(y,s) = u(\overline{x},\overline{t}) - \frac{|\overline{x}-y|^2}{\varepsilon^2} - \frac{|\overline{t}-s|^2}{\alpha^2} \; ;$$

hence $(\overline{y},\overline{s})$ is a minimum point of the function $v - \varphi^2$; but v is viscosity supersolution of (16) and $(\overline{y},\overline{s}) \in \Omega \times (0,T]$, therefore

$$\frac{\partial \varphi^2}{\partial s}(\overline{y},\overline{s}) + H\left(\overline{y},\overline{s}, D\varphi^2(\overline{y},\overline{s})\right) = \frac{2(\overline{t}-\overline{s})}{\alpha^2} + H\left(\overline{y},\overline{s}, \frac{2(\overline{x}-\overline{y})}{\varepsilon^2}\right) \geq 0 \; .$$

Then we subtract the two viscosity inequalities: recalling that $\overline{p} := \dfrac{2(\overline{x}-\overline{y})}{\varepsilon^2}$, we obtain

$$H\left(\overline{x},\overline{t},\overline{p}\right) - H\left(\overline{y},\overline{s},\overline{p}\right) \leq -\eta \; .$$

We can remark that a formal proof where we would assume that u et v are C^1 and where we could directly consider a maximum point of $u - v$, would have lead us to an analogous situation, the term $\overline{p}$ playing the role of "$Du = Dv$" at the maximum point; the fact that we keep such equality here is a key point in the proof. The only -rather important- difference is the one corresponding to the current points: $(\overline{x},\overline{t})$ for u, $(\overline{y},\overline{s})$ for v. This is where (H1) is going to play a central role.

We add and subtract the term $H(\overline{x},\overline{s},\overline{p})$ which allows us to rewrite the inequality as

$$\left(H(\overline{x},\overline{t},\overline{p}) - H(\overline{x},\overline{s},\overline{p})\right) - \left(H(\overline{y},\overline{s},\overline{p}) - H(\overline{x},\overline{s},\overline{p})\right) \leq -\eta \; .$$

In the left-hand side, the first term is related to the regularity of H in t and the second one to the regularity of H in x, namely (H1). For fixed ε, $\overline{p}$ remains bounded (say, by at most a K/ε for some constant $K > 0$) and denoting by m_H^ε the modulus of continuity of H on $\overline{Q} \times \overline{B(0,K/\varepsilon)}$, we are lead, using (H1) to

$$m_H^\varepsilon(|\overline{t}-\overline{s}|) + m\left(|\overline{x}-\overline{y}|(1+|\overline{p}|)\right) \leq -\eta \; .$$

But, on one hand, $|\overline{t}-\overline{s}| \to 0$ as $\alpha \to 0$ since the maximum point property implies that the penalisation term $\frac{|\overline{t}-\overline{s}|^2}{\alpha^2}$ is less than $R := \max(||u||_\infty, ||v||_\infty)$ (see the proof of Lemma 5.2 below) and therefore $|\overline{t}-\overline{s}| \leq (2R)^{1/2}\alpha$ while, on the other hand,

$$|\overline{x}-\overline{y}|(1+|\overline{p}|) = |\overline{x}-\overline{y}| + \frac{2|\overline{x}-\overline{y}|^2}{\varepsilon^2} \to 0 \quad \text{when } \varepsilon, \alpha \to 0 \; .$$

In order to conclude, we first fix ε and let α tend to 0 and then we let ε tend to 0. The above inequality and the properties we just recall lead us to a contradiction.

In the case when u or v is Lipschitz continuous in x, uniformly w.r.t. t, Lemma 5.2 implies that $|\overline{p}|$ is uniformly bounded and the contradiction just follows from the uniform continuity of H on $\overline{Q} \times \overline{B(0,2\tilde{K})}$, where $\tilde{K}$ denotes the Lipschitz constant of u or v, and the proof is complete.

Now we prove Lemma 5.2. Since $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ is a maximum point of $\psi_{\varepsilon,\alpha}$, we have, for any $(x,t), (y,s) \in \overline{Q}$

$$\psi_{\varepsilon,\alpha}(x,t,y,s) \leq \psi_{\varepsilon,\alpha}(\overline{x}, \overline{t}, \overline{y}, \overline{s}) = u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} = \overline{M} \ . \tag{20}$$

Choosing $x = y$ and $t = s$ in the left-hand side yields

$$u(x,t) - v(x,t) \leq \overline{M} \ , \quad \text{for all } (x,t) \in \overline{Q} \ ,$$

and, by considering the supremum in x, we obtain the inequality $M \leq \overline{M}$.

Since u, v are bounded, we can set as above $R := \max(||u||_\infty, ||v||_\infty)$ and we also have by arguing in an analogous way

$$M \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq 2R - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \ .$$

Recalling that we assume $M > 0$, we deduce

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq 2R \ .$$

In particular, $|\overline{x}-\overline{y}|, |\overline{t}-\overline{s}| \to 0$ as $\varepsilon, \alpha \to 0$.
Now we use again the inequality

$$M \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \leq u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) \ . \tag{21}$$

Since $\overline{Q}$ is compact, we may assume without loss of generality that $(\overline{x}, \overline{t}), (\overline{y}, \overline{s})$ converge and this is to the same point because $|\overline{x}-\overline{y}|, |\overline{t}-\overline{s}| \to 0$ as $\varepsilon, \alpha \to 0$. We deduce from this property and (21) that

$$M \leq \liminf(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) \leq \limsup(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) \leq M \ . \tag{22}$$

As a consequence $\lim(u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s})) = M$ and using again (21)

$$\overline{M} = u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) - \frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} - \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to M \ .$$

But, since $u(\overline{x}, \overline{t}) - v(\overline{y}, \overline{s}) \to M$, we immediately deduce that

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \to 0 \ ,$$

and we have proved the two first points of the lemma.

For the last one, it is enough to remark that, if (x,t) is a limit of a subsequence of $(\overline{x},\overline{t})$, $(\overline{y},\overline{s})$, then $u(x,t) - v(x,t) = M > 0$ and therefore (x,t) cannot be on $\partial_p Q$.

It just remains to prove the estimate on $\overline{p}$ if u or v is Lipschitz continuous in x, uniformly w.r.t. t. We assume, for instance, that u has this property with Lipschitz constant $\tilde{K}$, the proof with v being analogous.

We come back to (20) and we choose $x = y = \overline{y}$, $t = \overline{t}$ and $s = \overline{s}$; after straightforward computations, this yields

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} \le u(\overline{y},\overline{t}) - u(\overline{x},\overline{t}) \le \tilde{K}|\overline{x}-\overline{y}| \ .$$

Therefore $|\overline{p}| \le 2\tilde{K}$. This concludes the proof of lemma.

5.2 Several Variations

The first one concerns equations with a dependence in u

$$u_t + H(x,t,u,Du) = 0 \quad \text{in } \Omega \times (0,T) \ . \tag{23}$$

Of course, an assumption is needed in order to avoid Burgers type equations which do not fall into this kind of framework. The classical one is

(H2) For any $0 < R < +\infty$, there exists $\gamma_R \in \mathbb{R}$ such that, for any $(x,t) \in Q$, $-R \le v \le u \le R$ and $p \in \mathbb{R}^N$

$$H(x,t,u,p) - H(x,t,v,p) \ge \gamma_R(u-v) \ .$$

If $\gamma_R \ge 0$ for any R, then the proof follows exactly from the same arguments. Otherwise, the simplest way to reduce to this case is to make a change of variable $u \to u\exp(\gamma t)$ for some well-chosen $\gamma \in \mathbb{R}$, typically some γ_R for large enough R (larger than $||u||_\infty$). Finally we point out that, in general, (H1) is modified by allowing the modulus m to depend on R as γ_R in (H2).

Next we consider problems set in the whole space $\mathbb{R}^N$ where the lack of compactness of the domain creates additional problems. The following assumption is needed

(H3) H is uniformly continuous on $\mathbb{R}^N \times [0,T] \times \overline{B}_R$ for any $R > 0$.

We also introduce the space $BUC(\mathbb{R}^N \times [0,T])$ of the functions which are bounded, uniformly continuous on $\mathbb{R}^N \times [0,T]$. The result for (16) is the

Theorem 5.2. *Assume (H1) and (H3). If $u, v \in BUC(\mathbb{R}^N \times [0,T])$ are respectively sub and supersolution of* (16) *with $\Omega = \mathbb{R}^N$, then*

$$\sup_{\mathbb{R}^N\times[0,T]} (u - v) \leq \sup_{\mathbb{R}^N} (u(x,0) - v(x,0)) .$$

Moreover, the result remains true if we replace the hypothesis (H1) *by either "u is Lipschitz continuous in x" or by "v is Lipschitz continuous in x", uniformly w.r.t. t.*

We just sketch the proof since it follows the same ideas as the proof of Theorem 5.1: for $0 < \varepsilon, \alpha, \beta \ll 1$, we introduce the test-function

$$\psi(x,t,y,s) = u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \beta(|x|^2 + |y|^2) .$$

The main change is with the β-term: because of the non-compactness of the domain, such term is needed for the maximum of ψ to be achieved. Two technical remarks are enough to complete the proof:

1. From the proof of Lemma 5.2, it is clear that $\beta(|x|^2 + |y|^2) \leq R = \max(||u||_\infty, ||v||_\infty)$ and these terms produces derivatives which are small since $|2\beta x| = 2\beta^{1/2}(\beta(|x|^2)^{1/2} \leq 2\beta^{1/2}R^{1/2}$ and the same is (of course) true for $2\beta y$. (H3) takes care of these small perturbations.
2. The proof of Lemma 5.2 is not as simple as in the compact case because the result is not true in general for any continuous functions u and v. In fact, the behavior of the maximum of ψ depends on the way we play with the different parameters. The two extreme cases are:
   - If we fix β and let first ε and α tend to 0, the maximum of ψ actually converges to $\max_{\overline{Q}}(u(x,t) - v(x,t) - 2\beta|x|^2))$ and then, if we send β tend to 0, this maximum converges to the supremum of $u - v$.
   - But, if, on the contrary, we first let β tend to 0 by fixing ε and α and then we let ε and α tend to 0, the maximum of ψ does not converges to the supremum of $u - v$ but to $\limsup_{h\downarrow 0} \sup_{|(x,t)-(y,s)|\leq h} (u(x,t) - v(y,s))$.

In general these limits are different and therefore playing with the parameters may be delicate. This explains the assumption "u or v is in $BUC(\mathbb{R}^N \times [0,T])$" in Theorem 5.2: indeed all these limits are the same in this case. In the $BUC(\mathbb{R}^N \times [0,T])$ framework, the proof follows the one of Theorem 5.1 since (21) leads to

$$\begin{aligned}
\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} &\leq u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) - M \\
&\leq u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) + u(\overline{y},\overline{s}) - v(\overline{y},\overline{s}) - M \\
&\leq u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) ,
\end{aligned}$$

because $u(\overline{y},\overline{s}) - v(\overline{y},\overline{s}) \leq M$. If m_u denotes a modulus of continuity of u, we have $u(\overline{x},\overline{t}) - u(\overline{y},\overline{s}) \leq m_u(|(\overline{x},\overline{t}) - (\overline{y},\overline{s})|)$ and therefore

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \frac{|\bar{t}-\bar{s}|^2}{\alpha^2} \le m_u(|(\bar{x},\bar{t})-(\bar{y},\bar{s})|) \ .$$

Finally using that $|\bar{x}-\bar{y}| \le (2R)^{1/2}\varepsilon$ and $|\bar{t}-\bar{s}| \le (2R)^{1/2}\alpha$, we have a complete estimate of the penalisation terms.

Remark 5.3. In fact, there is a technical way which allows to avoid (partially) the above mentioned difficulty, assuming only that there exists $u_0 \in BUC(\mathbb{R}^N)$ such that

$$u(x,0) \le u_0(x) \le v(x,0) \quad \text{in } \mathbb{R}^N \ .$$

By a standard result (exercise!), the modulus m given by (H1) satisfies: for any $\eta > 0$, there exists C_η such that $m(\tau) \le C_\eta \tau + \eta/2$. We then change the test-function into

$$\psi(x,t,y,s) = u(x,t) - v(y,s) - \exp(C_\eta t)\frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \beta(|x|^2+|y|^2) \ .$$

The effect of the new "$\exp(C_\eta t)$"-term is to produce a positive $C_\eta \exp(C_\eta t)\frac{|x-y|^2}{\varepsilon^2}$ term in the inequality which allows to control the "bad" dependence in $\frac{|x-y|^2}{\varepsilon^2}$ and therefore allows to treat cases where we do not know that this quantity tends to 0. Clearly the α-penalisation term does not create any difficulty.

5.3 Finite Speed of Propagation

An important feature of time-dependent equations is the possibility of having "finite speed of propagation" type results which can be stated in the following way for $u, v \in C(\mathbb{R}^N \times [0,T])$ which are respectively sub and supersolution of (16) in $\mathbb{R}^N \times [0,T]$

There exists a constant $c > 0$ such that, if $u(x,0) \le v(x,0)$ in $B(0,R)$ for some R then $u(x,t) \le v(x,t)$ for any x in $B(0,R-ct)$, $ct \le R$.

The constant c is the "speed of propagation" and, of course, $B(0,R)$ can be replaced by any other ball $B(z,R)$. The key assumption for having such result is the

(H4) For any $x \in \mathbb{R}^N$, $t \in [0,T]$ and $p, q \in \mathbb{R}^N$

$$|H(x,t,p) - H(x,t,q)| \le C|p-q| \ .$$

Theorem 5.3. *Assume (H1) and (H4). Then we have a "finite speed of propagation" type results for (16) in $\mathbb{R}^N \times [0,T]$ with a speed of propagation equal to C.*

Before giving the proof of this result, we want to point out that such result may also be obtained for sub and supersolutions which are Lipschitz continuous in space, uniformly w.r.t. t by assuming only H to be locally Lipschitz continuous in p: indeed, in that case, only bounded p and q play a role and the inequality in (H4) is satisfied if H is locally Lipschitz continuous.

Proof of Theorem 5.3. We just sketch it since it is a long but easy proof which borrows a lot of arguments from the proof of Theorem 5.1.

Lemma 5.3. *If* $u, v \in C(\mathbb{R}^N \times [0,T])$ *are respectively sub and supersolution of* (16) *in* $\mathbb{R}^N \times [0,T]$*, the function* $w := u - v$ *is a subsolution of*

$$w_t - C|Dw| = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \ . \tag{24}$$

Formally the result is obvious since it suffices to subtract the inequalities for u and v and use (H4). But to show it in the viscosity sense is a little bit more technical. Again we just sketch the proof: if (x_0, t_0) is a *strict* maximum point of $w - \varphi$ where φ is a smooth test-function, we introduce the function

$$(x,t,y,s) \mapsto u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \varphi(x,t) \ .$$

If (x_0, t_0) is a *strict* maximum point of $w - \varphi$ in $\overline{B((x_0,t_0),r)}$, we look at maximum points of this function in $\overline{B((x_0,t_0),r)} \times \overline{B((x_0,t_0),r)}$. Because of the compactness of the domain, the maximum is achieved at a point $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ and one easily shows that $(\overline{x}, \overline{t}), (\overline{y}, \overline{s}) \to (x_0, t_0)$ as $\varepsilon, \alpha \to 0$; in particular $(\overline{x}, \overline{t}), (\overline{y}, \overline{s})$ are in $B((x_0,t_0),r)$ for ε, α small enough. Writing the viscosity inequalities, following the arguments of the proof of Theorem 5.1 and using (H4), one concludes easily.

The next step consists in showing that, if $w(x,0) \le 0$ in $B(0,R)$ for some R, then $w(x,t) \le 0$ for any x in $B(0, R - Ct)$, $Ct \le R$, which is equivalent to the "finite speed of propagation" type results. To do so, it is enough to build a suitable sequence of (smooth) supersolutions.

We introduce smooth functions $\chi_\delta : \mathbb{R} \to \mathbb{R}$ such that $\chi_\delta(r) \equiv 0$ for $r \le R - \delta$, $\chi_\delta(r) \equiv M$ for $r \ge R$, where $M = \max_{\overline{B(0,R)} \times [0,T]} w(x,t)$ and χ_δ is increasing in $\mathbb{R}$. Next we consider the functions $\chi_\delta(|x| + Ct)$; it is immediate to check that this function is a smooth solution of (24) for $Ct \le R - \delta$, i.e. for $t \le t_\delta := (R-\delta)/C$ and that, on $\partial B(0,R) \times [0, t_\delta]$ and $B(0,R) \times \{0\}$, $w(x,t) \le \chi_\delta(|x| + Ct)$. Applying Theorem 5.1 in $B(0,R) \times [0, t_\delta]$, we obtain that $w(x,t) \le \chi_\delta(|x| + Ct)$ in $B(0,R) \times [0, t_\delta]$ and therefore, by the properties of χ_δ, $w(x,t) \le 0$ for $|x| + Ct \le R + \delta$. Letting δ tend to 0 gives the complete answer.

Remark 5.4. In fact, we do not really need a comparison result, namely Theorem 5.1, to conclude: the last part of the proof follows from the definition of viscosity (sub)solution. Indeed the function $\chi_\delta(|x| + Ct) + \delta t$ is a smooth *strict* supersolution in $B(0,R) \times (0, t_\delta)$; this shows that $w(x,t) - (\chi_\delta(|x| + Ct) + \delta t)$ cannot achieve a maximum point in $B(0,R) \times (0,T]$, which immediately leads to the conclusion.

6 Discontinuous Viscosity Solutions, Discontinuous Nonlinearities and the "Half-Relaxed Limits" Method

The main objective of this section is to present a general method, based on the notion of discontinuous viscosity solutions, which allows passage to the limit in (fully) nonlinear pdes with just an L^∞-bounds on the solutions. To do so, we have to extend the notion of viscosity solution to the discontinuous setting. We refer to Ishii [30,31], Perthame and the author [8,9] for the notion of discontinuous viscosity solutions, the half-relaxed limits method being introduced in [8].

We use the following notations: if z is a locally bounded function (possibly discontinuous), we denote by z^* its upper semicontinuous (usc) envelope

$$z^*(x) = \limsup_{y\to x} \; z(y) \, ,$$

and by z_* its lower semicontinuous (lsc) envelope

$$z_*(x) = \liminf_{y\to x} \; z(y) \, .$$

6.1 Discontinuous Viscosity Solutions

The definition is the following.

Definition 6.1 (Discontinuous Viscosity Solutions). A locally bounded upper semicontinuous (usc in short) function u is a viscosity subsolution of the equation

$$G(y, u, Du, D^2u) = 0 \quad \text{on } \overline{\mathcal{O}} \tag{25}$$

if and only if, for any $\varphi \in C^2(\overline{\mathcal{O}})$, if $y_0 \in \overline{\mathcal{O}}$ is a maximum point of $u - \varphi$, one has

$$G_*(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \leq 0.$$

A locally bounded lower semicontinuous (lsc in short) function v is a viscosity supersolution of the (25) **if and only if**, for any $\varphi \in C^2(\overline{\mathcal{O}})$, if $y_0 \in \overline{\mathcal{O}}$ is a minimum point of $u - \varphi$, one has

$$G^*(y_0, u(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0 \, .$$

A (discontinuous) solution is a function whose usc and lsc envelopes are respectively viscosity sub and supersolution of the equation.

The first reason to introduce such a complicated formulation is to unify the convergence result we present in the next section: in fact, when $\mathcal{O}$ is an open subset different from $\mathbb{R}^N$, the function G may contain both the equation and the boundary condition. With such general formulation, we avoid to have a different result for each type of boundary conditions. The possibility of handling discontinuous sub and supersolutions is also a key point in the convergence proof.

To be more specific, we consider the problem

$$\begin{cases} F(y,u,Du,D^2u) = 0 & \text{in } \mathscr{O}, \\ B(y,u,Du) = 0 & \text{on } \partial\mathscr{O}, \end{cases}$$

where F, B are a given continuous functions.

In order to solve it, a classical idea consists in considering the vanishing viscosity method

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + F(y,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon) = 0 & \text{in } \mathscr{O}, \\ B(y,u_\varepsilon,Du_\varepsilon) = 0 & \text{on } \partial\mathscr{O}. \end{cases}$$

Indeed, by adding a $-\varepsilon\Delta$ term, we regularize the equation in the sense that one can expect to have more regular solutions for this approximate problem—typically in $C^2(\mathscr{O}) \cap C^1(\overline{\mathscr{O}})$.

If we assume that this is indeed the case, i.e. that this regularized problem has a smooth solution u_ε and that, moreover, $u_\varepsilon \to u$ in $C(\overline{\mathscr{O}})$. It is easy to see, by the arguments of Theorem 4.1, that the continuous function u satisfies in the viscosity sense

$$\begin{cases} F(y,u,Du,D^2u) = 0 & \text{in } \mathscr{O}, \\ \min(F(y,u,Du,D^2u), B(y,u,Du)) \leq 0 & \text{on } \partial\mathscr{O}, \\ \max(F(y,u,Du,D^2u), B(y,u,Du)) \geq 0 & \text{on } \partial\mathscr{O}, \end{cases}$$

where, for example, the "min" inequality on $\partial\mathscr{O}$ means: *for any* $\varphi \in C^2(\overline{\mathscr{O}})$*, if* $y_0 \in \partial\mathscr{O}$ *is a maximum point of* $u - \varphi$ *on* $\overline{\mathscr{O}}$*, one has*

$$\min(F(y_0,u(y_0),D\varphi(y_0),D^2\varphi(y_0)), B(y,u(y_0),Du(y_0))) \leq 0.$$

The interpretation of this new problem can be done by setting the equation in $\overline{\mathscr{O}}$ instead of $\mathscr{O}$. To do so, we introduce the function G defined by

$$G(y,u,p,M) = \begin{cases} F(y,u,p,M) & \text{if } y \in \mathscr{O}, \\ B(y,u,p) & \text{if } y \in \partial\mathscr{O}. \end{cases}$$

The above argument shows that the function u is a viscosity solution of

$$G(y,u,Du,D^2u) = 0 \quad \text{on } \overline{\mathscr{O}},$$

and in particular on $\overline{\mathscr{O}}$, if

$$G_*(y,u,Du,D^2u) \leq 0 \quad \text{on } \overline{\mathscr{O}}$$

$$G^*(y,u,Du,D^2u) \geq 0 \quad \text{on } \overline{\mathscr{O}}$$

where G_* and G^* stand respectively for the lower semicontinuous and upper semicontinuous envelopes of G. Indeed, the "min" and the "max" above are nothing but G_* and G^* on $\partial\mathcal{O}$.

6.2 Back to the Running Example (II): The Dirichlet Boundary Condition for the Value-Function

In this subsection, we show that the value function of the exit time control problem actually satisfy the Dirichlet boundary condition in the viscosity sense.

To do so, we use a more sophisticated version of the Dynamic Programming Principle.

Theorem 6.1. *Under the assumptions* **(CA)***, the value-function satisfies, for any* $x \in \overline{\Omega}$*,* $t > 0$ *and* $0 < S < t$

$$\mathbf{U}(x,t) = \inf_{v(.)}\left[\int_0^{S\wedge\tau} f(y_x(s),\alpha(s))ds + \mathbf{1}_{\{S<\tau\}}\mathbf{U}(y_x(S),t-S) + \mathbf{1}_{\{S\geq\tau\}}\varphi(y_x(\tau))\right]. \tag{26}$$

In order to understand why this formulation leads naturally to boundary conditions in the viscosity solutions sense, we consider $x \in \partial\Omega$, $0 < t < T$ and a sequence $(x_\varepsilon, t_\varepsilon)$ converging to (x,t) such that $\mathbf{U}(x_\varepsilon, t_\varepsilon) \to \mathbf{U}_*(x,t)$. We apply the Dynamic Programming Principle at the point $(x_\varepsilon, t_\varepsilon)$. We argue formally assuming that there exists an optimal control $\alpha_\varepsilon(\cdot)$ in such a way that we have

$$\begin{aligned}\mathbf{U}(x_\varepsilon,t_\varepsilon) = &\int_0^{S\wedge\tau_\varepsilon} f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{1}_{\{S<\tau_\varepsilon\}}\mathbf{U}(y_{x_\varepsilon}(S),t_\varepsilon-S)\\ &+ \mathbf{1}_{\{S\geq\tau_\varepsilon\}}\varphi(y_{x_\varepsilon}(\tau_\varepsilon))\ .\end{aligned}$$

Here there are two cases:

(i) Either $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$ and letting ε tends to 0, we obtain (formally) $\mathbf{U}_*(x,t) = \varphi(x)$.

(ii) Or τ_ε remains bounded away from 0 and by choosing S small enough, we have

$$\mathbf{U}(x_\varepsilon,t_\varepsilon) = \int_0^S f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{U}(y_{x_\varepsilon}(S),t_\varepsilon-S)\ ,$$

which, since $\mathbf{U} \geq \mathbf{U}_*$ on $\overline{\Omega}$ can be rewritten as

$$\mathbf{U}_*(x,t) + o_\varepsilon(1) \geq \int_0^S f(y_{x_\varepsilon}(s),\alpha_\varepsilon(s))ds + \mathbf{U}_*(y_{x_\varepsilon}(S),t_\varepsilon-S)\ ,$$

a similar situation to the case when $x \in \Omega$. Playing with ε and S (or fixing S and using relaxed controls to pass to the limit $\varepsilon \to 0$), it is easy to show that the supersolution inequality holds.

In conclusion, boundary conditions in the viscosity solutions sense are natural from the optimal control point of view since they take into account the strategy of the controller and/or the controlability properties of the system. Indeed, we obtain $\mathbf{U}_*(x,t) \geq \varphi(x)$ [i.e. we are in the case (i)] if either it is interesting in term of cost to pay φ (and if we can exit the domain to do it) or, on the contrary, if we are obliged to exit the domain, even if this cost is high. Case (ii) may arise either if we want to avoid paying the cost φ (and if some control allows to do it) or if we have no choice but to go away from the boundary.

These interpretations for the "min" and "max" inequalities are important since they connect the control problem and its properties with the equation and the boundary conditions.

6.3 The Half-Relaxed Limit Method

The first key point is a stability result for discontinuous viscosity solutions. To state it we use the following notations: if $(z_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions, the half-relaxed limits of $(z_\varepsilon)_\varepsilon$ are defined by

$$\limsup{}^* \; z_\varepsilon(y) \;=\; \limsup_{\substack{\tilde{y}\to y\\ \varepsilon\to 0}} z_\varepsilon(\tilde{y}) \quad \text{and} \quad \liminf{}_* \; z_\varepsilon(y) \;=\; \liminf_{\substack{\tilde{y}\to y\\ \varepsilon\to 0}} z_\varepsilon(\tilde{y}) \; .$$

Theorem 6.2. *Assume that, for $\varepsilon > 0$, u_ε is an usc viscosity subsolution (resp. a lsc supersolution) of the equation*

$$G_\varepsilon(y, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) = 0 \quad \text{on } \overline{\mathscr{O}} \; ,$$

where $(G_\varepsilon)_\varepsilon$ is a sequence of uniformly locally bounded functions in $\overline{\mathscr{O}} \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N$ which satisfy the ellipticity condition. If the functions u_ε are uniformly locally bounded on $\overline{\mathscr{O}}$, then $\overline{u} = \limsup^* \; u_\varepsilon$ (resp. $\underline{u} = \liminf_* \; u_\varepsilon$) is a subsolution (resp. a supersolution) of the equation

$$\underline{G}(y, u, Du, D^2 u) = 0 \quad \text{on } \overline{\mathscr{O}} \; ,$$

where $\underline{G} = \liminf_* \; G_\varepsilon$.
(resp. of the equation

$$\overline{G}(y, u, Du, D^2 u) = 0 \quad \text{on } \overline{\mathscr{O}} \; ,$$

where $\overline{G} = \limsup^* \; G_\varepsilon$).

Of course, the main interest of this result is to allow the passage to the limit in fully nonlinear, degenerate elliptic pdes with only a uniform local L^∞—bound on the solutions. This is a striking difference with Theorem 4.1 which requires far more informations on the u_ε's. The counterpart is that we do not have anymore a limit but two half-limits $\overline{u}$ and $\underline{u}$ which have to be connected in order to obtain a real convergence result.

This is the aim of the *half-relaxed limit method*:

1. One proves that the u_ε are uniformly bounded in L^∞ (locally or globally).
2. One applies the above discontinuous stability result.
3. By definition, we have $\underline{u} \leq \overline{u}$ on $\overline{\mathscr{O}}$.
4. To obtain the converse inequality, one uses a *Strong Comparison Result* (SCR in short) i.e. a comparison result which is valid for *discontinuous* sub and supersolutions. It yields

$$\overline{u} \leq \underline{u} \quad \text{in } \mathscr{O} \text{ (or on } \overline{\mathscr{O}}\text{)}.$$

5. From the SCR, we deduce $\overline{u} = \underline{u}$ in $\mathscr{O}$ (or on $\overline{\mathscr{O}}$). If we set $u := \overline{u} = \underline{u}$, then u is continuous (because $\overline{u}$ is usc and $\underline{u}$ is lsc) and it is easy to show that, on one hand, u is *the unique solution* of the limiting equation (using again the SCR) and, on the other hand, we have the convergence of u_ε to u in $C(\mathscr{O})$ (or in $C(\overline{\mathscr{O}})$).

It is clear that, in this method, SCR play a central role: we give in the next subsection few indications on how to prove such results and references on the existing SCR.

We first describe a typical example of the use of Theorem 6.2.

Example 6.1. We consider the problem

$$\begin{cases} -\varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = 1 \text{ in } (0,1) \\ \quad u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases}$$

Of course, it is expected that the solution of this problem converges to the solution of

$$\begin{cases} \quad u'(x) = 1 & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

But the solution of this problem does not seem to exist.

The solution u_ε can be computed explicitly

$$u_\varepsilon(x) = x - \frac{\exp(\varepsilon^{-1}(x-1)) - \exp(-\varepsilon^{-1})}{1 - \exp(-\varepsilon^{-1})} ,$$

and therefore we can also compute the half-relaxed limits of the sequence $(u_\varepsilon)_\varepsilon$

$$\overline{u}(x) = x \quad \text{and} \quad \underline{u}(x) = \begin{cases} x & \text{if } x \in [0,1) \\ 0 & \text{for } x = 1 \,. \end{cases}$$

By Theorem 6.2, these half-relaxed limits are respectively sub and supersolution of

$$u'(x) - 1 = 0 \quad \text{in } (0,1) \ ,$$
$$\min(u'(x) - 1, u) \leq 0 \quad \text{at } x = 0 \text{ and } 1 \ ,$$
$$\max(u'(x) - 1, u) \geq 0 \quad \text{at } x = 0 \text{ and } 1 \ .$$

The problem is, of course, at the point $x = 1$ where $\overline{u}$ is 1 while $\underline{u}$ is 0. Several remarks: this fact is a consequence of the *boundary layer* near 1 since u_ε looks like x but it has also to satisfies the Dirichlet boundary condition $u_\varepsilon(1) = 0$. A clear advantage of Theorem 6.2 is that we can pass to the limit despite of this boundary layer. Of course, there is no hope here to apply Theorem 4.1. But the price to pay is that $\overline{u}(1)$ is different from $\underline{u}(1)$.

In order to recover the right result, namely the convergence in $[0,1)$ of u_ε to x, the SCR has to take care of this difference and this is done by "erasing" the "wrong" value of $\underline{u}$ at 1. This explains why we wrote above that we can compare $\overline{u}$ and $\underline{u}$ either in $\mathcal{O}$ or on $\overline{\mathcal{O}}$: here we can do it only in $\mathcal{O} := (0,1)$ (and even in $[0,1)$).

Now we give the *Proof of Theorem 6.2*. We do it only for the subsolution case, the supersolution one being analogous.

It is based on the

Lemma 6.1. *Let $(v_\varepsilon)_\varepsilon$ be a sequence of uniformly bounded usc functions on $\overline{\mathcal{O}}$ and $\overline{v} = \limsup^* \ v_\varepsilon$. If $y \in \overline{\mathcal{O}}$ is a strict local maximum point of $\overline{v}$ on $\overline{\mathcal{O}}$, there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ of $(v_\varepsilon)_\varepsilon$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ of points in $\overline{\mathcal{O}}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $v_{\varepsilon'}$ in $\overline{\mathcal{O}}$, the sequence $(y_{\varepsilon'})_{\varepsilon'}$ converges to y and $v_{\varepsilon'}(y_{\varepsilon'}) \to \overline{v}(y)$.*

We first prove Theorem 6.2 by using the lemma. Let $\varphi \in C^2(\overline{\mathcal{O}})$ and let $y \in \overline{\mathcal{O}}$ be a strict local maximum point de $\overline{u} - \varphi$. We apply Lemma 6.1 to $v_\varepsilon = u_\varepsilon - \varphi$ and $\overline{v} = \overline{u} - \varphi = \limsup^* \ (u_\varepsilon - \varphi)$. There exists a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ and a sequence $(y_{\varepsilon'})_{\varepsilon'}$ such that, for all ε', $y_{\varepsilon'}$ is a local maximum point of $u_{\varepsilon'} - \varphi$ on $\overline{\mathcal{O}}$. But $u_{\varepsilon'}$ is a subsolution of the $G_{\varepsilon'}$-equation, therefore

$$G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'}), D^2\varphi(y_{\varepsilon'})) \leq 0.$$

Since $y_{\varepsilon'} \to x$ and since φ is smooth $D\varphi(y_{\varepsilon'}) \to D\varphi(y)$ and $D^2\varphi(y_{\varepsilon'}) \to D^2\varphi(y)$; but we have also $u_{\varepsilon'}(y_{\varepsilon'}) \to \overline{u}(y)$, therefore by definition of $\underline{G}$

$$\underline{G}(x, \overline{u}(y), D\varphi(y), D^2\varphi(y)) \leq \liminf \ G_{\varepsilon'}(y_{\varepsilon'}, u_{\varepsilon'}(y_{\varepsilon'}), D\varphi(y_{\varepsilon'}), D^2\varphi(y_{\varepsilon'})) \ .$$

This immediately yields

$$\underline{G}(x, \overline{u}(y), D\varphi(y), D^2\varphi(y)) \leq 0,$$

and the proof is complete.

Now we turn to the *Proof of Lemma 6.1*: since y is a strict local maximum point of $\overline{v}$ on $\overline{\mathcal{O}}$, there exists $r > 0$ such that

$$\forall z \in \overline{\mathcal{O}} \cap \overline{B}(y,r)\ , \quad \overline{v}(z) \leq \overline{v}(y)\ ,$$

the inequality being strict for $z \neq y$. But $\overline{\mathcal{O}} \cap \overline{B}(y,r)$ is compact and v_ε is usc, therefore, for all $\varepsilon > 0$, there exists a maximum point y^ε of v_ε on $\overline{\mathcal{O}} \cap \overline{B}(y,r)$. In other words

$$\forall z \in \overline{\mathcal{O}} \cap \overline{B}(y,r)\ , \quad v_\varepsilon(z) \leq v_\varepsilon(y^\varepsilon)\ . \tag{27}$$

Now we take the lim sup for $z \to y$ and $\varepsilon \to 0$: by the definition of the lim sup* , we obtain

$$\overline{v}(y) \leq \limsup_{\varepsilon}\ v_\varepsilon(y^\varepsilon)\ .$$

Next we consider the right-hand side of this inequality: extracting a subsequence denoted by ε', we have $\limsup_\varepsilon\ v_\varepsilon(y^\varepsilon) = \lim_{\varepsilon'} v_{\varepsilon'}(y_{\varepsilon'})$ and since $\overline{\mathcal{O}} \cap \overline{B}(y,r)$ is compact, we may also assume that $y_{\varepsilon'} \to \overline{y} \in \overline{\mathcal{O}} \cap \overline{B}(y,r)$. But using again the definition of the lim sup* at $\overline{y}$, we get

$$\overline{v}(y) \leq \limsup_{\varepsilon}\ v_\varepsilon(y^\varepsilon) = \lim_{\varepsilon'} v_{\varepsilon'}(y_{\varepsilon'}) \leq \overline{v}(\overline{y})\ .$$

Since y is a strict maximum point of $\overline{v}$ in $\overline{\mathcal{O}} \cap \overline{B}(y,r)$ and that $\overline{y} \in \overline{\mathcal{O}} \cap \overline{B}(y,r)$, this inequality implies that $\overline{y} = y$ and that $v_{\varepsilon'}(y_{\varepsilon'}) \to \overline{v}(y)$ and the proof is complete.

We conclude this subsection by the

Lemma 6.2. *If $\mathcal{K}$ is a compact subset of $\overline{\mathcal{O}}$ and if $\overline{u} = \underline{u}$ on $\mathcal{K}$ then u_ε converges uniformly to the function $u := \overline{u} = \underline{u}$ on $\mathcal{K}$.*

Proof of Lemma 6.2. Since $\overline{u} = \underline{u}$ on $\mathcal{K}$ and since $\overline{u}$ is usc and $\underline{u}$ is lsc on $\overline{\mathcal{O}}$, u is continuous on $\mathcal{K}$.

We first consider $M_\varepsilon = \sup_{\mathcal{K}} \left(u_\varepsilon^* - u\right)$. The function u_ε^* being usc and u being continuous, this supremum is in fact a maximum and is achieved at a point y^ε. The sequence $(u_\varepsilon)_\varepsilon$ being locally uniformly bounded, the sequence $(M_\varepsilon)_\varepsilon$ is also bounded and, $\mathcal{K}$ being compact, we can extract subsequences such that $M_{\varepsilon'} \to \limsup_\varepsilon M_\varepsilon$ and $y_{\varepsilon'} \to \overline{y} \in \mathcal{K}$. But by the definition of the lim sup* , $\limsup u_{\varepsilon'}^*(y_{\varepsilon'}) \leq \overline{u}(\overline{y})$ while we have also $u(y_{\varepsilon'}) \to u(\overline{y})$ by the continuity of u. We conclude that

$$\limsup_{\varepsilon}\ M_\varepsilon = \lim_{\varepsilon'}\ M_{\varepsilon'} = \lim_{\varepsilon'}\ u_{\varepsilon'}^*(y_{\varepsilon'}) - u(y_{\varepsilon'}) \leq \overline{u}(\overline{y}) - u(\overline{y}) = 0\ .$$

This part of the proof gives half of the uniform convergence, the other part being obtained analogously by considering $\tilde{M}_\varepsilon = \sup_{\mathcal{K}}\ (u - (u_\varepsilon)_*)$.

6.4 Strong Comparison Results

In general, this is clearly THE difficulty when applying the half-relaxed limit method.

The basic comparison result we have already proved, namely Theorem 5.1, is in fact a SCR: we use the continuity of u and v only once to obtain that $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) \to M$ and then an estimate on the penalization terms through the inequality

$$\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2} + \frac{|\overline{t}-\overline{s}|^2}{\alpha^2} \le u(\overline{x},\overline{t}) - v(\overline{y},\overline{s}) - M \to 0\ .$$

But, if $(\overline{x},\overline{t}),(\overline{y},\overline{s}) \to (x_0,t_0)$, we have $\limsup u(\overline{x},\overline{t}) \le u(x_0,t_0)$ because u is usc and $\liminf v(\overline{y},\overline{s}) \ge v(x_0,t_0)$ because v is lsc, and therefore $\limsup(u(\overline{x},\overline{t}) - v(\overline{y},\overline{s})) \le M$, which is enough to obtain both the convergence of $u(\overline{x},\overline{t}) - v(\overline{y},\overline{s})$ to M and the right property for the penalization terms.

For problem with boundary conditions:

(a) One has general SCR for Neumann BC (even for second-order equations): see [6, 34].
(b) Dirichlet boundary conditions present more difficulties, at least when they are not assumed in a classical sense: we refer to [5, 9, 10] for first-order problems and [12] for second-order problems.

We come back again to our running example and provide a Strong Comparison Result for the Dirichlet problem of the exit time control problem.

Theorem 6.3. *Under the above assumptions, if Ω is a $W^{2,\infty}$-domain and if there exists $\nu > 0$ such that, for any $x \in \partial\Omega$, there exists $\alpha_x^1, \alpha_x^2 \in V$ such that*

$$b(x,\alpha_x^1)\cdot n(x) \ge \nu \quad \textit{and} \quad b(x,\alpha_x^2)\cdot n(x) \le -\nu\ , \tag{28}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x, then we have a Strong Comparison Result for (7)–(9), namely if u and v are respectively sub and supersolution of (7)–(9), then

$$u \le v \quad \text{on } \Omega\ .$$

We first comment Assumption (28): it is a (partial) controlability assumption on the boundary; roughly speaking, it means that, in a neighborhood of each point $x \in \partial\Omega$, the controller has both the possibility to leave Ω by using α_x^1 or to stay inside Ω by using α_x^2.

It is also worth pointing out that we can compare u and v only in Ω: unfortunately, as Example 6.1 shows it, the boundary conditions in the viscosity sense (at least in the Dirichlet case) do not impose strong enough constraints on the boundary and one may have "artificial" values for u and/or v. This is why we have to redefine u and/or v on the boundary in the proof of the SCR and also why the result holds only in Ω.

The program to study such control problems and obtain that the value-function is continuous and the unique solution of the associated Bellman problem is the following:

(a) Show that one has a dynamic programming principle for the control problem: in general, this is easy for deterministic problems, more technical for stochastic ones because of measurability issues. An alternative solution consists in arguing by approximation.
(b) Deduce that, if $\mathbf{U}$ is the value function, then $\mathbf{U}^*$ and $\mathbf{U}_*$ are respectively viscosity sub and supersolution of the Bellman problem.
(c) Use the Strong Comparison Result to prove that $\mathbf{U}^* \leq \mathbf{U}_*$ which shows that $\mathbf{U} := \mathbf{U}^* = \mathbf{U}_*$ is continuous since it is both upper and lower semicontinuous.
(d) Use again the Strong Comparison Result to obtain the uniqueness result.

7 Existence of Viscosity Solutions: Perron's Method

Perron's method was introduced in the context of viscosity solutions by Ishii [31]. We present the main arguments in the case of (16) together with the initial data

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{29}$$

where $u_0 \in BUC(\mathbb{R}^N)$.

The result is the

Theorem 7.1. *Assume (H1), (H3) and that $u_0 \in BUC(\mathbb{R}^N)$. For any $T > 0$, there exists a unique viscosity solution u of* (16)–(29) *in $BUC(\mathbb{R}^N \times [0,T])$.*

Proof of Theorem 7.1. We denote by $M = ||u_0||_\infty$ and $C = \sup_{\mathbb{R}^N \times [0,T]} H(x,t,0)$.

The functions $\underline{u}(x,t) := -M - Ct$ and $\overline{u}(x,t) := M + Ct$ are respectively sub and supersolution of (16); moreover

$$\underline{u}(x,0) \leq u_0(x) \leq \overline{u}(x,0) \quad \text{in } \mathbb{R}^N .$$

We denote by $\mathscr{S}$ the set of all usc subsolutions w of (16) such that $\underline{u} \leq w \leq \overline{u}$ in $\mathbb{R}^N \times [0,T]$ and which satisfies $w(x,0) \leq u_0(x)$ in $\mathbb{R}^N$. Then we set

$$u(x,t) = \sup\{w(x,t) : \ w \in \mathscr{S}\} .$$

The first step consists in showing that u^* is a (possibly discontinuous) viscosity subsolution of (16). The proof of this claim comes from three types of arguments:

1. If u_1 and u_2 are usc functions then $D^{2,+}[\sup(u_1,u_2)] \subset D^{2,+}u_1 \cap D^{2,+}u_2$, a property which immediately yields that the supremum of two subsolutions (and then of a finite number of subsolutions) is a subsolution.

2. Next the discontinuous stability result allows to extend this result to a countable number of subsolutions. In this case, the supremum of a countable number of usc functions is not necessarily usc and one has to use an usc envelope: this is done automatically by the $\limsup^*$ operation.
3. In order to prove that u^* is a subsolution of (16), we have to extend Point 2 to any set of subsolutions. We remark that, for a given point (x,t), there exists a sequence $(w_n)_n$ of elements of $\mathscr{S}$ such that, if

$$v_n(y,s) := \sup_{0\leq k\leq n} w_k(y,s) \; ,$$

then

$$u^*(x,t) = \limsup{}^* \; v_n(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} v_n(y,s) = \left(\sup_{k\in\mathbb{N}} w_k(y,s)\right)^* \; .$$

This leads us to introduce the function $\tilde{u} := \limsup^* \; v_n$ which is a subsolution of (16) by Point 2. To conclude, we use an analogous argument to the one of Point 1. If u_1 and u_2 are usc functions such that $u_1 \leq u_2$ and $u_1(x,t) = u_2(x,t)$ for some point (x,t) then $D^{2,+}u_2(x,t) \subset D^{2,+}u_1(x,t)$. Applying this result with $u_1 = \tilde{u}$ and $u_2 = u^*$ shows that u^* satisfies the subsolution inequalities at (x,t) since $\tilde{u}$ does. Since this is true for any point (x,t), we have proved that u^* is a subsolution of (16) and also that u is usc since, by definition, $u \geq u^*$ because $u^* \in \mathscr{S}$.

The next step consists in showing that u_* is a viscosity supersolution of (16). To do so, we argue by contradiction assuming that there exists a smooth function ϕ such that $u_* - \phi$ has a global minimum point at some $(\overline{x},\overline{t})$ for $\overline{t} > 0$ and

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t}, D\phi(\overline{x},\overline{t})) < 0 \; . \tag{30}$$

We may assume without loss of generality that $u_*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t})$. For $\varepsilon > 0$, we consider the functions

$$w_\varepsilon(x,t) = \max\{u(x,t), \phi_\varepsilon(x,t)\},$$

where $\phi_\varepsilon(x,t) := \phi(x,t) + \varepsilon - |x-\overline{x}|^4 - |t-\overline{t}|^4$.

Since $\phi \leq u_* \leq u$ and $u_*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t})$, w_ε can differ from u only in a small neighborhood of $(\overline{x},\overline{t})$ and more precisely where $|x-\overline{x}|^4 + |t-\overline{t}|^4 \leq \varepsilon$. And we point out that this neighborhood becomes smaller and smaller with ε. Using (30), we see that ϕ and therefore ϕ_ε are subsolution of (16) in a small neighborhood of $(\overline{x},\overline{t})$. This implies that w_ε is still a subsolution of (16) as the supremun of two subsolutions, if we choose ε small enough.

Next we want to prove that $w_\varepsilon \in \mathscr{S}$ and to do so, it remains to show that $w_\varepsilon \leq \overline{u}$, at least if ε is small enough. Since this is true for u, we have just to check it for ϕ_ε and for $|x-\overline{x}|^4 + |t-\overline{t}|^4 \leq \varepsilon$, i.e. close enough to $(\overline{x},\overline{t})$.

By the same argument as in Point 3 above, we cannot have $u_*(\overline{x},\overline{t}) = \overline{u}(\overline{x},\overline{t})$: otherwise, since $u_* \leq \overline{u}$, $D^{2,-}u_*(\overline{x},\overline{t}) \subset D^{2,+}\overline{u}(\overline{x},\overline{t})$ and u_* would satisfies the supersolutions inequalities at $(\overline{x},\overline{t})$. Therefore $u_*(\overline{x},\overline{t}) = \phi(\overline{x},\overline{t}) = \phi_\varepsilon(\overline{x},\overline{t}) < \overline{u}(\overline{x},\overline{t})$ and, for ε small enough, the last inequality remains true in a neighborhood by the continuity of ϕ_ε and $\overline{u}$. Hence $w_\varepsilon \in \mathscr{S}$.

This fact is a contradiction with the definition of u: indeed,

$$u_*(\overline{x},\overline{t}) := \liminf_{(y,s)\to(\overline{x},\overline{t})} u(y,s) = \lim_k u(y_k,t_k)\ .$$

But, $w_\varepsilon(\overline{x},\overline{t}) = u_*(\overline{x},\overline{t}) + \varepsilon$ and by the continuity of w_ε, it is clear that, for k large enough, $u(y_k,t_k) < w_\varepsilon(y_k,t_k)$.

In fact, the above argument is not completely correct since we do not take into account the initial data. There are two ways to do it, the first one being simpler, the second one being more general.

The first solution consists in showing that u is, in fact, continuous at time $t=0$ and that $u(x,0) = u_0(x)$ for any $x \in \mathbb{R}^N$. To do so, we remark that, thanks to the property on the modulus of continuity recalled in Remark 5.3, since u_0 is uniformly continuous in $\mathbb{R}^N$, we have, for any $x,y \in \mathbb{R}^N$ and $\eta > 0$

$$u_0(x) - \eta/2 - C_\eta|x-y| \leq u_0(y) \leq u_0(x) + \eta/2 + C_\eta|x-y|\ ,$$

for some large constant $C_\eta > 0$. Then choosing a constant $\tilde{C}_\eta > 0$ large enough, the functions

$$u_\pm(y,t) = u_0(x) \pm \eta/2 \pm C_\eta|x-y| \pm \tilde{C}_\eta t\ ,$$

are respectively viscosity subsolution and supersolution of (16). We use these functions in the following way: on one hand, if $w \in \mathscr{S}$, $w \leq u_+$ in $\mathbb{R}^N \times [0,T]$; this inequality can be easily obtained by smoothing the term $|x-y|$ and remarking that u^+ being a strict supersolution of (16) for $\tilde{C}_\eta$ large enough, $w - u_+$ cannot achieved a maximum in $\mathbb{R}^N \times (0,T]$ (remark also that such maximum is achieved because $u_+(y,t) \to +\infty$ as $|y| \to +\infty$) and therefore it is achieved for $t=0$ where $w \leq u_+$. On the other hand, $\max(u_-,\underline{u}) \in \mathscr{S}$. Therefore, combining these properties with the definition of u, we have

$$u_- \leq \max(u_-,\underline{u}) \leq u \leq u_+ \quad \text{in } \mathbb{R}^N \times [0,T]\ ,$$

and, since $u_\pm$ are continuous, this yields $u_-(x,0) \leq u_*(x,0) \leq u^*(x,0) \leq u_+(x,0)$, i.e

$$u_0(x) - \eta/2 \leq u_*(x,0) \leq u^*(x,0) \leq u_0(x) + \eta/2\ .$$

This property being true for any $\eta > 0$ and $x \in \mathbb{R}^N$, we have $u^*(x,0) \le u_0(x)$ and $u_*(x,0) \ge u_0(x)$ in $\mathbb{R}^N$, which are the desired properties since they imply that u is continuous at $(x,0)$ and $u(x,0) = u_0(x)$.

The second method to treat the initial data consists in understanding this initial data in the viscosity solution sense, i.e.

$$\min(w_t + H(x,0,Dw), w - u_0) \le 0 \quad \text{in } \mathbb{R}^N, \tag{31}$$

and

$$\max(w_t + H(x,0,Dw), w - u_0) \ge 0 \quad \text{in } \mathbb{R}^N . \tag{32}$$

With few modifications, the above arguments can take into account, at the same time, the equation in the domain and the initial data in this viscosity sense.

Hence u satisfies (31)–(32) but then we use the

Lemma 7.1. *If w is an usc subsolution of* (16) *satisfying* (31) *(resp. a lsc supersolution of* (16) *satisfying* (32)*), we have $w(x,0) \le u_0(x)$ (resp. $u_0(x) \le w(x,0)$) in $\mathbb{R}^N$.*

Therefore, in non-singular situations, initial data in the viscosity sense always reduce to initial data in the classical sense.

Using this lemma, Remark 5.3 shows that we can compare the subsolution u^* and the supersolution u_*; therefore

$$u^*(x,t) \le u_*(x,t) \quad \text{in } \mathbb{R}^N \times [0,T] .$$

But, by definition, the opposite inequality holds and we can conclude that u is continuous, the BUC-property for u coming from a careful examination of the uniqueness proof. And the existence result is complete.

Proof of Lemma 7.1. We prove the result only in the subsolution case, the supersolution one being analogous. For $x \in \mathbb{R}^N$, we introduce the function

$$\chi(y,t) = w(y,t) - \frac{|y-x|^2}{\varepsilon} - C_\varepsilon t ,$$

where $\varepsilon > 0$ is a parameter devoted to tend to 0 and $C_\varepsilon > 0$ is a large constant to be chosen later on.

Standard argument shows that χ has a maximum point $(\overline{y},\overline{t})$ near $(x,0)$ for small enough ε and large enough C_ε. Since w is a subsolution of (16) satisfying (31), if $\overline{t} > 0$, we have

$$C_\varepsilon + H\left(\overline{y},\overline{t},\frac{2(\overline{y}-x)}{\varepsilon}\right) \le 0.$$

But this inequality cannot hold if C_ε is chosen large enough (the size depending on ε and H but neither on $\overline{y}$ nor on $\overline{t}$ since the term $\frac{|\overline{y}-x|^2}{\varepsilon}$ is bounded). Therefore $\overline{t} = 0$ and (31) holds. But since the above inequality cannot hold, (31) implies

$w(\overline{y}, 0) \leq u_0(\overline{y})$. We conclude by remarking that, as $\varepsilon \to 0$, $w(\overline{y}, 0) \to w(x, 0)$ by using the maximum point property and the upper-semicontinuity of w, while $u_0(\overline{y}) \to u_0(x)$ by the continuity of u_0.

8 Regularity Results

The aim of this section is to investigate further regularity properties for the solutions obtained through Theorem 7.1. To do so, we first strengthen assumption (H1) into

(H1-s) There exists $L_1, L_2 > 0$ such that, for any $x, y \in \Omega$, $t \in (0, T]$ and $p \in \mathbb{R}^N$

$$|H(x,t,p) - H(y,t,p)| \leq L_1|x-y||p| + L_2|x-y| \ .$$

Theorem 8.1. *Assume (H1-s), (H3) and that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then the solution of u of* (16)–(29) *given by Theorem 7.1 is Lipschitz continuous in x for any $t \in [0, T]$ and*

$$||Du(\cdot,t)||_\infty \leq \exp(L_1 t)||Du_0||_\infty + \frac{L_2}{L_1}(\exp(L_1 t) - 1) \ .$$

Proof of Theorem 8.1. The proof is similar to the proof of the comparison result and we just sketch it to avoid repeating the same arguments. We introduce the function $(x, y, t) \mapsto u(x, t) - u(y, t) - C(t)|x - y|$: the aim is to show that this function is negative for some well-chosen (smooth) function $C(\cdot)$; at least for $t = 0$, we can choose $C(0) = ||Du_0||_\infty$ to have this property.

To do so, we argue by contradiction, assuming that its supremum is strictly positive and in order to use viscosity solutions' arguments, we double the variables in time, namely

$$\psi(x,t,y,s) = u(x,t) - u(y,s) - C(t)|x-y| - \frac{|t-s|^2}{\alpha^2} - \beta(|x|^2 + |y|^2) \ .$$

For $\alpha, \beta > 0$ small enough, the maximum of ψ is still strictly positive and we denote by $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ a maximum point of ψ. We notice that we cannot have $\overline{x} = \overline{y}$, otherwise $\psi(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ would be negative. Dropping the β-terms which are not going to play any role and performing the same arguments as in the proof of Theorem 5.1, we are lead to the inequality

$$\frac{dC}{dt}(\overline{t})|\overline{x} - \overline{y}| + H\left(\overline{x}, \overline{t}, \overline{p}\right) - H\left(\overline{y}, \overline{s}, \overline{p}\right) \leq 0 \ ,$$

with $\overline{p} = C(\overline{t})\frac{\overline{x}-\overline{y}}{|\overline{x}-\overline{y}|}$. Writing this inequality as

$$\frac{dC}{dt}(\overline{t})|\overline{x} - \overline{y}| + H\left(\overline{x}, \overline{t}, \overline{p}\right) - H\left(\overline{y}, \overline{t}, \overline{p}\right) + H\left(\overline{y}, \overline{t}, \overline{p}\right) - H\left(\overline{y}, \overline{s}, \overline{p}\right) \leq 0 \ ,$$

and using (H1-s), we obtain

$$\frac{dC}{dt}(\bar{t})|\bar{x}-\bar{y}| - L_1 C(\bar{t})|\bar{x}-\bar{y}| - L_2|\bar{x}-\bar{y}| + o_\alpha(1) \leq 0 ,$$

where $o_\alpha(1) \to 0$ as $\alpha \to 0$. If $\frac{dC}{dt}(\bar{t}) - L_1 C(\bar{t}) - L_2 > 0$, we get the contradiction by letting α tend to 0.

Therefore it is enough to solve $\frac{dC}{dt}(\bar{t}) - L_1 C(\bar{t}) - L_2 = \delta$ for some $\delta > 0$ and with $C(0) = ||Du_0||_\infty$. This yields

$$C_\delta(\bar{t}) = \exp(L_1 t)||Du_0||_\infty + \frac{L_2+\delta}{L_1}(\exp(L_1 t) - 1) .$$

The above proof shows that $u(x,t) - u(y,t) - C_\delta(t)|x-y| \leq 0$ for all x, y, t and $\delta > 0$. Letting δ tends to 0, we obtain the right bound on $||Du(\cdot,t)||_\infty$.

An other way to get Lipschitz regularity is, for coercive Hamiltonians, through an estimate of u_t when H is independent of t. We recall that $H(x,p)$ is said to be coercive if it satisfies

(H5) $H(x,p) \to +\infty$ as $|p| \to +\infty$, uniformly in x.

Theorem 8.2. *Assume that H is independent of t and satisfies (H1), (H3) and (H5). If $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then the solution of u of (16)–(29) given by Theorem 7.1 is Lipschitz continuous in x for any $t \in [0,T]$ and*

$$||Du(\cdot,t)||_\infty \leq K(H,u_0) .$$

Proof of Theorem 8.2. By the comparison result, since $u(x,t)$ and $u(x,t+h)$ for $h > 0$ are solutions of the same equation, we have

$$||u(x,t+h) - u(x,t)||_\infty \leq ||u(x,h) - u(x,0)||_\infty .$$

But u_0 being Lipschitz continuous, if we set

$$R := ||Du_0||_\infty \quad \text{and} \quad C := \max_{\mathbb{R}^N \times B(0,R)} |H(x,p)| ,$$

then $u_0(x) - Ct$ and $u_0(x) + Ct$ are respectively viscosity sub and supersolution of the equation and therefore

$$u_0(x) - Ct \leq u(x,t) \leq u_0(x) + Ct \quad \text{in } \mathbb{R}^N \times [0,T] .$$

In particular, $||u(x,h) - u(x,0)||_\infty \leq Ch$ and therefore $||u(x,t+h) - u(x,t)||_\infty \leq Ch$, which implies that $||u_t||_\infty \leq C$.

In order to deduce the gradient bound in space, we consider any point (x,t), $t>0$ and we want to show that $u(y,t) \leq u(x,t) + K|y-x|$ for some large enough constant K. To do so, we consider the function

$$(y,s) \mapsto u(y,s) - u(x,t) - K|y-x| - \frac{(t-s)^2}{\alpha^2}\ .$$

The maximum of this function is achieved at some point $(\overline{y}, \overline{s})$ since u is bounded and $K|y-x| + \frac{(t-s)^2}{\alpha^2} \to +\infty$ if $|y-x| + |t-s| \to +\infty$. Moreover $\overline{s} \to t$ when $\alpha \to 0$.

If $\overline{y} \neq x$, then the function $(y,s) \mapsto u(x,t) + K|y-x| + \frac{(t-s)^2}{\alpha^2}$ is smooth at $(\overline{y}, \overline{s})$ and since u is a viscosity subsolution of (16) we have

$$2\frac{(\overline{s}-t)}{\alpha^2} + H(\overline{y}, \overline{p}) \leq 0,$$

with $\overline{p} = K\frac{\overline{y}-x}{|\overline{y}-x|}$.

Now we claim that $|2\frac{(\overline{s}-t)}{\alpha^2}| \leq 2C$: this can be proved in an analogous way as in the proof of Lemma 5.2 (point (3) for the estimate on $|2\frac{(\overline{x}-\overline{y})}{\varepsilon^2}|$). Using (H5) and the fact that $|\overline{p}| = K$, the above inequality can not hold if K is large enough, namely if $H(\overline{y}, \overline{p}) > 2C$. Therefore $\overline{y} = x$ for α small enough and also necessarily $\overline{s} = t$ (otherwise the value at the maximum would be less than the value at (x,t)). The maximum point property for $s=t$ yields

$$u(y,t) - u(x,t) - K|y-x| \leq 0,$$

which is the desired property.

We provide a last result on the semi-concavity of solutions when the Hamiltonian is convex in p and satisfies some smoothness assumption in (x,p). We recall that a function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is semi-concave (with a uniform constant of semi-concavity wrt t) if there exists a constant $\bar{k}$ such that, for any $x, h \in \mathbb{R}^N$ and $t \in [0,T]$

$$u(x+h,t) + u(x-h,t) - 2u(x,t) \leq \bar{k}|h|^2\ .$$

For the Hamiltonian H, we use the following assumption which is satisfied for example if H is $W^{2,\infty}$ in (x,p) uniformly in t and convex in p

(H6) There exists constants $k_1, k_2 > 0$ such that, for any $x, h, p, k \in \mathbb{R}^N$ and $t \in [0,T]$

$$H(x+h,t,p+k) + H(x-h,t,p-k) - 2H(x,t,p) \geq -k_1|h|^2 - k_2|h||k|\ .$$

The result is the

Theorem 8.3. *Assume that H satisfies (H1), (H3) and (H6). If $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ is semi-concave, then the solution of u of* (16)–(29) *given by Theorem 7.1 is semi-concave in x for any $t \in [0,T]$, with a uniform constant of semi-concavity.*

We just give a short *sketch of the proof of Theorem 8.3* which is tedious since it requires to triple the variables (each of them corresponding to either $x + h$, $x - h$ or x). Namely we introduce the function

$$u(x,t) + u(y,s) - 2u(z,\tau) - \frac{|x+y-2z|^2}{\varepsilon^2} - \frac{|x-z|^2}{\varepsilon^2} - \frac{|y-z|^2}{\varepsilon^2} - \frac{(t-\tau)^2}{\alpha^2} - \frac{(s-\tau)^2}{\alpha^2} - \cdots ,$$

where we have dropped the usual "β"-terms to penalize infinity. With this function, the proof follows from straightforward but tedious computations.

9 Convex Hamiltonians, Barron–Jensen Solutions

In this section, we describe additional properties of viscosity solutions of (16) in the case when H is convex in p. The main motivation is to extend the theory -and in particular the uniqueness results- to the case when the initial data is only lower semi-continuous, a natural framework for optimal control problems. The key ideas described in this section were introduced by Barron and Jensen [17, 18] who also consider the applications to optimal control. The simplified presentation we provide follows the one of [4].

Our first result is the following.

Theorem 9.1. *Assume that H is convex in p and (H3) holds. If $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ satisfies*

$$u_t + H(x,t,Du) \leq 0 \quad a.e \text{ in } \Omega \times (0,T) ,$$

then u is viscosity subsolution of (16).

Proof of Theorem 9.1. We are going to use a standard regularization argument. Let $(\rho_\varepsilon)_\varepsilon$ be a sequence of C^∞, positive, smoothing kernels in $\mathbb{R}^{N+1}$, with compact support in the ball of radius ε. For $\eta > 0$ small enough, we are going to show that

$$u_\varepsilon(x,t) := \int_{\mathbb{R}^{N+1}} u(y,s)\rho_\varepsilon(x-y,t-s)dyds ,$$

is an approximate C^1 subsolution of the equation in $\mathbb{R}^N \times (\eta, T-\eta)$ if $\varepsilon < \eta$.

To do so, for $x \in \mathbb{R}^N$, $t \in (\eta, T-\eta)$, we multiply the equation at the point (y,s) by $\rho_\varepsilon(x-y,t-s)$ and we integrate over $\mathbb{R}^{N+1}$ (or, in fact, over the ball of radius ε). By the properties of the convolution, we obtain

$$(u_\varepsilon)_t(x,t) + \int_{\mathbb{R}^{N+1}} H\left(y,s,Du(y,s)\right)\rho_\varepsilon(x-y,t-s)dyds \leq 0.$$

Using (H3), we can replace, in the integral, $H\ (y, s, Du(y, s))$ by $H\ (x, t, Du(y, s))$ with a small error in ε. This gives

$$(u_\varepsilon)_t(x,t) + \int_{\mathbb{R}^{N+1}} H\ (x,t,Du(y,s))\,\rho_\varepsilon(x-y,t-s)dyds \le o_\varepsilon(1)\ .$$

In order to conclude, we have just to apply Jensen's inequality which leads to

$$(u_\varepsilon)_t(x,t) + H\ (x,t,Du_\varepsilon(x,t))\,dyds \le o_\varepsilon(1)\ .$$

Therefore u_ε is a smooth subsolution of (16) in $\mathbb{R}^N \times (\eta, T-\eta)$, hence a viscosity subsolution of (16) in $\mathbb{R}^N \times (\eta, T-\eta)$ and so is u which is the uniform limit of u_ε, by Theorem 4.1. Since this is true for any η, the proof is complete.

This result has several consequences which are listed in the following

Theorem 9.2. *Assume that H is convex in p and that (H1), (H3) hold.*

(i) The function $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ is a viscosity subsolution (resp. solution) of (16) if and only if, for any smooth function φ, if (x,t) is a local minimum point of $u-\varphi$, one has

$$\varphi_t(x,t) + H(x,t,D\varphi(x,t)) \le 0 \quad (resp. = 0). \tag{33}$$

(ii) If $u_1, u_2 \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ are viscosity subsolutions (resp. solutions) of (16), then $\min(u_1,u_2)$ is also a subsolution (resp. solution) of (16).

(iii) If $u \in W^{1,\infty}(\mathbb{R}^N \times (0,T))$ is a viscosity subsolution of (16) and if (H1-s) holds then

$$u_\varepsilon(x,t) = \inf_{y\in\mathbb{R}^N} \left\{ u(y,t) + e^{-L_1 t}\frac{|x-y|^2}{\varepsilon^2} \right\},$$

is a viscosity subsolution of (16) within a $O(\varepsilon)$ error term which depends only on the L^∞-norm of u.

In (iii), the function u_ε is obtained through an *inf-convolution* procedure on u. The connections of such inf and sup-convolution with viscosity solutions were remarked by Lasry and Lions [35]. In general, an inf-convolution is a supersolution, while sup-convolutions are subsolutions. Therefore (iii) is a priori a rather surprising result.

Proof of Theorem 9.2. The proof of (i), (ii) and (iii) are easy: for (i), we may assume that (x,t) is a *strict* local minimum point of $u-\varphi$ and we can approximate this minimum point by minimum points $(x_\varepsilon, t_\varepsilon)$ of $u_\varepsilon - \varphi$ where u_ε is the sequence of smooth approximations of u built in the proof of Theorem 9.1. By the regularity of u_ε and φ, we have $(u_\varepsilon)_t(x_\varepsilon,t_\varepsilon) = \varphi_t(x_\varepsilon,t_\varepsilon)$ and $Du_\varepsilon(x_\varepsilon,t_\varepsilon) = D\varphi(x_\varepsilon,t_\varepsilon)$ and therefore, since u_ε is a C^1 subsolution of (16)

$$\varphi_t(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon,t_\varepsilon,D\varphi(x_\varepsilon,t_\varepsilon)) \le 0.$$

The conclusion follows by letting $\varepsilon \to 0$.

For (ii), we have just to use Stampacchia's Theorem together with Theorem 9.1: indeed

$$D[\min(u_1, u_2)] = Du_1 \quad \text{if } u_1 < u_2 \text{ and } D[\min(u_1, u_2)] = Du_2 \text{ otherwise,}$$

and $Du_1 = Du_2$ a.e. on the set $\{u_1 = u_2\}$; and the same is, of course, true for the time derivative. To get the subsolution property, we have just to argue in the a.e. sense while the supersolution property always holds since the minimum of supersolutions is a supersolution (exactly in the same way as the maximum of two subsolutions is a subsolution, cf. Perron's method).

For (iii), we just sketch the proof since it requires long but straightforward computations. Using (i), we have look at what happens at a minimum point (x, t) of $u_\varepsilon - \varphi$ where φ is a smooth function. Thanks to the definition of u_ε, this leads to consider minimum point of the function

$$(y, t, z, s) \mapsto u(y, t) + e^{-L_1 t} \frac{|z - y|^2}{\varepsilon^2} - \varphi(z, t) \ .$$

We see that we are in a framework which is close to the proof of the comparison result, and in the spirit of Remark 5.3. The computations are then easy using (i).

Theorem 9.2 provides all the necessary (technical) ingredients to extend the theory and to do so, we are first going to say that a lsc function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ is a Barron–Jensen (BJ for short) subsolution (or solution) of (16) if and only if it satisfies (33). Theorem 9.2 (i) shows that this is equivalent to the usual notion of viscosity solution when u is Lipschitz continuous (and it is also the case when u is continuous).

The extension to lsc subsolutions and solutions, and the uniqueness result are given by the

Theorem 9.3. *Assume that H is convex in p and that (H1), (H3) hold.*

(i) If $(u_\varepsilon)_\varepsilon$ is a sequence of BJ subsolution (resp. solution) of (16) *then* $\liminf_* u_\varepsilon$ *is a subsolution (resp. solution) of* (16).

(ii) Assume (H1-s), (H3) and that u_0 is a bounded lsc initial data. There exists a unique lsc BJ solution u of (16)–(29) *which satisfies*

$$\liminf_{\substack{(y,s)\to(x,0) \\ s>0}} u(y, s) = u_0(x) \ . \tag{34}$$

We just give a very brief sketch of this result. The proof of (i) follows immediately from the arguments of the proof of the (discontinuous) stability results. For (ii), if u is a lsc BJ solution (or even only a subsolution) of (16) then the result of Theorem 9.2 (iii) holds (even if u is just lsc) and (34) implies that $u_\varepsilon(x, 0) \leq u_0(x)$ in $\mathbb{R}^N$. But now u_ε is an approximate solution of (16), which is Lipschitz continuous in x (by its definition through the "inf-convolution" formula) and also in t (by the equation). If v is an other solution, we can compare u_ε and v: clearly $u_\varepsilon(x, 0) \leq v(x)$

in $\mathbb{R}^N$ and the Lipschitz continuity of u_ε allows to use the arguments of the proof of Theorem 5.2 in a rather easy way.

10 Large Time Behavior of Solutions of Hamilton–Jacobi Equations

10.1 Introduction

In this second part, we are interested in the behavior, as $t \to +\infty$, of the viscosity solutions of first-order Hamilton–Jacobi Equations of the form

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ , \tag{35}$$

with the initial data

$$u = u_0 \quad \text{in } \mathbb{R}^N, \tag{36}$$

in the case when the Hamiltonian $H(x, p)$ and the initial datum u_0 are $\mathbb{Z}^N$*-periodic* in x, i.e., for all $x, p \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$,

$$H(x + z, p) = H(x, p) \quad \text{and} \quad u_0(x + z) = u_0(x) \ . \tag{37}$$

and when H is *coercive*, namely

$$H(x, p) \to +\infty \quad \text{when } |p| \to +\infty, \text{ uniformly wrt } x \in \mathbb{R}^N. \tag{38}$$

In the last decade, the large time behavior of solutions of Hamilton–Jacobi Equation in compact manifold $\mathscr{M}$ (or in $\mathbb{R}^N$, mainly in the periodic case) has received much attention and general convergence results for solutions have been established by using two different types of methods: in his course, H. Ishii [this volume] describes the "weak Kam approach" which is an optimal control/dynamical system approach and both uses and provides formulas of representation, the ones for the asymptotic solutions being based on the notion of Aubry–Mather sets.

Our aim is to describe a second approach which relies only on partial differential equations methods: it provides results even when the Hamiltonians are not convex but it gives a slightly less precise description of the phenomenas compared to the "weak Kam approach".

In 1999, Namah and Roquejoffre [42] are the first to obtain convergence results in a general framework, by pde arguments which we describe below. They use the following additional assumptions

$$H(x, p) \geq H(x, 0) \text{ for all } (x, p) \in \mathscr{M} \times \mathbb{R}^N \quad \text{and} \quad \max_{\mathscr{M}} H(x, 0) = 0, \tag{39}$$

where $\mathscr{M}$ is a smooth compact N-dimensional manifold without boundary.

Then Fathi in [25] proved a different type of convergence result, by dynamical systems type arguments, introducing the "weak KAM theory". Contrarily to [42], the results of [25] use strict convexity (and smoothness) assumptions on $H(x,\cdot)$, i.e., $D_{pp}H(x,p) \geq \alpha I$ for all $(x,p) \in \mathscr{M} \times \mathbb{R}^N$ and $\alpha > 0$ (and also far more regularity) but do not require (39). Afterwards Roquejoffre [43] and Davini and Siconolfi in [24] refined the approach of Fathi and they studied the asymptotic problem for Hamilton–Jacobi Equations on $\mathscr{M}$ or N-dimensional torus.

The first author and Souganidis obtained in [15] more general results, for possibly non-convex Hamiltonians, by using an approach based on partial differential equations methods and viscosity solutions, which was not using in a crucial way the explicit formulas of representation of the solutions: this is the second main type of results we (partially) describe here.

All these results (except perhaps the Namah–Roquejoffre ones) use in a crucial way the compactness of the domain: indeed either they are stated on a compact manifold or they use periodicity which means that we are looking at equations set on the torus. We also refer to the articles [11, 27–29, 33] for the asymptotic problems in the whole domain $\mathbb{R}^N$ without the periodic assumptions in various situations.

Finally there also exists results on the asymptotic behavior of solutions of convex Hamilton–Jacobi Equation with boundary conditions. Mitake [38] studied the case of the state constraint boundary condition and then the Dirichlet boundary conditions [39, 40]. Roquejoffre in [43] was also dealing with solutions of the Cauchy–Dirichlet problem which satisfy the Dirichlet boundary condition pointwise (in the classical sense): this is a key difference with the results of [39, 40] where the solutions were satisfying the Dirichlet boundary condition in a generalized (viscosity solutions) sense. These results were slightly extended in [7] by using an extension of PDE approach of [15].

10.2 Existence and Regularity of the Solution

The first result concerns the (global) existence, uniqueness and regularity of the solution.

Theorem 10.1. *Assume that H satisfies* (37)–(38) *and that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ is a $\mathbb{Z}^N$-periodic function. Then there exists a unique solution of* (35)–(36) *which is (i) periodic in x and (ii) Lipschitz continuous in x and t on $\mathbb{R}^N \times [0,+\infty)$.*

We just sketch the *proof of Theorem 10.1* since it is an easy adaptation of the results given in the previous sections, which we can simplify here.

For the existence, we use Perron's method: assuming first that $u_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, the functions $-Ct + u_0(x)$ and $Ct + u_0(x)$ are respectively sub and supersolution of (35)–(36) if C is given by

$$R := ||Du_0||_\infty \quad \text{and} \quad C := \max_{\mathbb{R}^N \times \overline{B(0,R)}} |H(x,p)| \ .$$

Truncating $H(x, p)$ by replacing it by $H_K(x, p) := \min(H(x, p), K)$ for some large constant $K > 0$, we can apply readily Perron's method. We obtain the existence of a continuous solution u_K of the H_K-equation which satisfies

$$-Ct + u_0(x) \le u_K(x,t) \le Ct + u_0(x) \quad \text{for any } x \in \mathbb{R}^N,\ t > 0\,. \tag{40}$$

Periodicity comes directly from the construction since if w is a subsolution of (35)–(36), then it is also the case for $\sup_{z\in\mathbb{Z}^N}[w(\cdot + z)] \ge w(\cdot)$. Therefore the supremum of subsolutions is clearly achieved for a periodic subsolution.

The uniqueness is proved readily by the argument of the proof of Theorem 5.1 (at least if we assume periodicity) or by the slight adaptation for having the comparison in $BUC(\overline{Q})$.

The time derivative $(u_K)_t$ is bounded since, for any $h > 0$

$$||u_K(x,t+h) - u_K(x,t)||_\infty \le ||u_K(x,h) - u_0(x)||_\infty$$

and $-Ch+u_0(x) \le u_K(x,h) \le Ch+u_0(x)$ by construction. Therefore $|(u_K)_t| \le C$ and, if $K > C$, then u_K is a solution of the H-equation. We denote it by u.

Finally, since H is coercive and $H(x, Du) = -u_t$, we deduce immediately that Du is bounded as well. Using that u is Lipschitz continuous, a (slight) variant of Theorem 5.1 implies that it is the unique solution of (35)–(36).

10.3 Ergodic Behavior

The first step in the study of the large time behavior of u is the

Theorem 10.2. *Under the assumptions of Theorem 10.1, there exists a constant $c \in \mathbb{R}$ such that*

$$\frac{u(x,t)}{t} \to c \quad \text{as } t \to +\infty \text{ uniformly w.r.t. } x \in \mathbb{R}^N\,. \tag{41}$$

Proof. We set

$$m(t) := \max_{\mathbb{R}^N}(u(x,t) - u_0(x))\,.$$

We first have

$$m(t+s) \le \max_{\mathbb{R}^N}(u(x,t+s) - u(x,t)) + \max_{\mathbb{R}^N}(u(x,t) - u_0(x))\,,$$

and then by comparison

$$\max_{\mathbb{R}^N}(u(x,t+s) - u(x,t)) \le \max_{\mathbb{R}^N}(u(x,s) - u(x,0)) = m(s)\,.$$

Therefore $m(t+s) \leq m(t) + m(s)$ for any $t, s > 0$ but the Lipschitz continuity of u in t gives also $m(t) \geq -Ct$ for some constant C. A classical result on sub-additive functions implies

$$\frac{m(t)}{t} \to c := \inf_{t>0}\left(\frac{m(t)}{t}\right) .$$

Finally, it is easy to show that $u(x,t) - m(t)$ is bounded independently of x and t by using the periodicity and Lipschitz continuity in x of u, and the result follows.

For the convenience of the reader we sketch the proof of the result for m. Pick any $\tau > 0$. If $t > 0$, there exists $n \in \mathbb{N}$ such that $n\tau \leq t < (n+1)\tau$. Using the sub-additivity of m yields

$$m(t) \leq nm(\tau) + m(\varepsilon) ,$$

where $\varepsilon := t - n\tau \in [0,1)$. Dividing by $t = n\tau + \varepsilon$ gives

$$\frac{m(t)}{t} \leq \frac{nm(\tau)}{n\tau + \varepsilon} + \frac{m(\varepsilon)}{n\tau + \varepsilon} ,$$

and letting $t \to +\infty$, we obtain

$$\limsup_{t\to+\infty} \frac{m(t)}{t} \leq \frac{m(\tau)}{\tau} .$$

But this is true for any τ, hence

$$\limsup_{t\to+\infty}\left(\frac{m(t)}{t}\right) \leq \inf_{\tau}\left(\frac{m(\tau)}{\tau}\right) = c .$$

But obviously $\liminf\limits_{t\to+\infty} \dfrac{m(t)}{t} \geq c$, therefore $\dfrac{m(t)}{t} \to c$.

It is worth pointing out that the assumption "$m(t) \geq -Ct$" is just used to have a well-defined constant c.

Then we are led to several natural questions:

(a) Can we have a characterization of the constant c?
(b) Can we go further in the asymptotic behavior ? Namely: is $u(x,t) - ct$ bounded? does it converge to some function?

A first remark is the following: if, for large t, $u(x,t)$ looks like $\lambda t + v(x)$, then λ and v should satisfy the equation

$$H(x, Dv) + \lambda = 0 \quad \text{in } \mathbb{R}^N . \tag{42}$$

A key question is then: does this equation, where both the constant λ and the function v are unknown, have (periodic) solutions?

The answer is given by the following result of Lions et al., Homogenization of Hamilton–Jacobi equations, unpublished work.

Theorem 10.3. *Assume that H satisfies (37)–(38). There exists a unique constant λ such that (42) has a periodic, Lipschitz continuous solution.*

An immediate consequence of Theorem 10.3 is the

Corollary 10.1. *Assume that H satisfies (37)–(38). Then $c = \lambda$ and $u(x,t) - ct$ is bounded.*

The proof of this corollary is obvious since, if (λ, v) solves (42), then $v(x) + \lambda t$ is a solution of (35) and by comparison

$$||u(x,t) - (v(x) + \lambda t)||_\infty \leq ||u(x,0) - v(x)||_\infty \ .$$

Therefore $u(x,t) - \lambda t$ is bounded and dividing by t and letting $t \to +\infty$ shows that $c = \lambda$.

As a consequence, Theorem 10.3 gives a characterization of the ergodic constant c as the unique constant such that the "ergodic problem" (42) has a periodic (bounded) solution.

Proof of Theorem 10.3. For $0 < \alpha \ll 1$, we consider the equation

$$H(x, Dv_\alpha) + \alpha v_\alpha = 0 \quad \text{in } \mathbb{R}^N \ , \tag{43}$$

and we set $M := ||H(x,0)||_\infty$. In order to prove that this equation has a unique periodic solution v_α, we use Perron's method.

We first remark that $-\frac{1}{\alpha}M$ and $\frac{1}{\alpha}M$ are respectively sub and supersolution of this equation and we are looking for a solution which satisfies

$$-\frac{1}{\alpha}M \leq v_\alpha \leq \frac{1}{\alpha}M \quad \text{in } \mathbb{R}^N \ .$$

Since H does not a priori satisfy Assumption (H1), we have to argue either as in proof of Theorem 10.1, introducing some truncated Hamiltonians H_K or we remark that, because of (38), the subsolutions w which are bounded from below by $-\frac{1}{\alpha}M$ are equi-Lipschitz continuous: in this last case, we directly build a Lipschitz continuous solution of (43).

In any case, we build a solution v_α of (43) such that

$$||v_\alpha||_\infty \leq \frac{1}{\alpha}M \ ,$$

which is Lipschitz continuous and an easy modification of the proof of Theorem 5.1 shows that v_α is the unique periodic solution of (43).

Moreover, as a consequence of (38), since αv_α is bounded, $H(x, Dv_\alpha)$ is also bounded and therefore the v_α's are equi-Lipschitz continuous.

Using this property together with the periodicity of the v_α, the functions $w_\alpha(x) := v_\alpha(x) - v_\alpha(0)$ are equi-bounded and equi-Lipschitz continuous. By Ascoli's Theorem, they converge (up to a subsequence) to some function $v \in W^{1,\infty}(\mathbb{R}^N)$. And we may assume as well that the bounded constants $\alpha v_\alpha(0)$ converges to some constant λ.

We have $H(x, Dw_\alpha) + \alpha w_\alpha + \alpha v_\alpha(0) = 0$ in $\mathbb{R}^N$ and we can pass to the limit by using Theorem 4.1: λ and v solves (42).

For the *uniqueness* of λ, if (v, λ) and (v', λ') are solutions of the ergodic problem, we compare the solutions $v(x) + \lambda t$ and $v'(x) + \lambda' t$ of (35)

$$||(v(x) + \lambda t) - (v'(x) + \lambda' t)||_\infty \le ||v(x) - v'(x)||_\infty$$

or equivalently

$$||(v(x) - v'(x)) + (\lambda - \lambda')t)||_\infty \le ||v(x) - v'(x)||_\infty .$$

Dividing by t and letting $t \to +\infty$ gives $\lambda = \lambda'$.

10.4 Asymptotic Behavior of $u(x, \quad - ct$

By considering $H_c = H + c$ and $u_c(x,t) = u(x,t) - ct$, we may assume that $c = 0$ and the solutions u of (35) are uniformly bounded and Lipschitz continuous. We are going to do it from now on.

The main question of this section is: do the $u(x,t)$ always converge as $t \to +\infty$? or do we need additional assumptions? The following examples shows that the answer is not completely obvious.

Example 1. The function $u(x,t) := \sin(x - t)$ is a solution of the transport equation

$$u_t + u_x = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) ,$$

it satisfies very good regularity properties and uniform estimates but it does not converge as $t \to +\infty$. This shows that convergence is not only a question of estimates. But, of course, in this example the coercivity assumption is not satisfied.

Example 2. The same function is also a solution of

$$u_t + |u_x + 1| - 1 = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) .$$

In this example, the Hamiltonian is coercive and even convex but not strictly convex.

These two examples shows that the convergence as $t \to +\infty$ requires additional assumptions and/or a particular framework: we are going to show that the convergence holds in two cases:

(a) The Namah–Roquejoffre framework for which a typical example is

$$u_t + |Du| = f(x) \quad \text{in } \mathbb{R}^N \times (0, +\infty) ,$$

where $f(x) \geq 0$ and the set $\{x : \; f(x) = 0\}$ is non-empty.

(b) The "strictly convex" framework for which a typical example is

$$u_t + |Du + q(x)|^2 - |q(x)|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) ,$$

where q is (say) a periodic, Lipschitz continuous function.

Roughly speaking, the first framework is more restrictive on the structure of the Hamiltonians but it allows to take into account Hamiltonians $H(x, p)$ which are not strictly convex in p, contrarily to the second framework where the structure of the Hamiltonians is very general but where we have to impose strict convexity.

10.5 The Namah–Roquejoffre Framework

The main assumptions are the following. In the sequel, we refer to this assumptions as (NR).

- $H(x, p) \geq H(x, 0)$ for any $x, p \in \mathbb{R}^N$.
- $H(x, 0) \leq 0$ for any $x \in \mathbb{R}^N$ and the set $\mathscr{Z} = \{x \in \mathbb{R}^N ; \; H(x, 0) = 0\}$ is non-empty.
- For any $\alpha > 0$ (small) and for any $0 < \mu < 1$, there exists $\eta(\alpha, \mu) > 0$ such that

$$H(x, \mu p) \leq -\eta(\alpha, \mu) \quad \text{if } H(x, p) \leq 0 \text{ and if } d(x, \mathscr{Z}) \geq \alpha .$$

Remark 10.1. If $H(x, p) = |p| - f(x)$ where $f(x) \geq 0$ and the set $\mathscr{Z} := \{x : f(x) = 0\}$ is non-empty, these assumptions are satisfied since

$$\begin{aligned} H(x, \mu p) &= \mu|p| - f(x) = \mu(|p| - f(x)) - (1-\mu) f(x) . \\ &\leq -\eta(\alpha, \mu) := -(1-\mu) \min_{d(x,\mathscr{Z}) \geq \alpha} f(x) < 0 \quad \text{if } |p| - f(x) \leq 0. \end{aligned}$$

Theorem 10.4. *Assume that H satisfies (37)–(38) and (NR), then $c = 0$ and, for any $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, the solution u of (35)–(36) converges to a solution of the stationary equation.*

Proof of Theorem 10.4. To show that $c = 0$, we have first to solve the equation

$$H(x, Dv) = 0 \quad \text{in } \mathbb{R}^N .$$

We first remark that, because of (NR), 0 is a subsolution.

On the other hand, if $z \in \mathcal{Z}$, by the coercivity of H, $C|x-z|$ is a supersolution for C large enough: indeed this is obviously true for $x \neq z$ since the gradient of this function has norm C. And this is also clear for $x = z$ since, by (NR), $H(z, p) \geq H(z, 0) = 0$ for any p. As a consequence, $Cd(x, \mathcal{Z}) := \inf_{z \in \mathcal{Z}} C|x-z|$ is a (periodic) supersolution of the equation as the infimum of supersolutions.

We apply Perron's method which provides us with a discontinuous solution. To prove that this solution is continuous, we need a SCR.

Noticing that both the (continuous) sub and supersolution vanish on $\mathcal{Z}$, the value of the solution is imposed on $\mathcal{Z}$ (see the construction above) and we need a SCR for the Dirichlet problem set in the complementary of $\mathcal{Z}$, namely

$$\begin{cases} H(x, Du) = 0 & \text{dans } \mathcal{O} := \mathbb{R}^N \setminus \mathcal{Z} \\ u(x) = 0 & \text{sur } \partial\mathcal{O} . \end{cases}$$

To obtain it, we use ideas which are introduced in Ishii [32] (see also [5]). If v_1 is a subsolution of this problem and v_2 a supersolution with $v_1 \leq 0 \leq v_2$ on $\partial\mathcal{O}$, we pick some $\mu \in (0, 1)$, close to 1. Because of the last requirement in (NR), we have in the viscosity sense

$$H(x, D\mu v_1(\cdot)) \leq -\eta(\alpha, \mu) \quad \text{if } d(x, \mathcal{Z}) \geq \alpha ,$$

and following the arguments of the comparison proof, it is clear that the maximum of $\mu v_1 - v_2$ can be achieved only on $\mathcal{Z}$. Therefore $\mu v_1 - v_2 \leq 0$ and we conclude by letting μ tends to 1. Therefore we have a continuous solution of the stationary equation and $c = 0$.

Next we examine the behavior of the solution u of the evolution equation on $\mathcal{Z}$: since $H(x, p) \geq 0$ on $\mathcal{Z}$, we have $u_t \leq 0$ on $\mathcal{Z}$ and therefore $t \mapsto u(x, t)$ is decreasing. Recalling that u is Lipschitz continuous, this implies that $u(x, t) \to \varphi(x)$ uniformly on $\mathcal{Z}$ where φ is a Lipschitz continuous function.

It remains to show the global behavior: to do so, we use the half-relaxed limit method outside $\mathcal{Z}$. For $\varepsilon > 0$, we set

$$u_\varepsilon(x, t) := u\left(x, \frac{t}{\varepsilon}\right) \quad \text{in } \mathbb{R}^N \times (0, \infty) .$$

The function u_ε solves

$$\varepsilon \frac{\partial u_\varepsilon}{\partial t} + H(x, Du_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) .$$

We introduce (as usual) the half-relaxed limits

$$\overline{u}(x,t) = \limsup{}^* \, u_\varepsilon(x,t) \text{ and } \underline{u}(x,t) = \liminf{}_* \, u_\varepsilon(x,t) \ .$$

For any $t > 0$, $\overline{u}(\cdot,t)$ and $\underline{u}(\cdot,t)$ are respectively sub and supersolution of $H(x,Dw) = 0$ in $\mathbb{R}^N$. It is worth pointing out that, here, $\overline{u}$ and $\underline{u}$ are Lipschitz continuous in x for any t, because of the uniform Lipschitz properties of u.

A priori we do not have a strong comparison result for this equation in $\mathbb{R}^N$ but we can use the additional information that we have on $\mathscr{Z}$, namely $\overline{u}(\cdot,t) = \underline{u}(\cdot,t) = \varphi(\cdot)$ on $\mathscr{Z}$. Therefore we are lead to the same Dirichlet problem as above, except that the boundary condition is now φ instead of 0. Applying readily the same arguments with a slight modification due to the Dirichlet data φ, we conclude that, for any $s,t > 0$, $\overline{u}(\cdot,t) \leq \underline{u}(\cdot,s)$ in $\mathbb{R}^N$. This implies that $\overline{u}(\cdot,t) = \underline{u}(\cdot,s)$ for any $s,t > 0$ and, setting $w(\cdot) = \overline{u}(\cdot,t) = \underline{u}(\cdot,s)$, we have the uniform convergence of $u(\cdot,t)$ as $t \to +\infty$ to the continuous function w which is the unique solution of the Dirichlet problem with φ and also solves

$$H(x,Dw) = 0 \quad \text{in } \mathbb{R}^N \ .$$

Remark 10.2. This approach does not work for the equation

$$u_t + |Du + q(x)|^2 - |q(x)|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty)$$

which does not satisfy the (NR) assumptions.

10.6 The "Strictly Convex" Framework

In fact, like in the Namah–Roquejoffre framework, the assumptions on H we are going to use in this section does not really imply that H is strictly convex; the title of this section is just to fix ideas.

Our key assumption is the following.

(SCA) There exists $\eta_0 > 0$ such that, for any $\eta \in (0,\eta_0]$, there exists a constant $\psi_\eta > 0$ such that if $H(x,p+q) \geq \eta$ and $H(x,q) \leq 0$ for some $x,p,q \in \mathbb{R}^N$, then for any $\mu \in (0,1]$,

$$\mu H\left(x, \frac{p}{\mu} + q\right) \geq H(x,p+q) + \psi_\eta(1-\mu).$$

This assumption does not implies that H is convex but it implies that, for all x, the set $\{p : \ H(x,p) \leq 0\}$ is convex (Ishii, personal communication) and imposes the behavior of H in the set $\{p : \ H(x,p) \geq 0\}$.

Remark 10.3. If H is indeed a C^2, strictly convex function of p, i.e. if $D^2_{pp}H(x,p) \geq \nu Id$ for some $\nu > 0$, have, for any $\mu \in (0,1]$, $a, b \in \mathbb{R}^N$

$$H(x, \mu a + (1-\mu)b) \leq \mu H(x,a) + (1-\mu)H(x,b) - C(\nu)\mu(1-\mu)|a-b|^2.$$

Choose $a = \frac{p}{\mu} + q$, $b = q$, $\mu a + (1-\mu)b = p + q$ and therefore

$$H(x, p+q) \leq \mu H(x, \frac{p}{\mu} + q) + (1-\mu)H(x,q) - C(\nu)\mu(1-\mu)|\frac{p}{\mu}|^2 ,$$

i.e.

$$H(x, p+q) \leq \mu H(x, \frac{p}{\mu} + q) - C(\nu)\mu(1-\mu)|\frac{p}{\mu}|^2 ,$$

since $H(x,q) \leq 0$. But p is bounded away from 0 since $H(x, p+q) \geq \eta$ and $H(x,q) \leq 0$, therefore **(SCA)** holds.

Our result is the following.

Theorem 10.5. *Assume that H satisfies (37)–(38), $c = 0$ and* **(SCA)***, then, for any $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, the solution u of (35)–(36) converges to a solution of the stationary equation.*

It is worth recalling that, in this case, we actually assume that $c = 0$, it is not a consequence of the assumptions on H.

The key result is this approach is the

Theorem 10.6 (Asymptotically Monotone Property). *Under the assumption of Theorem 10.5, for any $\eta \in (0, \eta_0]$, there exists $\delta_\eta : [0,\infty) \to [0,1]$ such that*

$$\delta_\eta(s) \to 0 \quad \text{as } s \to \infty \quad \text{and}$$
$$u(x,s) - u(x,t) + \eta(s-t) \leq \delta_\eta(s)$$

for all $x \in \mathbb{R}^N$, $s, t \in [0,\infty)$ with $t \geq s$.

The meaning of Theorem 10.6 is that the solution u is becoming more and more increasing as $t \to \infty$. Why should this be true?

We can first consider the Oleinik–Lax Formula. The solution of

$$u_t + |Du|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty) ,$$

is given by

$$u(x,t) := \inf_{y \in \mathbb{R}^N} \left(u_0(y) + \frac{|x-y|^2}{4t} \right) .$$

Formally, if y is a minimum point in this formula

$$Du(x,t) = \frac{2(x-y)}{4t} \quad \text{and} \quad u_t(x,t) := -\frac{|x-y|^2}{4t^2} \ .$$

But we know that $\frac{|x-y|^2}{4t}$ remains bounded since u_0 is bounded, hence $u_t = O(t^{-1})$.

A more general remark can be made by assuming that H is strictly convex and

$$H_p(x,p)\cdot p - H(x,p) \geq cH(x,p) \quad \text{if } H(x,p) \geq 0 \ ,$$

for any $x, p \in \mathbb{R}^N$ and for some $c > 0$. For example, one can think about quadratic Hamiltonians like $|p+q|^2 - |q|^2$ or $|p|^2 - f(x)^2$.

In this case, we perform the Kruzkov's change $w = -\exp(-u)$. The function w solves

$$w_t - wH(x, -\frac{Dw}{w}) = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty) \ .$$

Then we set $z = w_t$ and $m(t) = \|z^-\|_\infty$. Differentiating the equation with respect to t, we find that z satisfies at the same time (dropping the arguments of H and its derivatives)

$$z_t + (H_p \cdot p - H)z + H_p \cdot Dz = 0 \ ,$$

$$z - wH = 0 \ .$$

Next looking at a (negative) minimum point of z (where $Dz = 0$), it follows

$$m'(t) + (H_p \cdot p - H)m(t) = 0.$$

But $H = z/w > 0$ and therefore $(H_p \cdot p - H) \geq cH = cz/w$. Hence

$$m'(t) + c[m(t)]^2/w = 0 \quad \text{which implies} \quad m'(t) \geq \tilde{c}[m(t)]^2 \ .$$

Recalling that $m(t) \leq 0$, this inequality yields a behavior like $m(t) = O(t^{-1})$.

We first *prove Theorem 10.5* by using the Asymptotically Monotone Property.

(a) Since the family $(u(\cdot,t))_{t\geq 0}$ is bounded in $W^{1,\infty}(\mathbb{R}^N)$, by Ascoli's Theorem, there exists a sequence $(u(\cdot,T_n))_{n\in\mathbb{N}}$ which converges uniformly on $\mathbb{R}^N$ as $n \to \infty$.

By comparison, we have

$$\|u(\cdot,T_n+\cdot) - u(\cdot,T_m+\cdot)\|_\infty \leq \|u(\cdot,T_n) - u(\cdot,T_m)\|_\infty$$

for any $n, m \in \mathbb{N}$. Therefore, $(u(\cdot,T_n+\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}^N \times (0,+\infty))$ and therefore it converges uniformly to a function denoted by $u^\infty \in C(\mathbb{R}^N \times (0,+\infty))$. Moreover u^∞ is a solution of (35), by stability.

(b) Fix any $x \in \mathbb{R}^N$ and $s, t \in [0, \infty)$ with $t \geq s$. By the Asymptotically Monotone Property, we have

$$u(x, s + T_n) - u(x, t + T_n) + \eta(s - t) \leq \delta_\eta(s + T_n)$$

for any $n \in \mathbb{N}$ and $\eta > 0$. Sending $n \to \infty$ and then $\eta \to 0$, we get, for any $t \geq s$

$$u^\infty(x, s) \leq u^\infty(x, t).$$

The functions $x \mapsto u^\infty(x, t)$ are uniformly bounded and equi-continuous, and they are also monotone in t. This implies that $u^\infty(x, t) \to w(x)$ uniformly on $\mathbb{R}^N$ as $t \to \infty$ for some $w \in W^{1,\infty}(\mathbb{R}^N)$ which is a solution of the stationary equation.

(c) Since $u(\cdot, T_n + \cdot) \to u^\infty$ *uniformly*[4] in $\mathbb{R}^N \times (0, +\infty)$ as $n \to \infty$, we have

$$-o_n(1) + u^\infty(x, t) \leq u(x, T_n + t) \leq u^\infty(x, t) + o_n(1),$$

where $o_n(1) \to \infty$ as $n \to \infty$, uniformly in x and t.

Taking the half-relaxed semi-limits as $t \to +\infty$, we get

$$-o_n(1) + w \leq \liminf_{t \to \infty}{}_* \; u \leq \limsup_{t \to \infty}{}^* \; u \leq w + o_n(1).$$

Sending $n \to \infty$ yields

$$w(x) = \liminf_{t \to \infty}{}_* \; u(x, t) = \limsup_{t \to \infty}{}^* \; u(x, t)$$

for all $x \in \mathbb{R}^N$. Therefore $u(x, t) \to w(x)$ uniformly as $t \to \infty$ and the proof is complete.

Now we turn to the *Proof of the Asymptotically Monotone Property*. Let v be a periodic, Lipschitz continuous solution of $H(x, Dv) = 0$.

Since u is bounded and since we can change v in $v - M$ for some large constant $M > 0$, we may assume that

$$u(x, t) - v(x) \geq 1 \quad \text{for any } x \in \mathbb{R}^N \text{ and } t > 0\ .$$

We introduce the function

$$\mu_\eta(s) := \min_{x \in \mathbb{R}^N, t \geq s} \left(\frac{u(x, t) - v(x) + \eta(t - s)}{u(x, s) - v(x)} \right) .$$

[4] This is a key point: the compactness of the domain (periodicity) plays a crucial role here since local uniform convergence is the same as global uniform convergence.

By the uniform continuity of u and v, $\mu_\eta \in C([0,\infty))$ and we have $0 \le \mu_\eta(s) \le 1$ for all $s \in [0,\infty)$ and $\eta \in (0,\eta_0]$.

Proposition 10.1. *Under the assumption of Theorem 10.5, $\mu_\eta(s) \to 1$ as $s \to \infty$ for any $\eta \in (0,\eta_0]$.*

As a consequence, for any $x \in \mathbb{R}^N$ and $t \ge s$,

$$\frac{u(x,t) - v(x) + \eta(t-s)}{u(x,s) - v(x)} \ge 1 + o_s(1) ,$$

where $o_s(1)$ depends on η and tends to 0 as $s \to \infty$.

A simple computation yields

$$u(x,t) - u(x,s) + \eta(t-s) \ge o_s(1) .$$

The proposition is a consequence of the following lemma.

Lemma 10.1. *Under the assumption of Theorem 10.6, for any $\eta \in (0,\eta_0]$, there exists a constant $C > 0$ such that the function μ_η is a supersolution of*

$$\max\left\{w(s) - 1, w'(s) + \frac{\psi_\eta}{C}(w(s) - 1)\right\} = 0 \text{ in } (0,\infty) .$$

Using the lemma, it is easy to prove the proposition since the solution of the variational inequality with initial data $\mu_\eta(0)$ is given by

$$w(s) := 1 - (\mu_\eta(0) + 1)\exp\left(-\frac{\psi_\eta}{C}s\right) .$$

and therefore, by comparison

$$\mu_\eta(s) \ge 1 - (\mu_\eta(0) + 1)\exp\left(-\frac{\psi_\eta}{C}s\right) ,$$

for any s. Recalling that $\mu_\eta(s) \le 1$, we have $\mu_\eta(s) \to 1$ as $s \to \infty$.

Proof of Lemma 10.1. We fix $\eta \in (0,\eta_0]$ and, to simplify the notations, we write μ for μ_η.

Let $\phi \in C^1((0,\infty))$ and $\overline{s} > 0$ be a strict local minimum of $\mu - \phi$.

Since there is nothing to check if $\mu(\overline{s}) = 1$, we assume that $\mu(\overline{s}) < 1$. We choose $\overline{x} \in \mathbb{R}^N$ and $\overline{t} \ge \overline{s}$ such that

$$\mu(\overline{s}) = \frac{u(\overline{x},\overline{t}) - v(\overline{x}) + \eta(\overline{t} - \overline{s})}{u(\overline{x},\overline{s}) - v(\overline{x})} .$$

For $0 < \varepsilon \ll 1$, we introduce the function

$$\Psi(x,y,z,t,s) := \frac{u(x,t) - v(z) + \eta(t-s)}{u(y,s) - v(z)} - \phi(s) + \frac{1}{\varepsilon^2}(|x-y|^2 + |x-z|^2) \\ + |x-\overline{x}|^2 + |t-\overline{t}|^2$$

The function Ψ achieve its minimum at a point (x,y,z,t,s) (depending on ε) and, by classical arguments, as $\varepsilon \to 0$, we have

$$x, y, z \to \overline{x} \quad \text{and} \quad t \to \overline{t},\ s \to \overline{s}\ .$$

Moreover, by the Lipschitz continuity in x of u and v

$$\frac{|x-y|}{\varepsilon^2} + \frac{|x-z|}{\varepsilon^2} \leq C\ ,$$

for some constant C.

With the notations

$$\tilde{\mu}_1 := u(y,s) - v(z), \quad \tilde{\mu}_2 := u(x,t) - v(z) + \eta(t-s), \quad \tilde{\mu} := \frac{\tilde{\mu}_2}{\tilde{\mu}_1}$$

and if we set

$$P := \frac{\tilde{\mu}_1}{\tilde{\mu}}\left(\frac{2(y-x)}{\varepsilon^2}\right) \quad \text{and} \quad Q := \frac{\tilde{\mu}_1}{1-\tilde{\mu}}\left(\frac{2(z-x)}{\varepsilon^2}\right),$$

we have formally,

$$\begin{aligned} D_x u(x,t) &= \tilde{\mu} P + (1-\tilde{\mu}) Q + o_\varepsilon(1)\ , \\ u_t(x,t) &= -\eta - 2\tilde{\mu}_1(t-\overline{t})\ , \\ D_y u(y,s) &= P\ , \\ u_s(y,s) &= -\tfrac{1}{\tilde{\mu}}(\eta + \tilde{\mu}_1 \phi'(s))\ , \\ D_z v(z) &= Q. \end{aligned}$$

By the definition of viscosity solutions

$$\begin{aligned} -\eta + o_\varepsilon(1) + H(x, \tilde{\mu} P + (1-\tilde{\mu}) Q + o_\varepsilon(1)) &\geq 0, \\ -\frac{1}{\tilde{\mu}}(\eta + \tilde{\mu}_1 \phi'(s)) + H(y,P) &\leq 0, \\ H(z,Q) &\leq 0. \end{aligned}$$

Since P and Q are bounded, we may even let ε tend to 0 and drop the $o_\varepsilon(1)$-terms.

With

$$\mu_1 := u(\overline{x}, \overline{s}) - v(\overline{x}), \quad \mu_2 := u(\overline{x}, \overline{t}) - v(\overline{x}) + \eta(\overline{t} - \overline{s}), \quad \mu = \frac{\mu_2}{\mu_1}$$

we end up with

$$\begin{aligned} -\eta + H(\overline{x}, \mu P + (1-\mu)Q) &\geq 0, \\ -\frac{1}{\mu}(\eta + \mu_1 \phi'(\overline{s})) + H(\overline{x}, P) &\leq 0, \\ H(\overline{x}, Q) &\leq 0. \end{aligned}$$

If $p := \mu(P - Q)$ and $q = Q$, we have $H(\overline{x}, p + q) \geq \eta$ and $H(\overline{x}, q) \leq 0$, and therefore, by **(SCA)**

$$\begin{aligned} \frac{1}{\mu}(\eta + \mu_1 \phi'(\overline{s})) \geq H(\overline{x}, P) &= H(\overline{x}, \tfrac{p}{\mu} + q) \\ &\geq \frac{1}{\mu}\left(H(\overline{x}, p + q) + \psi_\eta(1-\mu)\right) \\ &\geq \frac{1}{\mu}\left(\eta + \psi_\eta(1-\mu)\right) \ . \end{aligned}$$

This shows

$$\phi'(\overline{s}) \geq \frac{1}{\mu_1}\psi_\eta(1-\mu) \ ,$$

which is the desired conclusion.

10.7 Concluding Remarks

- The Asymptotically Monotone Property is true in a more general framework (problems set in the whole space or with boundary conditions ... etc) but, in general, it does not imply the convergence as $t \to \infty$. This shows the importance of the periodic framework (compactness) where local uniform convergence is equivalent to global uniform convergence.
- In the Namah–Roquejoffre case, periodicity is less important, even if one has to avoid the infinity to play a role (by assuming that $\limsup_{|x| \to +\infty} H(x, 0) < 0$). See, for example, [11].
- For problems set in the whole space, the behavior at infinity of u_0 may determine the asymptotic behavior as $t \to \infty$ of u, even at the level of the ergodic constant c (cf. [11]).

- If H is convex and if S_H, S_{H^+} denote respectively the semi-groups associated to H and H^+, we know that these semi-groups commutes, namely

$$S_H(t)S_{H^+}(s) = S_{H^+}(s)S_H(t)$$

for any $s, t > 0$.

For any u_0, $S_{H^+}(s)u_0$ converges to the maximal subsolution of $H = 0$ which is below u_0.

If we are in a framework where we have convergence for $S_H(t)$ as $t \to \infty$, i.e. $S_H(t)u_0 \to u_\infty$ as $t \to +\infty$, then

$$S_H(\infty)S_{H^+}(s)u_0 = S_{H^+}(s)S_H(\infty)u_0 = u_\infty$$

This shows that u_∞ is the same for u_0 and for maximal subsolution of $H = 0$ which is below u_0: in other words, given u_0, $u(x,t)$ converges to the minimal solution which is above the maximal subsolution which is below u_0.

For such properties of commutations of semi-groups, we refer the reader to Cardin and Viterbo [20], Motta and Rampazzo [41] and Tourin and the author [16].

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A Short Introduction to Viscosity Solutions and the Large Time Behavior of Solutions of Hamilton–Jacobi Equations

Hitoshi Ishii

In memory of Riichi Iino, my former adviser at Waseda University.

Abstract We present an introduction to the theory of viscosity solutions of first-order partial differential equations and a review on the optimal control/dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations, with the Neumann boundary condition. This article also includes some of basics of mathematical analysis related to the optimal control/dynamical approach for easy accessibility to the topics.

Introduction

This article is an attempt to present a brief introduction to viscosity solutions of first-order partial differential equations (PDE for short) and to review some aspects of the large time behavior of solutions of Hamilton–Jacobi equations with Neumann boundary conditions.

The notion of viscosity solution was introduced in [20] (see also [18]) by Crandall and Lions, and it has been widely accepted as the right notion of generalized solutions of the first-order PDE of the Hamilton–Jacobi type and fully nonlinear (possibly degenerate) elliptic or parabolic PDE. There have already been many nice contributions to overview of viscosity solutions of first-order and/or

H. Ishii (✉)
Faculty of Education and Integrated Arts and Sciences, Waseda University, Nishi-Waseda 1-6-1, Shinjuku, Tokyo 169–8050, Japan

Faculty of Science, King Abdulaziz University, P.O. Box 80203 Jeddah, 21589, Saudi Arabia
e-mail: hitoshi.ishii@waseda.jp

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second-order partial differential equations. The following list touches just a few of them [2, 6, 15, 19, 29, 31, 41, 42].

This article is meant to serve as a quick introduction for graduate students or young researchers to viscosity solutions and is, of course, an outcome of the lectures delivered by the author at the CIME school as well as at Waseda University, Collège de France, Kumamoto University, King Abdulaziz University and University of Tokyo. For its easy readability, it contains some of very basics of mathematical analysis which are usually left aside to other textbooks.

The first section is an introduction to viscosity solutions of first-order partial differential equations. As a motivation to viscosity solutions we take up an optimal control problem and show that the value function of the control problem is characterized as a unique viscosity solution of the associated Bellman equation. This choice is essentially the same as used in the book [42] by Lions as well as in [2, 6, 29].

In Sects. 2–5, we develop the theory of viscosity solutions of Hamilton–Jacobi equations with the linear Neumann boundary condition together with the corresponding optimal control problems, which we follow [8,38,39]. In Sect. 6, following [38], we show the convergence of the solution of Hamilton–Jacobi equation of evolution type with the linear Neumann boundary condition to a solution of the stationary problem.

The approach here to the convergence result depends heavily on the variational formula for solutions, that is, the representation of solutions as the value function of the associated control problem. There is another approach, due to [3], based on the asymptotic monotonicity of a certain functional of the solutions as time goes to infinity, which is called the PDE approach. The PDE approach does not depend on the variational formula for the solutions and provides a very simple proof of the convergence with sharper hypotheses. The approach taken here may be called the dynamical or optimal control one. This approach requires the convexity of the Hamiltonian, so that one can associate it with an optimal control problem. Although it requires lots of steps before establishing the convergence result, its merit is that one can get an interpretation to the convergence result through the optimal control representation.

The topics covered in this article are very close to the ones discussed by Barles [4]. Both are to present an introduction to viscosity solutions and to discuss the large time asymptotics for solutions of Hamilton–Jacobi equations. This article has probably a more elementary flavor than [4] in the part of the introduction to viscosity solutions, and the paper [4] describes the PDE-viscosity approach to the large time asymptotics while this article concentrates on the dynamical or optimal control approach.

The reference list covers only those papers which the author more or less consulted while he was writing this article, and it is far from a complete list of those which have contributed to the developments of the subject.

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Notation:

- When $\mathscr{F}$ is a set of real-valued functions on X, $\sup \mathscr{F}$ and $\inf \mathscr{F}$ denote the functions on X given, respectively, by

$$(\sup \mathscr{F})(x) := \sup\{f(x) \, : \, f \in \mathscr{F}\} \quad \text{and} \quad (\inf \mathscr{F})(x) := \inf\{f(x) \, : \, f \in \mathscr{F}\}.$$

- For any $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Also, we write $a_+ = a \vee 0$ and $a_- = (-a)_+$.
- A function $\omega \in C([0, R))$, with $0 < R \leq \infty$, is called a modulus if it is nondecreasing and satisfies $\omega(0) = 0$.
- For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $x \cdot y$ denotes the Euclidean inner product $x_1 y_1 + \cdots + x_n y_n$ of x and y.
- For any $x, y \in \mathbb{R}^n$ the line segment between x and y is denoted by $[x, y] := \{(1-t)x + ty \, : \, t \in [0, 1]\}$.
- For $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$, $C^k(\Omega, \mathbb{R}^m)$ (or simply, $C^k(\Omega, \mathbb{R}^m)$) denotes the collection of functions $f : \Omega \to \mathbb{R}^m$ (not necessarily open), each of which has an open neighborhood U of Ω and a function $g \in C^k(U)$ such that $f(x) = g(x)$ for all $x \in \Omega$.
- For $f \in C(\Omega, \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^n$, the support of f is defined as the closure of $\{x \in \Omega \, : \, f(x) \neq 0\}$ and is denoted by $\operatorname{supp} f$.
- $\mathrm{UC}(X)$ (resp., $\mathrm{BUC}(X)$) denotes the space of all uniformly continuous (resp., bounded, uniformly continuous) functions in a metric space X.
- We write $\mathbf{1}_E$ for the characteristic function of the set E. That is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise.
- The sup-norm of function f on a set Ω is denoted by $\|f\|_{\infty,\Omega} = \|f\|_\infty := \sup_\Omega |f|$.
- We write $\mathbb{R}_+$ for the interval $(0, \infty)$.
- For any interval $J \subset \mathbb{R}$, $\mathrm{AC}(J, \mathbb{R}^m)$ denotes the space of all absolutely continuous functions in J with value in $\mathbb{R}^m$.
- Given a convex Hamiltonian $H \in C(\overline{\Omega} \times \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is an open set, we denote by L the Lagrangian given by

$$L(x, \xi) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - H(x, p)) \quad \text{for } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

- Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$, $g \in C(\partial\Omega, \mathbb{R})$, $t > 0$ and $(\eta, v, l) \in L^1([0, t], \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ such that $\eta(s) \in \overline{\Omega}$ for all $s \in [0, t]$ and $l(s) = 0$ whenever $\eta(s) \in \Omega$. We write

$$\mathscr{L}(t, \eta, v, l) = \int_0^t [L(\eta(s), -v(s)) + g(\eta(s)) l(s)] \mathrm{d}s.$$

1 Introduction to Viscosity Solutions

We give the definition of viscosity solutions of first-order PDE and study their basic properties.

1.1 Hamilton–Jacobi Equations

Let Ω be an open subset of $\mathbb{R}^n$. Given a function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$, we consider the PDE

$$H(x, Du(x)) = 0 \quad \text{in } \Omega, \tag{1}$$

where Du denotes the gradient of u, that is,

$$Du := (u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \equiv (\partial u/\partial x_1, \ldots, \partial u/\partial x_n).$$

We also consider the PDE

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } \Omega \times (0, \infty). \tag{2}$$

Here the variable t may be regarded as the time variable and u_t denotes the time derivative $\partial u/\partial t$. The variable x is then regarded as the space variable and $D_x u$ (or, Du) denotes the gradient of u in the space variable x.

The PDE of the type of (1) or (2) are called Hamilton–Jacobi equations. A more concrete example of (1) is given by

$$|Du(x)| = k(x),$$

which appears in geometrical optics and describes the surface front of propagating waves. Hamilton–Jacobi equations arising in Mechanics have the form

$$|Du(x)|^2 + V(x) = 0,$$

where the terms $|Du(x)|^2$ and $V(x)$ correspond to the kinetic and potential energies, respectively.

More generally, the PDE of the form

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega \tag{3}$$

may be called Hamilton–Jacobi equations.

1.2 An Optimal Control Problem

We consider the function

$$X = X(t) = (X_1(t), X_2(t), \dots, X_n(t)) \in \mathbb{R}^n$$

of time $t \in \mathbb{R}$, and

$$\dot{X} = \dot{X}(t) = \frac{\mathrm{d}X}{\mathrm{d}t}(t)$$

denotes its derivative. Let $A \subset \mathbb{R}^m$ be a given set, let $g : \mathbb{R}^n \times A \to \mathbb{R}^n$, $f : \mathbb{R}^n \times A \to \mathbb{R}$ be given functions and $\lambda > 0$ be a given constant. We denote by $\mathbf{A}$ the set of all Lebesgue measurable $\alpha : [0, \infty) \to A$.

Fix any $x \in \mathbb{R}^n$ and $\alpha \in \mathbf{A}$, and consider the initial value problem for the ordinary differential equation (for short, ODE)

$$\begin{cases} \dot{X}(t) = g(X(t), \alpha(t)) \quad \text{for a.e. } t > 0, \\ X(0) = x. \end{cases} \tag{4}$$

The solution of (4) will be denoted by $X = X(t) = X(t; x, \alpha)$. The solution $X(t)$ may depend significantly on choices of $\alpha \in \mathbf{A}$. Next we introduce the functional

$$J(x, \alpha) = \int_0^\infty f(X(t), \alpha(t)) \mathrm{e}^{-\lambda t} \, \mathrm{d}t, \tag{5}$$

a function of x and $\alpha \in \mathbf{A}$, which serves a criterion to decide which choice of α is better. The best value of the functional J is given by

$$V(x) = \inf_{\alpha \in \mathbf{A}} J(x, \alpha). \tag{6}$$

This is an optimization problem, and the main theme is to select a control $\alpha = \alpha_x \in \mathbf{A}$ so that

$$V(x) = J(x, \alpha).$$

Such a control α is called an *optimal control*. The ODE in (4) is called the *dynamics* or *state equation*, the functional J given by (5) is called the *cost functional*, and the function V given by (6) is called the value function. The function f or $t \mapsto \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))$ is called the *running cost* and λ is called the *discount rate*.

In what follows, we assume that f, g are bounded continuous functions on $\mathbb{R}^n \times A$ and moreover, they satisfy the Lipschitz condition, i.e., there exists a constant $M > 0$ such that

$$\begin{cases} |f(x, a)| \le M, \qquad |g(x, a)| \le M, \\ |f(x, a) - f(y, a)| \le M|x - y|, \\ |g(x, a) - g(y, a)| \le M|x - y|. \end{cases} \tag{7}$$

A basic result in ODE theory guarantees that the initial value problem (4) has a unique solution $X(t)$.

There are two basic approaches in optimal control theory:

1. Pontryagin's Maximum Principle Approach.
2. Bellman's Dynamic Programming Approach.

Both of approaches have been introduced and developed since 1950s.

Pontryagin's maximum principle gives a necessary condition for the optimality of controls and provides a powerful method to design an optimal control.

Bellman's approach associates the optimization problem with a PDE, called the *Bellman equation*. In the problem, where the value function V is given by (6), the corresponding Bellman equation is the following.

$$\lambda V(x) + H(x, DV(x)) = 0 \quad \text{in } \mathbb{R}^n, \tag{8}$$

where H is a function given by

$$H(x, p) = \sup_{a \in A}\{-g(x, a) \cdot p - f(x, a)\},$$

with $x \cdot y$ denoting the Euclidean inner product in $\mathbb{R}^n$. Bellman's idea is to characterize the value function V by the Bellman equation, to use the characterization to compute the value function and to design an optimal control. To see how it works, we assume that (8) has a smooth bounded solution V and compute formally as follows. First of all, we choose a function $a : \mathbb{R}^n \to A$ so that

$$H(x, DV(x)) = -g(x, a(x)) \cdot DV(x) - f(x, a(x)),$$

and solve the initial value problem

$$\dot{X}(t) = g(X(t), a(X(t))), \qquad X(0) = x,$$

where x is a fixed point in $\mathbb{R}^n$. Next, writing $\alpha(t) = a(X(t))$, we have

$$\begin{aligned}
0 &= \int_0^\infty \mathrm{e}^{-\lambda t}\big(\lambda V(X(t)) + H(X(t), DV(X(t)))\big)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\big(\lambda V(X(t)) - g(X(t), \alpha(t)) \cdot DV(X(t)) - f(X(t), \alpha(t))\big)\,\mathrm{d}t \\
&= \int_0^\infty \left(-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\lambda t} V(X(t)) - \mathrm{e}^{\lambda t} f(X(t), \alpha(t))\right)\,\mathrm{d}t \\
&= V(X(0)) - \int_0^\infty \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\,\mathrm{d}t.
\end{aligned}$$

Thus we have

$$V(x) = J(x, \alpha).$$

If PDE (8) characterizes the value function, that is, the solution V is the value function, then the above equality says that the control $\alpha(t) = a(X(t))$ is an optimal control, which we are looking for.

In Bellman's approach PDE plays a central role, and we discuss this approach in what follows. The first remark is that the value function may not be differentiable at some points. A simple example is as follows.

Example 1.1. We consider the case where $n = 1$, $A = [-1, 1] \subset \mathbb{R}$, $f(x,a) = \mathrm{e}^{-x^2}$, $g(x,a) = a$ and $\lambda = 1$. Let $X(t)$ be the solution of (4) for some control $\alpha \in \mathbf{A}$, which means just to satisfy

$$|\dot{X}(t)| \le 1 \quad \text{a.e. } t > 0.$$

Let V be the value function given by (6). Then it is clear that $V(-x) = V(x)$ for all $x \in \mathbb{R}$ and that

$$V(x) = \int_0^\infty \mathrm{e}^{-t-(x+t)^2}\,\mathrm{d}t = \mathrm{e}^x \int_x^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t \quad \text{if } \ x > 0.$$

For $x > 0$, one gets

$$V'(x) = \mathrm{e}^x \int_x^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t - \mathrm{e}^{-x^2},$$

and

$$V'(0+) = \int_0^\infty \mathrm{e}^{-t-t^2}\,\mathrm{d}t - 1 < \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t - 1 = 0.$$

This together with the symmetry property, $V(-x) = V(x)$ for all $x \in \mathbb{R}$, shows that V is not differentiable at $x = 0$.

Value functions in optimal control do not have enough regularity to satisfy, in the classical sense, the corresponding Bellman equations in general as the above example shows.

We introduce the notion of viscosity solution of the first-order PDE

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{FE}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given continuous function.

Definition 1.1. (i) We call $u \in C(\Omega)$ a viscosity subsolution of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u - \phi) = (u - \phi)(z) \\ \implies F(z, u(z), D\phi(z)) \le 0. \end{cases}$$

(ii) We call $u \in C(\Omega)$ a viscosity supersolution of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \min_{\Omega}(u-\phi) = (u-\phi)(z) \\ \quad \Longrightarrow \ F(z, u(z), D\phi(z)) \geq 0. \end{cases}$$

(iii) We call $u \in C(\Omega)$ a viscosity solution of (FE) if u is both a viscosity subsolution and supersolution of (FE).

The viscosity subsolution or supersolution property is checked through smooth functions ϕ in the above definition, and such smooth functions ϕ are called test functions.

Remark 1.1. If we set $F^-(x,r,p) = -F(x,-r,-p)$, then it is obvious that $u \in C(\Omega)$ is a viscosity subsolution (resp., supersolution) of (FE) if and only if $u^-(x) := -u(x)$ is a viscosity supersolution (resp., subsolution) of

$$F^-(x, u^-(x), Du^-(x)) = 0 \quad \text{in } \Omega.$$

Note also that $(F^-)^- = F$ and $(u^-)^- = u$. With these observations, one property for viscosity subsolutions can be phrased as a property for viscosity supersolutions. In other words, every proposition concerning viscosity subsolutions has a counterpart for viscosity supersolutions.

Remark 1.2. It is easily seen by adding constants to test functions that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u-\phi) = (u-\phi)(z) = 0 \\ \quad \Longrightarrow \ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

One can easily formulate a counterpart of this proposition for viscosity supersolutions.

Remark 1.3. It is easy to see by an argument based on a partition of unity (see Appendix A.1) that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ u-\phi \text{ attains a local maximum at } z \\ \quad \Longrightarrow \ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

Remark 1.4. It is easily seen that $u \in C(\Omega)$ is a viscosity subsolution of (FE) if and only if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ u-\phi \text{ attains a strict maximum at } z \\ \quad \Longrightarrow \ F(z, \phi(z), D\phi(z)) \leq 0. \end{cases}$$

Similarly, one may replace “strict maximum” by “strict local maximum” in the statement. The idea to show these is to replace the function ϕ by $\phi(x) + |x - z|^2$ when needed.

Remark 1.5. The condition, $\phi \in C^1(\Omega)$, can be replaced by the condition, $\phi \in C^\infty(\Omega)$ in the above definition. The argument in the following example explains how to see this equivalence.

Example 1.2 (Vanishing viscosity method). The term “viscosity solution” originates to the vanishing viscosity method, which is one of classical methods to construct solutions of first-order PDE.

Consider the second-order PDE

$$- \varepsilon \Delta u^\varepsilon + F(x, u^\varepsilon(x), Du^\varepsilon(x)) = 0 \quad \text{in } \Omega, \tag{9}$$

where $\varepsilon > 0$ is a parameter to be sent to zero later on, Ω is an open subset of $\mathbb{R}^n$, F is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and Δ denotes the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We assume that functions $u^\varepsilon \in C^2(\Omega)$, with $\varepsilon \in (0, 1)$, and $u \in C(\Omega)$ are given and that

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x) \quad \text{locally uniformly on } \Omega.$$

Then the claim is that u is a viscosity solution of

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{FE}$$

In what follows, we just check that u is a viscosity subsolution of (FE). For this, we assume that

$$\phi \in C^1(\Omega), \quad \hat{x} \in \Omega, \quad \max_{\Omega}(u - \phi) = (u - \phi)(\hat{x}),$$

and moreover, this maximum is a strict maximum of $u - \phi$. We need to show that

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x})) \le 0. \tag{10}$$

First of all, we assume that $\phi \in C^2(\Omega)$, and show that (10) holds. Fix an $r > 0$ so that $\overline{B}_r(\hat{x}) \subset \Omega$. Let x_ε be a maximum point over $\overline{B}_r(\hat{x})$ of the function $u^\varepsilon - \phi$. We may choose a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1)$ so that $\lim_{j \to \infty} \varepsilon_j = 0$ and $\lim_{j \to \infty} x_{\varepsilon_j} = y$ for some $y \in \overline{B}_r(\hat{x})$. Observe that

$$
\begin{aligned}
(u-\phi)(\hat{x}) &\le (u^{\varepsilon_j}-\phi)(\hat{x}) + \|u-u^{\varepsilon_j}\|_{\infty,B_r(\hat{x})} \\
&\le (u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) + \|u-u^{\varepsilon_j}\|_{\infty,B_r(\hat{x})} \\
&\le (u-\phi)(x_{\varepsilon_j}) + 2\|u^{\varepsilon_j}-u\|_{\infty,B_r(\hat{x})} \\
&\to (u-\phi)(y) \quad \text{as } j \to \infty.
\end{aligned}
$$

Accordingly, since $\hat{x}$ is a strict maximum point of $u-\phi$, we see that $y = \hat{x}$. Hence, if j is sufficiently large, then $x_{\varepsilon_j} \in B_r(\hat{x})$. By the maximum principle from Advanced Calculus, we find that

$$
\frac{\partial}{\partial x_i}(u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x_i^2}(u^{\varepsilon_j}-\phi)(x_{\varepsilon_j}) \le 0 \quad \text{for all } i = 1,2,\ldots,n.
$$

Hence, we get

$$
Du^{\varepsilon_j}(x_{\varepsilon_j}) = D\phi(x_{\varepsilon_j}), \quad \Delta u^{\varepsilon_j}(x_{\varepsilon_j}) \le \Delta\phi(x_{\varepsilon_j}).
$$

These together with (9) yield

$$
-\varepsilon_j \Delta\phi(x_{\varepsilon_j}) + F(x_{\varepsilon_j}, u^{\varepsilon_j}(x_{\varepsilon_j}), D\phi(x_{\varepsilon_j})) \le 0.
$$

Sending $j \to \infty$ now ensures that (10) holds.

Finally we show that the C^2 regularity of ϕ can be relaxed, so that (10) holds for all $\phi \in C^1(\Omega)$. Let $r > 0$ be the constant as above, and choose a sequence $\{\phi_k\} \subset C^\infty(\Omega)$ so that

$$
\lim_{k\to\infty} \phi_k(x) = \phi(x) \quad \text{uniformly on } B_r(\hat{x}).
$$

Let $\{y_k\} \subset \overline{B}_r(\hat{x})$ be a sequence consisting of a maximum point of $u-\phi_k$. An argument similar to the above yields

$$
\lim_{k\to\infty} y_k = \hat{x}.
$$

If k is sufficiently large, then we have $y_k \in B_r(\hat{x})$ and, due to (10) valid for C^2 test functions,

$$
F(y_k, u(y_k), D\phi_k(y_k)) \le 0.
$$

Sending $k \to \infty$ allows us to conclude that (10) holds.

1.3 Characterization of the Value Function

In this subsection we are concerned with the characterization of the value function V by the Bellman equation

$$\lambda V(x) + H(x, DV(x)) = 0 \quad \text{in } \mathbb{R}^n, \tag{11}$$

where λ is a positive constant and

$$H(x, p) = \sup_{a \in A}\{-g(x, a) \cdot p - f(x, a)\}.$$

Recall that

$$V(x) = \inf_{\alpha \in \mathbf{A}} J(x, \alpha),$$

and

$$J(x, \alpha) = \int_0^\infty f(X(t), \alpha(t)) \mathrm{e}^{-\lambda t}\, \mathrm{d}t,$$

where $X(t) = X(t; x, \alpha)$ denotes the solution of the initial value problem

$$\begin{cases} \dot{X}(t) = g(X(t), \alpha(t)) & \text{for a.e. } t > 0, \\ X(0) = x. \end{cases}$$

Recall also that for all $(x, a) \in \mathbb{R}^n \times A$ and some constant $M > 0$,

$$\begin{cases} |f(x, a)| \le M, \qquad |g(x, a)| \le M, \\ |f(x, a) - f(y, a)| \le M|x - y|, \\ |g(x, a) - g(y, a)| \le M|x - y|. \end{cases} \tag{12}$$

The following lemma will be used without mentioning, the proof of which may be an easy exercise.

Lemma 1.1. *Let $h, k : A \to \mathbb{R}$ be bounded functions. Then*

$$\left|\sup_{a \in A} h(a) - \sup_{a \in A} k(a)\right| \vee \left|\inf_{a \in A} h(a) - \inf_{a \in A} k(a)\right| \le \sup_{a \in A} |h(a) - k(a)|.$$

In view of the above lemma, the following lemma is an easy consequence of (12), and the detail of the proof is left to the reader.

Lemma 1.2. *The Hamiltonian H satisfies the following inequalities:*

$$|H(x, p) - H(y, p)| \le M|x - y|(|p| + 1) \quad \text{for all } x, y, p \in \mathbb{R}^n,$$
$$|H(x, p) - H(x, q)| \le M|p - q| \quad \text{for all } x, p, q \in \mathbb{R}^n.$$

In particular, we have $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 1.1. *The inequality*

$$|V(x)| \le \frac{M}{\lambda}$$

holds for all $x \in \mathbb{R}^n$. Hence, the value function V is bounded on $\mathbb{R}^n$.

Proof. For any $(x, \alpha) \in \mathbb{R}^n \times \mathbf{A}$, we have

$$|J(x,\alpha)| \le \int_0^\infty \mathrm{e}^{-\lambda t}|f(X(t),\alpha(t))|\,\mathrm{d}t \le M\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t = \frac{M}{\lambda}.$$

Applying Lemma 1.1 yields

$$|V(x)| \le \sup_{\alpha\in\mathbf{A}} |J(x,\alpha)| \le \frac{M}{\lambda}.$$ □

Proposition 1.2. *The function V is Hölder continuous on $\mathbb{R}^n$.*

Proof. Fix any $x, y \in \mathbb{R}^n$. For any $\alpha \in \mathbf{A}$, we estimate the difference of $J(x,\alpha)$ and $J(y,\alpha)$. To begin with, we estimate the difference of $X(t) := X(t; x, \alpha)$ and $Y(t) := X(t; y, \alpha)$. Since

$$\begin{aligned}|\dot{X}(t) - \dot{Y}(t)| &= |g(X(t),\alpha(t)) - g(Y(t),\alpha(t))| \\ &\le M|X(t) - Y(t)| \quad \text{for a.e. } t \ge 0,\end{aligned}$$

we find that

$$\begin{aligned}|X(t) - Y(t)| &\le |X(0) - Y(0)| + \int_0^t |\dot{X}(s) - \dot{Y}(s)|\,\mathrm{d}s \\ &\le |x - y| + M\int_0^t |X(s) - Y(s)|\,\mathrm{d}s \quad \text{for all } t \ge 0.\end{aligned}$$

By applying Gronwall's inequality, we get

$$|X(t) - Y(t)| \le |x - y|\,\mathrm{e}^{Mt} \quad \text{for all } t \ge 0.$$

Next, since

$$|J(x,\alpha) - J(y,\alpha)| \le \int_0^\infty \mathrm{e}^{-\lambda s}|f(X(s),\alpha(s)) - f(Y(s),\alpha(s))|\,\mathrm{d}s,$$

if $\lambda > M$, then we have

$$\begin{aligned}|J(x,\alpha) - J(y,\alpha)| &\le \int_0^\infty \mathrm{e}^{-\lambda s} M|X(s) - Y(s)|\,\mathrm{d}s \\ &\le M\int_0^\infty \mathrm{e}^{-\lambda s}|x - y|\mathrm{e}^{Ms}\,\mathrm{d}s = \frac{M|x - y|}{\lambda - M},\end{aligned}$$

and

$$|V(x) - V(y)| \le \frac{M}{\lambda - M}|x - y|. \tag{13}$$

If $0 < \lambda < M$, then we select $0 < \theta < 1$ so that $\theta M < \lambda$, and calculate

$$\begin{aligned} |f(\xi, a) - f(\eta, a)| &\le |f(\xi, a) - f(\eta, a)|^{\theta + (1-\theta)} \\ &\le (M|\xi - \eta|)^{\theta}(2M)^{1-\theta} \quad \text{for all } \xi, \eta \in \mathbb{R}^n, a \in A, \end{aligned}$$

and

$$\begin{aligned} |J(x, \alpha) - J(y, \alpha)| &\le (2M)^{1-\theta} \int_0^{\infty} \mathrm{e}^{-\lambda s}(M|X(s) - Y(s)|)^{\theta}\, \mathrm{d}s \\ &\le (2M)^{1-\theta} \int_0^{\infty} \mathrm{e}^{-\lambda s}(M|x - y|)^{\theta}\mathrm{e}^{\theta M s}\, \mathrm{d}s \\ &\le 2M|x - y|^{\theta} \int_0^{\infty} \mathrm{e}^{-(\lambda - \theta M)s}\, \mathrm{d}s = \frac{2M|x - y|^{\theta}}{\lambda - \theta M}, \end{aligned}$$

which shows that

$$|V(x) - V(y)| \le \frac{2M|x - y|^{\theta}}{\lambda - \theta M}. \tag{14}$$

Thus we conclude from (13) and (14) that V is Hölder continuous on $\mathbb{R}^n$. □

Proposition 1.3 (Dynamic programming principle). Let $0 < \tau < \infty$ and $x \in \mathbb{R}^n$. Then

$$V(x) = \inf_{\alpha \in \mathbf{A}} \Big(\int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\, \mathrm{d}t + \mathrm{e}^{-\lambda \tau} V(X(\tau)) \Big),$$

where $X(t)$ denotes $X(t; x, \alpha)$.

Proof. Let $0 < \tau < \infty$ and $x \in \mathbb{R}^n$. Fix $\gamma \in \mathbf{A}$. We have

$$\begin{aligned} J(x, \gamma) &= \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \gamma(t))\, \mathrm{d}t + \int_{\tau}^{\infty} \mathrm{e}^{-\lambda t} f(X(t), \gamma(t))\, \mathrm{d}t \\ &= \int_0^{\tau} \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\, \mathrm{d}t + \mathrm{e}^{-\lambda \tau} \int_0^{\infty} \mathrm{e}^{-\lambda t} f(Y(t), \beta(t))\, \mathrm{d}t, \end{aligned} \tag{15}$$

where

$$\begin{aligned} &X(t) = X(t; x, \gamma), \quad \alpha(t) := \gamma(t), \quad \beta(t) := \gamma(t + \tau), \\ &Y(t) := X(t + \tau) = X(t; X(\tau), \beta). \end{aligned}$$

By (15), we get

$$J(x,\gamma) \geq \int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)),$$

from which we have

$$J(x,\gamma) \geq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big).$$

Consequently,

$$V(x) \geq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big). \tag{16}$$

Now, let $\alpha, \beta \in \mathbf{A}$. Define $\gamma \in \mathbf{A}$ by

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } 0 \leq t \leq \tau, \\ \beta(t-\tau) & \text{if } \tau < t. \end{cases}$$

Set

$$X(t) := X(t;x,\alpha) \quad \text{and} \quad Y(t) := X(t;X(\tau),\beta).$$

We have

$$\begin{cases} X(t) = X(t;x,\gamma) \ \text{ and } \ \alpha(t) = \gamma(t) \quad \text{for all } t \in [0,\,\tau], \\ \beta(t) = \gamma(t+\tau) \ \text{ and } \ Y(t) = X(t+\tau) \quad \text{for all } t \geq 0. \end{cases}$$

Hence, we have (15) and therefore,

$$V(x) \leq \int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} J(X(\tau),\beta).$$

Moreover, we get

$$V(x) \leq \int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)),$$

and

$$V(x) \leq \inf_{\alpha\in\mathbf{A}} \Big(\int_0^\tau \mathrm{e}^{-\lambda t} f(X(t),\alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda\tau} V(X(\tau)) \Big). \tag{17}$$

Combining (16) and (17) completes the proof. □

Theorem 1.1. *The value function V is a viscosity solution of* (11).

Proof. (Subsolution property) Let $\phi \in C^1(\mathbb{R}^n)$ and $\hat{x} \in \mathbb{R}^n$, and assume that

$$(V - \phi)(\hat{x}) = \max_{\mathbb{R}^n}(V - \phi) = 0.$$

Fix any $a \in A$ and set $\alpha(t) := a$, $X(t) := X(t; \hat{x}, \alpha)$. Let $0 < h < \infty$. Now, since $V \leq \phi$, $V(\hat{x}) = \phi(\hat{x})$, by Proposition 1.3 we get

$$\begin{aligned}\phi(\hat{x}) = V(\hat{x}) &\leq \int_0^h \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h} V(X(h)) \\ &\leq \int_0^h \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h}\phi(X(h)).\end{aligned}$$

From this, we get

$$\begin{aligned}0 &\leq \int_0^h \mathrm{e}^{-\lambda t} f(X(t), a)\,\mathrm{d}t + \int_0^h \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\lambda t}\phi(X(t)))\,\mathrm{d}t \\ &= \int_0^h \mathrm{e}^{-\lambda t}\left(f(X(t), a) - \lambda\phi(X(t)) + D\phi(X(t)) \cdot \dot{X}(t)\right)\,\mathrm{d}t \\ &= \int_0^h \mathrm{e}^{-\lambda t}\left(f(X(t), a) - \lambda\phi(X(t)) + D\phi(X(t)) \cdot g(X(t), a)\right)\mathrm{d}t.\end{aligned} \tag{18}$$

Noting that

$$|X(t) - \hat{x}| = \left|\int_0^t \dot{X}(s)\,\mathrm{d}s\right| \leq \int_0^t |g(X(s), a)|\,\mathrm{d}s \leq M\int_0^t \mathrm{d}s = Mt, \tag{19}$$

dividing (18) by h and sending $h \to 0$, we find that

$$0 \leq -\lambda\phi(\hat{x}) + f(\hat{x}, a) + g(\hat{x}, a) \cdot D\phi(\hat{x}).$$

Since $a \in A$ is arbitrary, we have $\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) \leq 0$.

(Supersolution property) Let $\phi \in C^1(\mathbb{R}^n)$ and $\hat{x} \in \mathbb{R}^n$, and assume that

$$(V - \phi)(\hat{x}) = \min_{\mathbb{R}^n}(V - \phi) = 0.$$

Fix $\varepsilon > 0$ and $h > 0$. By Proposition 1.3, we may choose $\alpha \in \mathbf{A}$ so that

$$V(\hat{x}) + \varepsilon h > \int_0^h \mathrm{e}^{-\lambda t} f(X(t), \alpha(t))\,\mathrm{d}t + \mathrm{e}^{-\lambda h} V(X(h)),$$

where $X(t) := X(t; \hat{x}, \alpha)$. Since $V \geq \phi$ in $\mathbb{R}^n$ and $V(\hat{x}) = \phi(\hat{x})$, we get

$$\phi(\hat{x}) + \varepsilon h > \int_0^h e^{-\lambda t} f(X(t), \alpha(t))\, dt + e^{-\lambda h}\phi(X(h)).$$

Hence we get

$$\begin{aligned} 0 \geq & \int_0^h e^{-\lambda t} f(X(t), \alpha(t))\, dt + \int_0^h \frac{d}{dt}(e^{-\lambda t}\phi(X(t)))\, dt - \varepsilon h \\ = & \int_0^h e^{-\lambda t} \big(f(X(t), \alpha(t)) - \lambda\phi(X(t)) + D\phi(X(t)) \cdot \dot{X}(t)\big)\, dt - \varepsilon h \\ = & \int_0^h e^{-\lambda t}\big(f(X(t), \alpha(t)) - \lambda\phi(X(t)) + D\phi(X(t)) \cdot g(X(t), \alpha(t))\big)\, dt - \varepsilon h. \end{aligned}$$

By the definition of H, we get

$$\int_0^h e^{-\lambda t}(\lambda\phi(X(t)) + H(X(t), D\phi(t))\, dt + \varepsilon h > 0. \tag{20}$$

As in (19), we have

$$|X(t) - \hat{x}| \leq Mt.$$

Dividing (20) by h and sending $h \to 0$ yield

$$\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) + \varepsilon \geq 0,$$

from which we get $\lambda\phi(\hat{x}) + H(\hat{x}, D\phi(\hat{x})) \geq 0$. The proof is now complete. □

Theorem 1.2. *Let $u \in \mathrm{BUC}(\mathbb{R}^n)$ and $v \in \mathrm{BUC}(\mathbb{R}^n)$ be a viscosity subsolution and supersolution of* (11)*, respectively. Then $u \leq v$ in $\mathbb{R}^n$.*

Proof. Let $\varepsilon > 0$, and define $u_\varepsilon \in C(\mathbb{R}^n)$ by $u_\varepsilon(x) = u(x) - \varepsilon(\langle x\rangle + M)$, where $\langle x\rangle = (|x|^2 + 1)^{1/2}$. A formal calculation

$$\begin{aligned} u_\varepsilon(x) + H(x, Du_\varepsilon(x)) \leq & \, u(x) - \varepsilon M + H(x, Du(x)) + \varepsilon M|D\langle x\rangle| \\ \leq & \, u(x) + H(x, Du(x)) \leq 0 \end{aligned}$$

reveals that u_ε is a viscosity subsolution of (11), which can be easily justified.

We show that the inequality $u_\varepsilon \leq v$ holds, from which we deduce that $u \leq v$ is valid. To do this, we assume that $\sup_{\mathbb{R}^n}(u_\varepsilon - v) > 0$ and will get a contradiction. Since

$$\lim_{|x|\to\infty} (u_\varepsilon - v)(x) = -\infty,$$

we may choose a constant $R > 0$ so that

$$\sup_{\mathbb{R}^n \setminus B_R} (u_\varepsilon - v) < 0.$$

The function $u_\varepsilon - v \in C(\overline{B}_R)$ then attains a maximum at a point in B_R, but not at any point in ∂B_R.

Let $\alpha > 1$ and consider the function

$$\Phi(x, y) = u_\varepsilon(x) - v(y) - \alpha|x - y|^2$$

on $K := \overline{B}_R \times \overline{B}_R$. Since $\Phi \in C(K)$, Φ attains a maximum at a point in K. Let $(x_\alpha, y_\alpha) \in K$ be its maximum point. Because K is compact, we may choose a sequence $\{\alpha_j\} \subset (1, \infty)$ diverging to infinity so that for some $(\hat{x}, \hat{y}) \in K$,

$$(x_{\alpha_j}, y_{\alpha_j}) \to (\hat{x}, \hat{y}) \quad \text{as } j \to \infty.$$

Note that

$$\begin{aligned} 0 < \max_{\overline{B}_R}(u_\varepsilon - v) &= \max_{x \in \overline{B}_R} \Phi(x, x) \le \Phi(x_\alpha, y_\alpha) \\ &= u_\varepsilon(x_\alpha) - v(y_\alpha) - \alpha|x_\alpha - y_\alpha|^2, \end{aligned} \tag{21}$$

from which we get

$$\alpha|x_\alpha - y_\alpha|^2 \le \sup_{\mathbb{R}^n} u_\varepsilon + \sup_{\mathbb{R}^n}(-v).$$

We infer from this that $\hat{x} = \hat{y}$. Once again by (21), we get

$$\max_{\overline{B}_R}(u_\varepsilon - v) \le u_\varepsilon(x_\alpha) - v(y_\alpha).$$

Setting $\alpha = \alpha_j$ and sending $j \to \infty$ in the above, since $u, v \in C(\mathbb{R}^n)$, we see that

$$\begin{aligned} \max_{\overline{B}_R}(u_\varepsilon - v) &\le \lim_{\alpha = \alpha_j, j \to \infty} (u_\varepsilon(x_\alpha) - v(y_\alpha)) \\ &= u_\varepsilon(\hat{x}) - v(\hat{x}). \end{aligned}$$

That is, the point $\hat{x}$ is a maximum point of $u_\varepsilon - v$. By (21), we have

$$\alpha|x_\alpha - y_\alpha|^2 \le u_\varepsilon(x_\alpha) - v(y_\alpha) - \max_{\overline{B}_R}(u - v),$$

and hence

$$\lim_{\alpha = \alpha_j, j \to \infty} \alpha|x_\alpha - y_\alpha|^2 = 0.$$

Since $\hat{x}$ is a maximum point of $u_\varepsilon - v$, by our choice of R we see that $\hat{x} \in B_R$. Accordingly, if $\alpha = \alpha_j$ and j is sufficiently large, then $x_\alpha, y_\alpha \in B_R$. By the viscosity property of u_ε and v, for $\alpha = \alpha_j$ and $j \in \mathbb{N}$ large enough, we have

$$u_\varepsilon(x_\alpha) + H(x_\alpha, 2\alpha(x_\alpha - y_\alpha)) \le 0, \qquad v(y_\alpha) + H(y_\alpha, 2\alpha(x_\alpha - y_\alpha)) \ge 0.$$

Subtracting one from the other yields

$$u_\varepsilon(x_\alpha) - v(y_\alpha) \le H(y_\alpha, 2\alpha(x_\alpha - y_\alpha)) - H(x_\alpha, 2\alpha(x_\alpha - y_\alpha)).$$

Using one of the properties of H from Lemma 1.2, we obtain

$$u_\varepsilon(x_\alpha) - v(y_\alpha) \le M|x_\alpha - y_\alpha|(2\alpha|x_\alpha - y_\alpha| + 1).$$

Sending $\alpha = \alpha_j \to \infty$, we get

$$u_\varepsilon(\hat{x}) - v(\hat{x}) \le 0,$$

which is a contradiction. □

1.4 Semicontinuous Viscosity Solutions and the Perron Method

Let $u, v \in C(\Omega)$ be a viscosity subsolutions of (FE) and set

$$w(x) = \max\{u(x), v(x)\} \quad \text{for} \quad x \in \Omega.$$

It is easy to see that w is a viscosity subsolution of (FE). Indeed, if $\phi \in C^1(\Omega)$, $y \in \Omega$ and $w - \phi$ has a maximum at y, then we have either $w(y) = u(y)$ and $(u - \phi)(x) \le (w - \phi)(x) \le (w - \phi)(y) = (u - \phi)(y)$ for all $x \in \Omega$, or $w(y) = v(y)$ and $(v - \phi)(x) \le (v - \phi)(y)$, from which we get $F(y, w(y), D\phi(y)) \le 0$. If $\{u_k\}_{k\in\mathbb{N}} \subset C(\Omega)$ is a uniformly bounded sequence of viscosity subsolutions of (FE), then the function w given by $w(x) = \sup_k u_k(x)$ defines a bounded function on Ω but it may not be continuous, a situation that the notion of viscosity subsolution does not apply.

We are thus led to extend the notion of viscosity solution to that for discontinuous functions.

Let $U \subset \mathbb{R}^n$, and recall that a function $f : U \to \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$ is *upper semicontinuous* if

$$\limsup_{y \to x} f(y) \le f(x) \quad \text{for all} \quad x \in U.$$

The totality of all such upper semicontinuous functions f will be denoted by $\mathrm{USC}(U)$. Similarly, we denote by $\mathrm{LSC}(U)$ the space of all *lower* semicontinuous functions on U. That is, $\mathrm{LSC}(U) := -\mathrm{USC}(U) = \{-f \,:\, f \in \mathrm{USC}(U)\}$.

Some basic observations regarding semicontinuity are the following three propositions.

Proposition 1.4. *Let $f : U \to [-\infty, \infty]$. Then, $f \in \mathrm{USC}(U)$ if and only if the set $\{x \in U \,:\, f(x) < a\}$ is a relatively open subset of U for any $a \in \mathbb{R}$.*

Proposition 1.5. *If $\mathscr{F} \subset \mathrm{LSC}(U)$, then $\sup \mathscr{F} \in \mathrm{LSC}(U)$. Similarly, if $\mathscr{F} \subset \mathrm{USC}(U)$, then $\inf \mathscr{F} \in \mathrm{USC}(U)$.*

Proposition 1.6. *Let K be a compact subset of $\mathbb{R}^n$ and $f \in \mathrm{USC}(K)$. Then f attains a maximum. Here the maximum value may be either $-\infty$ or ∞.*

Next, we define the upper (resp., lower) *semicontinuous envelopes* f^* (resp., f_*) of $f : U \to [-\infty, \infty]$ by

$$f^*(x) = \lim_{r\to 0+} \sup\{f(y) \,:\, y \in U \cap B_r(x)\}$$

(resp., $f_* = -(-f)^*$ or, equivalently, $f_*(x) = \lim_{r\to 0+} \inf\{f(y) \,:\, y \in U \cap B_r(x)\}$).

Proposition 1.7. *Let $f : U \to [-\infty, \infty]$. Then we have $f^* \in \mathrm{USC}(U)$, $f_* \in \mathrm{LSC}(U)$ and*

$$f^*(x) = \min\{g(x) \,:\, g \in \mathrm{USC}(U),\ g \geq f\} \quad \textit{for all } x \in U.$$

A consequence of the above proposition is that if $f \in \mathrm{USC}(U)$, then $f^* = f$ in U. Similarly, $f_* = f$ in U if $f \in \mathrm{LSC}(U)$.

We go back to

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{FE}$$

Here we assume neither that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous nor that $\Omega \subset \mathbb{R}^n$ is open. We just assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is locally bounded and that Ω is a subset of $\mathbb{R}^n$.

Definition 1.2. (i) A locally bounded function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of (FE) if

$$\begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \max_{\Omega}(u^* - \phi) = (u^* - \phi)(z) \\ \implies F_*(z, u^*(z), D\phi(z)) \leq 0 \end{cases}$$

$$\left(resp., \quad \begin{cases} \phi \in C^1(\Omega),\ z \in \Omega,\ \min_{\Omega}(u_* - \phi) = (u_* - \phi)(z) \\ \implies F^*(z, u_*(z), D\phi(z)) \geq 0 \end{cases} \right).$$

(ii) A locally bounded function $u : \Omega \to \mathbb{R}$ is a viscosity solution of (FE) if it is both a viscosity subsolution and supersolution of (FE).

We warn here that the envelopes F_* and F^* are taken in the full variables. For instance, if $\xi \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then

$$F_*(\xi) = \lim_{r \to 0+} \inf\{F(\eta) : \eta \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \ |\eta - \xi| < r\}.$$

We say conveniently that u is a viscosity solution (or subsolution) of $F(x, u(x), Du(x)) \leq 0$ in Ω if u is a viscosity subsolution of (FE). Similarly, we say that u is a viscosity solution (or supersolution) of $F(x, u(x), Du(x)) \geq 0$ in Ω if u is a viscosity supersolution of (FE). Also, we say that u satisfies $F(x, u(x), Du(x)) \leq 0$ in Ω (resp., $F(x, u(x), Du(x)) \geq 0$ in Ω) in the *viscosity sense* if u is a viscosity subsolution (resp., supersolution) of (FE).

Once we fix a PDE, like (FE), on a set Ω, we denote by $\mathscr{S}^-$ and $\mathscr{S}^+$ the sets of all its viscosity subsolutions and supersolutions, respectively.

The above definition differs from the one in [19]. As is explained in [19], the above one allows the following situation: let Ω be a nonempty open subset of $\mathbb{R}^n$ and suppose that the Hamilton–Jacobi equation (1) has a continuous solution $u \in C(\Omega)$. Choose two dense subsets U and V of Ω such that $U \cap V = \emptyset$ and $U \cup V \neq \Omega$. Select a function $v : \Omega \to \mathbb{R}$ so that $v(x) = u(x)$ if $x \in U$, $v(x) = u(x) + 1$ if $x \in V$ and $v(x) \in [u(x), u(x)+1]$ if $x \in \Omega \setminus (U \cup V)$. Then we have $v_*(x) = u(x)$ and $v^*(x) = u(x) + 1$ for all $x \in \Omega$. Consequently, v is a viscosity solution of (1). If $U \cup V \neq \Omega$, then there are infinitely many choices of such functions v.

The same remarks as Remarks 1.1–1.4 are valid for the above generalized definition.

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. The subdifferential $D^-u(x)$ and superdifferential $D^+u(x)$ of the function u at $x \in \Omega$ are defined, respectively, by

$$D^-u(x) = \{p \in \mathbb{R}^n : u(x+h) \geq u(x) + p \cdot h + o(|h|) \ \text{as} \ x + h \in \Omega, \ h \to 0\},$$
$$D^+u(x) = \{p \in \mathbb{R}^n : u(x+h) \leq u(x) + p \cdot h + o(|h|) \ \text{as} \ x + h \in \Omega, \ h \to 0\},$$

where $o(|h|)$ denotes a function on an interval $(0, \delta)$, with $\delta > 0$, having the property: $\lim_{h \to 0} o(|h|)/|h| = 0$.

We remark that $D^-u(x) = -D^+(-u)(x)$. If u is a convex function in $\mathbb{R}^n$ and $p \in D^-u(x)$ for some $x, p \in \mathbb{R}^n$, then

$$u(x+h) \geq u(x) + p \cdot h \quad \text{for all} \ \ h \in \mathbb{R}^n.$$

See Proposition B.1 for the above claim. In convex analysis, $D^-u(x)$ is usually denoted by $\partial u(x)$.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be locally bounded. Let $x \in \Omega$. Then*

$$D^+u(x) = \{D\phi(x) : \phi \in C^1(\Omega),\ u - \phi \text{ attains a maximum at } x\}.$$

As a consequence of the above proposition, we have the following: if u is locally bounded in Ω, then

$$\begin{aligned} D^-u(x) &= -D^+(-u)(x) \\ &= -\{D\phi(x) : \phi \in C^1(\Omega),\ -u - \phi \text{ attains a maximum at } x\} \\ &= \{D\phi(x) : \phi \in C^1(\Omega),\ u - \phi \text{ attains a minimum at } x\}. \end{aligned}$$

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^n$. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be locally bounded. Then u is a viscosity subsolution (resp., supersolution) of* (FE) *if and only if*

$$F_*(x, u^*(x), p) \le 0 \quad \text{for all } x \in \Omega,\ p \in D^+u^*(x)$$
$$(\text{ resp., } \ F^*(x, u_*(x), p) \ge 0 \quad \text{for all } x \in \Omega,\ p \in D^-u_*(x)\ ).$$

This corollary (or Remark 1.3) says that the viscosity properties of a function, i.e., the properties that the function be a viscosity subsolution, supersolution, or solution are of local nature. For instance, under the hypotheses of Corollary 1.1, the function u is a viscosity subsolution of (FE) if and only if for each $x \in \Omega$ there exists an open neighborhood U_x, in $\mathbb{R}^n$, of x such that u is a viscosity subsolution of (FE) in $U_x \cap \Omega$.

Proof. Let $\phi \in C^1(\Omega)$ and $y \in \Omega$, and assume that $u - \phi$ has a maximum at y. Then

$$(u - \phi)(y + h) \le (u - \phi)(y) \quad \text{if } y + h \in \Omega,$$

and hence, as $y + h \in \Omega,\ h \to 0$,

$$u(y + h) \le u(y) + \phi(y + h) - \phi(y) = u(y) + D\phi(y) \cdot h + o(|h|).$$

This shows that

$$\{D\phi(y) : \phi \in C^1(\Omega),\ u - \phi \text{ attains a maximum at } y\} \subset D^+u(y).$$

Next let $y \in \Omega$ and $p \in D^+u(y)$. Then we have

$$u(y + h) \le u(y) + p \cdot h + \omega(|h|)|h| \quad \text{if } y + h \in \Omega \text{ and } |h| \le \delta$$

for some constant $\delta > 0$ and a function $\omega \in C([0, \delta])$ satisfying $\omega(0) = 0$. We may choose ω to be nondecreasing in $[0, \delta]$. In the above inequality, we want to replace the term $\omega(|h|)|h|$ by a C^1 function $\psi(h)$ having the property: $\psi(h) = o(|h|)$. Following [23], we define the function $\gamma : [0, \delta/2] \to \mathbb{R}$ by

$$\gamma(r) = \int_0^{2r} \omega(t)\,\mathrm{d}t.$$

Noting that

$$\gamma(r) \geq \int_r^{2r} \omega(t)\,\mathrm{d}t \geq \omega(r)r \quad \text{for } r \in [0,\ \delta/2],$$

we see that

$$u(y+h) \leq u(y) + p \cdot h + \gamma(|h|) \quad \text{if } y+h \in \Omega \text{ and } |h| \leq \delta/2.$$

It immediate to see that $\gamma \in C^1([0,\ \delta/2])$ and $\gamma(0) = \gamma'(0) = 0$. We set $\psi(h) = \gamma(|h|)$ for $h \in B_{\delta/2}(0)$. Then $\psi \in C^1(B_{\delta/2}(0))$, $\psi(0) = 0$ and $D\psi(0) = 0$. It is now clear that if we set

$$\phi(x) = u(y) + p \cdot (x-y) + \psi(x-y) \quad \text{for } x \in B_{\delta/2}(y),$$

then the function $u - \phi$ attains a maximum over $\Omega \cap B_{\delta/2}(y)$ at y and $D\phi(y) = p$. □

Now, we discuss a couple of stability results concerning viscosity solutions.

Proposition 1.9. *Let $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathscr{S}^-$. Assume that Ω is locally compact and $\{u_\varepsilon\}$ converges locally uniformly to a function u in Ω as $\varepsilon \to 0$. Then $u \in \mathscr{S}^-$.*

Proof. Let $\phi \in C^1(\Omega)$. Assume that $u^* - \phi$ attains a strict maximum at $\hat{x} \in \Omega$. We choose a constant $r > 0$ so that $K := \overline{B}_r(\hat{x}) \cap \Omega$ is compact. For each $\varepsilon \in (0,\ 1)$, we choose a maximum point (over K) x_ε of $u_\varepsilon^* - \phi$.

Next, we choose a sequence $\{\varepsilon_j\} \subset (0,\ 1)$ converging to zero such that $x_{\varepsilon_j} \to z$ for some $z \in K$ as $j \to \infty$. Next, observe in view of the choice of x_ε that

$$\begin{aligned}(u^* - \phi)(x_{\varepsilon_j}) &\geq (u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) - \|u^* - u_{\varepsilon_j}^*\|_{\infty,K} \\ &\geq (u^* - \phi)(x_{\varepsilon_j}) - 2\|u^* - u_{\varepsilon_j}^*\|_{\infty,K} \\ &\geq (u^* - \phi)(\hat{x}) - 2\|u^* - u_{\varepsilon_j}^*\|_{\infty,K}.\end{aligned}$$

Sending $j \to \infty$ yields

$$(u^* - \phi)(z) \geq \limsup(u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) \geq \liminf_{j\to\infty}(u_{\varepsilon_j}^* - \phi)(x_{\varepsilon_j}) \geq (u^* - \phi)(\hat{x}),$$

which shows that $z = \hat{x}$ and $\lim_{j\to\infty} u_{\varepsilon_j}^*(x_{\varepsilon_j}) = u^*(\hat{x})$. For $j \in \mathbb{N}$ sufficiently large, we have $x_{\varepsilon_j} \in B_r(\hat{x})$ and, since $u_{\varepsilon_j} \in \mathscr{S}^-$,

$$F_*(x_{\varepsilon_j}, u^*_{\varepsilon_j}(x_{\varepsilon_j}), D\phi(x_{\varepsilon_j})) \le 0.$$

If we send $j \to \infty$, we find that $u \in \mathscr{S}^-$. □

Proposition 1.10. *Let Ω be locally compact. Let $\mathscr{F} \subset \mathscr{S}^-$. That is, $\mathscr{F}$ is a family of viscosity subsolutions of* (FE). *Assume that* $\sup \mathscr{F}$ *is locally bounded in Ω. Then we have* $\sup \mathscr{F} \in \mathscr{S}^-$.

Remark 1.6. By definition, the set Ω is locally compact if for any $x \in \Omega$, there exists a constant $r > 0$ such that $\Omega \cap \overline{B}_r(x)$ is compact. For instance, every open subset and closed subset of $\mathbb{R}^n$ are locally compact. The set $A := (0, 1) \times [0, 1] \subset \mathbb{R}^2$ is locally compact, but the set $A \cup \{(0, 0)\}$ is not locally compact.

Remark 1.7. Similarly to Remark 1.5, if Ω is locally compact, then the C^1 regularity of the test functions in the Definition 1.2 can be replaced by the C^∞ regularity.

Proof. Set $u = \sup \mathscr{F}$. Let $\phi \in C^1(\Omega)$ and $\hat{x} \in \Omega$, and assume that

$$\max_{\Omega}(u^* - \phi) = (u^* - \phi)(\hat{x}) = 0.$$

We assume moreover that $\hat{x}$ is a strict maximum point of $u^* - \phi$. That is, we have $(u^* - \phi)(x) < 0$ for all $x \ne \hat{x}$. Choose a constant $r > 0$ so that $W := \Omega \cap \overline{B}_r(\hat{x})$ is compact.

By the definition of u^*, there are sequences $\{y_k\} \subset W$ and $\{v_k\} \subset \mathscr{F}$ such that

$$y_k \to \hat{x}, \qquad v_k(y_k) \to u^*(\hat{x}) \quad \text{as} \quad k \to \infty.$$

Since W is compact, for each $k \in \mathbb{N}$ we may choose a point $x_k \in W$ such that

$$\max_{W}(v_k^* - \phi) = (v_k^* - \phi)(x_k).$$

By passing to a subsequence if necessary, we may assume that $\{x_k\}$ converges to a point $z \in W$. We then have

$$\begin{aligned} 0 = (u^* - \phi)(\hat{x}) &\ge (u^* - \phi)(x_k) \ge (v_k^* - \phi)(x_k) \\ &\ge (v_k^* - \phi)(y_k) \ge (v_k - \phi)(y_k) \to (u^* - \phi)(\hat{x}) = 0, \end{aligned}$$

and consequently

$$\lim_{k\to\infty} u^*(x_k) = \lim_{k\to\infty} v_k^*(x_k) = u^*(\hat{x}).$$

In particular, we see that

$$(u^* - \phi)(z) \ge \lim_{k\to\infty} (u^* - \phi)(x_k) = 0,$$

which shows that $z = \hat{x}$. That is, $\lim_{k\to\infty} x_k = \hat{x}$.

Thus, we have $x_k \in B_r(\hat{x})$ for sufficiently large $k \in \mathbb{N}$. Since $v_k \in \mathscr{S}^-$, we get

$$F_*(x_k, v_k^*(x_k), D\phi(x_k)) \le 0$$

if k is large enough. Hence, sending $k \to \infty$ yields

$$F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \le 0,$$

which proves that $u \in \mathscr{S}^-$. □

Theorem 1.3. *Let Ω be a locally compact subset of $\mathbb{R}^n$. Let $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{F_\varepsilon\}_{\varepsilon\in(0,1)}$ be locally uniformly bounded collections of functions on Ω and $\Omega \times \mathbb{R} \times \mathbb{R}^n$, respectively. Assume that for each $\varepsilon \in (0, 1)$, u_ε is a viscosity subsolution of*

$$F_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x)) \le 0 \quad \text{in } \Omega.$$

Set

$$\bar{u}(x) = \lim_{r\to 0+} \sup\{u_\varepsilon(y) \,:\, y \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\},$$

$$\underline{F}(\xi) = \lim_{r\to 0+} \inf\{F_\varepsilon(\eta) \,:\, \eta \in \Omega \times \mathbb{R} \times \mathbb{R}^n,\ |\eta - \xi| < r,\ \varepsilon \in (0, r)\}.$$

Then $\bar{u}$ is a viscosity subsolution of

$$\underline{F}(x, \bar{u}(x), D\bar{u}(x)) \le 0 \quad \text{in } \Omega.$$

Remark 1.8. The function $\bar{u}$ is upper semicontinuous in Ω. Indeed, we have

$$\bar{u}(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\}$$

for all $x \in \Omega$ and $y \in B_r(x) \cap \Omega$. This yields

$$\limsup_{\Omega\ni y\to x} \bar{u}(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega,\ \varepsilon \in (0, r)\}$$

for all $x \in \Omega$. Hence,

$$\limsup_{\Omega\ni y\to x} \bar{u}(y) \le \bar{u}(x) \quad \text{for all } x \in \Omega.$$

Similarly, the function $\underline{F}$ is lower semicontinuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Proof. It is easily seen that for all $x \in \Omega$, $r > 0$ and $y \in B_r(x) \cap \Omega$,

$$u_\varepsilon^*(y) \le \sup\{u_\varepsilon(z) \,:\, z \in B_r(x) \cap \Omega\}.$$

From this we deduce that

$$\bar{u}(x) = \lim_{r\to 0+} \sup\{u_\varepsilon^*(y) \;:\; y \in B_r(x) \cap \Omega,\, 0 < \varepsilon < r\} \quad \text{for all } \ x \in \Omega.$$

Hence, we may assume by replacing u_ε by u_ε^* if necessary that $u_\varepsilon \in \mathrm{USC}(\Omega)$. Similarly, we may assume that $F_\varepsilon \in \mathrm{LSC}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$.

Let $\phi \in C^1(\Omega)$ and $\hat{x} \in \Omega$. Assume that $\bar{u} - \phi$ has a strict maximum at $\hat{x}$. Let $r > 0$ be a constant such that $\overline{B}_r(\hat{x}) \cap \Omega$ is compact.

For each $k \in \mathbb{N}$ we choose $y_k \in B_{r/k}(\hat{x}) \cap \Omega$ and $\varepsilon_k \in (0,\ 1/k)$ so that

$$|\bar{u}(\hat{x}) - u_{\varepsilon_k}(y_k)| < 1/k,$$

and then choose a maximum point $x_k \in B_r(\hat{x}) \cap \Omega$ of $u_{\varepsilon_k} - \phi$ over $\overline{B}_r(\hat{x}) \cap \Omega$.

Since

$$(u_{\varepsilon_k} - \phi)(x_k) \geq (u_{\varepsilon_k} - \phi)(y_k),$$

we get

$$\limsup_{k\to 0} (u_{\varepsilon_k} - \phi)(x_k) \geq (\bar{u} - \phi)(\hat{x}),$$

which implies that

$$\lim_{k\to\infty} x_k = \hat{x} \quad \text{and} \quad \lim_{k\to\infty} u_{\varepsilon_k}(x_k) = \bar{u}(\hat{x}).$$

If $k \in \mathbb{N}$ is sufficiently large, we have $x_k \in B_r(\hat{x}) \cap \Omega$ and hence

$$F_{\varepsilon_k}(x_k, u_{\varepsilon_k}(x_k), D\phi(x_k)) \leq 0.$$

Thus, we get

$$\underline{F}(\hat{x}, \bar{u}(\hat{x}), D\phi(\hat{x})) \leq 0. \qquad \square$$

Proposition 1.9 can be seen now as a direct consequence of the above theorem. The following proposition is a consequence of the above theorem as well.

Proposition 1.11. *Let Ω be locally compact. Let $\{u_k\}$ be a sequence of viscosity subsolutions of (FE). Assume that $\{u_k\} \subset \mathrm{USC}(\Omega)$ and that $\{u_k\}$ is a nonincreasing sequence of functions on Ω, i.e., $u_k(x) \geq u_{k+1}(x)$ for all $x \in \Omega$ and $k \in \mathbb{N}$. Set*

$$u(x) = \lim_{k\to\infty} u_k(x) \quad \textit{for} \ \ x \in \Omega.$$

Assume that u is locally bounded on Ω. Then $u \in \mathscr{S}^-$.

Let us introduce the (outer) normal cone $N(z, \Omega)$ at $z \in \Omega$ by

$$N(z, \Omega) = \{p \in \mathbb{R}^n \;:\; 0 \geq p \cdot (x - z) + o(|x - z|) \ \text{ as } \ \Omega \ni x \to z\}.$$

Another definition equivalent to the above is the following:

$$N(z, \Omega) = -D^+\mathbf{1}_\Omega(z),$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω. Note that if $z \in \Omega$ is an interior point of Ω, then $N(z, \Omega) = \{0\}$.

We say that (FE) or the pair (F, Ω) is *proper* if $F(x, r, p + q) \geq F(x, r, p)$ for all $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and all $q \in N(x, \Omega)$.

Proposition 1.12. *Assume that* (FE) *is proper. If* $u \in C^1(\Omega)$ *is a classical subsolution of* (FE)*, then* $u \in \mathscr{S}^-$.

Proof. Let $\phi \in C^1(\Omega)$ and assume that $u - \phi$ attains a maximum at $z \in \Omega$. We may assume by extending the domain of definition of u and ϕ that u and ϕ are defined and of class C^1 in $B_r(z)$ for some $r > 0$. By reselecting $r > 0$ small enough if needed, we may assume that

$$(u - \phi)(x) < (u - \phi)(z) + 1 \quad \text{for all } x \in B_r(z).$$

It is clear that the function $u - \phi + \mathbf{1}_\Omega$ attains a maximum over $B_r(z)$ at z, which shows that $D\phi(z) - Du(z) \in D^+\mathbf{1}_\Omega(z)$. Setting $q = -D\phi(z) + Du(z)$, we have $Du(z) = D\phi(z) + q$ and

$$0 \geq F(z, u(z), D\phi(z) + q) \geq F(z, u(z), D\phi(z)) \geq F_*(z, u(z), D\phi(z)),$$

which completes the proof. □

Proposition 1.13 (Perron method). *Let* $\mathscr{F}$ *be a nonempty subset of* $\mathscr{S}^-$ *having the properties:*

(P1) $\sup \mathscr{F} \in \mathscr{F}$.
(P2) *If* $v \in \mathscr{F}$ *and* $v \notin \mathscr{S}^+$*, then there exists a* $w \in \mathscr{F}$ *such that* $w(y) > v(y)$ *at some point* $y \in \Omega$.

Then $\sup \mathscr{F} \in \mathscr{S}$.

Proof. We have $\sup \mathscr{F} \in \mathscr{F} \subset \mathscr{S}^-$. That is, $\sup \mathscr{F} \in \mathscr{S}^-$. If we suppose that $\sup \mathscr{F} \notin \mathscr{S}^+$, then, by (P2), we have $w \in \mathscr{F}$ such that $w(y) > (\sup \mathscr{F})(y)$ for some $y \in \Omega$, which contradicts the definition of $\sup \mathscr{F}$. Hence, $\sup \mathscr{F} \in \mathscr{S}^+$. □

Theorem 1.4. *Assume that* Ω *is locally compact and that* (FE) *is proper. Let* $f \in \mathrm{LSC}(\Omega) \cap \mathscr{S}^-$ *and* $g \in \mathrm{USC}(\Omega) \cap \mathscr{S}^+$*. Assume that* $f \leq g$ *in* Ω*. Set*

$$\mathscr{F} = \{v \in \mathscr{S}^- \,:\, f \leq v \leq g \textit{ in } \Omega\}.$$

Then $\sup \mathscr{F} \in \mathscr{S}$.

In the above theorem, the semicontinuity requirement on f, g is "opposite" in a sense: the lower (resp., upper) semicontinuity for the subsolution f (resp., supersolution g). This choice of semicontinuities is convenient in practice since in the construction of supersolution f, for instance, one often takes the infimum of a collection of continuous supersolutions and the resulting function is automatically upper semicontinuous.

Of course, under the same hypotheses of the above theorem, we have following conclusion as well: if we set $\mathscr{F}^+ = \{v \in \mathscr{S}^+ : f \le v \le g \text{ in } \Omega\}$, then $\inf \mathscr{F}^+ \in \mathscr{S}$.

Lemma 1.3. *Assume that Ω is locally compact and that* (FE) *is proper. Let $u \in \mathscr{S}^-$ and $y \in \Omega$, and assume that u is not a viscosity supersolution of* (FE) *at y, that is,*

$$F^*(y, u_*(y), p) < 0 \quad \text{for some } \ p \in D^- u_*(y).$$

Let $\varepsilon > 0$ and U be a neighborhood of y. Then there exists a $v \in \mathscr{S}^-$ such that

$$\begin{cases} u(x) \le v(x) \le \max\{u(x), u_*(y) + \varepsilon\} & \text{for all } \ x \in \Omega, \\ v = u & \text{in } \ \Omega \setminus U, \\ v_*(y) > u_*(y). \end{cases} \tag{22}$$

Furthermore, if u is continuous at y, then there exist an open neighborhood V of y and a constant $\delta > 0$ such that v is a viscosity subsolution of

$$F(x, v(x), Dv(x)) = -\delta \quad \text{in } V \cap \Omega. \tag{23}$$

Proof. By assumption, there exists a function $\phi \in C^1(\Omega)$ such that $u_*(y) = \phi(y)$, $u_*(x) > \phi(x)$ for all $x \ne y$ and

$$F^*(y, u_*(y), D\phi(y)) < 0.$$

Thanks to the upper semicontinuity of F^*, there exists a $\delta \in (0, \varepsilon)$ such that

$$F^*(x, \phi(x) + t, D\phi(x)) < -\delta \quad \text{for all } \ (x, t) \in (\overline{B}_\delta(y) \cap \Omega) \times [0, \delta], \tag{24}$$

and $\overline{B}_\delta(y) \cap \Omega$ is a compact subset of U.

By replacing $\delta > 0$ by a smaller number if needed, we may assume that

$$\phi(x) + \delta \le u_*(y) + \varepsilon \quad \text{for all } \ x \in \overline{B}_\delta(y) \cap \Omega. \tag{25}$$

Since $u_* - \phi$ attains a strict minimum at y and the minimum value is zero, if $(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y) \ne \emptyset$, then the constant

$$m := \min_{(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y)} (u_* - \phi)$$

is positive. Of course, in this case, we have

$$u_*(x) \geq \phi(x) + m \quad \text{for all } x \in (\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y).$$

Set $\lambda = \min\{m, \delta\}$ if $(\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y) \neq \emptyset$ and $\lambda = \delta$ otherwise, and observe that

$$u_*(x) \geq \phi(x) + \lambda \quad \text{for all } x \in (\Omega \cap \overline{B}_\delta(y)) \setminus B_{\delta/2}(y). \tag{26}$$

We define $v : \Omega \to \mathbb{R}$ by

$$v(x) = \begin{cases} \max\{u(x), \phi(x) + \lambda\} & \text{if } x \in B_\delta(y), \\ u(x) & \text{if } x \notin B_\delta(y). \end{cases}$$

If we set $\psi(x) = \phi(x) + \lambda$ for $x \in B_\delta(y) \cap \Omega$, by (24), ψ is a classical subsolution of (FE) in $B_\delta(y) \cap \Omega$. Since (FE) is proper, ψ is a viscosity subsolution of (FE) in $B_\delta(y) \cap \Omega$. Hence, by Proposition 1.10, we see that v is a viscosity subsolution of (FE) in $B_\delta(y) \cap \Omega$.

According to (26) and the definition of v, we have

$$v(x) = u(x) \quad \text{for all } x \in \Omega \setminus B_{\delta/2}(y),$$

and, hence, v is a viscosity subsolution of (FE) in $\Omega \setminus \overline{B}_{\delta/2}(y)$ Thus, we find that $v \in \mathscr{S}^-$.

Since $v = u$ in $\Omega \setminus B_\delta(y)$ by the definition of v, it follows that $v = u$ in $\Omega \setminus U$. It is clear by the definition of v that $v \geq u$ in Ω. Moreover, by (25) we get

$$v(x) \leq \max\{u(x),\, u_*(y) + \varepsilon\} \quad \text{for all } x \in \Omega \cap B_\delta(y).$$

Also, observe that

$$v_*(y) = \max\{u_*(y),\, u_*(y) + \lambda\} = u_*(y) + \lambda > u_*(y).$$

Thus, (22) is valid.

Now, we assume that u is continuous at y. Then we find an open neighborhood $V \subset B_\delta(y)$ of y such that

$$u(x) < \phi(x) + \lambda \quad \text{for all } \; x \in V \cap \Omega,$$

and hence, we have $v(x) = \phi(x) + \lambda$ for all $x \in V \cap \Omega$. Now, by (24) we see that v is a classical (and hence viscosity) subsolution of (23). □

Proof (Theorem 1.4). We have $\mathscr{F} \neq \emptyset$ since $f \in \mathscr{F}$. In view of Proposition 1.13, we need only to show that the set $\mathscr{F}$ satisfies (P1) and (P2).

By Proposition 1.10, we see immediately that $\mathscr{F}$ satisfies (P1).

To check property (P2), let $v \in \mathscr{F}$ be not a viscosity supersolution of (FE). There is a point $y \in \Omega$ where v is not a viscosity supersolution of (FE). That is, for some $p \in D^- v_*(y)$, we have

$$F^*(y, v_*(y), p) < 0. \tag{27}$$

Noting $v_* \le g_*$ in Ω, there are two possibilities: $v_*(y) = g_*(y)$ or $v_*(y) < g_*(y)$. If $v_*(y) = g_*(y)$, then $p \in D^- g_*(y)$. Since $g \in \mathscr{S}^+$, we have

$$F^*(y, g_*(y), p) \ge 0,$$

which contradicts (27). If $v_*(y) < g_*(y)$, then we choose a constant $\varepsilon > 0$ and a neighborhood V of y so that

$$v_*(y) + \varepsilon < g_*(x) \quad \text{for all } x \in V \cap \Omega. \tag{28}$$

Now, Lemma 1.3 guarantees that there exist $w \in \mathscr{S}^-$ such that $v \le w \le \max\{v, v_*(y) + \varepsilon\}$ in Ω, $v = w$ in $\Omega \setminus V$ and $w_*(y) > v_*(y)$. For any $x \in \Omega \cap V$, by (28) we have

$$w(x) \le \max\{v(x), g_*(x)\} \le g(x).$$

For any $x \in \Omega \setminus V$, we have

$$w(x) = v(x) \le g(x).$$

Thus, we find that $w \in \mathscr{F}$. Since $w_*(y) > v_*(y)$, it is clear that $w(z) > v(z)$ at some point $z \in \Omega$. Hence, $\mathscr{F}$ satisfies (P2). □

1.5 An Example

We illustrate the use of the stability properties established in the previous subsection by studying the solvability of the Dirichlet problem for the eikonal equation

$$|Du(x)| = k(x) \qquad \text{in } \Omega, \tag{29}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega, \tag{30}$$

where Ω is a bounded, open, connected subset of $\mathbb{R}^n$ and $k \in C(\overline{\Omega})$ is a positive function in $\overline{\Omega}$, i.e., $k(x) > 0$ for all $x \in \overline{\Omega}$.

Note that the constant function $f(x) := 0$ is a classical subsolution of (29). Set $M = \max_{\overline{\Omega}} k$. We observe that for each $y \in \partial\Omega$ the function $g_y(x) := M|x - y|$ is a classical supersolution of (29). We set

$$g(x) = \inf\{g_y(x) \,:\, y \in \partial\Omega\} \quad \text{for } x \in \overline{\Omega}.$$

By Proposition 1.10 (its version for supersolutions), we see that g is a viscosity supersolution of (29). Also, by applying Lemma 1.1, we find that g is Lipschitz continuous in $\overline{\Omega}$.

An application of Theorem 1.4 ensures that there is a viscosity solution $u : \Omega \to \mathbb{R}$ of (29) such that $f \leq u \leq g$ in Ω. Since $f(x) = g(x) = 0$ on $\partial\Omega$, if we set $u(x) = 0$ for $x \in \partial\Omega$, then the resulting function u is continuous at points on the boundary $\partial\Omega$ and satisfies the Dirichlet condition (30) in the classical sense.

Note that $u^* \leq g$ in Ω, which clearly implies that $u = u^* \in \mathrm{USC}(\Omega)$. Now, if we use the next proposition, we find that u is locally Lipschitz continuous in Ω and conclude that $u \in C(\overline{\Omega})$. Thus, the Dirichlet problem (29)–(30) has a viscosity solution $u \in C(\overline{\Omega})$ which satisfies (30) in the classical (or pointwise) sense.

Proposition 1.14. *Let $R > 0$, $C > 0$ and $u \in \mathrm{USC}(B_R)$. Assume that u is a viscosity solution of*

$$|Du(x)| \leq C \quad \text{in } B_R.$$

Then u is Lipschitz continuous in B_R with C being a Lipschitz bound. That is, $|u(x) - u(y)| \leq C|x - y|$ for all $x, y \in B_R$.

Proof. Fix any $z \in B_R$ and set $r = (R - |z|)/4$. Fix any $y \in B_r(z)$. Note that $B_{3r}(y) \subset B_R$. Choose a function $f \in C^1([0, 3r))$ so that $f(t) = t$ for all $t \in [0, 2r]$, $f'(t) \geq 1$ for all $t \in [0, 3r)$ and $\lim_{t\to 3r-} f(t) = \infty$. Fix any $\varepsilon > 0$, and we claim that

$$u(x) \leq v(x) := u(y) + (C + \varepsilon)f(|x - y|) \quad \text{for all } x \in B_{3r}(y). \tag{31}$$

Indeed, if this were not the case, we would find a point $\xi \in B_{3r}(y) \setminus \{y\}$ such that $u - v$ attains a maximum at ξ, which yields together with the viscosity property of u

$$C \geq |Dv(\xi)| = (C + \varepsilon)f'(|\xi - y|) \geq C + \varepsilon.$$

This is a contradiction. Thus we have (31).

Note that if $x \in B_r(z)$, then $x \in B_{2r}(y)$ and $f(|x - y|) = |x - y|$. Hence, from (31), we get

$$u(x) - u(y) \leq (C + \varepsilon)|x - y| \quad \text{for all } x, y \in B_r(z).$$

By symmetry, we see that

$$|u(x) - u(y)| \leq (C + \varepsilon)|x - y| \quad \text{for all } x, y \in B_r(z),$$

from which we deduce that

$$|u(x) - u(y)| \leq C|x - y| \quad \text{for all } x, y \in B_r(z), \tag{32}$$

Now, let $x, y \in B_R$ be arbitrary points. Set $r = \frac{1}{4}\min\{R - |x|,\ R - |y|\}$, and choose a finite sequence $\{z_i\}_{i=0}^N$ of points on the line segment $[x,\ y]$ so that $z_0 = x$, $z_N = y$, $|z_i - z_{i-1}| < r$ for all $i = 1, \ldots, N$ and $\sum_{i=1}^N |z_i - z_{i-1}| = |x - y|$. By (32), we get

$$|u(z_i) - u(z_{i-1})| \le C|z_i - z_{i-1}| \quad \text{for all } i = 1, \ldots, N.$$

Summing these over $i = 1, \ldots, N$ yields the desired inequality. □

1.6 Sup-convolutions

Sup-convolutions and inf-convolutions are basic and important tools for regularizing or analyzing viscosity solutions. In this subsection, we recall some properties of sup-convolutions.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function and $\varepsilon \in \mathbb{R}_+$. The standard sup-convolution $u^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ and inf-convolution $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ are defined, respectively, by

$$u^\varepsilon(x) = \sup_{y \in \mathbb{R}^n} \left(u(y) - \frac{1}{2\varepsilon}|y - x|^2 \right) \quad \text{and} \quad u_\varepsilon(x) = \inf_{y \in \mathbb{R}^n} \left(u(y) + \frac{1}{2\varepsilon}|y - x|^2 \right).$$

Note that

$$u_\varepsilon(x) = -\sup \left(-u(y) - \frac{1}{2\varepsilon}|y - x|^2 \right) = -(-u)^\varepsilon(x).$$

This relation immediately allows us to interpret a property of sup-convolutions into the corresponding property of inf-convolutions.

In what follows we assume that u is bounded and upper semicontinuous in $\mathbb{R}^n$. Let $M > 0$ be a constant such that $|u(x)| \le M$ for all $x \in \mathbb{R}^n$.

Proposition 1.15. (i) *We have*

$$-M \le u(x) \le u^\varepsilon(x) \le M \quad \textit{for all } x \in \mathbb{R}^n.$$

(ii) *Let $x \in \mathbb{R}^n$ and $p \in D^+u^\varepsilon(x)$. Then*

$$|p| \le 2\sqrt{\frac{M}{\varepsilon}} \quad \textit{and} \quad p \in D^+u(x + \varepsilon p).$$

Another important property of sup-convolutions is that the sup-convolution u^ε is semiconvex in $\mathbb{R}^n$. More precisely, the function

$$u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2 = \sup_{y \in \mathbb{R}^n} \left(u(y) - \frac{1}{2\varepsilon}|y|^2 + \frac{1}{\varepsilon} y \cdot x \right)$$

is convex in $\mathbb{R}^n$ (see Appendix A.2) as is clear from the form of the right hand side of the above identity.

Proof. To show assertion (i), we just check that for all $x \in \mathbb{R}^n$,

$$u^\varepsilon(x) \le \sup_{y\in\mathbb{R}^n} u(y) \le M,$$

and

$$u^\varepsilon(x) \ge u(x) \ge -M.$$

Next, we prove assertion (ii). Let $\hat{x} \in \mathbb{R}^n$ and $\hat{p} \in D^+u^\varepsilon(\hat{x})$. Choose a point $\hat{y} \in \mathbb{R}^n$ so that

$$u^\varepsilon(\hat{x}) = u(\hat{y}) - \frac{1}{2\varepsilon}|\hat{y}-\hat{x}|^2.$$

(Such a point $\hat{y}$ always exists under our assumptions on u.) It is immediate to see that

$$\frac{1}{2\varepsilon}|\hat{y}-\hat{x}|^2 = u(\hat{y}) - u^\varepsilon(\hat{x}) \le 2M. \tag{33}$$

We may choose a function $\phi \in C^1(\mathbb{R}^n)$ so that $D\phi(\hat{x}) = \hat{p}$ and $\max_{\mathbb{R}^n}(u^\varepsilon - \phi) = (u^\varepsilon - \phi)(\hat{x})$. Observe that the function

$$\mathbb{R}^{2n} \ni (x,y) \mapsto u(y) - \frac{1}{2\varepsilon}|y-x|^2 - \phi(x)$$

attains a maximum at $(\hat{x},\hat{y})$. Hence, both the functions

$$\mathbb{R}^n \ni x \mapsto -\frac{1}{2\varepsilon}|\hat{y}-x|^2 - \phi(x)$$

and

$$\mathbb{R}^n \ni x \mapsto u(x+\hat{y}-\hat{x}) - \phi(x)$$

attain maximum values at $\hat{x}$. Therefore, we find that

$$\frac{1}{\varepsilon}(\hat{x}-\hat{y}) + D\phi(\hat{x}) = 0 \quad \text{and} \quad D\phi(\hat{x}) \in D^+u(\hat{y}),$$

which shows that

$$\hat{p} = \frac{1}{\varepsilon}(\hat{y}-\hat{x}) \in D^+u(\hat{y}).$$

From this, we get $\hat{y} = \hat{x} + \varepsilon\hat{p}$, and, moreover, $\hat{p} \in D^+u(\hat{x}+\varepsilon\hat{p})$. Also, using (33), we get $|\hat{p}| \le 2\sqrt{M/\varepsilon}$. Thus we see that (ii) holds. □

The following observations illustrate a typical use of the above proposition. Let Ω be an open subset of $\mathbb{R}^n$. Let $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be bounded and upper semicontinuous. Let $M > 0$ be a constant such that $|u(x)| \le M$ for all $x \in \Omega$. Let $\varepsilon > 0$. Set $\delta = 2\sqrt{\varepsilon M}$ and $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$.

Define u^ε as above with u extended to $\mathbb{R}^n$ by setting $u(x) = -M$ for $x \in \mathbb{R}^n \setminus \Omega$. (Or, in a slightly different and more standard way, one may define u^ε by

$$u^\varepsilon(x) = \sup_{y\in\Omega}\left(u(y) - \frac{1}{2\varepsilon}|x-y|^2\right).)$$

By applying Proposition 1.15, we deduce that if u is a viscosity subsolution of

$$H(x, Du(x)) \leq 0 \quad \text{in } \Omega,$$

then u^ε is a viscosity subsolution of both

$$H(x + \varepsilon Du^\varepsilon(x),\, Du^\varepsilon(x)) \leq 0 \quad \text{in } \Omega_\delta, \tag{34}$$

and

$$|Du^\varepsilon(x)| \leq 2\sqrt{\frac{M}{\varepsilon}} \quad \text{in } \Omega_\delta. \tag{35}$$

If we set

$$G(x, p) = \inf_{y\in B_\delta} H(x + y, p) \quad \text{for } x \in \Omega_\delta,$$

then (34) implies that u^ε is a viscosity subsolution of

$$G(x, Du^\varepsilon(x)) \leq 0 \quad \text{in } \Omega_\delta.$$

If we apply Proposition 1.14 to u^ε, we see from (35) that u^ε is locally Lipschitz in Ω_δ.

2 Neumann Boundary Value Problems

We assume throughout this section and the rest of this article that $\Omega \subset \mathbb{R}^n$ is *open*.

We will be concerned with the initial value problem for the Hamilton–Jacobi equation of evolution type

$$\frac{\partial u}{\partial t}(x, t) + H(x, D_x u(x, t)) = 0 \quad \text{in } \Omega \times (0, \infty),$$

and the asymptotic behavior of its solutions $u(x, t)$ as $t \to \infty$.

The stationary problem associated with the above Hamilton–Jacobi equation is stated as

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \text{boundary condition on } \partial\Omega. \end{cases} \tag{36}$$

In this article we will be focused on the Neumann boundary value problem among other possible choices of boundary conditions like periodic, Dirichlet, state-constraints boundary conditions.

We are thus given two functions $\gamma \in C(\partial\Omega, \mathbb{R}^n)$ and $g \in C(\partial\Omega, \mathbb{R})$ which satisfy

$$\nu(x) \cdot \gamma(x) > 0 \quad \text{for all } x \in \partial\Omega, \tag{37}$$

where $\nu(x)$ denotes the outer unit normal vector at x, and the boundary condition posed on the unknown function u is stated as

$$\gamma(x) \cdot Du(x) = g(x) \quad \text{for } x \in \partial\Omega.$$

This condition is called the (inhomogeneous, linear) *Neumann boundary condition*. We remark that if $u \in C^1(\overline{\Omega})$, then the directional derivative $\partial u/\partial\gamma$ of u in the direction of γ is given by

$$\frac{\partial u}{\partial \gamma}(x) = \gamma(x) \cdot Du(x) = \lim_{t\to 0} \frac{u(x + t\gamma(x)) - u(x)}{t} \quad \text{for } x \in \partial\Omega.$$

(Note here that u is assumed to be defined in a neighborhood of x.)

Our boundary value problem (36) is now stated precisely as

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{SNP}$$

Let U be an open subset of $\mathbb{R}^n$ such that $U \cap \Omega \neq \emptyset$. At this stage we briefly explain the *viscosity formulation* of a more general boundary value problem

$$\begin{cases} F(x, u(x), Du(x)) = 0 & \text{in } U \cap \Omega, \\ B(x, u(x), Du(x)) = 0 & \text{on } U \cap \partial\Omega, \end{cases} \tag{38}$$

where the functions $F : (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $B : (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $u : (U \cap \overline{\Omega}) \to \mathbb{R}$ are assumed to be locally bounded in their domains of definition. The function u is said to be a *viscosity subsolution* of (38) if the following requirements are fulfilled:

$$\begin{cases} \phi \in C^1(\overline{\Omega}), \quad \hat{x} \in \overline{\Omega}, \quad \max_{\overline{\Omega}}(u^* - \phi) = (u^* - \phi)(\hat{x}) \\ \Longrightarrow \\ \text{(i)} \quad F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \leq 0 & \text{if } \hat{x} \in U \cap \Omega, \\ \text{(ii)} \quad F_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \wedge B_*(\hat{x}, u^*(\hat{x}), D\phi(\hat{x})) \leq 0 & \text{if } \hat{x} \in U \cap \partial\Omega. \end{cases}$$

The upper and lower semicontinuous envelopes are taken in all the variables. That is, for $x \in U \cap \Omega$, $\xi \in (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n$ and $\eta \in (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n$,

$$u^*(x) = \lim_{r \to 0+} \sup\{u(y) \,:\, y \in B_r(x) \cap U \cap \overline{\Omega})\},$$

$$F_*(\xi) = \lim_{r \to 0+} \inf\{F(X) \,:\, X \in (U \cap \Omega) \times \mathbb{R} \times \mathbb{R}^n,\ |X - \xi| < r\},$$

$$B_*(\eta) = \lim_{r \to 0+} \inf\{B(Y) \,:\, Y \in (U \cap \partial\Omega) \times \mathbb{R} \times \mathbb{R}^n,\ |Y - \eta| < r\}.$$

The definition of viscosity supersolutions of the boundary value problem (38) is given by reversing the upper and lower positions of $*$, the inequalities, and "sup" and "inf" (including $\wedge$ and $\vee$), respectively. Then viscosity solutions of (38) are defined as those functions which are both viscosity subsolution and supersolution of (38).

Here, regarding the above definition of boundary value problems, we point out the following: define the function $G \,:\, (U \cap \overline{\Omega}) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$G(x, u, p) = \begin{cases} F(x, u, p) & \text{if } x \in \Omega, \\ B(x, u, p) & \text{if } x \in \partial\Omega, \end{cases} \tag{39}$$

and note that the lower (resp., upper) semicontinuous envelope G_* (resp., G^*) of G is given by

$$G_*(x, u, p) = \begin{cases} F_*(x, u, p) & \text{if } x \in \Omega, \\ F_*(x, u, p) \wedge B_*(x, u, p) & \text{if } x \in \partial\Omega \end{cases}$$

$$\left(\text{resp., } G^*(x, u, p) = \begin{cases} F^*(x, u, p) & \text{if } x \in \Omega, \\ F^*(x, u, p) \vee B^*(x, u, p) & \text{if } x \in \partial\Omega \end{cases} \right).$$

Thus, the above definition of viscosity subsolutions, supersolutions and solutions of (38) is the same as that of Definition 1.2 with F and Ω replaced by G defined by (39) and $U \cap \overline{\Omega}$, respectively. Therefore, the propositions in Sect. 1.4 are valid as well to viscosity subsolutions, supersolutions and solutions of (38). In order to apply the above definition to (SNP), one may take $\mathbb{R}^n$ as U or any open neighborhood of $\overline{\Omega}$.

In Sect. 1.4 we have introduced the notion of properness of PDE (FE). The following example concerns this property.

Example 2.1. Consider the boundary value problem (38) in the case where $n = 1$, $\Omega = (0, 1)$, $U = \mathbb{R}$, $F(x, p) = p - 1$ and $B(x, p) = p - 1$. The function $u(x) = x$ on $[0,\ 1]$ is a classical solution of (38). But this function u is not a viscosity subsolution of (38). Indeed, if we take the test function $\phi(x) = 2x$, then $u - \phi$ takes a maximum at $x = 0$ while we have $B(0, \phi'(0)) = F(0, \phi'(0)) = 2 - 1 = 1 > 0$.

However, if we reverse the direction of derivative at 0 by replacing the above B by the function

$$B(x, p) = \begin{cases} p - 1 & \text{for } x = 1, \\ -p + 1 & \text{for } x = 0, \end{cases}$$

then the function u is a classical solution of (38) as well as a viscosity solution of (38).

Definition 2.1. The domain Ω is said to be of class C^1 (or simply $\Omega \in C^1$) if there is a function $\rho \in C^1(\mathbb{R}^n)$ which satisfies

$$\Omega = \{x \in \mathbb{R}^n \,:\, \rho(x) < 0\},$$
$$D\rho(x) \neq 0 \quad \text{for all } x \in \partial\Omega.$$

The functions ρ having the above properties are called defining functions of Ω.

Remark 2.1. If ρ is chosen as in the above definition, then the outer unit normal vector $\nu(x)$ at $x \in \partial\Omega$ is given by

$$\nu(x) = \frac{D\rho(x)}{|D\rho(x)|}.$$

Indeed, we have

$$N(x, \Omega) = \{t\nu(x) \,:\, t \geq 0\} \quad \text{for all } x \in \partial\Omega.$$

To see this, observe that if $t \geq 0$, then $\mathbf{1}_{\overline{\Omega}} + t\rho$ as a function in $\mathbb{R}^n$ attains a local maximum at any point $x \in \partial\Omega$, which shows that

$$t|D\rho(x)|\nu(x) \in -D^+\mathbf{1}_{\overline{\Omega}}(x) = N(x, \overline{\Omega}).$$

Next, let $z \in \partial\Omega$ and $\phi \in C^1(\mathbb{R}^n)$ be such that $\mathbf{1}_{\overline{\Omega}} - \phi$ attains a strict maximum over $\mathbb{R}^n$ at z. Observe that $-\phi$ attains a strict maximum over $\overline{\Omega}$ at x. Fix a constant $r > 0$ and, for each $k \in \mathbb{N}$, choose a maximum (over $\overline{B}_r(z)$) point $x_k \in \overline{B}_r(z)$ of $-\phi - k\rho^2$, and observe that $-(\phi + k\rho^2)(x_k) \geq -(\phi + k\rho^2)(z) = -\phi(z)$ for all $k \in \mathbb{N}$ and that $x_k \to z$ as $k \to \infty$. For $k \in \mathbb{N}$ sufficiently large we have

$$D(\phi + k\rho^2)(x_k) = 0,$$

and hence

$$D\phi(x_k) = -2k\rho(x_k)D\rho(x_k),$$

which shows in the limit as $k \to \infty$ that

$$D\phi(z) = -tD\rho(z) = -t|D\rho(z)|\nu(z),$$

where $t = \lim_{k\to\infty} 2k\rho(x_k) \in \mathbb{R}$. Noting that $-(\phi + k\rho^2)(x) < -\phi(x) \le -\phi(z)$ for all $x \in \Omega$, we find that $x_k \notin \overline{B}_r(z) \setminus \Omega$ for all $k \in \mathbb{N}$. Hence, we have $t \ge 0$. Thus, we see that $N(z, \overline{\Omega}) \subset \{t\nu(z) \,:\, t \ge 0\}$ and conclude that $N(z, \overline{\Omega}) = \{t\nu(z) \,:\, t \ge 0\}$

Henceforth in this section we assume that Ω is of class C^1.

Proposition 2.1. *If $u \in C^1(\overline{\Omega})$ is a classical solution (resp., subsolution, or supersolution) of* (SNP)*, then it is a viscosity solution (resp., subsolution, or supersolution) of* (SNP)*.*

Proof. Let G be the function given by (39), with $B(x,u,p) = \gamma(x)\cdot p - g(x)$. According to the above discussion on the equivalence between the notion of viscosity solution for (SNP) and that for PDE $G(x, Du(x)) = 0$ in $\overline{\Omega}$ and Proposition 1.12, it is enough to show that the pair $(G, \overline{\Omega})$ is proper. From the above remark, we know that for any $x \in \partial\Omega$ we have $N(x, \overline{\Omega}) = \{t\nu(x) \,:\, t \ge 0\}$ and

$$G(x, p + t\nu(x)) = \gamma(x)\cdot(p + t\nu(x)) \ge \gamma(x)\cdot p = G(x,p) \quad \text{for all } t \ge 0.$$

As we noted before, we have $N(x, \overline{\Omega}) = \{0\}$ if $x \in \Omega$. Thus, we have for all $x \in \overline{\Omega}$ and all $q \in N(x, \overline{\Omega})$,

$$G(x, p+q) \ge G(x,p). \qquad \square$$

We may treat in the same way the evolution problem

$$\begin{cases} u_t(x,t) + H(x,t,D_x u(x,t)) = 0 & \text{in } \Omega \times J, \\ \dfrac{\partial u}{\partial \gamma}(x,t) = g(x,t) & \text{on } \partial\Omega \times J, \end{cases} \tag{40}$$

where J is an open interval in $\mathbb{R}$, $H : \overline{\Omega} \times J \times \mathbb{R}^n \to \mathbb{R}$ and $g : \partial\Omega \times J \to \mathbb{R}$. If we set $\widetilde{\Omega} = \Omega \times \mathbb{R}$, $U = \mathbb{R}^n \times J$,

$$F(x,t,p,q) = q + H(x,p) \quad \text{for } (x,t,p,q) \in \overline{\Omega} \times J \times \mathbb{R}^n \times \mathbb{R},$$

and

$$B(x,t,p,q) = \gamma(x)\cdot p - g(x,t) \quad \text{for } (x,t,p,q) \in \partial\Omega \times J \times \mathbb{R}^n \times \mathbb{R},$$

then the viscosity formulation for (38) applies to (40), with Ω replaced by $\widetilde{\Omega}$.

We note here that if ρ is a defining function of Ω, then it, as a function of (x,t), is also a defining function of the "cylinder" $\Omega\times\mathbb{R}$. Hence, if we set $\tilde{\gamma}(x,t) = (\gamma(x), 0)$ and $\tilde{\nu}(x,t) = (\nu(x), 0)$ for $(x,t) \in \partial(\Omega \times \mathbb{R}) = \partial\Omega \times \mathbb{R}$, then $\tilde{\nu}(x,t)$ is the outer unit normal vector at $(x,t) \in \partial\Omega \times \mathbb{R}$. Moreover, if γ satisfies (37), then we have $\tilde{\gamma}(x,t)\cdot\tilde{\nu}(x,t) = \gamma(x)\cdot\nu(x) > 0$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. Thus, as Proposition 2.1 says, if (37) holds, then any classical solution (resp., subsolution or supersolution) of (40) is a viscosity solution (resp., subsolution or supersolution) of (40).

Before closing this subsection, we add two lemmas concerning C^1 domains.

Lemma 2.1. *Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$. Assume that Ω is of class C^1. Then there exists a constant $C > 0$ and, for each $x, y \in \overline{\Omega}$ with $x \neq y$, a curve $\eta \in \mathrm{AC}([0, t(x, y)])$, with $t(x, y) > 0$, such that $t(x, y) \leq C|x - y|$, $\eta(s) \in \Omega$ for all $s \in (0, t(x, y))$, and $|\dot{\eta}(s)| \leq 1$ for a.e. $s \in [0, t(x, y)]$.*

Lemma 2.2. *Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$. Assume that Ω is of class C^1. Let $M > 0$ and $u \in C(\Omega)$ be a viscosity subsolution of $|Du(x)| \leq M$ in Ω. Then the function u is Lipschitz continuous in Ω.*

The proof of these lemmas is given in Appendix A.3.

3 Initial-Boundary Value Problem for Hamilton–Jacobi Equations

We study the initial value problem for Hamilton–Jacobi equations with the Neumann boundary condition.

To make the situation clear, we collect our assumptions on Ω, γ and H.

(A1) Ω is bounded open connected subset of $\mathbb{R}^n$.
(A2) Ω is of class C^1.
(A3) $\gamma \in C(\partial\Omega, \mathbb{R}^n)$ and $g \in C(\partial\Omega, \mathbb{R})$.
(A4) $\gamma(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$D.
(A5) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$.
(A6) H is coercive, i.e.,

$$\lim_{R\to\infty} \inf\{H(x, p) \,:\, (x, p) \in \overline{\Omega} \times \mathbb{R}^n,\ |p| \geq R\} = \infty.$$

In what follows, we assume always that (A1)–(A6) hold.

3.1 Initial-Boundary Value Problems

Given a function $u_0 \in C(\overline{\Omega})$, we consider the problem of evolution type

$$\begin{cases} u_t + H(x, D_x u) = 0 & \text{in } \Omega \times (0, \infty), \\ \gamma(x) \cdot D_x u = g(x) & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{ENP}$$

$$u(x, 0) = u_0(x) \qquad \text{for } x \in \overline{\Omega}. \tag{ID}$$

Here $u = u(x, t)$ is a function of $(x, t) \in \overline{\Omega} \times [0, \infty)$ and represents the unknown function.

When we say u is a (viscosity) solution of (ENP)–(ID), u is assumed to satisfy the initial condition (ID) in the pointwise (classical) sense.

Henceforth Q denotes the set $\overline{\Omega} \times (0, \infty)$.

Theorem 3.1 (Comparison). *Let $u \in \mathrm{USC}(\overline{Q})$ and $v \in \mathrm{LSC}(\overline{Q})$ be a viscosity subsolution and supersolution of (ENP), respectively. Assume furthermore that $u(x,0) \le v(x,0)$ for all $x \in \overline{\Omega}$. Then $u \le v$ in Q.*

To proceed, we concede the validity of the above theorem and will come back to its proof in Sect. 3.3.

Remark 3.1. The above theorem guarantees that if u is a viscosity solution of (ENP)–(ID) and continuous for $t = 0$, then it is unique.

Theorem 3.2 (Existence). *There exists a viscosity solution u of (ENP)–(ID) in the space $C(\overline{Q})$.*

Proof. Fix any $\varepsilon \in (0, 1)$. Choose a function $u_{0,\varepsilon} \in C^1(\overline{\Omega})$ so that

$$|u_{0,\varepsilon}(x) - u_0(x)| < \varepsilon \quad \text{for all } x \in \overline{\Omega}.$$

Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. Since

$$D\rho(x) = |D\rho(x)|\nu(x) \quad \text{for } x \in \partial\Omega,$$

we may choose a constant $M_\varepsilon > 0$ so large that

$$M_\varepsilon \gamma(x) \cdot D\rho(x) \ge \max_{\partial\Omega}(|g| + |\gamma \cdot Du_{0,\varepsilon}|) \quad \text{for all } x \in \partial\Omega.$$

Next choose a function $\zeta \in C^1(\mathbb{R})$ so that

$$\begin{cases} \zeta'(0) = 1, & \\ -1 \le \zeta(r) \le 0 & \text{for } r \le 0, \\ 0 \le \zeta'(r) \le 1 & \text{for } r \le 0. \end{cases}$$

Setting

$$\chi_\varepsilon(x) = \varepsilon\zeta(M_\varepsilon \rho(x)/\varepsilon),$$

we have

$$\begin{cases} -\varepsilon \le \chi_\varepsilon(x) \le 0 & \text{for all } x \in \overline{\Omega}, \\ \gamma(x) \cdot D\chi_\varepsilon(x) \ge |g(x)| + |\gamma(x) \cdot Du_{0,\varepsilon}(x)| & \text{for all } x \in \partial\Omega, \end{cases}$$

and we may choose a constant $C_\varepsilon > 0$ such that

$$|D\chi_\varepsilon(x)| \le C_\varepsilon \quad \text{for all } x \in \overline{\Omega}.$$

Then define the functions $f_\varepsilon^\pm \in C^1(\overline{\Omega})$ by

$$f_\varepsilon^\pm(x) = u_{0,\varepsilon}(x) \pm (\chi_\varepsilon(x) + 2\varepsilon),$$

and observe that

$$\begin{cases} u_0(x) \le f_\varepsilon^+(x) \le u_0(x) + 3\varepsilon & \text{for all } x \in \overline{\Omega}, \\ u_0(x) \ge f_\varepsilon^-(x) \ge u_0(x) - 3\varepsilon & \text{for all } x \in \overline{\Omega}, \\ \gamma(x) \cdot Df_\varepsilon^+(x) \ge g(x) & \text{for all } x \in \partial\Omega, \\ \gamma(x) \cdot Df_\varepsilon^-(x) \le g(x) & \text{for all } x \in \partial\Omega. \end{cases}$$

Now, we choose a constant $A_\varepsilon > 0$ large enough so that

$$|H(x, Df_\varepsilon^\pm(x))| \le A_\varepsilon \quad \text{for all } x \in \overline{\Omega},$$

and set

$$g_\varepsilon^\pm(x,t) = f_\varepsilon^\pm(x) \pm A_\varepsilon t \quad \text{for } (x,t) \in \overline{Q}.$$

The functions g_ε^+, $g_\varepsilon^- \in C^1(\overline{Q})$ are a viscosity supersolution and subsolution of (ENP), respectively.

Setting

$$h^+(x,t) = \inf\{g_\varepsilon^+(x,t) \,:\, \varepsilon \in (0,\,1)\},$$
$$h^-(x,t) = \sup\{g_\varepsilon^-(x,t) \,:\, \varepsilon \in (0,\,1)\},$$

we observe that $h^+ \in \mathrm{USC}(\overline{Q})$ and $h^- \in \mathrm{LSC}(\overline{Q})$ are, respectively, a viscosity supersolution and subsolution of (ENP). Moreover we have

$$u_0(x) = h^\pm(x,0) \quad \text{for all } x \in \overline{\Omega},$$
$$h^-(x,t) \le u_0(x) \le h^+(x,t) \quad \text{for all } (x,t) \in \overline{Q}.$$

By Theorem 1.4, we find that there exists a viscosity solution u of (ENP) which satisfies

$$h^-(x,t) \le u(x,t) \le h^+(x,t) \quad \text{for all } (x,t) \in \overline{Q}.$$

Applying Theorem 3.1 to u^* and u_* yields

$$u^* \le u_* \quad \text{for all } (x,t) \in \overline{Q},$$

while $u_* \le u^*$ in $\overline{Q}$ by definition, which in particular implies that $u \in C(\overline{Q})$. The proof is complete. □

Theorem 3.3 (Uniform continuity). *The viscosity solution $u \in C(\overline{Q})$ of (ENP)–(ID) is uniformly continuous in $\overline{Q}$. Furthermore, if $u_0 \in \mathrm{Lip}(\overline{\Omega})$, then $u \in \mathrm{Lip}(\overline{Q})$.*

Lemma 3.1. *Let* $u_0 \in \mathrm{Lip}(\overline{\Omega})$. *Then there is a constant* $C > 0$ *such that the functions* $u_0(x) + Ct$ *and* $u_0(x) - Ct$ *are, respectively, a viscosity supersolution and subsolution of* (ENP)–(ID).

Proof. Let ρ and ζ be the functions which are used in the proof of Theorem 3.2. Choose the collection $\{u_{0,\varepsilon}\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ of functions so that

$$\begin{cases} \lim_{\varepsilon\to 0} \|u_{0,\varepsilon} - u_0\|_{\infty,\Omega} = 0, \\ \sup_{\varepsilon\in(0,1)} \|Du_{0,\varepsilon}\|_{\infty,\Omega} < \infty. \end{cases}$$

As in the proof of Theorem 3.2, we may fix a constant $M > 0$ so that

$$\begin{aligned} M\gamma(x)\cdot D\rho(x) &= M|D\rho(x)|\nu(x)\cdot\gamma(x) \\ &\geq |g(x)| + |\gamma(x)\cdot Du_{0,\varepsilon}(x)| \quad \text{for all } x \in \partial\Omega. \end{aligned}$$

Next set

$$R = \sup_{\varepsilon\in(0,1)} \|Du_{0,\varepsilon}\|_{\infty,\Omega} + M\|D\rho\|_{\infty,\Omega},$$

and choose $C > 0$ so that

$$\max_{\overline{\Omega}\times\overline{B}_R} |H| \leq C.$$

Now, we put

$$v_\varepsilon^{\pm}(x,t) = u_{0,\varepsilon}(x) \pm (M\varepsilon\zeta(\rho(x)/\varepsilon) + Ct) \quad \text{for } (x,t) \in \overline{Q},$$

and note that v_ε^+ and v_ε^- are a classical supersolution and subsolution of (ENP). Sending $\varepsilon \to 0+$, we conclude by Proposition 1.9 that the functions $u_0(x) + Ct$ and $u_0(x) - Ct$ are a viscosity supersolution and subsolution of (ENP), respectively. □

Proof (Theorem 3.3). We first assume that $u_0 \in \mathrm{Lip}(\overline{\Omega})$, and show that $u \in \mathrm{Lip}(\overline{Q})$. According to Lemma 3.1, there exists a constant $C > 0$ such that the function $u_0(x) - Ct$ is a viscosity subsolution of (ENP). By Theorem 3.1, we get

$$u(x,t) \geq u_0(x) - Ct \quad \text{for all } (x,t) \in \overline{Q}.$$

Fix any $t > 0$, and apply Theorem 3.1 to the functions $u(x, t+s)$ and $u(x,s) - Ct$ of (x,s), both of which are viscosity solutions of (ENP), to get

$$u(x,t+s) \geq u(x,s) - Ct \quad \text{for all } (x,s) \in \overline{Q}.$$

Hence, if $(p,q)\in D^{+}u(x,s)$, then we find that as $t\to 0+$,

$$u(x,s)\le u(x,s+t)+Ct\le u(x,s)+qt+Ct+o(t),$$

and consequently, $q\ge -C$. Moreover, if $x\in\Omega$, we have

$$0\ge q+H(x,p)\ge H(x,p)-C.$$

Due to the coercivity of H, there exists a constant $R>0$ such that

$$p\in B_R.$$

Therefore, we get

$$q\le -H(x,p)\le \max_{\overline{\Omega}\times\overline{B}_R}|H|.$$

Thus, if $(x,s)\in\Omega\times(0,\infty)$ and $(p,q)\in D^{+}u(x,s)$, then we have

$$|p|+|q|\le M:=R+C+\max_{\overline{\Omega}\times\overline{B}_R}|H|.$$

Thanks to Proposition 1.14, we conclude that u is Lipschitz continuous in Q.

Next, we show in the general case that $u\in \mathrm{UC}(\overline{Q})$. Let $\varepsilon\in(0,\,1)$, and choose a function $u_{0,\varepsilon}\in \mathrm{Lip}(\overline{\Omega})$ so that

$$\|u_{0,\varepsilon}-u_0\|_\infty\le\varepsilon.$$

Let u_ε be the viscosity solution of (ENP) satisfying the initial condition

$$u_\varepsilon(x,0)=u_{0,\varepsilon}(x)\quad \text{for all }\ x\in\overline{\Omega}.$$

As we have shown above, we know that $u_\varepsilon\in \mathrm{Lip}(\overline{Q})$. Moreover, by Theorem 3.1 we have

$$\|u_\varepsilon-u\|_{\infty,\overline{Q}}\le\varepsilon.$$

It is now obvious that $u\in \mathrm{UC}(\overline{Q})$. □

3.2 Additive Eigenvalue Problems

Under our hypotheses (A1)–(A6), the boundary value problem

$$\begin{cases} H(x,Du)=0 & \text{in } \Omega,\\ \gamma(x)\cdot Du=g(x) & \text{on } \partial\Omega \end{cases} \tag{SNP}$$

may not have a viscosity solution. For instance, the Hamiltonian $H(x, p) = |p|^2 + 1$ satisfies (A5) and (A6), but, since $H(x, p) > 0$, (SNP) does not have any viscosity subsolution.

Instead of (SNP), we consider the *additive eigenvalue* problem

$$\begin{cases} H(x, Dv) = a & \text{in } \Omega, \\ \gamma(x) \cdot Dv = g(x) & \text{on } \partial\Omega. \end{cases} \tag{EVP}$$

This is a problem to seek for a pair $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ such that v is a viscosity solution of the stationary problem (EVP). If $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ is such a pair, then a and v are called an (additive) *eigenvalue* and *eigenfunction* of (EVP), respectively. This problem is often called the *ergodic problem* in the viewpoint of ergodic optimal control.

Theorem 3.4. (i) *There exists a solution* $(a, v) \in \mathbb{R} \times \mathrm{Lip}(\overline{\Omega})$ *of* (EVP).
(ii) *The eigenvalue of* (EVP) *is unique. That is, if* $(a, v), (b, w) \in \mathbb{R} \times C(\overline{\Omega})$ *are solutions of* (EVP)*, then* $a = b$.

The above result has been obtained by Lions et al., Homogenization of Hamilton-Jacobi equations, unpublished.

In what follows we write $c^\#$ for the unique eigenvalue a of (EVP).

Corollary 3.1. *Let* $u \in C(\overline{Q})$ *be the solution of* (ENP)–(ID)*. Then the function* $u(x, t) + c^\# t$ *is bounded on* $\overline{Q}$.

Corollary 3.2. *We have*

$$c^\# = \inf\{a \in \mathbb{R} \, : \, (\text{EVP}) \textit{ has a viscosity subsolution } v\}.$$

Lemma 3.2. *Let* $b, c \in \mathbb{R}$ *and* $v, w \in C(\overline{\Omega})$*. Assume that* v *(resp.,* w*) is a viscosity supersolution (resp., subsolution) of* (EVP) *with* $a = b$ *(resp.,* $a = c$*). Then* $b \le c$.

Remark 3.2. As the following proof shows, the assertion of the above lemma is valid even if one replaces the continuity of v and w by the boundedness.

Proof. By adding a constant to v if needed, we may assume that $v \ge w$ in $\overline{\Omega}$. Since the functions $v(x) - bt$ and $w(x) - ct$ are a viscosity supersolution and subsolution of (ENP), by Theorem 3.1 we get

$$v(x) - bt \ge w(x) - ct \quad \text{for all } (x, t) \in \overline{Q},$$

from which we conclude that $b \le c$. □

Proof (Theorem 3.4). Assertion (ii) is a direct consequence of Lemma 3.2.

We prove assertion (i). Consider the boundary value problem

$$\begin{cases} \lambda v + H(x, Dv) = 0 & \text{in } \Omega, \\ \gamma(x) \cdot Dv = g & \text{on } \partial\Omega, \end{cases} \tag{41}$$

where $\lambda > 0$ is a given constant. We will take the limit as $\lambda \to 0$ later on.

We fix $\lambda \in (0, 1)$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of the domain Ω. Select a constant $A > 0$ so large that $A\gamma(x) \cdot D\rho(x) \geq |g(x)|$ for all $x \in \partial\Omega$, and then $B > 0$ so large that $B \geq A|\rho(x)| + |H(x, \pm AD\rho(x))|$ for all $x \in \overline{\Omega}$. Observe that the functions $A\rho(x) + B/\lambda$ and $-A\rho(x) - B/\lambda$ are a classical supersolution and subsolution of (41), respectively.

The Perron method (Theorem 1.4) guarantees that there is a viscosity solution v_λ of (41) which satisfies

$$|v_\lambda(x)| \leq A\rho(x) + B/\lambda \leq B/\lambda \quad \text{for all } x \in \overline{\Omega}.$$

Now, since

$$-\lambda v_\lambda(x) \leq B \quad \text{for all } x \in \overline{\Omega},$$

v_λ satisfies in the viscosity sense

$$H(x, Dv_\lambda(x)) \leq B \quad \text{for all } x \in \Omega,$$

which implies, together with the coercivity of H, the equi-Lipschitz continuity of $\{v_\lambda\}_{\lambda \in (0,1)}$. Thus the collections $\{v_\lambda - \inf_\Omega v_\lambda\}_{\lambda \in (0,1)}$ and $\{\lambda v_\lambda\}_{\lambda \in (0,1)}$ of functions on $\overline{\Omega}$ are relatively compact in $C(\overline{\Omega})$. We may select a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset (0, 1)$ such that

$$\begin{aligned} &\lambda_j \to 0, \\ &v_{\lambda_j}(x) - \inf_\Omega v_{\lambda_j} \to v(x), \\ &\lambda_j v_{\lambda_j}(x) \to w(x) \end{aligned}$$

for some functions $v, w \in C(\overline{\Omega})$ as $j \to \infty$, where the convergences to v and w are uniform on $\overline{\Omega}$. Observe that for all $x \in \overline{\Omega}$,

$$\begin{aligned} w(x) &= \lim_{j \to \infty} \lambda_j v_{\lambda_j}(x) \\ &= \lim_{j \to \infty} \lambda_j \big[(v_{\lambda_j}(x) - \inf_\Omega v_{\lambda_j}) + \inf_\Omega v_{\lambda_j}\big] \\ &= \lim_{j \to \infty} \lambda_j \inf_\Omega v_{\lambda_j}, \end{aligned}$$

which shows that w is constant on $\overline{\Omega}$. If we write this constant as a, then we see by Proposition 1.9 that v is a viscosity solution of (EVP). This completes the proof of (i). □

Proof (Corollary 3.1). Let $v \in C(\overline{\Omega})$ be an eigenfunction of (EVP). That is, v is a viscosity solution of (EVP), with $a = c^\#$. Then, for any constant $C \in \mathbb{R}$, the function $w(x,t) := v(x) - c^\# t + C$ is a viscosity solution of (ENP). We may choose constants C_i, $i = 1, 2$, so that $v(x) + C_1 \le u_0(x) \le v(x) + C_2$ for all $x \in \overline{\Omega}$. By Theorem 3.1, we see that

$$v(x) - c^\# t + C_1 \le u(x,t) \le v(x) - c^\# t + C_2 \quad \text{for all } (x,t) \in \overline{Q},$$

which shows that the function $u(x,t) + c^\# t$ is bounded on $\overline{Q}$. □

Proof (Corollary 3.2). It is clear that

$$c^\# \ge c^\star := \inf\{a \in \mathbb{R} \,:\, \text{(EVP) has a viscosity subsolution } v\}.$$

To show that $c^\# \le c^\star$, we suppose by contradiction that $c^\# > c^\star$. By the definition of $c^\star$, there is a $b \in [c^\star, c^\#)$ such that (EVP), with $a = b$, has a viscosity subsolution ψ. Let v be a viscosity solution of (EVP), with $a = c^\#$. Since $b < c^\#$, v is a viscosity supersolution of (EVP), with $a = b$. We may assume that $\psi \le v$ in $\overline{\Omega}$. Theorem 1.4 now guarantees the existence of a viscosity solution of (EVP), which contradicts Theorem 3.4, (ii) (see Remark 3.2). □

Example 3.1. We consider the case where $n = 1$, $\Omega = (-1, 1)$, $H(x,p) = H(p) := |p|$ and $\gamma(\pm 1) = \pm 1$, respectively, and evaluate the eigenvalue $c^\#$. We set $g_{\min} = \min\{g(-1), g(1)\}$. Assume first that $g_{\min} \ge 0$. In this case, the function $v(x) = 0$ is a classical subsolution of (SNP) and, hence, $c^\# \le 0$. On the other hand, since $H(p) \ge 0$ for all $p \in \mathbb{R}$, we have $c^\# \ge 0$. Thus, $c^\# = 0$. We next assume that $g_{\min} < 0$. It is easily checked that if $g(1) = g_{\min}$, then the function $v(x) = g_{\min} x$ is a viscosity solution of (EVP), with $a = |g_{\min}|$. (Notice that

$$-D^+ v(-1) = (-\infty, -|g_{\min}|\,] \cup [-|g_{\min}|, |g_{\min}|\,],$$
$$-D^- v(-1) = [\,|g_{\min}|, \infty).)$$

Similarly, if $g(-1) = g_{\min}$, then the function $v(x) = |g_{\min}| x$ is a viscosity solution of (EVP), with $a = |g_{\min}|$. These observations show that $c^\# = |g_{\min}|$.

3.3 Proof of Comparison Theorem

This subsection will be devoted to the proof of Theorem 3.1.

We begin with the following two lemmas.

Lemma 3.3. *Let u be the function from Theorem 3.1. Set $P = \Omega \times (0, \infty)$. Then, for every $(x,t) \in \partial\Omega \times (0, \infty)$, we have*

$$u(x,t) = \limsup_{P \ni (y,s) \to (x,t)} u(y,s). \tag{42}$$

Proof. Fix any $(x, t) \in \partial\Omega \times (0, \infty)$. To prove (42), we argue by contradiction, and suppose that

$$\limsup_{P \ni (y,s) \to (x,t)} u(y, s) < u(x, t).$$

We may choose a constant $r \in (0, t)$ so that

$$u(y, s) + r < u(x, t) \quad \text{for all } (y, s) \in P \cap (\overline{B}_r(x) \times [t - r, t + r]). \tag{43}$$

Note that

$$P \cap (\overline{B}_r(x) \times [t - r, t + r]) = (\Omega \cap \overline{B}_r(x)) \times [t - r, t + r].$$

Since u is bounded on $\overline{\Omega} \times [t - r, t + r]$, we may choose a constant $\alpha > 0$ so that for all $(y, s) \in \overline{\Omega} \times [t - r, t + r]$,

$$u(y, s) + r - \alpha(|y - x|^2 + (s-t)^2) < u(x, t) \quad \text{if } |y-x| \geq r/2 \text{ or } |s - t| \geq r/2. \tag{44}$$

Let ρ be a defining function of Ω. Let ζ be the function on $\mathbb{R}$ introduced in the proof of Theorem 3.2. For $k \in \mathbb{N}$ we define the function $\psi \in C^1(\mathbb{R}^{n+1})$ by

$$\psi(y, s) = k^{-1}\zeta(k^2\rho(y)) + \alpha(|y - x|^2 + (s - t)^2).$$

Consider the function

$$u(y, s) - \psi(y, s)$$

on the set $\big(\overline{\Omega} \cap \overline{B}_r(x)\big) \times [t - r, t + r]$. Let $(y_k, s_k) \in \big(\overline{\Omega} \cap \overline{B}_r(x)\big) \times [t - r, t + r]$ be a maximum point of the above function. Assume that $k > r^{-1}$.

Using (43) and (44), we observe that for all $(y, s) \in (\overline{\Omega} \cap \overline{B}_r(x)) \times [t - r, t + r]$,

$$u(y, s) - \psi(y, s) < u(x, t) = u(x, t) - \psi(x, t)$$

if either $y \in \Omega$, $|y - x| \geq r/2$, or $|s - t| \geq r/2$. Accordingly, we have

$$(y_k, s_k) \in \big(\partial\Omega \cap \overline{B}_{r/2}(x)\big) \times (t - r/2, t + r/2).$$

Hence, setting

$$p_k = kD\rho(y_k) + 2\alpha(y_k - x) \quad \text{and} \quad q_k = 2\alpha(s_k - t),$$

we have

$$\min\{q_k + H(y_k, p_k), \gamma(y_k) \cdot p_k - g(y_k)\} \leq 0.$$

If we note that

$$\gamma(y_k) \cdot D\rho(y_k) \geq \min_{\partial\Omega} \gamma \cdot D\rho > 0,$$

then, by sending $k \to \infty$, we get a contradiction. □

Lemma 3.4. *Let $y, z \in \mathbb{R}^n$, and assume that $y \cdot z > 0$. Then there exists a quadratic function ζ in $\mathbb{R}^n$ which satisfies:*

$$\begin{cases} \zeta(tx) = t^2\zeta(x) & \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \zeta(x) > 0 & \text{if } x \neq 0, \\ z \cdot D\zeta(x) = 2(y \cdot z)(y \cdot x) & \text{for all } x \in \mathbb{R}^n. \end{cases}$$

Proof. We define the function ζ by

$$\zeta(x) = \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 + (y \cdot x)^2.$$

We observe that for any $t \in \mathbb{R}$,

$$\begin{aligned} \zeta(x + tz) &= \left| x + tz - \frac{y \cdot (x + tz)}{y \cdot z} z \right|^2 + (y \cdot (x + tz))^2 \\ &= \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 + (y \cdot x)^2 + 2t(y \cdot x)(y \cdot z) + t^2(y, z)^2, \end{aligned}$$

from which we find that

$$z \cdot D\zeta(x) = 2(y \cdot z)(y \cdot x).$$

If $\zeta(x) = 0$, then $y \cdot x = 0$ and

$$0 = \zeta(x) = \left| x - \frac{y \cdot x}{y \cdot z} z \right|^2 = |x|^2.$$

Hence, we have $x = 0$ if $\zeta(x) = 0$, which shows that $\zeta(x) > 0$ if $x \neq 0$. It is obvious that the function ζ is homogeneous of degree two. The function ζ has the required properties. □

For the proof of Theorem 3.1, we argue by contradiction: we suppose that

$$\sup_{\overline{\Omega} \times [0,\infty)} (u - v) > 0,$$

and, to conclude the proof, we will get a contradiction.

Reduction 1: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

$$u \text{ is a viscosity subsolution of}$$
$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times J, \end{cases} \tag{45}$$

$$\max_{\overline{\Omega}\times\overline{J}} (u - v) > 0 > \max_{\overline{\Omega}\times\partial J} (u - v), \tag{46}$$

and

$$u \text{ and } v \text{ are bounded on } \overline{\Omega} \times \overline{J}. \tag{47}$$

Proof. We choose a $T > 0$ so that $\sup_{\overline{\Omega}\times(0,\,T)} (u - v) > 0$ and set

$$u_\varepsilon(x,t) = u(x,t) - \frac{\varepsilon}{T - t} \quad \text{for } (x,t) \in \overline{\Omega} \times [0,\, T),$$

where $\varepsilon > 0$ is a constant. It is then easy to check that u_ε is a viscosity subsolution of

$$\begin{cases} u_{\varepsilon,t} + H(x, D_x u_\varepsilon(x,t)) \le -\dfrac{\varepsilon}{T^2} & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x,t) \le g(x) & \text{on } \partial\Omega \times (0,\, T). \end{cases}$$

Choosing $\varepsilon > 0$ sufficiently small, we have

$$\sup_{\overline{\Omega}\times[0,T)} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega}\times\{0\}} (u_\varepsilon - v).$$

If we choose $\alpha > 0$ sufficiently small, then

$$\max_{\overline{\Omega}\times[0,T-\alpha]} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega}\times\partial[0,T-\alpha]} (u_\varepsilon - v).$$

Thus, if we set $J = (0,\, T - \alpha)$ and replace u by u_ε, then we are in the situation of (45)–(47). □

We may assume furthermore that $u \in \mathrm{Lip}(\overline{\Omega} \times \overline{J})$ as follows.

Reduction 2: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

u is a viscosity subsolution of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times J, \end{cases} \tag{48}$$

$$\max_{\overline{\Omega}\times\overline{J}}(u - v) > 0 > \max_{\overline{\Omega}\times\partial J}(u - v), \tag{49}$$

and

$$u \in \mathrm{Lip}(\overline{\Omega} \times \overline{J}) \ \text{ and } \ v \ \text{ is bounded on } \ \overline{\Omega} \times \overline{J}. \tag{50}$$

Proof. Let J be as in Reduction 1. We set $J = (a, b)$. Let $M > 0$ be a bound of $|u|$ on $\overline{\Omega} \times [a, b]$.

For each $\varepsilon > 0$ we define the sup-convolution in the t-variable

$$u_\varepsilon(x,t) = \max_{s\in[a,b]} \Big(u(x,s) - \frac{(t-s)^2}{2\varepsilon} \Big).$$

We note as in Sect. 1.6 that

$$M \ge u_\varepsilon(x,t) \ge u(x,t) \ge -M \quad \text{for all } \ (x,t) \in \overline{\Omega} \times [a, b].$$

Noting that

$$\frac{1}{2\varepsilon}(t-s)^2 \le 2M \iff |t-s| \le 2\sqrt{\varepsilon M} \tag{51}$$

and setting $m_\varepsilon = 2\sqrt{\varepsilon M}$, we find that

$$u_\varepsilon(x,t) = \max_{a<s<b} \Big(u(x,s) - \frac{(t-s)^2}{2\varepsilon} \Big) \quad \text{for all } \ (x,t) \in \overline{\Omega} \times (a + m_\varepsilon,\, b - m_\varepsilon).$$

Let $(x,t) \in \overline{\Omega} \times (a + m_\varepsilon,\, b - m_\varepsilon)$. Choose an $s \in (a, b)$ so that

$$u_\varepsilon(x,t) = u(x,s) - \frac{(t-s)^2}{2\varepsilon}.$$

Note by (51) that

$$|t-s| \le m_\varepsilon.$$

Let $(p,q) \in D^+ u_\varepsilon(x,t)$ and choose a function $\phi \in C^1(\overline{\Omega} \times (a, b))$ so that $D\phi(x,t) = (p,q)$ and $\max(u_\varepsilon - \phi) = (u_\varepsilon - \phi)(x,t)$. Observe as in Sect. 1.6 that

$$(p, (s-t)/\varepsilon) \in D^+ u(x,s) \quad \text{and} \quad \frac{(t-s)}{\varepsilon} + q = 0.$$

Hence,

$$(p, q) \in D^+u(x, s).$$

Therefore, we have

$$\begin{cases} q + H(x, p) + \delta \le 0 & \text{if } \ x \in \Omega, \\ \min\{q + H(x, p) + \delta, \ \gamma(x) \cdot p - g(x)\} \le 0 & \text{if } \ x \in \partial\Omega. \end{cases} \tag{52}$$

Moreover, we see that

$$|q| = \frac{|t - s|}{\varepsilon} \le \frac{m_\varepsilon}{\varepsilon},$$

and

$$H(x, p) \le -q \le \frac{m_\varepsilon}{\varepsilon} \quad \text{if } x \in \Omega.$$

Hence, by the coercivity of H, we have

$$|q| + |p| \le R(\varepsilon) \quad \text{if } x \in \Omega, \tag{53}$$

for some constant $R(\varepsilon) > 0$.

Thus, we conclude from (52) that u_ε is a viscosity subsolution of

$$\begin{cases} u_t + H(x, D_x u) \le -\delta & \text{in } \ \overline{\Omega} \times (a + m_\varepsilon, \ b - m_\varepsilon), \\ \gamma \cdot D_x u \le g & \text{on } \ \partial\Omega \times (a + m_\varepsilon, \ b - m_\varepsilon), \end{cases}$$

and from (53) that u_ε is Lipschitz continuous in $\Omega \times (a + m_\varepsilon, \ b - m_\varepsilon)$. By Lemma 3.3, we have

$$u_\varepsilon(x, t) = \limsup_{\Omega \times (a+m_\varepsilon, b-m_\varepsilon) \ni (y,s) \to (x,t)} u_\varepsilon(y, s) \quad \text{for all } \ (x, t) \in \partial\Omega \times (a + m_\varepsilon, \ b - m_\varepsilon).$$

Since $u_\varepsilon \in \mathrm{Lip}(\Omega \times (a + m_\varepsilon, \ b - m_\varepsilon))$, the limsup operation in the above formula can be replaced by the limit operation. Hence,

$$u_\varepsilon \in C(\overline{\Omega} \times (a + m_\varepsilon, \ b - m_\varepsilon)),$$

which guarantees that u_ε is Lipschitz continuous in $\overline{\Omega} \times (a + m_\varepsilon, \ b - m_\varepsilon)$.

Finally, if we replace u and J by u_ε and $(a + 2m_\varepsilon, \ b - 2m_\varepsilon)$, respectively, and select $\varepsilon > 0$ small enough so that

$$\max_{\overline{\Omega} \times [a+2m_\varepsilon, b-2m_\varepsilon]} (u_\varepsilon - v) > 0 > \max_{\overline{\Omega} \times \partial[a+2m_\varepsilon, b-2m_\varepsilon]} (u_\varepsilon - v),$$

then conditions (48)–(50) are satisfied. □

Reduction 3: We may assume that there exist a constant $\delta > 0$ and a finite open interval $J \subset (0, \infty)$ such that

$$u \text{ is a viscosity subsolution of}$$
$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le -\delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x u(x,t) \le g(x) - \delta & \text{on } \partial\Omega \times J, \end{cases} \tag{54}$$

$$v \text{ is a viscosity supersolution of}$$
$$\begin{cases} v_t(x,t) + H(x, D_x v(x,t)) \ge \delta & \text{in } \Omega \times J, \\ \gamma(x) \cdot D_x v(x,t) \ge g(x) + \delta & \text{on } \partial\Omega \times J, \end{cases} \tag{55}$$

$$\max_{\overline{\Omega}\times\overline{J}}(u - v) > 0 > \max_{\overline{\Omega}\times\partial J}(u - v), \tag{56}$$

and

$$u \in \mathrm{Lip}(\overline{\Omega} \times \overline{J}) \ \text{ and } \ v \ \text{ is bounded on } \ \overline{\Omega} \times \overline{J}. \tag{57}$$

Proof. Let u, v, J be as in Reduction 2. Set $J = (a, b)$. Let ρ be a defining function of Ω as before. Let $0 < \varepsilon < 1$. We set

$$u_\varepsilon(x,t) = u(x,t) - \varepsilon\rho(x) \ \text{ and } \ v_\varepsilon(x,t) = v(x,t) + \varepsilon\rho(x) \quad \text{for } \ (x,t) \in \overline{\Omega} \times \overline{J},$$

and

$$H_\varepsilon(x, p) = H(x, p - \varepsilon D\rho(x)) + \varepsilon \quad \text{for } \ (x, p) \in \overline{\Omega} \times \mathbb{R}^n.$$

Let $(x,t) \in \overline{\Omega} \times J$ and $(p,q) \in D^- v_\varepsilon(x,t)$. Then we have

$$(p - \varepsilon D\rho(x), q) \in D^- v(x,t).$$

Since v is a viscosity supersolution of (ENP), if $x \in \Omega$, then

$$q + H(x, p - \varepsilon D\rho(x)) \ge 0.$$

If $x \in \partial\Omega$, then either

$$q + H(x, p - \varepsilon D\rho(x)) \ge 0,$$

or

$$\begin{aligned} \gamma(x) \cdot p &= \gamma(x) \cdot (p - \varepsilon D\rho(x)) + \varepsilon\gamma(x) \cdot D\rho(x) \\ &\ge g(x) + \varepsilon\gamma(x) \cdot D\rho(x) \ge g(x) + \lambda\varepsilon, \end{aligned}$$

where

$$\lambda = \min_{\partial\Omega} \gamma \cdot D\rho \ (> 0).$$

Now let $(p, q) \in D^+u_\varepsilon(x,t)$. Note that $(p + \varepsilon D\rho(x), q) \in D^+u(x,t)$. Since $u \in \mathrm{Lip}(\overline{\Omega} \times [a,\, b])$, we have a bound $C_0 > 0$ such that

$$|q| \le C_0.$$

If $x \in \Omega$, then

$$\begin{aligned} q + H(x, p - \varepsilon D\rho(x)) &\le q + H(x, p + \varepsilon D\rho(x)) + \omega(2\varepsilon|D\rho(x)|) \\ &\le -\delta + \omega(2\varepsilon C_1), \end{aligned}$$

where

$$C_1 = \max_{\overline{\Omega}} |D\rho|,$$

and ω denotes the modulus of continuity of H on the set $\overline{\Omega} \times \overline{B}_{R+2C_1}$, with $R > 0$ being chosen so that

$$\min_{\overline{\Omega}\times(\mathbb{R}^n\setminus B_R)} H > C_0.$$

(Here we have used the fact that $H(x, p + \varepsilon D\rho(x)) \le C_0$, which implies that $|p + \varepsilon D\rho(x)| \le R$.)

If $x \in \partial\Omega$, then either

$$q + H(x, p - \varepsilon D\rho(x)) \le -\delta + \omega(2\varepsilon C_1),$$

or

$$\gamma(x)\cdot p \le \gamma(x)\cdot(p + \varepsilon D\rho(x)) - \varepsilon\gamma(x)\cdot D\rho(x) \le g(x) - \lambda\varepsilon.$$

Thus we see that v_ε is a viscosity supersolution of

$$\begin{cases} v_{\varepsilon,t} + H_\varepsilon(x, D_x v_\varepsilon) \ge \varepsilon & \text{in } \Omega \times J, \\ \gamma(x)\cdot D_x v_\varepsilon(x,t) \ge g(x) + \lambda\varepsilon & \text{on } \partial\Omega \times J, \end{cases}$$

and u_ε is a viscosity subsolution of

$$\begin{cases} u_{\varepsilon,t} + H_\varepsilon(x, D_x u_\varepsilon) \le -\delta + \omega(2C_1\varepsilon) + \varepsilon & \text{in } \Omega \times J, \\ \gamma\cdot Du_\varepsilon \le g(x) - \lambda\varepsilon & \text{on } \partial\Omega \times J, \end{cases}$$

If we replace u, v, H and δ by u_ε, v_ε, H_ε and

$$\min\{\varepsilon, \lambda\varepsilon, \delta - \omega(2C_1\varepsilon) - \varepsilon\},$$

respectively, and choose $\varepsilon > 0$ sufficiently small, then conditions (54)–(57) are satisfied. □

Final step: Let u, v, J and δ be as in Reduction 3. We choose a maximum point $(z,\tau) \in \overline{\Omega} \times \overline{J}$ of the function $u - v$. Note that $\tau \in J$, that is, $\tau \notin \partial J$.

By replacing u, if necessary, by the function

$$u(x,t) - \varepsilon|x-z|^2 - \varepsilon(t-\tau)^2,$$

where $\varepsilon > 0$ is a small constant, we may assume that (z,τ) is a strict maximum point of $u - v$.

By making a change of variables, we may assume that $z = 0$ and

$$\Omega \cap B_{2r} = \{x = (x_1, \ldots, x_n) \in B_{2r} \ : \ x_n < 0\},$$

while we may assume as well that $[\tau - r, \ \tau + r] \subset J$.

We set $\hat{\gamma} = \gamma(0)$ and apply Lemma 3.4, with $y = (0, \ldots, 0, 1) \in \mathbb{R}^n$ and $z = \hat{\gamma}$, to find a quadratic function ζ so that

$$\begin{cases} \zeta(t\xi) = t^2\zeta(\xi) & \text{for all } \ (\xi, t) \in \mathbb{R}^n \times \mathbb{R}, \\ \zeta(\xi) > 0 & \text{if } \ \xi \neq 0, \\ \hat{\gamma} \cdot D\zeta(\xi) = 2\hat{\gamma}_n \xi_n & \text{for all } \ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \end{cases}$$

where $\hat{\gamma}_n$ denotes the n-th component of the n-tuple $\hat{\gamma}$.

By replacing ζ by a constant multiple of ζ, we may assume that

$$\begin{aligned} &\zeta(\xi) \geq |\xi|^2 && \text{for all } \xi \in \mathbb{R}^n, \\ &|D\zeta(\xi)| \leq C_0|\xi| && \text{for all } \xi \in \mathbb{R}^n, \\ &\hat{\gamma} \cdot D\zeta(\xi) \begin{cases} \geq 0 & \text{if } \xi_n \geq 0, \\ \leq 0 & \text{if } \xi_n \leq 0, \end{cases} \end{aligned}$$

where $C_0 > 0$ is a constant.

Let $M > 0$ be a Lipschitz bound of the function u. Set

$$\hat{g} = g(0), \quad \mu = \hat{g}\frac{\hat{\gamma}}{|\hat{\gamma}|^2} \quad \text{and} \quad M_1 = M + |\mu|.$$

We may assume by replacing r by a smaller positive constant if needed that for all $x \in B_r \cap \partial\Omega$,

$$|\gamma(x) - \hat{\gamma}| < \frac{\delta}{2(|\mu| + C_0 M_1)} \quad \text{and} \quad |g(x) - \hat{g}| < \frac{\delta}{2}. \tag{58}$$

For $\alpha > 1$ we consider the function

$$\Phi(x,t,y,s) = u(x,t) - v(y,s) - \mu \cdot (x-y) - \alpha\zeta(x-y) - \alpha(t-s)^2$$

on $K := \big((\overline{\Omega} \cap \overline{B}_r(0,\tau) \times [\tau - r, \tau + r]\big)^2$. Let $(x_\alpha, t_\alpha, y_\alpha, s_\alpha)$ be a maximum point of the function Φ. By the inequality $\Phi(y_\alpha, s_\alpha, y_\alpha, s_\alpha) \le \Phi(x_\alpha, t_\alpha, y_\alpha, s_\alpha)$, we get

$$\begin{aligned}\alpha(|x_\alpha - y_\alpha|^2 + (t_\alpha - s_\alpha)^2) &\le \alpha(\zeta(x_\alpha - y_\alpha) + (t_\alpha - s_\alpha)^2)\\ &\le u(x_\alpha, t_\alpha) - u(y_\alpha, s_\alpha) + |\mu||x_\alpha - y_\alpha|\\ &\le M_1(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2},\end{aligned}$$

and hence

$$\alpha(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2} \le M_1. \tag{59}$$

As usual we may deduce that as $\alpha \to \infty$,

$$\begin{cases}(x_\alpha, \tau_\alpha),\ (y_\alpha, s_\alpha) \to (0, \tau),\\ u(x_\alpha, t_\alpha) \to u(0, \tau),\\ v(y_\alpha, s_\alpha) \to v(0, \tau).\end{cases}$$

Let $\alpha > 1$ be so large that

$$(x_\alpha, t_\alpha), (y_\alpha, s_\alpha) \in (\overline{\Omega} \cap B_r) \times (\tau - r,\ \tau + r).$$

Accordingly, we have

$$\begin{aligned}(\mu + \alpha D\zeta(x_\alpha - y_\alpha),\ 2\alpha(t_\alpha - s_\alpha)) &\in D^+u(x_\alpha, t_\alpha),\\ (\mu + \alpha D\zeta(x_\alpha - y_\alpha),\ 2\alpha(t_\alpha - s_\alpha)) &\in D^-v(y_\alpha, s_\alpha).\end{aligned}$$

Using (59), we have

$$\alpha|D\zeta(x_\alpha - y_\alpha)| \le C_0\alpha|x_\alpha - y_\alpha| \le C_0 M_1. \tag{60}$$

If $x_\alpha \in \partial\Omega$, then $x_{\alpha,n} = 0$ and $(x_\alpha - y_\alpha)_n \ge 0$. Hence, in this case, we have

$$\hat{\gamma} \cdot D\zeta(x_\alpha - y_\alpha) \ge 0,$$

and moreover, in view of (58) and (60),

$$\begin{aligned}\gamma(x_\alpha) \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha)) &\ge \hat{\gamma} \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha))\\ &\quad - |\gamma(x_\alpha) - \hat{\gamma}|(|\mu| + C_0 M_1)\\ &> g(x_\alpha) - |\hat{g} - g(x_\alpha)| - \frac{\delta}{2} > g(x_\alpha) - \delta.\end{aligned}$$

Now, by the viscosity property of u, we obtain

$$2\alpha(t_\alpha - s_\alpha) + H(x_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \leq -\delta,$$

which we certainly have when $x_\alpha \in \Omega$.

If $y_\alpha \in \partial\Omega$, then $(x_\alpha - y_\alpha)_n \leq 0$ and

$$\hat{\gamma} \cdot D\zeta(x_\alpha - y_\alpha) \leq 0.$$

As above, we find that if $y_\alpha \in \partial\Omega$, then

$$\gamma(y_\alpha) \cdot (\mu + \alpha D\zeta(x_\alpha - y_\alpha)) < \delta,$$

and hence, by the viscosity property of v,

$$2(t_\alpha - s_\alpha) + H(y_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \geq \delta,$$

which is also valid in case when $y_\alpha \in \Omega$.

Thus, we always have

$$\begin{cases} 2\alpha(t_\alpha - s_\alpha) + H(x_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \leq -\delta, \\ 2(t_\alpha - s_\alpha) + H(y_\alpha, \mu + \alpha D\zeta(x_\alpha - y_\alpha)) \geq \delta. \end{cases}$$

Sending $\alpha \to \infty$ along a sequence, we obtain

$$q + H(0, \mu + p) \leq -\delta \quad \text{and} \quad q + H(0, \mu + p) \geq \delta$$

for some $p \in \overline{B}_{C_0 M_1}$ and $q \in [-2M_1, 2M_1]$, which is a contradiction. This completes the proof of Theorem 3.1. □

4 Stationary Problem: Weak KAM Aspects

In this section we discuss some aspects of weak KAM theory for Hamilton–Jacobi equations with the Neumann boundary condition. We refer to Fathi [25, 27], E [22] and Evans [24] for origins and developments of weak KAM theory.

Throughout this section we assume that (A1)–(A6) and the following (A7) hold:

(A7) The Hamiltonian H is convex. That is, the function $p \mapsto H(x, p)$ is convex in $\mathbb{R}^n$ for any $x \in \overline{\Omega}$.

As in Sect. 2 we consider the stationary problem

$$\begin{cases} H(x, Du(x)) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{SNP}$$

As remarked before this boundary value problem may have no solution in general, but, due to Theorem 3.4, if we replace H by $H - a$ with the right choice of $a \in \mathbb{R}$, the problem (SNP) has a viscosity solution. Furthermore, if we replace H by $H - a$ with a sufficiently large $a \in \mathbb{R}$, the problem (SNP) has a viscosity subsolution. With a change of Hamiltonians of this kind in mind, we make the following hypothesis throughout this section:

(A8) The problem (SNP) has a viscosity subsolution.

4.1 Aubry Sets and Representation of Solutions

We start this subsection by the following Lemma.

Lemma 4.1. *Let $u \in \mathrm{USC}(\overline{\Omega})$ be a viscosity subsolution of (SNP). Then $u \in \mathrm{Lip}(\overline{\Omega})$. Moreover, u has a Lipschitz bound which depends only on H and Ω.*

Proof. By the coercivity of H, there exists a constant $M > 0$ such that $H(x, p) > 0$ for all $(x, p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_M)$. Fix such a constant $M > 0$ and note that u is a viscosity subsolution of $|Du(x)| \le M$ in Ω. Accordingly, we see by Lemma 2.2 that $u \in \mathrm{Lip}(\Omega)$. Furthermore, if $C > 0$ is the constant from Lemma 2.1, then we have $|u(x) - u(y)| \le CM|x - y|$ for all $x, y \in \Omega$. (See also Appendix A.3.)

Since the function $u(x)$, as a function of (x, t), is a viscosity subsolution of (ENP), Lemma 3.3 guarantees that u is continuous up to the boundary $\partial\Omega$. Thus, we get $|u(x) - u(y)| \le CM|x - y|$ for all $x, y \in \overline{\Omega}$, which completes the proof. □

We introduce the distance-like function $d : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ by

$$d(x, y) = \sup\{v(x) - v(y) \,:\, v \in \mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-\},$$

where $\mathscr{S}^- = \mathscr{S}^-(\overline{\Omega})$ has been defined as the set of all viscosity subsolutions of (SNP). By (A8), we have $\mathscr{S}^- \neq \emptyset$ and hence $d(x, x) = 0$ for all $x \in \overline{\Omega}$. Since $\mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-$ is equi-Lipschitz continuous on $\overline{\Omega}$ by Lemma 4.1, we see that the functions $(x, y) \mapsto v(x) - v(y)$, with $v \in \mathrm{USC}(\overline{\Omega}) \cap \mathscr{S}^-$, are equi-Lipschitz continuous and d is Lipschitz continuous on $\overline{\Omega} \times \overline{\Omega}$. By Proposition 1.10, the functions $x \mapsto d(x, y)$, with $y \in \overline{\Omega}$, are viscosity subsolutions of (SNP). Hence, by the definition of $d(x, z)$ we get

$$d(x, y) - d(z, y) \le d(x, z) \quad \text{for all } x, y, z \in \overline{\Omega}.$$

We set

$$\mathscr{F}_y = \{v(x) - v(y) \; : \; v \in \mathscr{S}^-\}, \quad \text{with} \;\; y \in \overline{\Omega},$$

and observe by using Proposition 1.10 and Lemma 1.3 that $\mathscr{F}_y$ satisfies (P1) and (P2), with Ω replaced by $\overline{\Omega} \setminus \{y\}$, of Proposition 1.13. Hence, by Proposition 1.13, the function $d(\cdot, y) = \sup \mathscr{F}_y$ is a viscosity solution of (SNP) in $\overline{\Omega} \setminus \{y\}$.

The following proposition collects these observations.

Proposition 4.1. *We have:*

(i) $d(x, x) = 0$ *for all* $x \in \overline{\Omega}$.
(ii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \overline{\Omega}$.
(iii) $d(\cdot, y) \in \mathscr{S}^-(\overline{\Omega})$ *for all* $y \in \overline{\Omega}$.
(iv) $d(\cdot, y) \in \mathscr{S}(\overline{\Omega} \setminus \{y\})$ *for all* $y \in \overline{\Omega}$.

The *Aubry set* (or Aubry–Mather set) $\mathscr{A}$ associated with (SNP) is defined by

$$\mathscr{A} = \{y \in \overline{\Omega} \; : \; d(\cdot, y) \in \mathscr{S}(\overline{\Omega})\}.$$

Example 4.1. Let $n = 1$, $\Omega = (-1, 1)$, $H(x, p) = |p| - f(x)$, $f(x) = 1 - |x|$, $\gamma(\pm 1) = \pm 1$ and $g(\pm 1) = 0$. The function $v \in C^1([-1, 1])$ given by

$$v(x) = \begin{cases} 1 - \frac{1}{2}(x+1)^2 & \text{if } x \leq 0, \\ \frac{1}{2}(x-1)^2 & \text{if } x \geq 0 \end{cases}$$

is a classical solution of (SNP). We show that $d(x, 1) = v(x)$ for all $x \in [-1, 1]$. It is enough to show that $d(x, 1) \leq v(x)$ for all $x \in [-1, 1]$. To prove this, we suppose by contradiction that $\max_{x \in [-1,1]}(d(x, 1) - v(x)) > 0$. We may choose a constant $\varepsilon > 0$ so small that $\max_{x \in [-1,1]}(d(x, 1) - v(x) - \varepsilon(1 - x)) > 0$. Let $x_\varepsilon \in [-1, 1]$ be a maximum point of the function $d(x, 1) - v(x) - \varepsilon(1 - x)$. Since this function vanishes at $x = 1$, we have $x_\varepsilon \in [-1, 1)$. If $x_\varepsilon > -1$, then we have

$$0 \geq H(x_\varepsilon, v'(x_\varepsilon) - \varepsilon) = |v'(x_\varepsilon)| + \varepsilon - f(x_\varepsilon) = \varepsilon > 0,$$

which is impossible. Here we have used the fact that $v'(x) = |x| - 1 \leq 0$ for all $x \in [-1, 1]$. On the other hand, if $x_\varepsilon = -1$, then we have

$$0 \geq \min\{H(-1, v'(-1) - \varepsilon), \; -(v'(-1) - \varepsilon)\} = \min\{\varepsilon, \; \varepsilon\} = \varepsilon > 0,$$

which is again impossible. Thus we get a contradiction. That is, we have $d(x, 1) \leq v(x)$ and hence $d(x, 1) = v(x)$ for all $x \in [-1, 1]$. Arguments similar to the above show moreover that

$$d(x, -1) = \begin{cases} \frac{1}{2}(x+1)^2 & \text{if } x \leq 0, \\ 1 - \frac{1}{2}(x-1)^2 & \text{if } x \geq 0, \end{cases}$$

and

$$d(x,y) = \begin{cases} d(x,1) - d(y,1) & \text{if } x \le y, \\ d(x,-1) - d(y,-1) & \text{if } x \ge y. \end{cases}$$

Since two functions $d(x, \pm 1)$ are classical solutions of (SNP), we see that $\pm 1 \in \mathscr{A}$. Noting that $d(x,y) \ge 0$ and $d(x,x) = 0$ for all $x, y \in [-1, 1]$, we find that for each fixed $y \in [-1, 1]$ the function $x \mapsto d(x,y)$ has a minimum at $x = y$. If $y \in (-1, 1)$, then $H(y, 0) = -f(y) < 0$. Hence, we see that the interval $(-1, 1)$ does not intersect $\mathscr{A}$. Thus, we conclude that $\mathscr{A} = \{-1, 1\}$.

A basic observation on $\mathscr{A}$ is the following:

Proposition 4.2. *The Aubry set $\mathscr{A}$ is compact.*

Proof. It is enough to show that $\mathscr{A}$ is a closed subset of $\overline{\Omega}$. Note that the function d is Lipschitz continuous in $\overline{\Omega} \times \overline{\Omega}$. Therefore, if $\{y_k\}_{k\in\mathbb{N}} \subset \mathscr{A}$ converges to $y \in \overline{\Omega}$, then the sequence $\{d(\cdot, y_k)\}_{k\in\mathbb{N}}$ converges to the function $d(\cdot, y)$ in $C(\overline{\Omega})$. By the stability of the viscosity property under the uniform convergence, we see that $d(\cdot, y) \in \mathscr{S}$. Hence, we have $y \in \mathscr{A}$. □

The main assertion in this section is the following and will be proved at the end of the section.

Theorem 4.1. *Let $u \in C(\overline{\Omega})$ be a viscosity solution of* (SNP). *Then*

$$u(x) = \inf\{u(y) + d(x,y) \,:\, y \in \mathscr{A}\} \quad \textit{for all } x \in \overline{\Omega}. \tag{61}$$

We state the following approximation result on viscosity subsolutions of (SNP).

Theorem 4.2. *Let $u \in C(\overline{\Omega})$ be a viscosity subsolution of* (SNP). *There exists a collection $\{u^\varepsilon\}_{\varepsilon\in(0,\,1)} \subset C^1(\overline{\Omega})$ such that for any $\varepsilon \in (0,\,1)$,*

$$\begin{cases} H(x, Du^\varepsilon(x)) \le \varepsilon & \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \le g(x) & \text{on } \partial\Omega, \end{cases}$$

and

$$\|u^\varepsilon - u\|_{\infty,\Omega} < \varepsilon.$$

A localized version of the above theorem is in [39] (see also Appendix A.4 and [8]) and the above theorem seems to be new in the global nature.

As a corollary, we get the following theorem.

Theorem 4.3. *Let $f_1, f_2 \in C(\overline{\Omega})$ and $g_1, g_2 \in C(\partial\Omega)$. Let $u, v \in C(\overline{\Omega})$ be viscosity solutions of*

$$\begin{cases} H(x, Du) \le f_1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} \le g_1 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} H(x, Dv) \le f_2 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma} \le g_2 & \text{on } \partial\Omega, \end{cases}$$

respectively. Let $0 < \lambda < 1$ *and set* $w = (1-\lambda)u + \lambda v$. *Then* w *is a viscosity subsolution of*

$$\begin{cases} H(x, Dw) \le (1-\lambda) f_1 + \lambda f_2 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \gamma} \le (1-\lambda) g_1 + \lambda g_2 & \text{on } \partial\Omega, \end{cases} \tag{62}$$

Proof. By Theorem 4.2, for each $\varepsilon \in (0, 1)$ there are functions $u^\varepsilon, v^\varepsilon \in C^1(\overline{\Omega})$ such that

$$\|u^\varepsilon - u\|_{\infty,\Omega} + \|v^\varepsilon - v\|_{\infty,\Omega} < \varepsilon,$$

$$\begin{cases} H(x, Du^\varepsilon(x)) \le f_1(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \le g_1(x) & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} H(x, Dv^\varepsilon(x)) \le f_2(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le g_2(x) & \text{on } \partial\Omega. \end{cases}$$

If we set $w^\varepsilon = (1-\lambda)u^\varepsilon + \lambda v^\varepsilon$, then we get with use of (A7)

$$\begin{cases} H(x, Dw^\varepsilon(x)) \le (1-\lambda) f_1(x) + \lambda f_2(x) + \varepsilon & \text{in } \overline{\Omega}, \\ \dfrac{\partial w^\varepsilon}{\partial \gamma}(x) \le (1-\lambda) g_1(x) + \lambda g_2(x) & \text{on } \partial\Omega. \end{cases}$$

Thus, in view of the stability property (Proposition 1.9), we see in the limit as $\varepsilon \to 0$ that w is a viscosity subsolution of (62). □

The following theorem is also a consequence of (A7), the convexity of H, and Theorem 4.2.

Theorem 4.4. *Let* $\mathscr{F} \subset USC(\overline{\Omega})$ *be a nonempty collection of viscosity subsolutions of* (SNP). *Assume that* $u(x) := \inf \mathscr{F}(x) > -\infty$ *for all* $x \in \overline{\Omega}$. *Then* $u \in \mathrm{Lip}(\overline{\Omega})$ *and it is a viscosity subsolution of* (SNP).

This theorem may be regarded as part of the theory of Barron–Jensen's lower semicontinuous viscosity solutions. There are at least two approaches to this theory: the original one by Barron–Jensen [11] and the other due to Barles [5]. The following proof is close to Barles' approach.

Proof. By Lemma 4.1, the collection $\mathscr{F}$ is equi-Lipschitz in $\overline{\Omega}$. Hence, u is a Lipschitz continuous function in $\overline{\Omega}$. For each $x \in \overline{\Omega}$ there is a sequence $\{u_{x,k}\}_{k\in\mathbb{N}} \subset \mathscr{F}$ such that $\lim_{k\to\infty} u_{x,k}(x) = u(x)$. Fix such sequences $\{u_{x,k}\}_{k\in\mathbb{N}}$, with $x \in \overline{\Omega}$ and select a countable dense subset $Y \subset \overline{\Omega}$. Observe that $Y \times \mathbb{N}$ is a countable set and

$$u(x) = \inf\{u_{y,k}(x) \,:\, (y,k) \in Y \times \mathbb{N}\} \quad \text{for all } \ x \in \overline{\Omega}.$$

Thus we may assume that $\mathscr{F}$ is a sequence.

Let $\mathscr{F} = \{u_k\}_{k\in\mathbb{N}}$. Then we have

$$u(x) = \lim_{k\to\infty} (u_1 \wedge u_2 \wedge \cdots \wedge u_k)(x) \quad \text{for all } \ x \in \overline{\Omega}.$$

We show that $u_1 \wedge u_2 \wedge \cdots \wedge u_k$ is a viscosity subsolution of (SNP) for every $k \in \mathbb{N}$. It is enough to show that if v and w are viscosity subsolutions of (SNP), then so is the function $v \wedge w$.

Let v and w be viscosity subsolutions of (SNP). Fix any $\varepsilon > 0$. In view of Theorem 4.2, we may select functions v_ε, $w_\varepsilon \in C^1(\overline{\Omega})$ so that both for $(\phi_\varepsilon, \phi) = (v_\varepsilon, v)$ and $(\phi_\varepsilon, \phi) = (w_\varepsilon, w)$, we have

$$\begin{cases} H(x, D\phi_\varepsilon(x)) \leq \varepsilon & \text{for all } \ x \in \Omega, \\ \dfrac{\partial \phi_\varepsilon}{\partial \gamma}(x) \leq g(x) & \text{for all } \ x \in \partial\Omega, \\ \|\phi_\varepsilon - \phi\|_{\infty,\Omega} < \varepsilon. \end{cases}$$

Note that $(v_\varepsilon \wedge w_\varepsilon)(x) = v_\varepsilon(x) - (v_\varepsilon - w_\varepsilon)_+(x)$. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset C^1(\mathbb{R})$ be such that

$$\begin{cases} \eta_k(r) \to r_+ & \text{uniformly on } \ \mathbb{R} \ \text{ as } \ k \to \infty, \\ 0 \leq \eta_k'(r) \leq 1 & \text{for all } \ r \in \mathbb{R},\ k \in \mathbb{N}. \end{cases}$$

We set $z_{\varepsilon,k} = v_\varepsilon - \eta_k \circ (v_\varepsilon - w_\varepsilon)$ and observe that

$$Dz_{\varepsilon,k}(x) = \left(1 - \eta_k'(v_\varepsilon(x) - w_\varepsilon(x))\right) Dv_\varepsilon(x) + \eta_k'(v_\varepsilon(x) - w_\varepsilon(x)) Dw_\varepsilon(x).$$

By the convexity of H, we see easily that $z_{\varepsilon,k}$ satisfies

$$\begin{cases} H(x, Dz_{\varepsilon,k}(x)) \leq \varepsilon & \text{for all } \ x \in \Omega, \\ \dfrac{\partial z_{\varepsilon,k}}{\partial \gamma}(x) \leq g(x) & \text{for all } \ x \in \partial\Omega. \end{cases}$$

Since $v \wedge w$ is a uniform limit of $z_{\varepsilon,k}$ in $\overline{\Omega}$ as $k \to \infty$ and $\varepsilon \to 0$, we see that $v \wedge w$ is a viscosity subsolution of (SNP).

By the Ascoli–Arzela theorem or Dini's lemma, we deduce that the convergence

$$u(x) = \lim_{k\to\infty} (u_1 \wedge \cdots \wedge u_k)(x)$$

is uniform in $\overline{\Omega}$. Thus we conclude that u is a viscosity subsolution of (SNP). □

Remark 4.1. Theorem 4.2 has its localized version which concerns viscosity subsolutions of

$$\begin{cases} H(x, Du(x)) \le 0 & \text{in } U \cap \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) \le g(x) & \text{on } U \cap \partial\Omega, \end{cases}$$

where U is an open subset of $\mathbb{R}^n$ having nonempty intersection with Ω. More importantly, it has a version for the Neumann problem for Hamilton–Jacobi equations of evolution type, which concerns solutions of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le 0 & \text{in } U \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u}{\partial \gamma}(x,t) \le g(x) & \text{on } U \cap (\partial\Omega \times \mathbb{R}_+), \end{cases}$$

where U is an open subset of $\mathbb{R}^n \times \mathbb{R}_+$, with $U \cap (\Omega \times \mathbb{R}_+) \ne \emptyset$. Consequently, Theorems 4.3 and 4.4 are valid for these problems with trivial modifications. For these, see Appendix A.4.

Theorem 4.5. *We have*

$$c^{\#} = \inf\left\{ \max_{x\in\overline{\Omega}} H(x, D\psi(x)) \ :\ \psi \in C^1(\overline{\Omega}),\ \partial\psi/\partial\gamma \le g \text{ on } \partial\Omega \right\}.$$

Remark 4.2. A natural question here is if there is a function $\psi \in C^1(\overline{\Omega})$ which attains the infimum in the above formula. See [12, 28].

Proof. Let $c^\star$ denote the right hand side of the above minimax formula. By the definition of $c^\star$, it is clear that for any $a > c^\star$, there is a classical subsolution of (EVP). Hence, by Corollary 3.2, we see that $c^{\#} \le c^\star$.

On the other hand, by Theorem 3.4, there is a viscosity solution v of (EVP), with $a = c^{\#}$. By Theorem 4.2, for any $a > c^{\#}$ there is a classical subsolution of (EVP). That is, we have $c^\star \le c^{\#}$. Thus we conclude that $c^{\#} = c^\star$. □

Theorem 4.6 (Comparison). *Let $v, w \in C(\overline{\Omega})$ be a viscosity subsolution and supersolution of* (SNP)*, respectively. Assume that $v \le w$ on $\mathscr{A}$. Then $v \le w$ in $\overline{\Omega}$.*

For the proof of the above theorem, we need the following lemma.

Lemma 4.2. *Let K be a compact subset of $\overline{\Omega} \setminus \mathscr{A}$. Then there exists a function $\psi \in C^1(U \cap \overline{\Omega})$, where U is an open neighborhood of K in $\mathbb{R}^n$, and a positive constant $\delta > 0$ such that*

$$\begin{cases} H(x, D\psi(x)) \leq -\delta & \text{in } U \cap \Omega, \\ \dfrac{\partial \psi}{\partial \gamma}(x) \leq g(x) - \delta & \text{on } U \cap \partial\Omega. \end{cases} \tag{63}$$

We assume temporarily the validity of the above lemma and complete the proof of Theorems 4.6 and 4.1. The proof of the above lemma will be given in the sequel.

Proof (Theorem 4.6). By contradiction, we suppose that $M := \sup_{\overline{\Omega}}(v - w) > 0$. Let

$$K = \{x \in \overline{\Omega} \,:\, (v - w)(x) = M\},$$

which is a compact subset of $\overline{\Omega} \setminus \mathscr{A}$. According to Lemma 4.2, there are $\delta > 0$ and $\psi \in C^1(U \cap \overline{\Omega})$, where U is an open neighborhood of K such that ψ is a subsolution of (63).

According to Theorem 4.2, for each $\varepsilon \in (0,\,1)$ there is a function $v^\varepsilon \in C^1(\overline{\Omega})$ such that

$$\begin{cases} H(x, Dv^\varepsilon(x)) \leq \varepsilon & \text{in } \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \leq g(x) & \text{on } \partial\Omega, \end{cases}$$

and

$$\|v^\varepsilon - v\|_{\infty,\Omega} < \varepsilon.$$

We fix a $\lambda \in (0,\,1)$ so that $\delta_\varepsilon := -(1-\lambda)\varepsilon + \delta\lambda > 0$ and set

$$u_\varepsilon(x) = (1-\lambda)v^\varepsilon(x) + \lambda\psi(x).$$

This function satisfies

$$\begin{cases} H(x, Du_\varepsilon(x)) \leq -\delta_\varepsilon & \text{in } U \cap \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x) \leq g(x) - \delta_\varepsilon & \text{on } U \cap \partial\Omega. \end{cases}$$

This contradicts the viscosity property of the function w if $u_\varepsilon - w$ attains a maximum at a point $z \in U \cap \overline{\Omega}$. Hence, we have

$$\max_{\overline{U} \cap \overline{\Omega}}(u_\varepsilon - w) = \max_{\partial U \cap \overline{\Omega}}(u_\varepsilon - w).$$

Sending $\varepsilon \to 0$ and then $\lambda \to 0$ yields

$$\max_{\overline{U} \cap \overline{\Omega}}(v - w) = \max_{\partial U \cap \overline{\Omega}}(v - w),$$

that is,

$$M = \max_{\partial U \cap \overline{\Omega}} (v - w).$$

This is a contradiction. □

Remark 4.3. Obviously, the continuity assumption on v, w in the above lemma can be replaced by the assumption that $v \in \mathrm{USC}(\overline{\Omega})$ and $w \in \mathrm{LSC}(\overline{\Omega})$.

Proof (Theorem 4.1). We write $w(x)$ for the right hand side of (61) in this proof. By the definition of d, we have

$$u(x) - u(y) \le d(x, y) \quad \text{for all } x, y \in \overline{\Omega},$$

from which we see that $u(x) \le w(x)$.

By the definition of w, for every $x \in \mathscr{A}$, we have

$$w(x) \le u(x) + d(x, x) = u(x).$$

Hence, we have $w = u$ on $\mathscr{A}$.

Now, by Proposition 1.10 (its version for supersolutions), we see that w is a viscosity supersolution of (SNP) while Theorem 4.4 guarantees that w is a viscosity subsolution of (SNP). We invoke here Theorem 4.6, to see that $u = w$ in $\overline{\Omega}$. □

Proof (Lemma 4.2). In view of Theorem 4.2, it is enough to show that there exist functions $w \in \mathrm{Lip}(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ such that

$$\begin{cases} f(x) \ge 0 & \text{in } \Omega, \\ f(x) > 0 & \text{in } K, \end{cases}$$

and w is a viscosity subsolution of

$$\begin{cases} H(x, Dw(x)) \le -f(x) & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \gamma}(x) \le g(x) & \text{on } \partial\Omega. \end{cases}$$

For any $z \in \overline{\Omega} \setminus \mathscr{A}$, the function $x \mapsto d(x, z)$ is not a viscosity supersolution of (SNP) at z while it is a viscosity subsolution of (SNP). Hence, according to Lemma 1.3, there exist a function $\psi_z \in \mathrm{Lip}(\overline{\Omega})$, a neighborhood U_z of z in $\mathbb{R}^n$ and a constant $\delta_z > 0$ such that ψ_z is a viscosity subsolution of (SNP) and it is moreover a viscosity subsolution of

$$\begin{cases} H(x, D\psi_z(x)) \le -\delta_z & \text{in } U_z \cap \Omega, \\ \dfrac{\partial \psi_z}{\partial \gamma}(x) \le g(x) - \delta_z & \text{on } U_z \cap \partial\Omega. \end{cases}$$

We choose a function $f_z \in C(\overline{\Omega})$ so that $0 < f_z(x) \le \delta$ for all $x \in \overline{\Omega} \cap U_z$ and $f_z(x) = 0$ for all $x \in \overline{\Omega} \setminus U_z$, and note that ψ_z is a viscosity subsolution of

$$\begin{cases} H(x, D\psi_z(x)) \le -f_z(x) & \text{in } \Omega, \\ \dfrac{\partial \psi_z}{\partial \gamma}(x) \le g(x) - f_z(x) & \text{on } \partial\Omega. \end{cases}$$

We select a finite number of points $z_1, \ldots, z_k$ of K so that $\{U_{z_i}\}_{i=1}^k$ covers K.

Now, we define the function $\psi \in \mathrm{Lip}(\overline{\Omega})$ by

$$\psi(x) = \frac{1}{k} \sum_{i=1}^{k} \psi_{z_i}(x),$$

and observe by Theorem 4.3 that ψ is a viscosity subsolution of

$$\begin{cases} H(x, D\psi(x)) \le -f(x) & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial \gamma}(x) \le g(x) - f(x) & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\overline{\Omega})$ is given by

$$f(x) = \frac{1}{k} \sum_{i=1}^{k} f_{z_i}(x).$$

Finally, we note that $\inf_K f > 0$. □

4.2 Proof of Theorem 4.2

We give a proof of Theorem 4.2 in this subsection.

We begin by choosing continuous functions on $\mathbb{R}^n$ which extend the functions g, γ and ν. We denote them again by the same symbols g, γ and ν.

The following proposition guarantees the existence of test functions which are convenient to prove Theorem 4.2.

Theorem 4.7. *Let $\varepsilon > 0$ and $M > 0$. Then there exist a constant $\Lambda > 0$ and moreover, for each $R > 0$, a neighborhood U of $\partial\Omega$, a function $\chi \in C^1((\Omega \cup U) \times \mathbb{R}^n)$ and a constant $\delta > 0$ such that for all $(x, \xi) \in (\Omega \cup U) \times \mathbb{R}^n$,*

$$M|\xi| \le \chi(x, \xi) \le \Lambda(|\xi| + 1),$$

and for all $(x, \xi) \in U \times B_R$,

$$\gamma(x) \cdot D_\xi \chi(x,\xi) \begin{cases} \le g(x) + 2\varepsilon & \textit{if } \nu(x)\cdot\xi \le \delta, \\ \ge g(x) + \dfrac{\varepsilon}{2} & \textit{if } \nu(x)\cdot\xi \ge -\delta. \end{cases}$$

It should be noted that the constant Λ in the above statement does not depend on R while U, χ and δ do.

We begin the proof with two Lemmas.

We fix $r > 1$ and set

$$\mathbb{R}_r^{2n} = \{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, y \cdot z \ge r^{-1},\ \max\{|y|,|z|\} \le r\}.$$

We define the function $\zeta \in C^\infty(\mathbb{R}_r^{2n} \times \mathbb{R}^n)$ by

$$\zeta(y,z,\xi) = \left|\xi - \frac{y\cdot\xi}{y\cdot z} z\right|^2 + (y\cdot\xi)^2.$$

Lemma 4.3. *The function ζ has the properties:*

$$\begin{cases} \zeta(y,z,t\xi) = t^2\zeta(y,z,\xi) & \textit{for all } (y,z,\xi,t) \in \mathbb{R}_r^{2n} \times \mathbb{R}^n \times \mathbb{R}, \\ \zeta(y,z,\xi) > 0 & \textit{for all } (y,z,\xi) \in \mathbb{R}_r^{2n} \times (\mathbb{R}^n \setminus \{0\}), \\ z \cdot D_\xi\zeta(y,z,\xi) = 2(y\cdot z)(y\cdot\xi) & \textit{for all } (y,z,\xi) \in \mathbb{R}_r^{2n} \times \mathbb{R}^n. \end{cases}$$

This is a version of Lemma 3.4, the proof of which is easily adapted to the present case.

We define the function $\phi \,:\, \mathbb{R}_r^{2n} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(y,z,\xi) = (\zeta(y,z,\xi) + 1)^{1/2}.$$

Lemma 4.4. *There exists a constant $\Lambda > 1$, which depends only on r, such that for all $(y,z,\xi) \in \mathbb{R}_r^{2n} \times \mathbb{R}^n$,*

$$\begin{cases} z \cdot D_\xi\phi(y,z,\xi) = \phi(y,z,\xi)^{-1}(y\cdot z)(y\cdot\xi), \\ \max\{\Lambda^{-1}|\xi|,\,1\} \le \phi(y,z,\xi) \le \Lambda(|\xi|+1), \\ \max\{|D_y\phi(y,z,\xi)|, |D_z\phi(y,z,\xi)|\} \le \Lambda, \\ |D_\xi\phi(y,z,\xi)| \le \Lambda. \end{cases}$$

Proof. It is clear by the definition of ϕ that

$$\phi(y,z,\xi) \ge 1.$$

We may choose a constant $C > 1$ so that for all $(y, z, \xi) \in \mathbb{R}^{2n}_r \times S^{n-1}$,

$$\max\{\zeta(y, z, \xi), \zeta(y, z, \xi)^{-1}, |D_y\zeta(y, z, \xi)|, |D_z\zeta(y, z, \xi)|, |D_\xi\zeta(y, z, \xi)|\} \le C,$$

where $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. By the homogeneity of the function $\zeta(y, z, \xi)$ in ξ, we have

$$\begin{aligned} \max\{\zeta(y, z, \xi), |D_y\zeta(y, z, \xi)|, |D_z\zeta(y, z, \xi)|\} \le C|\xi|^2, \\ |D_\xi\zeta(y, z, \xi)| \le C|\xi|, \\ \zeta(y, z, \xi) \ge C^{-1}|\xi|^2 \end{aligned} \tag{64}$$

for all $(y, z, \xi) \in \mathbb{R}^{2n}_r \times \mathbb{R}^n$. From this it follows that

$$C^{-1/2}|\xi| \le \phi(y, z, \xi) \le C^{1/2}(|\xi| + 1).$$

By a direct computation, we get

$$D_x\phi(y, z, \xi) = \frac{D_x\zeta(y, z, \xi)}{2\phi(y, z, \xi)} \quad \text{for } x = y, z, \xi.$$

Hence, using (64), we get

$$|D_y\phi(y, z, \xi)| \le \frac{C|\xi|^2}{2\phi(y, z, \xi)} \le C^{3/2}|\xi|.$$

In the same way, we get

$$|D_z\phi(y, z, \xi)| \le C^{3/2}|\xi|.$$

Also, we get

$$|D_\xi\phi(y, z, \xi)| \le \frac{C|\xi|^2}{2\phi(y, z, \xi)} \le C^{3/2}|\xi|.$$

We observe that

$$z \cdot D_\xi\phi(y, z, \xi) = \frac{z \cdot D_\xi\zeta(y, z, \xi)}{2\phi(y, z, \xi)} = \frac{(y \cdot z)(y \cdot \xi)}{\phi(y, z, \xi)}.$$

By setting $\Lambda = C^{3/2}$, we conclude the proof. □

Let $\alpha > 0$. For any $W \subset \mathbb{R}^n$ we denote by W^α the α–neighborhood of W, that is,

$$W^\alpha = \{x \in \mathbb{R}^n : \operatorname{dist}(x, W) < \alpha\}.$$

For each $\delta \in (0, 1)$ we select $\nu_\delta \in C^1(\Omega^1, \mathbb{R}^n)$, $\gamma_\delta \in C^1(\Omega^1, \mathbb{R}^n)$ and $g_\delta \in C^1(\Omega^1, \mathbb{R})$ so that for all $x \in \Omega^1$,

$$\max\{|\nu_\delta(x) - \nu(x)|,\ |\gamma_\delta(x) - \gamma(x)|,\ |g_\delta(x) - g(x)|\} < \delta. \tag{65}$$

(Just to be sure, note that $\Omega^1 = \{x \in \mathbb{R}^n \ :\ \mathrm{dist}(x, \Omega) < 1\}$.)

By assumption, we have

$$\nu(x) \cdot \gamma(x) > 0 \quad \text{for all } x \in \partial\Omega.$$

Hence, we may fix $\delta_0 \in (0,\ 1)$ so that

$$\inf\{\nu_\delta(x) \cdot \gamma_\delta(x)\ :\ x \in (\partial\Omega)^{\delta_0},\ \delta \in (0,\ \delta_0)\} > 0.$$

We choose a constant $r > 1$ so that if $\delta \in (0,\ \delta_0)$, then

$$\begin{cases} \min\{\nu_\delta(x) \cdot \gamma_\delta(x),\ |\gamma_\delta(x)|\} \geq r^{-1}, \\ \max\{|\nu_\delta(x)|,\ |\gamma_\delta(x)|\} \leq r, \\ |g_\delta(x)| + 1 < r. \end{cases} \tag{66}$$

for all $x \in (\partial\Omega)^{\delta_0}$. In particular, we have

$$(\nu_\delta(x),\ \gamma_\delta(x)) \in \mathbb{R}_r^{2n} \quad \text{for all } \ x \in (\partial\Omega)^{\delta_0} \ \text{ and } \ \delta \in (0,\ \delta_0). \tag{67}$$

To proceed, we fix any $\varepsilon \in (0,\ 1)$, $M > 0$ and $R > 0$. For each $\delta \in (0,\ \delta_0)$ we define the function $\psi_\delta \in C^1((\partial\Omega)^{\delta_0} \times \mathbb{R}^n)$ by

$$\psi_\delta(x, \xi) = (g_\delta(x) + \varepsilon)\frac{\gamma_\delta(x) \cdot \xi}{|\gamma_\delta(x)|^2},$$

choose a cut-off function $\eta_\delta \in C_0^1(\mathbb{R}^n)$ so that

$$\begin{cases} \operatorname{supp} \eta_\delta \subset (\partial\Omega)^\delta, \\ 0 \leq \eta_\delta(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^n, \\ \eta_\delta(x) = 1 \quad \text{for all } x \in (\partial\Omega)^{\delta/2}, \end{cases}$$

and define the function $\chi_\delta \in C^1(\Omega^{\delta_0})$ by

$$\chi_\delta(x, \xi) = M\langle\xi\rangle(1 - \eta_\delta(x)) + \eta_\delta(x)\left[\psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi)\right],$$

where Λ and ϕ are the constant and function from Lemma 4.4, $\langle\xi\rangle := (|\xi|^2 + 1)^{1/2}$ and $\phi_\delta(x, \xi) := \phi(\nu_\delta(x), \gamma_\delta(x), \xi)$. Since $\operatorname{supp} \eta_\delta \subset (\partial\Omega)^{\delta_0}$ for all $\delta \in (0,\ \delta_0)$, in view of (67) we see that χ_δ is well-defined.

Proof (Theorem 4.7). Let $\delta_0 \in (0,\ 1)$ and $\psi_\delta,\ \phi_\delta,\ \chi_\delta \in C^1(\Omega^{\delta_0} \times \mathbb{R}^n)$ be as above.

Let $\delta \in (0, \delta_0)$, which will be fixed later on. It is obvious that for all $(x, \xi) \in (\Omega)^{\delta_0} \times \mathbb{R}^n$,

$$\begin{cases} \gamma_\delta(x) \cdot D_\xi \psi_\delta(x, \xi) = g_\delta(x) + \varepsilon, \\ |\psi_\delta(x, \xi)| \le r^2|\xi|. \end{cases} \tag{68}$$

For any $(x, \xi) \in (\partial\Omega)^\delta \times \mathbb{R}^n$, using (66), (68) and Lemma 4.4, we get

$$\psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi) \ge -r^2|\xi| + (r^2 + M)|\xi| \ge M|\xi|,$$

and

$$\begin{aligned} \psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi) &\le r^2|\xi| + (r^2 + M)\Lambda^2(|\xi| + 1) \\ &\le (2r^2 + M)\Lambda^2(|\xi| + 1). \end{aligned}$$

Thus, we have

$$M|\xi| \le \chi_\delta(x, \xi) \le (2r^2 + M)\Lambda^2(|\xi| + 1) \quad \text{for all } \ (x, \xi) \in \Omega^\delta \times \mathbb{R}^n. \tag{69}$$

Now, note that if $(x, \xi) \in (\partial\Omega)^{\delta/2} \times \mathbb{R}^n$, then

$$\chi_\delta(x, \xi) = \psi_\delta(x, \xi) + (r^2 + M)\Lambda\phi_\delta(x, \xi).$$

Hence, by Lemma 4.4 and (68), we get

$$\gamma_\delta(x) \cdot D_\xi \chi_\delta(x, \xi) = g_\delta(x) + \varepsilon + (r^2 + M)\Lambda \frac{(\nu_\delta(x) \cdot \gamma_\delta(x))(\nu_\delta(x) \cdot \xi)}{\phi_\delta(x, \xi)}$$

for all $(x, \xi) \in (\partial\Omega)^{\delta/2} \times \mathbb{R}^n$.

Next, let $(x, \xi) \in \Omega^\delta \times \mathbb{R}^n$. Since

$$D_\xi \chi_\delta(x, \xi) = M(1 - \eta_\delta(x))D\langle\xi\rangle + \eta_\delta(x)\left[D_\xi \psi_\delta(x, \xi) + (r^2 + M)\Lambda D_\xi \phi_\delta(x, \xi)\right],$$

using Lemma 4.4, we get

$$\begin{aligned} |D_\xi \chi_\delta(x, \xi)| &\le \max\left\{M|D\langle\xi\rangle|,\ \frac{|g_\delta(x) + \varepsilon|}{|\gamma_\delta(x)|} + (r^2 + M)\Lambda|D_\xi \phi_\delta(x, \xi)|\right\} \\ &\le \max\{M,\ r^2 + (r^2 + M)\Lambda^2\} = (2r^2 + M)\Lambda^2. \end{aligned} \tag{70}$$

Let $(x, \xi) \in (\partial\Omega)^{\delta/2} \times B_R$. Note by (65) and (70) that

$$\left|(\gamma_\delta(x) - \gamma(x)) \cdot D_\xi \chi_\delta(x, \xi)\right| \le \delta(2r^2 + M)\Lambda^2.$$

Note also that if $\nu(x)\cdot\xi \le \delta$, then

$$\begin{aligned}(r^2+M)\Lambda&\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu_\delta(x)\cdot\xi)}{\phi_\delta(x,\xi)}\\ &\le (r^2+M)\Lambda\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu(x)\cdot\xi)}{\phi_\delta(x,\xi)}+(r^2+M)\Lambda r^2R\delta\\ &\le (r^2+M)\Lambda r^2\delta(1+R).\end{aligned}$$

Hence, if $\nu(x)\cdot\xi\le\delta$, then

$$\begin{aligned}&\gamma(x)\cdot D_\xi\chi_\delta(x,\xi)\le\gamma_\delta(x)\cdot D_\xi\chi_\delta(x,\xi)+\delta(2r^2+M)\Lambda^2\\ &\le\delta(2r^2+M)\Lambda^2+g_\delta(x)+\varepsilon+(r^2+M)\Lambda\frac{(\nu_\delta(x)\cdot\gamma_\delta(x))(\nu_\delta(x)\cdot\xi)}{\phi_\delta(x,\xi)}\\ &\le g(x)+\varepsilon+\delta\left[1+(2r^2+M)\Lambda^2r^2+(r^2+M)\Lambda r^2(1+R)\right].\end{aligned}$$

Similarly, we see that if $\nu(x)\cdot\xi\ge-\delta$, then

$$\gamma(x)\cdot D_\xi\chi_\delta(x,\xi)\ge g(x)+\varepsilon-\delta\left[1+(2r^2+M)\Lambda^2r^2+(r^2+M)\Lambda r^2(1+R)\right].$$

If we select $\delta\in(0,\delta_0)$ so that

$$\delta\left[1+(2r^2+M)\Lambda^2r^2+(r^2+M)\Lambda r^2(1+R)\right]\le\frac{\varepsilon}{2},$$

then we have for all $(x,\xi)\in(\partial\Omega)^{\delta/2}\times B_R$,

$$\gamma(x)\cdot D_\xi\chi_\delta(x,\xi)\begin{cases}\le g(x)+2\varepsilon & \text{if } \nu(x)\cdot\xi\le\delta,\\ \ge g(x)+\frac{\varepsilon}{2} & \text{if } \nu(x)\cdot\xi\ge-\delta.\end{cases}$$

Thus, the function $\chi=\chi_\delta$ has the required properties, with $(\partial\Omega)^{\delta/2}$ and $(2r^2+M)\Lambda^2$ in place of U and Λ, respectively. □

We are ready to prove the following theorem.

Theorem 4.8. *Let $\varepsilon>0$ and $u\in\mathrm{Lip}(\overline{\Omega})$ be a viscosity subsolution of (SNP). Then there exist a neighborhood U of $\partial\Omega$ and a function $u_\varepsilon\in C^1(\Omega\cup U)$ such that*

$$\begin{cases}H(x,Du_\varepsilon(x))\le\varepsilon & \textit{for all } x\in\Omega\cup U,\\ \gamma(x)\cdot Du_\varepsilon(x)\le g(x)+\varepsilon & \textit{for all } x\in U,\\ \|u_\varepsilon-u\|_{\infty,\Omega}\le\varepsilon.\end{cases}\tag{71}$$

Proof. Fix any $\varepsilon > 0$ and a constant $M > 1$ so that $M - 1$ is a Lipschitz bound of the function u. With these constants ε and M, let $\Lambda > 0$ be the constant from Theorem 4.7. Set $R = M + 2\Lambda$, and let U, χ and δ be as in Theorem 4.7.

Let $\alpha > 0$. We define the sup-convolution $u_\alpha \in C(\Omega \cup U)$ by

$$u_\alpha(x) = \max_{y \in \overline{\Omega}} (u(y) - \alpha\chi(x, (y-x)/\alpha)).$$

Let $x \in \Omega \cup U$, $p \in D^+ u_\alpha(x)$ and $y \in \overline{\Omega}$ be a maximum point in the definition of u_α, that is,

$$u_\alpha(x) = u(y) - \alpha\chi(x, (y-x)/\alpha). \tag{72}$$

It is easily seen that

$$\begin{cases} D_\xi \chi(x, (y-x)/\alpha) \in D^+ u(y), \\ p = D_\xi \chi(x, (y-x)/\alpha) - \alpha D_x \chi(x, (y-x)/\alpha). \end{cases} \tag{73}$$

Fix an $\alpha_0 \in (0,\ 1)$ so that

$$\overline{(\partial\Omega)^{\alpha_0^2}} \subset U.$$

Here, of course, $\overline{V}$ denotes the closure of V. For $\alpha \in (0, \alpha_0)$ we set $U_\alpha = (\partial\Omega)^{\alpha^2}$ and $V_\alpha = \Omega \cup U_\alpha = \Omega^{\alpha^2}$. Note that $\chi \in C^1(\overline{V}_\alpha \times \mathbb{R}^n)$. We set $W_\alpha = \{(x, y) \in V_\alpha \times \overline{\Omega} \,:\, (72) \text{ holds}\}$.

Now, we fix any $\alpha \in (0, \alpha_0)$. Let $(x, y) \in W_\alpha$. We may choose a point $z \in \overline{\Omega}$ so that $|x - z| < \alpha^2$. Note that

$$u(y) - \alpha\chi(x, (y-x)/\alpha) = u_\alpha(x) \geq u(z) - \alpha\chi(x, (z-x)/\alpha).$$

Hence,

$$\alpha\chi(x, (y-x)/\alpha) \leq (M-1)|z-y| + \alpha\chi(x, (z-x)/\alpha).$$

Now, since $M|\xi| \leq \chi(x, \xi) \leq \Lambda(|\xi| + 1))$ for all $(x, \xi) \in V_\alpha \times \mathbb{R}^n$ and $|x - z| \leq \alpha^2 < \alpha$, we get

$$\begin{aligned} M|x-y| &\leq (M-1)(|x-y| + \alpha^2) + \alpha\Lambda(|z-x|/\alpha + 1) \\ &\leq (M-1)|x-y| + \alpha(M + 2\Lambda). \end{aligned}$$

Consequently,

$$|y - x| \leq \alpha(M + 2\Lambda) = R\alpha \quad \text{for all } (x, y) \in W_\alpha. \tag{74}$$

Next, we choose a constant $C > 0$ so that

$$|D_x\chi(x, \xi)| + |D_\xi\chi(x, \xi)| \leq C \quad \text{for all } (x, \xi) \in V_{\alpha_0} \times B_R.$$

Let $(x, y) \in W_\alpha$ and $z \in B_{R\alpha}(x) \cap V_{\alpha_0}$. Assume moreover that $x \in U$. In view of (74) and the choice of χ and δ, we have

$$\gamma(x) \cdot D_\xi \chi(x, (y-x)/\alpha) \begin{cases} \le g(x) + 2\varepsilon & \text{if } \nu(x) \cdot (y-x) \le \alpha\delta, \\ \ge g(x) + \dfrac{\varepsilon}{2} & \text{if } \nu(x) \cdot (y-x) \ge -\alpha\delta. \end{cases}$$

We observe that

$$\nu(x) \cdot (y-x) \begin{cases} \le \dfrac{\alpha\delta}{2} + \omega_\nu(R\alpha)R\alpha & \text{if } \nu(z) \cdot (y-x) \le \dfrac{\alpha\delta}{2}, \\ \ge \dfrac{\alpha\delta}{2} - \omega_\nu(R\alpha)R\alpha & \text{if } \nu(z) \cdot (y-x) \ge -\dfrac{\alpha\delta}{2}, \end{cases}$$

where ω_ν denotes the modulus of continuity of the function ν on V_{α_0}. Observe as well that

$$\begin{aligned} \big|\gamma(z) \cdot D_\xi \chi(x, (y-x)/\alpha) - \gamma(x) \cdot D_\xi \chi(x, (y-x)/\alpha)\big| &\le C\omega_\gamma(R\alpha), \\ |g(z) - g(x)| &\le \omega_g(R\alpha), \end{aligned}$$

where ω_γ and ω_g denote the moduli of continuity of the functions γ and g on the set V_{α_0}, respectively.

We may choose an $\alpha_1 \in (0, \alpha_0)$ so that

$$\omega_\nu(R\alpha_1)R < \frac{\delta}{2} \quad \text{and} \quad C\omega_\gamma(R\alpha_1) + \omega_g(R\alpha_1) < \frac{\varepsilon}{4},$$

and conclude from the above observations that for all $(x, y) \in W_\alpha$ and $z_i \in B_{R\alpha}(x) \cap V_{\alpha_0}$, with $i = 1, 2, 3$, if $x \in U$ and $\alpha < \alpha_1$, then

$$\gamma(z_1) \cdot D_\xi \chi(x, (y-x)/\alpha) \begin{cases} \le g(z_2) + 3\varepsilon & \text{if } \nu(z_3) \cdot (y-x) \le \alpha\delta/2, \\ \ge g(z_2) + \dfrac{\varepsilon}{4} & \text{if } \nu(z_3) \cdot (y-x) \ge -\alpha\delta/2. \end{cases} \tag{75}$$

We may assume, by reselecting $\alpha_1 > 0$ small enough if necessary, that

$$(\partial\Omega)^{R\alpha_1} \subset U. \tag{76}$$

In what follows we assume that $\alpha \in (0, \alpha_1)$. Let $(x, y) \in W_\alpha$ and $p \in D^+u_\alpha(x)$. By (73) and (74), we have

$$\max\{|p|, |D_\xi \chi(x, (y-x)/\alpha)|\} \le C(1+\alpha). \tag{77}$$

Let ω_H denote the modulus of continuity of H on $V_{\alpha_0} \times B_{C(1+\alpha_0)}$.

We now assume that $y \in \partial\Omega$. By (74) and (76), we have $x \in U$. Let ρ be a defining function of Ω. We may assume that $|D\rho| \le 1$ in V_{α_0} and $\rho_0 := \inf_{U_{\alpha_0}} |D\rho| > 0$. Observe that

$$\alpha^2 > \rho(x) = \rho(x) - \rho(y) = D\rho(z) \cdot (x - y) = |D\rho(z)|\nu(z) \cdot (x - y)$$

for some point z on the line segment $[x, y]$. Hence, we get

$$\nu(z) \cdot (x - y) \le \rho_0^{-1}\alpha^2.$$

If $\alpha \le \rho_0\delta/2$, then

$$\nu(z) \cdot (y - x) \ge -\alpha\delta/2.$$

Hence, noting that $|z - x| \le |x - y| < R\alpha$, by (75), we get

$$\gamma(y) \cdot D_\xi \chi(x, (y - x)/\alpha) \ge g(y) + \frac{\varepsilon}{4},$$

and, by the viscosity property of u,

$$0 \ge H(y, D_\xi \chi(x, (y - x)/\alpha)) \ge H(x, p) - \omega_H((R + C)\alpha).$$

Thus, if $\omega_H((R + C)\alpha) < \varepsilon$ and $\alpha \le \rho_0\delta/2$, then we have

$$H(x, p) \le \varepsilon.$$

On the other hand, if $y \in \Omega$, then, by the viscosity property of u, we have

$$0 \ge H(y, D_\xi \chi(x, (y - x)/\alpha)).$$

Therefore, if $\omega_H((R + C)\alpha) < \varepsilon$, then

$$H(x, p) \le \varepsilon.$$

We may henceforth assume by selecting $\alpha_1 > 0$ small enough that

$$\omega_H((R + C)\alpha_1) < \varepsilon \quad \text{and} \quad \alpha_1 \le \rho_0\delta/2,$$

and we conclude that u_α is a viscosity subsolution of

$$H(x, Du_\alpha(x)) \le \varepsilon \quad \text{in } V_\alpha. \tag{78}$$

As above, let $(x, y) \in W_\alpha$ and $p \in D^+u_\alpha(x)$. We assume that $x \in U_\alpha$. Then

$$-\alpha^2 < \rho(x) \le \rho(x) - \rho(y) \le D\rho(z) \cdot (x - y)$$

for some $z \in [x, y]$, which yields

$$\nu(z) \cdot (y - x) < |D\rho(z)|^{-1}\alpha^2 \leq \rho_0^{-1}\alpha^2.$$

Hence, if $\alpha \leq \rho_0\delta/2$, then

$$\nu(z) \cdot (y - x) \leq \frac{\delta\alpha}{2},$$

and, by (75), we get

$$\gamma(x) \cdot D_\xi \chi(x, (y - x)/\alpha) \leq g(x) + 3\varepsilon.$$

Furthermore,

$$\begin{aligned}\gamma(x) \cdot p &\leq \gamma(x) \cdot D_\xi \chi(x, (y - x)/\alpha) + \alpha C \|\gamma\|_{\infty, U_{\alpha_0}} \\ &\leq g(x) + 3\varepsilon + \alpha C \|\gamma\|_{\infty, U_{\alpha_0}}.\end{aligned}$$

We may assume again by selecting $\alpha_1 > 0$ small enough that

$$\alpha_1 C \|\gamma\|_{\infty, U_{\alpha_0}} < \varepsilon.$$

Thus, u_α is a viscosity subsolution of

$$\gamma(x) \cdot Du_\alpha(x) \leq g(x) + 4\varepsilon \quad \text{in } U_\alpha. \tag{79}$$

Let $(x, y) \in W_\alpha$ and observe by using (74) that if $x \in \overline{\Omega}$, then

$$|u(x) - u_\alpha(x)| \leq |u(x) - u(y)| + \alpha|\chi(x, (y - x)/\alpha)| \leq \alpha(MR + C).$$

We fix $\alpha \in (0, \alpha_1)$ so that $\alpha_1(MR + C) < \varepsilon$, and conclude that u_α is a viscosity subsolution of (78) and (79) and satisfies

$$\|u_\alpha - u\|_{\infty, \Omega} \leq \varepsilon.$$

The final step is to mollify the function u_α. Let $\{k_\lambda\}_{\lambda>0}$ be a collection of standard mollification kernels.

We note by (77) or (78) that u_α is Lipschitz continuous on any compact subset of V_α. Fix any $\lambda \in (0, \alpha^2/4)$. We note that the closure of $V_{\alpha/2} + B_\lambda$ is a compact subset of V_α. Let $M_1 > 0$ be a Lipschitz bound of the function u_α on $V_{\alpha/2} + B_\lambda$.

We set

$$u^\lambda(x) = u_\alpha * k_\lambda(x) \quad \text{for } x \in V_{\alpha/2}.$$

In view of Rademacher's theorem (see Appendix A.6), we have

$$\begin{aligned}&H(x, Du_\alpha(x)) \leq \varepsilon &&\text{for a.e. } x \in V_\alpha, \\ &\gamma(x) \cdot Du_\alpha(x) \leq g(x) + 4\varepsilon &&\text{for a.e. } x \in U_\alpha.\end{aligned}$$

Here Du_α denotes the distributional derivative of u_α, and we have

$$Du^\lambda = k_\lambda * Du_\alpha \quad \text{in } V_{\alpha/2}.$$

By Jensen's inequality, we get

$$\begin{aligned} H(x, Du^\lambda(x)) &\le \int_{B_\lambda} H(x, Du_\alpha(x-y))k_\lambda(y)\,\mathrm{d}y \\ &\le \int_{B_\lambda} H(x-y, Du_\alpha(x-y))k_\lambda(y)\,\mathrm{d}y + \omega_H(\lambda) \\ &\le \varepsilon + \omega_H(\lambda), \end{aligned}$$

where ω_H is the modulus of continuity of H on the set $V_\alpha \times B_{M_1}$. Similarly, we get

$$\gamma(x)\cdot Du^\lambda(x) \le g(x) + 4\varepsilon + \omega_g(\lambda) + M_1\omega_\gamma(\lambda),$$

where ω_g and ω_γ are the moduli of continuity of the functions g and γ on V_α, respectively. If we choose $\lambda > 0$ small enough, then (71) holds with $u^\lambda \in C^1(V_{\alpha/2})$, $U_{\alpha/2}$ and 5ε in place of u_ε, U and ε, respectively. The proof is complete. □

Proof (Theorem 4.2). Let $\varepsilon > 0$ and $u \in \mathrm{Lip}(\overline{\Omega})$ be a viscosity subsolution of (SNP). Let ρ be a defining function of Ω. We may assume that

$$D\rho(x)\cdot\gamma(x) \ge 1 \quad \text{for all } x \in \partial\Omega.$$

For $\delta > 0$ we set

$$u^\delta(x) = u(x) - \delta\rho(x) \quad \text{for } x \in \overline{\Omega}.$$

It is easily seen that if $\delta > 0$ is small enough, then u^δ is a viscosity subsolution of

$$\begin{cases} H(x, Du^\delta(x)) \le \varepsilon & \text{in } \Omega, \\ \gamma(x)\cdot Du^\delta(x) \le g(x) - \delta & \text{on } \partial\Omega, \end{cases}$$

and the following inequality holds:

$$\|u^\delta - u\|_{\infty,\Omega} \le \varepsilon.$$

Then, Theorem 4.8, with $\min\{\varepsilon,\delta\}$, u^δ, $H-\varepsilon$ and $g-\delta$ in place of ε, u, H and g, respectively, ensures that there are a neighborhood U of $\partial\Omega$ and a function $u_\varepsilon \in C^1(\Omega\cup U)$ such that

$$\begin{cases} H(x, Du_\varepsilon(x)) \le 2\varepsilon & \text{in } \Omega \cup U, \\ \gamma(x) \cdot Du_\varepsilon(x) \le g(x) & \text{in } U, \\ \|u_\varepsilon - u\|_{\infty,\Omega} \le 2\varepsilon, \end{cases}$$

which concludes the proof. □

5 Optimal Control Problem Associated with (ENP)–(ID)

In this section we introduce an optimal control problem associated with the initial-boundary value problem (ENP)–(ID),

5.1 Skorokhod Problem

In this section, following [39, 44], we study the Skorokhod problem. We recall that $\mathbb{R}_+$ denotes the interval $(0, \infty)$, so that $\overline{\mathbb{R}}_+ = [0, \infty)$. We denote by $L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^k)$ (resp., $\mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^k)$) the space of functions $v : \overline{\mathbb{R}}_+ \to \mathbb{R}^k$ which are integrable (resp., absolutely continuous) on any bounded interval $J \subset \overline{\mathbb{R}}_+$.

Given $x \in \overline{\Omega}$ and $v \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$, the Skorokhod problem is to seek for a pair of functions, $(\eta, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R})$, such that

$$\begin{cases} \eta(0) = x, & \\ \eta(t) \in \overline{\Omega} & \text{for all } t \in \overline{\mathbb{R}}_+, \\ \dot{\eta}(t) + l(t)\gamma(\eta(t)) = v(t) & \text{for a.e. } t \in \overline{\mathbb{R}}_+, \\ l(t) \ge 0 & \text{for a.e. } t \in \overline{\mathbb{R}}_+, \\ l(t) = 0 \ \text{ if } \eta(t) \in \Omega & \text{for a.e. } t \in \overline{\mathbb{R}}_+. \end{cases} \tag{80}$$

Regarding the solvability of the Skorokhod problem, our main claim is the following.

Theorem 5.1. *Let $v \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and $x \in \overline{\Omega}$. Then there exits a pair $(\eta, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R})$ such that (80) holds.*

We refer to [44] and references therein for more general viewpoints (especially, for applications to stochastic differential equations with reflection) on the Skorokhod problem.

A natural question arises whether uniqueness of the solution (η, l) holds or not in the above theorem. On this issue we just give the following counterexample and do not discuss it further.

Example 5.1. Let $n = 2$ and $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$. (For simplicity of presentation, we consider the case where Ω is unbounded.) Define $\gamma \in C(\partial\Omega, \mathbb{R}^2)$ and $v \in L^\infty(\overline{\mathbb{R}}_+, \mathbb{R}^2)$ by

$$\gamma(0, x_2) = (-1, -3|x_2|^{-1/3}x_2) \quad \text{and} \quad v(t) = (-1, 0).$$

Set

$$\eta^\pm(t) = (0, \pm t^3) \quad \text{for all } t \ge 0.$$

Then the pairs $(\eta^+, 1)$ and $(\eta^-, 1)$ are both solutions of (80), with $\eta^\pm(0) = (0, 0)$.

We first establish the following assertion.

Theorem 5.2. *Let $v \in L^\infty(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and $x \in \overline{\Omega}$. Then there exits a pair $(\eta, l) \in$ $\mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^\infty(\overline{\mathbb{R}}_+, \mathbb{R})$ such that (80) holds.*

Proof. We may assume that γ is defined and continuous on $\mathbb{R}^n$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. We may assume that $\liminf_{|x|\to\infty} \rho(x) > 0$ and that $D\rho$ is bounded on $\mathbb{R}^n$. We may select a constant $\delta > 0$ so that for all $x \in \mathbb{R}^n$,

$$\gamma(x) \cdot D\rho(x) \ge \delta|D\rho(x)| \quad \text{and} \quad |D\rho(x)| \ge \delta \quad \text{if } 0 \le \rho(x) \le \delta.$$

We set $q(x) = (\rho(x) \vee 0) \wedge \delta$ for $x \in \mathbb{R}^n$ and observe that $q(x) = 0$ for all $x \in \overline{\Omega}$ and $q(x) > 0$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$.

Fix $\varepsilon > 0$ and $x \in \overline{\Omega}$. We consider the initial value problem for the ODE

$$\dot{\xi}(t) + \frac{1}{\varepsilon} q(\xi(t))\gamma(\xi(t)) = v(t) \ \text{ for a.e. } t \in \mathbb{R}_+, \quad \xi(0) = x. \tag{81}$$

By the standard ODE theory, there is a solution $\xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+)$ of (81). Fix such a solution $\xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ in what follows.

Note that $(\mathrm{d}\, q \circ \xi/\mathrm{d}t)(t) = D\rho(\xi(t)) \cdot \dot{\xi}(t)$ a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in (0, \delta)\}$. Moreover, noting that $q \circ \xi \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \mathbb{R})$ and hence it is differentiable a.e., we deduce that $(\mathrm{d}\, q \circ \xi/\mathrm{d}t)(t) = 0$ a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in \{0, \delta\}\}$.

Let $m \ge 2$. We multiply the ODE of (81) by $mq(\xi(t))^{m-1}D\rho(\xi(t))$, to get

$$\frac{\mathrm{d}}{\mathrm{d}t} q(\xi(t))^m + \frac{m}{\varepsilon} q(\xi(t))^m Dq(\xi(t)) \cdot \gamma(\xi(t)) = mq(\xi(t))^{m-1} Dq(\xi(t)) \cdot v(t)$$

a.e. in the set $\{t \in \mathbb{R}_+ : \rho \circ \xi(t) \in (0, \delta)\}$. For any $T \in \mathbb{R}_+$, integration over $E_T := \{t \in [0, T] : \rho \circ \xi(t) \in (0, \delta)\}$ yields

$$\begin{aligned} q(\xi(T))^m - q(\xi(0))^m &+ \frac{m}{\varepsilon} \int_{E_T} q(\xi(s))^m \gamma(\xi(s)) \cdot D\rho(\xi(s))\mathrm{d}s \\ &= m \int_{E_T} q(\xi(s))^{m-1} D\rho(\xi(s)) \cdot v(s)\mathrm{d}s. \end{aligned}$$

Here we note

$$\int_{E_T} q(\xi(s))^m \gamma(\xi(s)) \cdot D\rho(\xi(s))\mathrm{d}s \geq \delta \int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s,$$

and

$$\int_{E_T} q(\xi(s))^{m-1} D\rho(\xi(s)) \cdot v(s)\mathrm{d}s$$
$$\leq \left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s)|\mathrm{d}s\right)^{1-\frac{1}{m}} \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}.$$

Combining these, we get

$$q(\xi(T))^m + \frac{m\delta}{\varepsilon}\int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s$$
$$\leq m\left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s)|\mathrm{d}s\right)^{1-\frac{1}{m}} \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}.$$

Hence,

$$\frac{\delta}{\varepsilon}\left(\int_{E_T} q(\xi(s))^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}} \leq \left(\int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s\right)^{\frac{1}{m}}$$

and

$$q(\xi(T))^m \leq \left(\frac{\varepsilon}{\delta}\right)^{m-1} m \int_{E_T} |v(s)|^m |D\rho(\xi(s))|\mathrm{d}s.$$

Thus, setting $C_0 = \|D\rho\|_{L^\infty(\mathbb{R}^n)}$, we find that for any $T \in \mathbb{R}_+$,

$$q(\xi(t))^m \leq \left(\frac{\varepsilon}{\delta}\right)^{m-1} mC_0T\|v\|^m_{L^\infty(0,T)} \quad \text{for all } t \in [0,\, T]. \tag{82}$$

We henceforth write ξ_ε for ξ, in order to indicate the dependence on ε of ξ, and observe from (82) that for any $T > 0$,

$$\lim_{\varepsilon\to 0+}\max_{t\in[0,\,T]} \mathrm{dist}(\xi_\varepsilon(t),\, \Omega) = 0. \tag{83}$$

Also, (82) ensures that for any $T > 0$,

$$\frac{\delta}{\varepsilon}\|q \circ \xi_\varepsilon\|_{L^\infty(0,T)} \leq \left(\frac{\delta m C_0 T}{\varepsilon}\right)^{\frac{1}{m}} \|v\|_{L^\infty(0,T)}.$$

Sending $m \to \infty$, we find that $(\delta/\varepsilon)\|q \circ \xi_\varepsilon\|_{L^\infty(0,T)} \le \|v\|_{L^\infty(0,T)}$, and moreover

$$\frac{\delta}{\varepsilon}\|q \circ \xi_\varepsilon\|_{L^\infty(\mathbb{R}_+)} \le \|v\|_{L^\infty(\mathbb{R}_+)}. \tag{84}$$

We set $l_\varepsilon = (1/\varepsilon)q \circ \xi_\varepsilon$. Thanks to (84), we may choose a sequence $\varepsilon_j \to 0+$ (see Lemma E.1) so that $l_{\varepsilon_j} \to l$ weakly-star in $L^\infty(\mathbb{R}_+)$ as $j \to \infty$ for a function $l \in L^\infty(\mathbb{R}_+)$. It is clear that $l(s) \ge 0$ for a.e. $s \in \mathbb{R}_+$.

ODE (81) together with (84) guarantees that $\{\dot{\xi}_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_+)$. Hence, we may assume as well that ξ_{ε_j} converges locally uniformly on $\overline{\mathbb{R}}_+$ to a function $\eta \in \mathrm{Lip}(\mathbb{R}_+)$ as $j \to \infty$. It is then obvious that $\eta(0) = x$ and the pair (η, l) satisfies

$$\eta(t) + \int_0^t \big(l(s)\gamma(\eta(s)) - v(s)\big)\mathrm{d}s = x \quad \text{for all } t \in \mathbb{R}_+,$$

from which we get

$$\dot{\eta}(t) + l(t)\gamma(\eta(t)) = v(t) \quad \text{for a.e. } t \in \mathbb{R}_+.$$

It follows from (83) that $\eta(t) \in \overline{\Omega}$ for $t \ge 0$.

In order to show that the pair (η, l) is a solution of (80), we need only to prove that for a.e. $t \in \mathbb{R}_+$, $l(t) = 0$ if $\eta(t) \in \Omega$. Set $A = \{t \ge 0 \,:\, \eta(t) \in \Omega\}$. It is clear that A is an open subset of $[0, \infty)$. We can choose a sequence $\{I_k\}_{k\in\mathbb{N}}$ of closed finite intervals of A such that $A = \bigcup_{k\in\mathbb{N}} I_k$. Note that for each $k \in \mathbb{N}$, the set $\eta(I_k)$ is a compact subset of Ω and the convergence of $\{\xi_{\varepsilon_j}\}$ to η is uniform on I_k. Hence, for any fixed $k \in \mathbb{N}$, we may choose $J \in \mathbb{N}$ so that $\xi_{\varepsilon_j}(t) \in \Omega$ for all $t \in I_k$ and $j \ge J$. From this, we have $q(\xi_{\varepsilon_j}(t)) = 0$ for $t \in I_k$ and $j \ge J$. Moreover, in view of the weak-star convergence of $\{l_{\varepsilon_j}\}$, we find that for any $k \in \mathbb{N}$,

$$\int_{I_k} l(t)\mathrm{d}t = \lim_{j\to\infty} \int_{I_k} \frac{1}{\varepsilon_j} q(\xi_j(t))\mathrm{d}t = 0,$$

which yields $l(t) = 0$ for a.e. $t \in I_k$. Since $A = \bigcup_{k\in\mathbb{N}} I_k$, we see that $l(t) = 0$ a.e. in A. The proof is now complete. □

For $x \in \overline{\Omega}$, let $\mathrm{SP}(x)$ denote the set of all triples

$$(\eta, v, l) \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n) \times L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$$

which satisfies (80). We set $\mathrm{SP} = \bigcup_{x\in\overline{\Omega}} \mathrm{SP}(x)$.

We remark that for any $x, y \in \overline{\Omega}$ and $T \in \mathbb{R}_+$, there exists a triple $(\eta, v, l) \in \mathrm{SP}(x)$ such that $\eta(T) = y$. Indeed, given $x, y \in \overline{\Omega}$ and $T \in \mathbb{R}_+$, we choose a curve $\eta \in \mathrm{Lip}([0,\, T], \overline{\Omega})$ (see Lemma 2.1) so that $\eta(0) = x$, $\eta(T) = y$ and $\eta(t) \in \overline{\Omega}$ for all $t \in [0, T]$. We extend the domain of definition of η to $\overline{\mathbb{R}}_+$ by setting $\eta(t) = y$

for $t > T$. If we set $v(t) = \dot{\eta}(t)$ and $l(t) = 0$ for $t \geq 0$, we have $(\eta, v, l) \in \mathrm{SP}(x)$, which has the property, $\eta(T) = y$.

We note also that problem (80) has the following *semi-group* property: for any $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$ and $(\eta_1, v_1, l_1), (\eta_2, v_2, l_2) \in \mathrm{SP}$, if $\eta_1(0) = x$ and $\eta_2(0) = \eta_1(t)$ hold and if (η, v, l) is defined on $\overline{\mathbb{R}}_+$ by

$$(\eta(s), v(s), l(s)) = \begin{cases} (\eta_1(s), v_1(s), l_1(s)) & \text{for } s \in [0, t), \\ (\eta_2(s-t), v_2(s-t), l_2(s-t)) & \text{for } s \in [t, \infty), \end{cases}$$

then $(\eta, v, l) \in \mathrm{SP}(x)$.

The following proposition concerns a stability property of sequences of points in SP.

Proposition 5.1. *Let* $\{(\eta_k, v_k, l_k)\}_{k\in\mathbb{N}} \subset \mathrm{SP}$. *Let* $x \in \overline{\Omega}$ *and* $(w, v, l) \in L_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^{2n+1})$. *Assume that as* $k \to \infty$,

$$\eta_k(0) \to x,$$

$$(\dot{\eta}_k, v_k, l_k) \to (w, v, l) \quad \textit{weakly in } L^1([0, T], \mathbb{R}^{2n+1})$$

for every $T \in \mathbb{R}_+$. *Set*

$$\eta(s) = x + \int_0^s w(r)\mathrm{d}r \quad \textit{for } s \geq 0.$$

Then $(\eta, v, l) \in \mathrm{SP}(x)$.

Proof. For all $t > 0$ and $k \in \mathbb{N}$, we have

$$\eta_k(t) = \eta_k(0) + \int_0^t \dot{\eta}_k(s)\mathrm{d}s = \eta_k(0) + \int_0^t (v_k(s) - l_k(s)\gamma(\eta_k(s)))\,\mathrm{d}s.$$

First, we observe that as $k \to \infty$,

$$\eta_k(t) \to \eta(t) \quad \text{locally uniformly on } \overline{\mathbb{R}}_+,$$

and then we get in the limit as $k \to \infty$,

$$\eta(t) = x + \int_0^t (v(s) - l(s)\gamma(\eta(s)))\,\mathrm{d}s \quad \text{for all } t > 0.$$

This shows that $\eta \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ and

$$\dot{\eta}(s) + l(s)\gamma(\eta(s)) = v(s) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

It is clear that $\eta(0) = x$, $\eta(s) \in \overline{\Omega}$ for all $s \in \mathbb{R}_+$ and $l(s) \geq 0$ for a.e. $s \in \mathbb{R}_+$.

To show that $(\eta, v, l) \in \mathrm{SP}(x)$, it remains to prove that for a.e. $t \in \mathbb{R}_+$, $l(t) = 0$ if $\eta(t) \in \Omega$. As in the last part of the proof of Theorem 5.2, we set $A = \{t \geq 0 : \eta(t) \in \Omega\}$ and choose a sequence $\{I_j\}_{j\in\mathbb{N}}$ of closed finite intervals of A such that $A = \bigcup_{j\in\mathbb{N}} I_j$. Fix any $j \in \mathbb{N}$ and choose $K \in \mathbb{N}$ so that $\eta_k(t) \in \Omega$ for all $t \in I_j$ and $k \geq K$. From this, we have $l_k(t) = 0$ for a.e. $t \in I_j$ and $k \geq K$. Moreover, in view of the weak convergence of $\{l_k\}$, we find that

$$\int_{I_j} l(t)\mathrm{d}t = \lim_{k\to\infty} \int_{I_j} l_k(t)\mathrm{d}t = 0,$$

which yields $l(t) = 0$ for a.e. $t \in I_j$. Since j is arbitrary, we see that $l(t) = 0$ a.e. in $A = \bigcup_{j\in\mathbb{N}} I_j$. □

Proposition 5.2. *There is a constant $C > 0$, depending only on Ω and γ, such that for all $(\eta, v, l) \in \mathrm{SP}$,*

$$|\dot\eta(s)| \vee l(s) \leq C|v(s)| \quad \textit{for a.e. } s \geq 0.$$

An immediate consequence of the above proposition is that for $(\eta, v, l) \in \mathrm{SP}$, if $v \in L^p(\mathbb{R}_+, \mathbb{R}^n)$ (resp., $v \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$), with $1 \leq p \leq \infty$, then $(\dot\eta, l) \in L^p(\mathbb{R}_+, R^{n+1})$ (resp., $(\dot\eta, l) \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^{n+1})$).

Proof. Thanks to hypothesis (A4), there is a constant $\delta_0 > 0$ such that $\nu(x)\cdot\gamma(x) \geq \delta_0$ for $x \in \partial\Omega$. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω.

Let $s \in \mathbb{R}_+$ be such that $\eta(s) \in \partial\Omega$, η is differentiable at s, $l(s) \geq 0$ and $\dot\eta(s) + l(s)\gamma(\eta(s)) = v(s)$. Observe that the function $\rho \circ \eta$ attains a maximum at s. Hence,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}s}\rho(\eta(s)) = D\rho(\eta(s)) \cdot \dot\eta(s) = |D\rho(\eta(s))|\nu(\eta(s)) \cdot \dot\eta(s) \\ &= |D\rho(\eta(s))|\nu(\eta(s)) \cdot \big(v(s) - l(s)\gamma(\eta(s))\big) \\ &\leq |D\rho(\eta(s))|\big(\nu(\eta(s)) \cdot v(s) - l(s)\delta_0\big). \end{aligned}$$

Thus, we get

$$l(s) \leq \delta_0^{-1}\nu(\eta(s)) \cdot v(s) \leq \delta_0^{-1}|v(s)|$$

and

$$\begin{aligned} |\dot\eta(s)| &= |v(s) - l(s)\gamma(\eta(s))| \leq |v(s)| + l(s)\|\gamma\|_{\infty,\partial\Omega} \\ &\leq (1 + \delta_0^{-1}\|\gamma\|_{\infty,\partial\Omega})|v(s)|, \end{aligned}$$

which completes the proof. □

5.2 Value Function I

We define the function $L \in \mathrm{LSC}(\overline{\Omega} \times \mathbb{R}^n,\ (-\infty, \infty])$, called the *Lagrangian* of H, by

$$L(x, \xi) = \sup_{p \in \mathbb{R}^n} \big(\xi \cdot p - H(x, p)\big).$$

For each x the function $\xi \mapsto L(x, \xi)$ is the convex conjugate of the function $p \mapsto H(x, p)$. See Appendix A.2 for properties of conjugate convex functions.

We consider the optimal control with the dynamics given by (80), the running cost $(L,\ g)$ and the pay-off u_0, and its value function V on $\overline{Q}$, where $Q = \overline{\Omega} \times \mathbb{R}_+$, is given by

$$\begin{aligned} V(x, t) = \inf\Big\{ \int_0^t \big(L(\eta(s), -v(s)) + g(\eta(s))l(s)\big)\mathrm{d}s \\ + u_0(\eta(t)) \ :\ (\eta, v, l) \in \mathrm{SP}(x)\Big\} \quad \text{for } (x, t) \in Q, \end{aligned} \tag{85}$$

and $V(x, 0) = u_0(x)$ for all $x \in \overline{\Omega}$.

For $t > 0$ and $(\eta, v, l) \in \mathrm{SP} = \bigcup_{x \in \overline{\Omega}} \mathrm{SP}(x)$, we write

$$\mathscr{L}(t, \eta, v, l) = \int_0^t \big(L(\eta(s), -v(s)) + g(\eta(s))l(s)\big)\mathrm{d}s$$

for notational simplicity, and then formula (85) reads

$$V(x, t) = \inf\big\{\mathscr{L}(t, \eta, v, l) + u_0(\eta(t)) \ :\ (\eta, v, l) \in \mathrm{SP}(x)\big\}.$$

Under our hypotheses, the Lagrangian L may take the value ∞ and, on the other hand, if we set $C = \min_{x \in \overline{\Omega}}(-H(x, 0))$, then we have

$$L(x, \xi) \ge C \quad \text{for all } \ (x,\ \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Thus, it is reasonable to interpret

$$\int_0^t L(\eta(s), -v(s))\mathrm{d}s = \infty$$

if the function: $s \mapsto L(\eta(s), -v(s))$ is not integrable, which we adopt here.

It is easily checked as in the proof of Proposition 1.3 that the value function V satisfies the dynamic programming principle: given a point $(x, t) \in Q$ and a nonanticipating mapping $\tau\ :\ \mathrm{SP}(x) \to [0,\ t]$, we have

$$V(x, t) = \inf\big\{\mathscr{L}(\tau(\alpha), \alpha) + V(\eta(\tau(\alpha)),\ t - \tau(\alpha)) \ :\ \alpha = (\eta, v, l) \in \mathrm{SP}(x)\big\}. \tag{86}$$

Here a mapping $\tau : \mathrm{SP}(x) \to [0,\, t]$ is called *nonanticipating* if $\tau(\alpha) = \tau(\beta)$ whenever $\alpha(s) = \beta(s)$ a.e. in the interval $[0,\, \tau(\alpha)]$.

We here digress to recall the state-constraint problem, whose Bellman equation is given by the Hamilton–Jacobi equation

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

and to make a comparison between (ENP) and the state-constraint problem. For $x \in \overline{\Omega}$ let $\mathrm{SC}(x)$ denote the collection of all $\eta \in \mathrm{AC}_{\mathrm{loc}}(\overline{\mathbb{R}}_+, \mathbb{R}^n)$ such that $\eta(0) = x$ and $\eta(s) \in \overline{\Omega}$ for all $s \in \overline{\mathbb{R}}_+$. The value function $\hat{V} : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ of the state-constraint problem is given by

$$\hat{V}(x,t) = \inf\Big\{ \int_0^t L(\eta(s), -\dot{\eta}(s))\mathrm{d}s + u_0(\eta(t)) \,:\, \eta \in \mathrm{SC}(x)\Big\}.$$

Observe that if $\eta \in \mathrm{SC}(x)$, with $x \in \overline{\Omega}$, then $(\eta, \dot{\eta}, 0) \in \mathrm{SP}(x)$. Hence, we have

$$\begin{aligned}\hat{V}(x,t) &= \inf\big\{\mathscr{L}(t, \eta, \dot{\eta}, 0) + u_0(\eta(t)) \,:\, \eta \in \mathrm{SC}(x)\big\}\\ &\geq V(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}_+.\end{aligned}$$

Heuristically it is obvious that if $g(x) \approx \infty$, then

$$V(x,t) \approx \hat{V}(x,t).$$

In terms of PDE the above state-constraint problem is formulated as follows: the value function $\hat{V}$ is a unique viscosity solution of

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \leq 0 & \text{in } \Omega \times \mathbb{R}_+,\\ u_t(x,t) + H(x, D_x u(x,t)) \geq 0 & \text{in } \overline{\Omega} \times \mathbb{R}_+.\end{cases}$$

See [48] for a proof of this result in this generality. We refer to [17, 55] for state-constraint problems. The corresponding additive eigenvalue problem is to find $(a, v) \in \mathbb{R} \times C(\overline{\Omega})$ such that v is a viscosity solution of

$$\begin{cases} H(x, Dv(x)) \leq a & \text{in } \Omega,\\ H(x, Dv(x)) \geq a & \text{in } \overline{\Omega}.\end{cases} \tag{87}$$

We refer to [17, 40, 48] for this eigenvalue problem.

Example 5.2. We recall (see [48]) that the additive eigenvalue $\hat{c}$ for (87) is given by

$$\hat{c} = \inf\{a \in \mathbb{R} \,:\, (87) \text{ has a viscosity subsolution } v\},$$

For a comparison between the Neumann problem and the state-constraint problem, we go back to the situation of Example 3.1. Then it is easy to see that $\hat{c} = 0$. Thus, we have $c^{\#} = \hat{c} = 0$ if and only if $\min\{g(-1), g(1)\} \geq 0$.

We here continue the above example with some more generality. Let $c^{\#}$ and $\hat{c}$ denote, as above, the eigenvalues of (EVP) and (87), respectively. It is easily seen that if $\psi \in C(\bar{\Omega})$ is a subsolution of (EVP) with $a = c^{\#}$, then it is also a subsolution of (87) with $a = c^{\#}$, which ensures that $\hat{c} \leq c^{\#}$.

Next, note that the subsolutions of (87) with $a = \hat{c}$ are equi-Lipschitz continuous on $\bar{\Omega}$. That is, there exists a constant $M > 0$ such that for any subsolution ψ of (87) with $a = \hat{c}$, $|\psi(x) - \psi(y)| \leq M|x - y|$ for all $x, y \in \bar{\Omega}$. Let ψ be any subsolution of (87) with $a = \hat{c}$, $y \in \partial\Omega$ and $p \in D^{+}\psi(y)$. Choose a $\phi \in C^{1}(\bar{\Omega})$ so that $D\phi(y) = p$ and $\psi - \phi$ has a maximum at y. If $t > 0$ is sufficiently small, then we have $y - t\gamma(y) \in \Omega$ and, moreover, $\psi(y - t\gamma(y)) - \psi(y) \leq \phi(y - t\gamma(y)) - \phi(y)$. By the last inequality, we deduce that $\gamma(y) \cdot p \leq M|\gamma(y)|$. Accordingly, we have $\gamma(y) \cdot p \leq M|\gamma(y)|$ for all $p \in D^{+}\psi(y)$. Thus, we see that if $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then any subsolution ψ of (87) with $a = \hat{c}$ is a subsolution of (EVP) with $a = \hat{c}$. This shows that if $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then $c^{\#} \leq \hat{c}$. As we have already seen above, we have $\hat{c} \leq c^{\#}$, and, therefore, $c^{\#} = \hat{c}$, provided that $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$.

Now, assume that $c^{\#} = \hat{c}$ and let $a = c^{\#} = \hat{c}$. It is easily seen that

$$\{\psi \,:\, \psi \text{ is a subsolution of (EVP)}\} \subset \{\psi \,:\, \psi \text{ is a subsolution of (87)}\},$$

which guarantees that $d_N \leq d_S$ on $\overline{\Omega}^2$, where $d_N(\cdot, y) = \sup \mathscr{F}_y^N$, $d_S(\cdot, y) = \sup \mathscr{F}_y^S$, and

$$\mathscr{F}_y^N \,(\text{resp.}, \mathscr{F}_y^S) = \{\psi - \psi(y) \,:\, \psi \text{ is a subsolution of (EVP) (resp., (87))}\}.$$

Let $\mathscr{A}_N$ and $\mathscr{A}_S$ denote the Aubry sets associated with (EVP) and (87), respectively. That is,

$$\mathscr{A}_N = \{y \in \overline{\Omega} \,:\, d_N(\cdot, y) \text{ is a solution of (EVP)}\},$$

$$\mathscr{A}_S = \{y \in \overline{\Omega} \,:\, d_S(\cdot, y) \text{ is a solution of (87)}\}.$$

The above inequality and the fact that $d_N(y, y) = d_S(y, y) = 0$ for all $y \in \overline{\Omega}$ imply that $D_x^- d_N(x, y)|_{x=y} \subset D_x^- d_S(x, y)|_{x=y}$. From this inclusion, we easily deduce that $\mathscr{A}_S \subset \mathscr{A}_N$.

Thus the following proposition holds.

Proposition 5.3. *With the above notation, we have:*

(i) $\hat{c} \leq c^{\#}$.
(ii) *If $M > 0$ is a Lipschitz bound of the subsolutions of* (87) *with $a = \hat{c}$ and $g(x) \geq M|\gamma(x)|$ for all $x \in \partial\Omega$, then $\hat{c} = c^{\#}$.*
(iii) *If $\hat{c} = c^{\#}$, then $d_N \leq d_S$ on $\bar{\Omega}^2$ and $\mathscr{A}_S \subset \mathscr{A}_N$.*

5.3 Basic Lemmas

In this subsection we present a proof of the sequential lower semicontinuity of the functional $(\eta, v, l) \mapsto \mathscr{L}(T, \eta, v, l)$ (see Theorem 5.3 below). We will prove an existence result (Theorem 5.6) for the variational problem involving the functional $\mathscr{L}$ in Sect. 5.4. These results are variations of Tonelli's theorem in variational problems. For a detailed description of the theory of one-dimensional variational problems, with a central focus on Tonelli's theorem, we refer to [14].

Lemma 5.1. *For each $A > 0$ there exists a constant $C_A \geq 0$ such that*

$$L(x, \xi) \geq A|\xi| - C_A \quad \textit{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Proof. Fix any $A > 0$ and observe that

$$\begin{aligned} L(x, \xi) &\geq \max_{p \in \overline{B}_A} (\xi \cdot p - H(x, p)) \\ &\geq A|\xi| + \min_{p \in \overline{B}_A} (-H(x, p)) \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \end{aligned}$$

Hence, setting $C_A \geq \max_{\overline{\Omega} \times \overline{B}_A} |H|$, we get

$$L(x, \xi) \geq A|\xi| - C_A \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \qquad \square$$

Lemma 5.2. *There exist constants $\delta > 0$ and $C_0 > 0$ such that*

$$L(x, \xi) \leq C_0 \quad \textit{for all } (x, \xi) \in \overline{\Omega} \times B_\delta.$$

Proof. By the continuity of H, there exists a constant $M > 0$ such that $H(x, 0) \leq M$ for all $x \in \overline{\Omega}$. Also, by the coercivity of H, there exists a constant $R > 0$ such that $H(x, p) > M + 1$ for all $(x, p) \in \overline{\Omega} \times \partial B_R$. We set $\delta = R^{-1}$. Let $(x, \xi) \in \overline{\Omega} \times B_\delta$. Let $q \in \overline{B}_R$ be the minimum point of the function $f(p) := H(x, p) - \xi \cdot p$ on $\overline{B}_R$. Noting that $f(0) = H(x, 0) \leq M$ and $f(p) > -\delta R + M + 1 = M$ for all $p \in \partial B_R$, we see that $q \in B_R$ and hence $\xi \in D_p^- H(x, q)$, where $D_p^- H(x, q)$ denotes the subdifferential at q of the function $p \mapsto H(x, p)$. Thanks to the convexity of H, this implies (see Theorem B.2) that $L(x, \xi) = \xi \cdot q - H(x, q)$. Consequently, we get

$$L(x, \xi) \leq \delta R + \max_{\overline{\Omega} \times \overline{B}_R} |H|.$$

Thus we have the desired inequality with $C_0 = \delta R + \max_{\overline{\Omega} \times \overline{B}_R} |H|$. $\square$

For later convenience, we formulate the following lemma, whose proof is left to the reader.

Lemma 5.3. *For each $i \in \mathbb{N}$ define the function L_i on $\overline{\Omega} \times \mathbb{R}^n$ by*

$$L_i(x, \xi) = \max_{p \in \overline{B}_i} (\xi \cdot p - H(x, p)).$$

Then $L_i \in \mathrm{UC}(\overline{\Omega} \times \mathbb{R}^n)$,

$$L_i(x, \xi) \le L_{i+1}(x, \xi) \le L(x, \xi) \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n \text{ and } i \in \mathbb{N},$$

and for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$,

$$L_i(x, \xi) \to L(x, \xi) \quad \text{as } i \to \infty.$$

The following lemma is a consequence of the Dunford–Pettis theorem.

Lemma 5.4. *Let $J = [a, b]$, with $-\infty < a < b < \infty$. Let $\{f_j\}_{j \in \mathbb{N}} \subset L^1(J, \mathbb{R}^m)$ be uniformly integrable in J. That is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable $E \subset J$ and $j \in \mathbb{N}$, we have*

$$\int_E |f_j(t)| \mathrm{d}t < \varepsilon \quad \text{if } |E| < \delta,$$

where $|E|$ denotes the Lebesgue measure of E. Then $\{f_j\}$ has a subsequence which converges weakly in $L^1(J, \mathbb{R}^m)$.

See Appendix A.5 for a proof of the above lemma.

Lemma 5.5. *Let $J = [0, T]$ with $T \in \mathbb{R}_+$, $(\eta, v) \in L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$, $i \in \mathbb{N}$ and $\varepsilon > 0$. Let $L_i \in \mathrm{UC}(\overline{\Omega} \times \mathbb{R}^n)$ be the function defined in Lemma 5.3. Assume that $\eta(s) \in \overline{\Omega}$ for all $s \in J$. Then there exists a function $q \in L^\infty(J, \mathbb{R}^n)$ such that for a.e. $s \in J$,*

$$q(s) \in \overline{B}_i \quad \text{and} \quad H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) \le -v(s) \cdot q(s) + \varepsilon.$$

Proof. Note that for each $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ there is a point $q = q(x, \xi) \in \overline{B}_i$ such that $L_i(x, \xi) = \xi \cdot q - H(x, q)$. By the continuity of the functions H and L_i, there exists a constant $r = r(x, \xi) > 0$ such that

$$L_i(y, z) + H(y, q) \le z \cdot q + \varepsilon \quad \text{for all } (y, z) \in (\overline{\Omega} \cap B_r(x)) \times B_r(\xi).$$

Hence, as $\overline{\Omega} \times \mathbb{R}^n$ is σ-compact, we may choose a sequence $\{(x_k, \xi_k, q_k, r_k)\}_{k \in \mathbb{N}} \subset \overline{\Omega} \times \mathbb{R}^n \times \overline{B}_i \times \mathbb{R}_+$ such that

$$\overline{\Omega} \times \mathbb{R}^n \subset \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k) \times B_{r_k}(\xi_k)$$

and for all $k \in \mathbb{N}$,

$$L_i(y,z) + H(y,q_k) \le z \cdot q_k + \varepsilon \quad \text{for all } (y,z) \in B_{r_k}(x_k) \times B_{r_k}(\xi_k).$$

Now we set $U_k = (\overline{\Omega} \cap B_{r_k}(x_k)) \times B_{r_k}(\xi_k)$ for $k \in \mathbb{N}$ and define the function $P : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(x,\xi) = q_k \quad \text{for all } (x,\xi) \in U_k \setminus \bigcup_{j<k} U_j \text{ and all } k \in \mathbb{N}.$$

It is clear that P is Borel measurable in $\overline{\Omega} \times \mathbb{R}^n$. Moreover we have $P(x,\xi) \in \overline{B}_i$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and

$$L_i(x,\xi) + H(x, P(x,\xi)) \le \xi \cdot P(x,\xi) + \varepsilon \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n. \tag{88}$$

We define the function $q \in L^\infty(J, \mathbb{R}^n)$ by setting $q(s) = P(\eta(s), -v(s))$. From (88), we see that $q(s) \in \overline{B}_i$ and

$$L_i(\eta(s), -v(s)) + H(\eta(s), q(s)) \le -v(s) \cdot q(s) + \varepsilon \quad \text{for a.e. } s \in J. \qquad \square$$

Lemma 5.6. *Let $J = [0, T]$ with $T \in \mathbb{R}_+$, $\varepsilon > 0$, $i \in \mathbb{N}$, $q \in L^\infty(J,\mathbb{R}^n)$ and $\eta \in C(J,\mathbb{R}^n)$ such that $\eta(s) \in \overline{\Omega}$ for all $s \in J$. Assume that $\|q\|_{L^\infty(J)} < i$. Let L_i be the function defined in Lemma 5.3. Then there exists a function $v \in L^\infty([0, T],\mathbb{R}^n)$ such that*

$$H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) < -v(s) \cdot q(s) + \varepsilon \quad \textit{for a.e. } s \in [0, T]. \tag{89}$$

Before going into the proof we remark that for any $x \in \overline{\Omega}$ the function $L_i(x,\cdot)$ is the convex conjugate of the function $\tilde{H}(x,\cdot)$ given by $\tilde{H}(x,p) = H(x,p)$ if $p \in \overline{B}_i$ and $\tilde{H}(x,p) = \infty$ otherwise.

Proof. The same construction as in the proof of Lemma 5.5, with the roles of H and L_i being exchanged, yields a measurable function $v : [0, T] \to \mathbb{R}^n$ for which (89) holds. Set $C = \max_{\overline{\Omega}\times\overline{B}_i} |H|$ and observe that

$$L_i(x,\xi) \ge i|\xi| - C \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

We combine this with (89), to get

$$\varepsilon + \|q\|_{L^\infty(J)}|v(s)| > i|v(s)| - 2C \quad \text{for a.e. } s \in J.$$

Hence,

$$\|v\|_{L^\infty(J)} \le \frac{\varepsilon + 2C}{i - \|q\|_{L^\infty(J)}}. \qquad \square$$

The following proposition concerns the lower semicontinuity of the functional

$$(\eta, v) \mapsto \int_0^T L(\eta(s), -v(s))\mathrm{d}s.$$

Theorem 5.3. *Let $J = [0, T]$ with $T \in \mathbb{R}_+$, $\{(\eta_k, v_k)\}_{k\in\mathbb{N}} \subset L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$ and $(\eta, v) \in L^\infty(J, \mathbb{R}^n) \times L^1(J, \mathbb{R}^n)$. Assume that $\eta_k(s) \in \overline{\Omega}$ for all $(s, k) \in J \times \mathbb{N}$ and that as $k \to \infty$,*

$$\begin{aligned} &\eta_k(s) \to \eta(s) && \textit{uniformly for } \ s \in J, \\ &v_k \to v && \textit{weakly in } \ L^1(J, \mathbb{R}^n). \end{aligned}$$

Let ψ be a function in $L^\infty(J, \mathbb{R})$ such that $\psi(s) \ge 0$ for a.e. $s \in J$. Then

$$\int_J \psi(s)L(\eta(s), -v(s))\mathrm{d}s \le \liminf_{k\to\infty} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s. \tag{90}$$

Proof. Fix any $i \in \mathbb{N}$. Due to Lemma 5.5, there is a function $q \in L^\infty(J, \mathbb{R}^n)$ such that $q(s) \in \overline{B}_i$ and

$$H(\eta(s), q(s)) + L_i(\eta(s), -v(s)) < -v(s) \cdot q(s) + \frac{1}{i} \quad \text{for a.e. } \ s \in J. \tag{91}$$

Note that for all $k \in \mathbb{N}$,

$$\begin{aligned} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s &\ge \int_J \psi(s)L_i(\eta_k(s), -v_k(s))\mathrm{d}s \\ &\ge \int_J \psi(s)[-v_k(s) \cdot q(s) - H(\eta_k(s), q(s))]\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} &\lim_{k\to\infty} \int_J \psi(s)[-v_k(s) \cdot q(s) - H(\eta_k(s), q(s))]\mathrm{d}s \\ &\quad = \int_J \psi(s)[-v(s) \cdot q(s) - H(\eta(s), q(s))]\mathrm{d}s. \end{aligned}$$

Hence, using (91), we get

$$\begin{aligned} \liminf_{k\to\infty} \int_J \psi(s)L(\eta_k(s), -v_k(s))\mathrm{d}s &\ge \int_J \psi(s)[-v(s) \cdot q(s) - H(\eta(s), q(s))]\mathrm{d}s \\ &\ge \int_J \psi(s)[L_i(\eta(s), -v(s)) - 1/i]\mathrm{d}s. \end{aligned}$$

By the monotone convergence theorem, we conclude that (90) holds. □

Corollary 5.1. *Under the hypotheses of the above theorem, let* $\{f_k\} \subset L^1(J, \mathbb{R})$ *be a sequence of functions converging weakly in* $L^1(J, \mathbb{R})$ *to* f*. Assume furthermore that for all* $k \in \mathbb{N}$*,*

$$L(\eta_k(s), -v_k(s)) \le f_k(s) \quad \textit{for a.e.} \ \ s \in J.$$

Then

$$L(\eta(s), -v(s)) \le f(s) \quad \textit{for a.e.} \ \ s \in J.$$

Proof. Set $E = \{s \in J \ : \ L(\eta(s), -v(s)) > f(s)\}$. By Theorem 5.3, we deduce that

$$\begin{aligned} 0 &\ge \liminf_{k\to\infty} \int_J \mathbf{1}_E(s)[L(\eta_k(s), -v_k(s)) - f_k(s)]\mathrm{d}s \\ &\ge \int_J \mathbf{1}_E(s)[L(\eta(s), -v(s)) - f(s)]\mathrm{d}s \\ &= \int_J [L(\eta(s), -v(s)) - f(s)]_+ \mathrm{d}s, \end{aligned}$$

where $[\cdots]_+$ denotes the positive part of $[\cdots]$. Thus we see that $L(\eta(s), -v(s)) \le f(s)$ for a.e. $s \in J$. □

Lemma 5.7. *Let* $J = [0, T]$*, with* $T \in \mathbb{R}_+$*, and* $q \in C(\overline{\Omega} \times J, \mathbb{R}^n)$*. Let* $x \in \overline{\Omega}$*. Then there exists a triple* $(\eta, v, l) \in \mathrm{SP}(x)$ *such that*

$$H(\eta(s),\, q(\eta(s), s)) + L(\eta(s),\, -v(s)) = -v(s) \cdot q(\eta(s), s) \quad \textit{for a.e.}\ s \in J.$$

Proof. Fix $k \in \mathbb{N}$. Set $\delta = T/k$ and $s_j = (j-1)\delta$ for $j = 1, 2, \ldots, k+1$. We define inductively a sequence $\{(x_j, \eta_j, v_j, l_j)\}_{j=1}^k \subset \overline{\Omega} \times \mathrm{SP}$. We set $x_1 = x$ and choose a $\xi_1 \in \mathbb{R}^n$ so that

$$H(x_1, q(x_1, 0)) + L(x_1, -\xi_1) \le -\xi_1 \cdot q(x_1, 0) + 1/k.$$

Set $v_1(s) = \xi_1$ for $s \ge 0$ and choose a pair $(\eta_1, l_1) \in \mathrm{Lip}(\overline{\mathbb{R}}_+,\, \overline{\Omega}) \times L^\infty(\mathbb{R}_+,\, \mathbb{R})$ so that $(\eta_1, v_1, l_1) \in \mathrm{SP}(x_1)$. In fact, Theorem 5.2 guarantees the existence of such a pair.

We argue by induction and now suppose that $k \ge 2$ and we are given (x_i, η_i, v_i, l_i) for all $i = 1, \ldots, j-1$ and some $2 \le j \le k$. Then set $x_j = \eta_{j-1}(\delta)$, choose a $\xi_j \in \mathbb{R}^n$ so that

$$H(x_j, q(x_j, s_j)) + L(x_j, -\xi_j) \le -\xi_j \cdot q(x_j, s_j) + 1/k, \tag{92}$$

set $v_j(s) = \xi_j$ for $s \ge 0$, and select a pair $(\eta_j, l_j) \in \mathrm{Lip}(\overline{\mathbb{R}}_+, \overline{\Omega}) \times L^\infty(\mathbb{R}_+, \mathbb{R})$ so that $(\eta_j, v_j, l_j) \in \mathrm{SP}(x_j)$. Thus, by induction, we can select a sequence

$\{(x_j, \eta_j, v_j, l_j)\}_{j=1}^{k} \subset \overline{\Omega} \times \mathrm{SP}$ such that $x_1 = \eta_1(0)$, $x_j = \eta_{j-1}(\delta) = \eta_j(0)$ for $j = 2, \ldots, k$ and for each $j = 1, 2, \ldots, k$, (92) holds with $\xi_j = v_j(s)$ for all $s \geq 0$. We set $\alpha_j = (\eta_j, v_j, l_j)$ for $j = 1, \ldots, k$.

Note that the choice of x_j, η_j, v_j, l_j, with $j = 1, \ldots, k$, depends on k, which is not explicit in our notation. We define $\bar{\alpha}_k = (\bar{\eta}_k, \bar{v}_k, \bar{l}_k) \in \mathrm{SP}(x)$ by setting

$$\bar{\alpha}_k(s) = \alpha_j(s - s_j) \quad \text{for } s \in [s_j, s_{j+1}) \text{ and } j = 1, \ldots, k.$$

and

$$\bar{\alpha}_k(s) = (\eta_k(\delta), 0, 0) \quad \text{for } s \geq s_{k+1} = T.$$

Also, we define $\bar{x}_k, \bar{q}_k \in L^\infty(J, \mathbb{R}^n)$ by

$$\bar{x}_k(s) = x_j \quad \text{and} \quad \bar{q}_k(s) = q(x_j, s_j) \quad \text{for } s \in [s_j, s_{j+1}) \text{ and } j = 1, \ldots, k.$$

Now we observe by (92) that for all $j = 1, \ldots, k$,

$$L(x_j, -\xi_j) \leq |\xi_j| R + \max_{\overline{\Omega} \times \overline{B}_R} |H| + 1,$$

where $R > 0$ is such a constant that $R \geq \max_{\overline{\Omega} \times J} |q|$. Combining this estimate with Lemma 5.1, we see that there is a constant $C_1 > 0$, independent of k, such that

$$\max_{s \geq 0} |\bar{v}_k(s)| = \max_{1 \leq j \leq k} |\xi_j| \leq C_1.$$

By Proposition 5.2, we find a constant $C_2 > 0$, independent of k, such that $\|\dot{\bar{\eta}}_k\|_{L^\infty(\mathbb{R}_+)} \vee \|\bar{l}_k\|_{L^\infty(\mathbb{R}_+)} \leq C_2$.

We may invoke standard compactness theorems, to find a triple $(\eta, v, l) \in \mathrm{Lip}(J, \mathbb{R}^n) \times L^\infty(J, \mathbb{R}^{n+1})$ and a subsequence of $\{(\bar{\eta}_k, \bar{v}_k, \bar{l}_k)\}_{k \in \mathbb{N}}$, which will be denoted again by the same symbol, so that for every $0 < S < \infty$, as $k \to \infty$,

$$\bar{\eta}_k \to \eta \quad \text{uniformly on } [0, S],$$
$$(\dot{\bar{\eta}}_k, \bar{v}_k, \bar{l}_k) \to (\dot{\eta}, v, l) \quad \text{weakly-star in } L^\infty([0, S], \mathbb{R}^{2n+1}).$$

By Proposition 5.1, we see that $(\eta, v, l) \in \mathrm{SP}(x)$. It follows as well that $\bar{x}_k(s) \to \eta(s)$ and $\bar{q}_k(s) \to q(\eta(s), s)$ uniformly for $s \in J$ as $k \to \infty$.

Now, the inequalities (92), $1 \leq j \leq k$, can be rewritten as

$$L(\bar{x}_k(s), -\bar{v}_k(s)) \leq -\bar{v}_k(s) \cdot \bar{q}_k(s) - H(\bar{x}_k(s), \bar{q}_k(s)) + 1/k \quad \text{for all } s \in [0, T).$$

It is obvious to see that the sequence of functions

$$-\bar{v}_k(s) \cdot q_k(s) + 1/k - H(\bar{x}_k(s), \bar{q}_k(s))$$

on J converges weakly-star in $L^\infty(J,\mathbb{R})$ to the function

$$-v(s)\cdot q(\eta(s),s) - H(\eta(s),q(\eta(s),s)).$$

Hence, by Corollary 5.1, we conclude that

$$H(\eta(s),q(\eta(s),s)) + L(\eta(s),-v(s)) \le -v(s)\cdot q(\eta(s),s) \quad \text{for a.e. } s\in J,$$

which implies the desired equality. □

Theorem 5.4. *Let $J=[0,\,T]$, with $T\in\mathbb{R}_+$, and $\{(\eta_k,\,v_k,\,l_k)\}_{k\in\mathbb{N}}\subset \mathrm{SP}$. Assume that there is a constant $C>0$, independent of $k\in\mathbb{N}$, such that*

$$\mathscr{L}(T,\eta_k,v_k,l_k)\le C \quad \textit{for all } k\in\mathbb{N}.$$

Then there exists a triple $(\eta,\,v,\,l)\in \mathrm{SP}$ such that

$$\mathscr{L}(T,\eta,v,l)\le \liminf_{k\to\infty}\mathscr{L}(T,\eta_k,v_k,l_k).$$

Moreover, there is a subsequence $\{(\eta_{k_j},\,v_{k_j},\,l_{k_j})\}_{j\in\mathbb{N}}$ of $\{(\eta_k,\,v_k,\,l_k)\}$ such that as $j\to\infty$,

$$\eta_{k_j}(s)\to\eta(s) \quad \textit{uniformly on } J,$$
$$(\dot\eta_{k_j},\,v_{k_j},\,l_{k_j})\to(\dot\eta,\,v,\,l) \quad \textit{weakly in } L^1(J,\mathbb{R}^{2n+1}).$$

Proof. We may assume without loss of generality that $\eta_k(t)=\eta_k(T)$, $v_k(t)=0$ and $l_k(t)=0$ for all $t\ge T$ and all $k\in\mathbb{N}$.

According to Proposition 5.2, there is a constant $C_0>0$ such that for any $(\eta,\,v,\,l)\in\mathrm{SP}$, $|\dot\eta(t)|\vee|l(t)|\le C_0|v(t)|$ for a.e. $t\ge 0$. Note by Lemma 5.1 that for each $A>0$ there is a constant $C_A>0$ such that $L(x,\xi)\ge A|\xi|-C_A$ for all $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$. From this lower bound of L, it is obvious that for all $(x,\xi,r)\in\partial\Omega\times\mathbb{R}^n\times\overline{\mathbb{R}}_+$, if $r\le C_0|\xi|$, then

$$L(x,\,\xi)+g(x)r\ge\left(A-C_0\max_{\partial\Omega}|g|\right)|\xi|-C_A, \tag{93}$$

which ensures that there is a constant $C_1>0$ such that for $(\eta,\,v,\,l)\in\mathrm{SP}$,

$$L(\eta(s),-v(s))+g(\eta(s))l(s)+C_1\ge 0 \quad \text{for a.e. } s\ge 0. \tag{94}$$

Set

$$\Lambda=\liminf_{k\to\infty}\mathscr{L}(T,\eta_k,v_k,l_k),$$

and note by (94) that $-C_1 T \le \Lambda \le C$. We may choose a subsequence $\{(\eta_{k_j}, v_{k_j}, l_{k_j})\}_{j\in\mathbb{N}}$ of $\{(\eta_k, v_k, l_k)\}$ so that

$$\Lambda = \lim_{j\to\infty} \mathscr{L}(T, \eta_{k_j}, v_{k_j}, l_{k_j}).$$

Using (94), we obtain for any measurable $E \subset [0,\ T]$,

$$\begin{aligned}\int_E \big(L(\eta_k(s), -v_k(s)) + g(\eta_k(s))l_k(s) + C_1\big)\,\mathrm{d}s \\ \le \int_0^T \big(L(\eta_k(s), -v_k(s)) + g(\eta_k(s))l_k(s) + C_1\big)\,\mathrm{d}s \le C + C_1 T.\end{aligned}$$

This together with (93) yields

$$\Big(A - C_0 \max_{\partial\Omega} |g|\Big) \int_E |v_k(s)|\,\mathrm{d}s \le C_A|E| + C + C_1 T \quad \text{for all } A > 0.$$

This shows that the sequence $\{v_k\}$ is uniformly integrable on $[0,\ T]$. Since $|\dot\eta_k(s)| \vee |l_k(s)| \le C_0|v_k(s)|$ for a.e. $s \ge 0$ and $v_k(s) = 0$ for all $s > T$, we see easily that the sequence $\{(\dot\eta_k, v_k, l_k)\}$ is uniformly integrable on $\overline{\mathbb{R}}_+$.

Due to Lemma 5.4, we may assume by reselecting the subsequence $\{(\eta_{k_j}, v_{k_j}, l_{k_j})\}$ if necessary that as $j \to \infty$,

$$(\dot\eta_{k_j}, v_{k_j}, l_{k_j}) \to (w, v, l) \quad \text{weakly in } L^1([0,\ S], \mathbb{R}^{2n+1})$$

for every $S > 0$ and some $(w, v, l) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+,\ \mathbb{R}^{2n+1})$. We may also assume that $\eta_{k_j}(0) \to x$ as $j \to \infty$ for some $x \in \overline{\Omega}$. By Proposition 5.1, if we set $\eta(s) = x + \int_0^s w(r)\mathrm{d}r$ for $s \ge 0$, then $(\eta, v, l) \in \mathrm{SP}(x)$ and, as $j \to \infty$,

$$\eta_{k_j}(s) \to \eta(s) \quad \text{locally uniformly on } \overline{\mathbb{R}}_+.$$

We apply Theorem 5.3, with the function $\psi(s) \equiv 1$, to find that

$$\int_J L(\eta(s), -v(s))\mathrm{d}s \le \liminf_{j\to\infty} \int_J L(\eta_{k_j}(s), -v_{k_j}(s))\mathrm{d}s.$$

Consequently, we have

$$\mathscr{L}(T, \eta, v, l) \le \liminf_{j\to\infty} \mathscr{L}(T, \eta_{k_j}, v_{k_j}, l_{k_j}) = \Lambda,$$

which completes the proof. □

5.4 Value Function II

Theorem 5.5. *Let $u \in \mathrm{UC}(\overline{\Omega} \times \overline{\mathbb{R}}_+)$ be the viscosity solution of* (ENP)–(ID). *Then $V = u$ in $\overline{\Omega} \times \mathbb{R}_+$.*

This is a version of classical observations on the value functions in optimal control, and, in this regard, we refer for instance to [43, 45]. The above theorem has been established in [39]. The above theorem gives a variational formula for the unique solution of (ENP)–(ID). This variational formula is sometimes called the Lax–Oleinik formula.

For the proof of Theorem 5.5, we need the following three lemmas.

Lemma 5.8. *Let $U \subset \mathbb{R}^n$ be an open set and $J = [a, b]$ a finite subinterval of $\overline{\mathbb{R}}_+$. Let $\psi \in C^1((\overline{U} \cap \overline{\Omega}) \times J)$ and assume that*

$$\psi_t(x,t) + H(x, D_x\psi(x,t)) \le 0 \qquad \text{for all } (x,t) \in (U \cap \Omega) \times J, \tag{95}$$

$$\frac{\partial \psi}{\partial \gamma}(x,t) \le g(x) \qquad \text{for all } (x,t) \in (U \cap \partial\Omega) \times J, \tag{96}$$

$$\psi(x,t) \le V(x,t) \qquad \text{for all } (x,t) \in (\partial U \cap \overline{\Omega}) \times J, \tag{97}$$

$$\psi(x,a) \le V(x,a) \qquad \text{for all } x \in \overline{U} \cap \overline{\Omega}. \tag{98}$$

Then $\psi \le V$ in $(U \cap \overline{\Omega}) \times J$.

We note that the following inclusion holds: $\partial(U \cap \overline{\Omega}) \subset [\partial U \cap \overline{\Omega}] \cup (U \cap \partial\Omega)$.

Proof. Let $(x,t) \in (U \cap \overline{\Omega}) \times J$. Define the mapping $\tau : \mathrm{SP}(x) \to [0, t-a]$ by

$$\tau(\eta, v, l) = \inf\{s \ge 0 : \eta(s) \notin U\} \wedge (t-a).$$

It is clear that τ is nonanticipating. Let $\alpha = (\eta, v, l) \in \mathrm{SP}(x)$, and observe that $\eta(s) \in U$ for all $s \in [0, \tau(\alpha))$ and that $\eta(\tau(\alpha)) \in \partial U$ if $\tau(\alpha) < t - a$. In particular, we find from (97) and (98) that

$$\psi(\eta(\tau(\alpha)), t - \tau(\alpha)) \le V(\eta(\tau(\alpha)), t - \tau(\alpha)). \tag{99}$$

Fix any $\alpha = (\eta, v, l) \in \mathrm{SP}(x)$. Note that

$$\begin{aligned}
&\psi(\eta(\tau(\alpha)), t-\tau(\alpha)) - \psi(x,t)\\
&\quad = \int_0^{\tau(\alpha)} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\eta(s), t-s)\mathrm{d}s\\
&\quad = \int_0^{\tau(\alpha)} \big(D_x\psi(\eta(s), t-s)\cdot\dot{\eta}(s) - \psi_t(\eta(s), t-s)\big)\mathrm{d}s\\
&\quad = \int_0^{\tau(\alpha)} \big(D_x\psi(\eta(s), t-s)\cdot(v(s) - l(s)\gamma(\eta(s))) - \psi_t(\eta(s), t-s)\big)\mathrm{d}s.
\end{aligned}$$

Now, using (95), (96) and (99), we get

$$\begin{aligned}
&\psi(x,t) - V(\eta(\tau(\alpha)), t-\tau(\alpha)) \\
&\quad \le \int_0^{\tau(\alpha)} \big(- D_x\psi(\eta(s), t-s)\cdot v(s) + l(s) D_x\psi(\eta(s))\cdot\gamma(\eta(s)) \\
&\qquad + \psi_t(\eta(s), t-s)\big)\mathrm{d}s \\
&\quad \le \int_0^{\tau(\alpha)} \big(H(\eta(s), D_x\psi(\eta(s), t-s)) + L(\eta(s), -v(s)) + l(s)g(\eta(s)) \\
&\qquad + \psi_t(\eta(s), t-s)\big)\mathrm{d}s \\
&\le \mathscr{L}(\tau(\alpha), \eta, v, l),
\end{aligned}$$

which immediately shows that

$$\psi(x,t) \le \inf\left(\mathscr{L}(\tau(\alpha), \eta, v, l) + V(\eta(\tau(\alpha)), t-\tau(\alpha))\right),$$

where the infimum is taken over all $\alpha = (\eta, v, l) \in \mathrm{SP}(x)$. Thus, by (86), we get $\psi(x,t) \le V(x,t)$. □

Lemma 5.9. *For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $V(x,t) \ge u_0(x) - \varepsilon - C_\varepsilon t$ for $(x,t) \in \overline{Q}$.*

Proof. Fix any $\varepsilon > 0$. According to the proof of Theorem 3.2, there are a function $f \in C^1(\overline{\Omega})$ and a constant $C > 0$ such that if we set $\psi(x,t) = f(x) - Ct$ for $(x,t) \in \overline{Q}$, then ψ is a classical subsolution of (ENP) and $u_0(x) \ge f(x) \ge u_0(x) - \varepsilon$ for all $x \in \overline{\Omega}$.

We apply Lemma 5.8, with $U = \mathbb{R}^n$, $a = 0$, arbitrary $b > 0$, to obtain

$$V(x,t) \ge \psi(x,t) \ge -\varepsilon + u_0(x) - Ct \quad \text{for all } (x,t) \in Q,$$

which completes the proof. □

Lemma 5.10. *There is a constant $C > 0$ such that $V(x,t) \le u_0(x) + Ct$ for $(x,t) \in Q$.*

Proof. Let $(x,t) \in Q$. Set $\eta(s) = x$, $v(s) = 0$ and $l(s) = 0$ for $s \ge 0$. Then $(\eta, v, l) \in \mathrm{SP}(x)$. Hence, we have

$$V(x,t) \le u_0(x) + \int_0^t L(x,0)\mathrm{d}s = u_0(x) + tL(x,0) \le u_0(x) - t\min_{p\in\mathbb{R}^n} H(x,p).$$

Setting $C = -\min_{\overline{\Omega}\times\mathbb{R}^n} H$, we get $V(x,t) \le u_0(x) + Ct$. □

Proof (Theorem 5.5). By Lemmas 5.9 and 5.10, there is a constant $C > 0$ and for each $\varepsilon > 0$ a constant $C_\varepsilon > 0$ such that

$$-\varepsilon - C_\varepsilon t \le V(x,t) - u_0(x) \le Ct \quad \text{for all } (x,t) \in Q.$$

This shows that V is locally bounded on $\overline{Q}$ and that

$$\lim_{t \to 0+} V(x,t) = u_0(x) \quad \text{uniformly for } x \in \overline{\Omega}.$$

In particular, we have $V_*(x,0) = V^*(x,0) = u_0(x)$ for all $x \in \overline{\Omega}$.

We next prove that V is a subsolution of (ENP). Let $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^1(\overline{Q})$. Assume that $V^* - \phi$ attains a strict maximum at $(\hat{x},\hat{t})$. We want to show that if $\hat{x} \in \Omega$, then

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) \le 0,$$

and if $\hat{x} \in \partial\Omega$, then either

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) \le 0 \quad \text{or} \quad \gamma(\hat{x}) \cdot D_x\phi(\hat{x},\hat{t}) \le g(\hat{x}).$$

We argue by contradiction and thus suppose that

$$\phi_t(\hat{x},\hat{t}) + H(\hat{x}, D_x\phi(\hat{x},\hat{t})) > 0$$

and furthermore

$$\gamma(\hat{x}) \cdot D_x\phi(\hat{x},\hat{t}) > g(\hat{x}) \quad \text{if } \hat{x} \in \partial\Omega.$$

By continuity, we may choose a constant $r \in (0,\hat{t}\,)$ so that

$$\phi_t(x,t) + H(x, D_x\phi(x,t)) > 0 \quad \text{for all } (x,t) \in (\overline{B}_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}, \tag{100}$$

where $\hat{J} = [\hat{t} - r,\ \hat{t} + r]$, and

$$\gamma(x) \cdot D_x\phi(x,t) > g(x) \quad \text{for all } (x,t) \in (\overline{B}_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \tag{101}$$

(Of course, if $\hat{x} \in \Omega$, we can choose r so that $\overline{B}_r(\hat{x}) \cap \partial\Omega = \emptyset$.)

We may assume that $(V^* - \phi)(\hat{x},\hat{t}) = 0$. Set

$$B = \left(\left(\partial B_r(\hat{x}) \cap \overline{\Omega}\right) \times \hat{J}\right) \cup \left(\left(B_r(\hat{x}) \cap \overline{\Omega}\right) \times \{\hat{t} - r\}\right),$$

and $m = -\max_B(V^* - \phi)$. Note that $m > 0$ and $V(x,t) \le \phi(x,t) - m$ for $(x,t) \in B$.

We set $\varepsilon = r/2$. In view of the definition of V^*, we may choose a point $(\bar{x},\bar{t}) \in \overline{\Omega} \cap B_\varepsilon(\hat{x}) \times (\hat{t} - \varepsilon,\ \hat{t} + \varepsilon)$ so that $(V - \phi)(\bar{x},\bar{t}) > -m$. Set $a = \bar{t} - \hat{t} + r$, and note that $a > \varepsilon$ and $\operatorname{dist}(\bar{x}, \partial B_r(\hat{x})) > \varepsilon$. For each $\alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x})$ we set

$$S(\alpha) = \{s \ge 0 \,:\, \eta(s) \in \partial B_r(\hat{x})\} \quad \text{and} \quad \tau = a \wedge \inf S(\alpha).$$

Clearly, the mapping $\tau : \mathrm{SP}(\bar{x}) \to [0, a]$ is nonanticipating. Observe also that if $\tau(\alpha) < a$, then $\eta(\tau(\alpha)) \in \partial B_r(\hat{x})$ or, otherwise, $\bar{t} - \tau(\alpha) = \bar{t} - a = \hat{t} - r$. That is, we have

$$(\eta(\tau(\alpha)), \bar{t} - \tau(\alpha)) \in B \quad \text{for all } \alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x}). \tag{102}$$

Note as well that $(\eta(s), \bar{t} - s) \in \overline{B}_r(\hat{x}) \times \hat{J}$ for all $s \in [0, \tau(\alpha)]$.

We apply Lemma 5.7, with $J = [0, a]$ and the function $q(x, s) = D\phi(x, \bar{t} - s)$, to find a triple $\alpha = (\eta, v, l) \in \mathrm{SP}(\bar{x})$ such that for a.e. $s \in [0, a]$,

$$H(\eta(s), D_x\phi(\eta(s), \bar{t} - s)) + L(\eta(s), -v(s)) \leq -v(s) \cdot D_x\phi(\eta(s), \bar{t} - s) \tag{103}$$

For this α, we write $\tau = \tau(\alpha)$ for simplicity of notation. Using (102), by the dynamic programming principle, we have

$$\begin{aligned} \phi(\bar{x}, \bar{t}) &< V(\bar{x}, \bar{t}) + m \\ &\leq \mathscr{L}(\tau, \eta, v, l) + V(\tau, \bar{t} - \tau) + m \\ &\leq \mathscr{L}(\tau, \eta, v, l) + \phi(\eta(\tau), \bar{t} - \tau). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} 0 &< \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) + \frac{\mathrm{d}}{\mathrm{d}s}\phi(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &\leq \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) + D_x\phi(\eta(s), \bar{t} - s) \cdot \dot{\eta}(s) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &\leq \int_0^\tau \big(L(\eta(s), -v(s)) + g(\eta(s))l(s) \\ &\quad + D_x\phi(\eta(s), \bar{t} - s) \cdot (v(s) - l(s)\gamma(\eta(s)) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s. \end{aligned}$$

Now, using (103), (100) and (101), we get

$$\begin{aligned} 0 &< \int_0^\tau \big(- H(\eta(s), D_x\phi(\eta(s), \bar{t} - s)) + g(\eta(s))l(s) \\ &\quad - l(s)D_x\phi(\eta(s), \bar{t} - s) \cdot \gamma(\eta(s)) - \phi_t(\eta(s), \bar{t} - s)\big)\mathrm{d}s \\ &< \int_0^\tau l(s)\big(g(\eta(s)) - \gamma(\eta(s)) \cdot D_x\phi(\eta(s), \bar{t} - s)\big)\mathrm{d}s \leq 0, \end{aligned}$$

which is a contradiction. We thus conclude that V is a viscosity subsolution of (ENP).

Now, we turn to the proof of the supersolution property of V. Let $\phi \in C^1(\overline{Q})$ and $(\hat{x}, \hat{t}) \in \overline{\Omega} \times \mathbb{R}_+$. Assume that $V_* - \phi$ attains a strict minimum at $(\hat{x}, \hat{t})$. As usual, we assume furthermore that $\min_{\overline{Q}}(V_* - \phi) = 0$.

We need to show that if $\hat{x} \in \Omega$, then

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) \geq 0,$$

and if $\hat{x} \in \partial\Omega$, then

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) \geq 0 \quad \text{or} \quad \gamma(\hat{x}) \cdot D_x\phi(\hat{x}, \hat{t}) \geq g(\hat{x}).$$

We argue by contradiction and hence suppose that this were not the case. That is, we suppose that

$$\phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x\phi(\hat{x}, \hat{t})) < 0,$$

and moreover

$$\gamma(\hat{x}) \cdot D_x\phi(\hat{x}, \hat{t}) < g(\hat{x}) \quad \text{if } \hat{x} \in \partial\Omega.$$

We may choose a constant $r \in (0, \hat{t})$ so that

$$\phi_t(x, t) + H(x, D_x\phi(x, t)) < 0 \quad \text{for all } (x, t) \in (B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J},$$

where $\hat{J} = [\hat{t} - r, \hat{t} + r]$, and

$$\gamma(x) \cdot D_x\phi(x, t) < g(x) \quad \text{for all } (x, t) \in (B_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \tag{104}$$

We set

$$R = \Big((\partial B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}\Big) \cup \Big((B_r(\hat{x}) \cap \overline{\Omega}) \times \{\hat{t} - r\}\Big) \quad \text{and} \quad m = \min_R (V_* - \phi),$$

and define the function $\psi \in C^1((\overline{B}_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J})$ by $\psi(x, t) = \phi(x, t) + m$. Note that $m > 0$, $\inf_{(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}} (V_* - \psi) = -m < 0$ and $V(x, t) \geq \psi(x, t)$ for all $(x, t) \in R$. Observe moreover that

$$\begin{aligned} &\psi_t(x, t) + H(x, D_x\psi(x, t)) < 0 && \text{for all } (x, t) \in (B_r(\hat{x}) \cap \Omega) \times \hat{J} \\ &\frac{\partial\psi}{\partial\gamma}(x, t) < g(x) && \text{for all } (x, t) \in (B_r(\hat{x}) \cap \partial\Omega) \times \hat{J}. \end{aligned}$$

We invoke Lemma 5.8, to find that $\psi \leq V$ in $(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}$. This means that $\inf_{(B_r(\hat{x}) \cap \overline{\Omega}) \times \hat{J}} (V_* - \psi) \geq 0$. This contradiction shows that V is a viscosity supersolution of (ENP).

We apply Theorem 3.1 to V_*, u and V^*, to obtain $V^* \leq u \leq V_*$ in $\overline{Q}$, from which we conclude that $u = V$ in $\overline{Q}$. □

Our control problem always has an optimal "control" in SP:

Theorem 5.6. *Let* $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$. *Then there exists a triple* $(\eta, v, l) \in \mathrm{SP}(x)$ *such that*

$$V(x,t) = \mathscr{L}(t,\eta,v,l) + u_0(\eta(t)).$$

If, in addition, $V \in \mathrm{Lip}(\overline{\Omega} \times J, \mathbb{R})$*, with* J *being an interval of* $[0,\ t]$*, then the triple* $(\eta,\ v,\ l)$*, restricted to* $\tilde{J}_t := \{s \in [0,\ t]\ :\ t - s \in J\}$*, belongs to* $\mathrm{Lip}(\tilde{J}_t, \mathbb{R}^n) \times L^\infty(\tilde{J}_t, \mathbb{R}^{n+1})$.

Proof. We may choose a sequence $\{(\eta_k,\ v_k,\ l_k)\} \subset \mathrm{SP}(x)$ such that

$$V(x,t) = \lim_{k\to\infty} \mathscr{L}(t,\eta_k,v_k,l_k) + u_0(\eta_k(t)).$$

In view of Theorem 5.4, we may assume by replacing the sequence $\{(\eta_k,\ v_k,\ l_k)\}$ by a subsequence if needed that for some $(\eta,v,l) \in \mathrm{SP}(x)$, $\eta_k(s) \to \eta(s)$ uniformly on $[0,\ t]$ as $k \to \infty$ and

$$\mathscr{L}(t,\eta,v,l) \le \liminf_{k\to\infty} \mathscr{L}(t,\eta_k,v_k,l_k).$$

It is then easy to see that

$$V(x,t) = \mathscr{L}(r,\eta,v,l) + u_0(\eta(t)). \tag{105}$$

Note by (105) that for all $r \in (0,\ t)$,

$$V(x,t) \ge \mathscr{L}(r,\eta,v,l) + V(\eta(r),t-r),$$

which yields together with the dynamic programming principle

$$V(x,t) = \mathscr{L}(r,\eta,v,l) + V(\eta(r),t-r) \tag{106}$$

for all $r \in (0,\ t)$.

Now, we assume that $V \in \mathrm{Lip}(\overline{\Omega} \times J)$, where $J \subset [0,t]$ is an interval. Observe by (106) that for a.e. $r \in \tilde{J}_t$,

$$\begin{aligned} L(\eta(r),-v(r)) + l(r)g(\eta(r)) &= \lim_{\varepsilon\to 0} \frac{V(\eta(r),t-r) - V(\eta(r+\varepsilon),t-r-\varepsilon)}{\varepsilon} \\ &\le M(|\dot{\eta}(r)|^2+1)^{1/2} \le M(|\dot{\eta}(r)|+1), \end{aligned}$$

where $M > 0$ is a Lipschitz bound of the function V on $\overline{\Omega} \times J$. Let $C > 0$ be the constant from Proposition 5.2, so that $|\dot{\eta}(s)| \vee l(s) \le C|v(s)|$ for a.e. $s \ge 0$. By Lemma 5.1, for each $A > 0$, we may choose a constant $C_A > 0$ so that $L(y,\xi) \ge A|\xi| - C_A$ for $(y,\ \xi) \in \overline{\Omega} \times \mathbb{R}^n$. Accordingly, for any $A > 0$, we get

$$\begin{aligned} A|v(r)| &\le L(\eta(r),-v(r)) + C_A \le -l(r)g(\eta(r)) + M(|\dot{\eta}(r)|+1) + C_A \\ &\le C(\|g\|_{\infty,\partial\Omega} + M)|v(r)| + M + C_A \quad \text{for a.e. } r \in \tilde{J}_t. \end{aligned}$$

This implies that $v \in L^\infty(\tilde J_t, \mathbb{R}^n)$ and moreover that $\eta \in \mathrm{Lip}(\tilde J_t, \mathbb{R}^n)$ and $l \in L^\infty(\tilde J_t, \mathbb{R})$. The proof is complete. □

Corollary 5.2. *Let $u \in \mathrm{Lip}(\overline{\Omega})$ be a viscosity solution of* (SNP) *and $x \in \overline{\Omega}$. Then there exists a $(\eta, v, l) \in \mathrm{SP}(x)$ such that for all $t > 0$,*

$$u(x) - u(\eta(t)) = \mathscr{L}(t, \eta, v, l). \tag{107}$$

Proof. Note that the function $u(x)$, as a function of (x, t), is a viscosity solution of (ENP). In view of Theorem 5.6, we may choose a sequence $\{(\eta_j, v_j, l_j)\}_{j\in\mathbb{N}}$ so that $\eta_1(0) = x$, $\eta_{j+1}(0) = \eta_j(1)$ for all $j \in \mathbb{N}$ and

$$u(\eta_j(0)) - u(\eta_j(1)) = \mathscr{L}(1, \eta_j, v_j, l_j) \quad \text{for all } \ j \in \mathbb{N}.$$

We define $(\eta, v, l) \in \mathrm{SP}(x)$ by

$$(\eta(s), v(s), l(s)) = (\eta_j(s - j + 1), v_j(s - j + 1), l_j(s - j + 1))$$

for all $s \in [j - 1, \ j)$ and $j \in \mathbb{N}$. By using the dynamic programming principle, we see that (107) holds for all $t > 0$. □

5.5 Distance-Like Function d

We assume throughout this subsection that (A8) holds, and discuss a few aspects of weak KAM theory related to (SNP).

Proposition 5.4. *We have the variational formula for the function d introduced in Sect. 4.1: for all $x, y \in \overline{\Omega}$,*

$$d(x, y) = \inf\big\{\mathscr{L}(t, \eta, v, l) \ : \ t > 0, \ (\eta, v, l) \in \mathrm{SP}(x) \quad \textit{such that} \quad \eta(t) = y\big\}. \tag{108}$$

We use the following lemma for the proof of the above proposition.

Lemma 5.11. *Let $u_0 \in C(\overline{\Omega})$ and $u \in \mathrm{UC}(\overline{Q})$ be the viscosity solution of* (ENP)–(ID)*. Set*

$$v(x, t) = \inf_{r>0} u(x, t + r) \quad \textit{for} \ \ x \in \overline{Q}.$$

Then $v \in \mathrm{UC}(\overline{Q})$ and it is a viscosity solution of (ENP)*. Moreover, for each $t > 0$, the function $v(\cdot, t)$ is a viscosity subsolution of* (SNP)*.*

Proof. By assumption (A8), there is a viscosity subsolution ψ of (SNP). Note that the function $(x, t) \mapsto \psi(x)$ is a viscosity subsolution of (ENP) as well.

We may assume by adding a constant to ψ if needed that $\psi \le u_0$ in $\overline{\Omega}$. By Theorem 3.1, we have $u(x, t) \ge \psi(x) > -\infty$ for all $(x, t) \in \overline{Q}$. Since $u \in \mathrm{UC}(\overline{Q})$, we see immediately that $v \in \mathrm{UC}(\overline{Q})$. Applying a version for (ENP) of Theorem 4.4, which can be proved based on Theorem D.2, to the collection of viscosity solutions

$(x,t) \mapsto u(x,t+r)$, with $r > 0$, of (ENP), we find that v is a viscosity subsolution of (ENP). Also, by Proposition 1.10 (its version for supersolutions), we see that v is a viscosity supersolution of (ENP). Thus, the function v is a viscosity solution of (ENP).

Next, note that for each $x \in \overline{\Omega}$, the function $v(x,\cdot)$ is nondecreasing in $\mathbb{R}_+$. Let $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^1(\overline{\Omega})$. Assume that the function $\overline{\Omega} \ni x \mapsto v(x,\hat{t}) - \phi(x)$ attains a strict maximum at $\hat{x}$. Let $\alpha > 0$ and consider the function

$$v(x,t) - \phi(x) - \alpha(t-\hat{t})^2 \quad \text{on } \overline{\Omega} \times [0,\, \hat{t}+1].$$

Let (x_α, t_α) be a maximum point of this function. It is easily seen that $(x_\alpha, t_\alpha) \to (\hat{x},\hat{t})$ as $\alpha \to \infty$. For sufficiently large α, we have $t_\alpha > 0$ and either

$$x_\alpha \in \partial\Omega \quad \text{and} \quad \gamma(x_\alpha)\cdot D\phi(x_\alpha) \le g(x_\alpha),$$

or

$$2\alpha(t_\alpha - \hat{t}) + H(x_\alpha, D\phi(x_\alpha)) \le 0.$$

By the monotonicity of $v(x,t)$ in t, we see easily that $2\alpha(t_\alpha - \hat{t}) \ge 0$. Hence, sending $\alpha \to \infty$, we conclude that the function $v(\cdot,\hat{t})$ is a viscosity subsolution of (SNP). □

Proof (Proposition 5.4). We write $W(x,y)$ for the right hand side of (108).

Fix any $y \in \overline{\Omega}$. For each $k \in \mathbb{N}$ let $u_k \in \mathrm{Lip}(\overline{Q})$ be the unique viscosity solution of (ENP)–(ID), with u_0 defined by $u_0(x) = k|x-y|$. By Theorem 5.5, we have the formula:

$$u_k(x,t) = \inf\big\{\mathscr{L}(t,\eta,v,l) + k|\eta(t)-y| \,:\, (\eta,v,l) \in \mathrm{SP}(x)\big\}.$$

It is then easy to see that

$$\inf_{t>0} u_k(x,t) \le W(x,y) \quad \text{for all } (x,k) \in \overline{\Omega}\times\mathbb{N}. \tag{109}$$

Since $d(\cdot,y) \in \mathrm{Lip}(\overline{\Omega})$, if k is sufficiently large, say $k \ge K$, we have $d(\cdot,y) \le k|x-y|$ for all $x \in \overline{\Omega}$. Noting that the function $(x,t) \mapsto d(x,y)$ is a viscosity subsolution of (ENP) and applying Theorem 3.1, we get $d(x,y) \le u_k(x,t)$ for all $(x,t) \in \overline{Q}$ if $k \ge K$. Combining this and (109), we find that $d(x,y) \le W(x,y)$ for all $x \in \overline{\Omega}$.

Next, we give an upper bound on W. According to Lemma 2.1, there exist a constant $C_1 > 0$ and a function $\tau : \overline{\Omega} \to \overline{\mathbb{R}}_+$ such that $\tau(x) \le C_1|x-y|$ for all $x \in \overline{\Omega}$ and, for each $x \in \overline{\Omega}$, there is a curve $\eta_x \in \mathrm{Lip}([0,\,\tau(x)])$ having the properties: $\eta_x(0) = x$, $\eta_x(\tau(x)) = y$, $\eta_x(s) \in \overline{\Omega}$ for all $s \in [0,\,\tau(x)]$ and $|\dot{\eta}_x(s)| \le 1$ for a.e. $s \in [0,\tau(x)]$. We fix such a function τ and a collection $\{\eta_x\}$ of curves. Thanks to Lemma 5.2, we may choose constants $\delta > 0$ and $C_0 > 0$ such that

$$L(x,\xi) \le C_0 \quad \text{for all } (x,\xi) \in \overline{\Omega}\times\overline{B}_\delta.$$

Fix any $x \in \overline{\Omega} \setminus \{y\}$ and define $(\eta, v, l) \in \mathrm{SP}(x)$ by setting $\eta(s) = \eta_x(\delta s)$ for $s \in [0,\ \tau(x)/\delta]$, $\eta(s) = y$ for $s > \tau(x)/\delta$ and $(v(s),\ l(s)) = (\dot\eta(s), 0)$ for $s \in \overline{\mathbb{R}}_+$. Observe that

$$\begin{aligned}\mathscr{L}(\tau(x)/\delta, \eta, v, l) &= \int_0^{\tau(x)/\delta} L(\eta_x(\delta s),\ \delta\dot\eta_x(\delta s))\mathrm{d}s \\ &= \delta^{-1}\int_0^{\tau(x)} L(\eta_x(s), -\delta\dot\eta_x(s))\mathrm{d}s \\ &\le \delta^{-1}C_0\tau(x) \le \delta^{-1}C_0C_1|x-y|,\end{aligned}$$

which yields

$$W(x, y) \le \delta^{-1}C_0C_1|x-y|. \tag{110}$$

We define the function $w : \overline{Q} \to \mathbb{R}$ by

$$w(x,t) = \inf\left\{\mathscr{L}(r, \eta, v, l)\ :\ r > t,\ (\eta, v, l) \in \mathrm{SP}(x) \quad \text{such that} \quad \eta(r) = y\right\}.$$

It is clear by the above definition that

$$W(x, y) = \inf_{t>0} w(x,t) \quad \text{for all } x \in \overline{\Omega}. \tag{111}$$

Also, the dynamic programming principle yields

$$w(x,t) = \inf\left\{\mathscr{L}(t, \eta, v, l) + W(\eta(t), y)\ :\ (\eta, v, l) \in \mathrm{SP}(x)\right\}.$$

(We leave it to the reader to prove this identity.) In view of (110), we fix a $k \in \mathbb{N}$ so that $\delta^{-1}C_0C_1 \le k$ and note that for all $(x,t) \in Q$,

$$w(x,t) \le \inf\left\{\mathscr{L}(t, \eta, v, l) + k|\eta(t) - y|\ :\ (\eta, v, l) \in \mathrm{SP}(x)\right\} = u_k(x,t).$$

Consequently, we have

$$\inf_{t>0} w(x,t) \le \inf_{t>0} u_k(x,t) \quad \text{for all}\ \ x \in \overline{\Omega},$$

which together with (111) yields

$$W(x, y) \le \inf_{t>0} u_k(x,t) \quad \text{for all}\ \ x \in \overline{\Omega}.$$

By Lemma 5.11, if we set $v(x) = \inf_{t>0} u_k(x,t)$ for $x \in \overline{\Omega}$, then $v \in C(\overline{\Omega})$ is a viscosity subsolution of (SNP). Moreover, since $v(x) \le u_k(x, 0) = k|x - y|$ for all $x \in \overline{\Omega}$, we have $v(y) \le 0$. Thus, we find that $v(x) \le v(y) + d(x, y) \le d(x, y)$ for all $x \in \overline{\Omega}$. We now conclude that $W(x, y) \le v(x) \le d(x, y)$ for all $x \in \overline{\Omega}$. The proof is complete. □

Proposition 5.5. *Let $y \in \overline{\Omega}$ and $\delta > 0$. Then we have $y \in \mathscr{A}$ if and only if*

$$\inf\left\{\mathscr{L}(t,\eta,v,l)\ :\ t>\delta,\ (\eta,v,l)\in \mathrm{SP}(y)\quad \textit{such that}\quad \eta(t)=y\right\}=0. \tag{112}$$

Proof. First of all, we define the function $u \in \mathrm{UC}(\overline{Q})$ as the viscosity solution of (ENP)–(ID), with $u_0 = d(\cdot, y)$. By Theorem 5.5, we have

$$u(x,t)=\inf\left\{\mathscr{L}(t,\eta,v,l)+d(\eta(t),y)\ :\ (\eta,v,l)\in \mathrm{SP}(x)\right\}\quad \text{for all } (x,t)\in Q.$$

We combine this formula and Proposition 5.4, to get

$$u(x,t)=\inf\left\{\mathscr{L}(r,\eta,v,l)\ : r>t,\ (\eta,v,l)\in \mathrm{SP}(x)\ \text{ such that }\ \eta(r)=y\right\} \\ \text{for all } (x,t)\in Q. \tag{113}$$

Now, we assume that $y \in \mathscr{A}$. The function $d(\cdot, y)$ is then a viscosity solution of (SNP) and u is a viscosity solution of (ENP)–(ID), with $u_0 = d(\cdot, y)$. Hence, by Theorem 3.1, we have $d(x, y) = u(x,t)$ for all $(x,t) \in \overline{Q}$. Thus,

$$0=d(y,y)=\inf\left\{\mathscr{L}(r,\eta,v,l)\ :\ r>t,(\eta,v,l)\in \mathrm{SP}(y)\quad \text{such that}\quad \eta(r)=y\right\} \\ \text{for all } t>0.$$

This shows that (112) is valid.

Now, we assume that (112) holds. This assumption and (113) show that $u(y,\delta) = 0$. Formula (113) shows as well that for each $x \in \overline{\Omega}$, the function $u(x,\cdot)$ is nondecreasing in $\overline{\mathbb{R}}_+$. In particular, we have $d(x,y) \le u(x,t)$ for all $(x,t) \in Q$. Let $p \in D_x^- d(x,y)|_{x=y}$. Then we have $(p,0) \in D^- u(y,\delta)$ and

$$\begin{cases} H(y,p)\ge 0 & \text{if } y\in\Omega,\\ \max\{H(y,p),\ \gamma(y)\cdot p-g(y)\}\ge 0 & \text{if } y\in\partial\Omega.\end{cases}$$

This shows that $d(\cdot, y)$ is a viscosity solution of (SNP). Hence, we have $y \in \mathscr{A}$. □

6 Large-Time Asymptotic Solutions

We discuss the large-time behavior of solutions of (ENP)–(ID) following [8, 38, 39].

There has been much interest in the large time behavior of solutions of Hamilton–Jacobi equations since Namah and Roquejoffre in [53] have first established a general convergence result for solutions of

$$u_t(x,t)+H(x,D_xu(x,t))=0\quad \text{in }\ (x,t)\in\Omega\times\mathbb{R}_+ \tag{1.2}$$

under (A5), (A6) and the assumptions

$$\begin{aligned} &H(x,p) \geq H(x,0) \quad \text{for all } (x,p) \in \Omega \times \mathbb{R}^n, \\ &\max_{\Omega} H(x,0) = 0, \end{aligned} \tag{114}$$

where Ω is a smooth compact n-dimensional manifold without boundary. Fathi in [26] has then established a similar convergence result but under different type hypotheses, where (114) replaced by a strict convexity of the Hamiltonian $H(x,p)$ in p, by the dynamical approach based on weak KAM theory [25, 27]. Barles and Souganidis have obtained in [3] more general results in the periodic setting (i.e., in the case where Ω is n-dimensional torus), for possibly non-convex Hamiltonians, by using a PDE-viscosity solutions approach, which does not depend on the variational formula for the solutions like the one in Theorem 5.5. We refer to [7] for a recent view on this approach.

The approach of Fathi has been later modified and refined by Roquejoffre [54], Davini and Siconolfi in [21], and others. The same asymptotic problem in the whole domain $\mathbb{R}^n$ has been investigated by Barles and Roquejoffre in [10], Fujita et al., Ichihara and the author in [30, 34–37] in various situations.

There has been as well a considerable interest in the large time asymptotic behavior of solutions of Hamilton–Jacobi equation with boundary conditions. The investigations in this direction are papers: Mitake [48] (the state-constraint boundary condition), Roquejoffre [54] (the Dirichlet boundary condition in the classical sense), Mitake [49, 50] (the Dirichlet boundary condition in the viscosity framework). More recent studies are due to Barles, Mitake and the author in [8, 9, 38], where the Neumann boundary conditions including the dynamical boundary conditions are treated. In [8, 9], the PDE-viscosity solutions approach of Barles–Souganidis is adapted to problems with boundary conditions.

Yokoyama et al. in [58] and Giga et al. in [32, 33] have obtained some results on the large time behavior of solutions of Hamilton–Jacobi equations with noncoercive Hamiltonian which is motivated by a crystal growth model.

We also refer to the articles [13, 54] and to [16, 51, 52] for the large time behavior of solutions, respectively, of time-dependent Hamilton–Jacobi equations and of weakly coupled systems of Hamilton–Jacobi equations.

As before, we assume throughout this section that hypotheses (A1)–(A7) hold and that $u_0 \in C(\overline{\Omega})$. Moreover, we assume that $c^{\#} = 0$. Throughout this section $u = u(x,t)$ denotes the viscosity solution of (ENP)–(ID).

We set

$$Z = \{(x,p) \in \overline{\Omega} \times \mathbb{R}^n \, : \, H(x,p) = 0\}.$$

(A9)$_\pm$ There exists a function $\omega_0 \in C([0,\infty))$ satisfying $\omega_0(r) > 0$ for all $r > 0$ such that if $(x,p) \in Z$, $\xi \in D_p^- H(x,p)$ and $q \in \mathbb{R}^n$, then

$$H(x, p+q) \geq \xi \cdot q + \omega_0((\xi \cdot q)_\pm).$$

The following proposition describes the long time behavior of solutions of (ENP)–(ID).

Theorem 6.1. *Assume that either* (A9)$_+$ *or* (A9)$_-$ *holds. Then there exists a viscosity solution* $w \in \mathrm{Lip}(\overline{\Omega})$ *of* (SNP) *for which*

$$\lim_{t\to\infty} u(x,t) = w(x) \quad \textit{uniformly on } \overline{\Omega}. \tag{115}$$

The following example is an adaptation of the one from Barles–Souganidis to the Neumann problem, which shows the necessity of a stronger condition like (A9)$_\pm$ beyond the convexity assumption (A7) in order to have the asymptotic behavior described in the above theorem.

Example 6.1. Let $n = 2$ and $\Omega = B_4$. Let $\eta, \zeta \in C^1(\overline{\mathbb{R}}_+)$ be functions such that $0 \le \eta(r) \le 1$ for all $r \in \overline{\mathbb{R}}_+$, $\eta(r) = 1$ for all $r \in [0, 1]$, $\eta(r) = 0$ for all $r \in [2,\infty)$, $\zeta(r) \ge 0$ for all $r \in \overline{\mathbb{R}}_+$, $\zeta(r) = 0$ for all $r \in [0,2] \cup [3,\infty)$ and $\zeta(r) > 0$ for all $r \in (2, 3)$. Fix a constant $M > 0$ so that $M \ge \|\zeta'\|_{\infty,\mathbb{R}_+}$. We consider the Hamiltonian $H : \overline{\Omega} \times \mathbb{R}^2$ given by

$$\begin{aligned} H(x,y,p,q) = {} & |-yp + xq + \zeta(r)| - \zeta(r) \\ & + \eta(r)\sqrt{p^2+q^2} + (1-\eta(r))\left(\left|\frac{x}{r}p + \frac{y}{r}q\right| - M\right)_+, \end{aligned}$$

where $r = r(x,y) := \sqrt{x^2+y^2}$. Let $u \in C^1(\overline{\Omega} \times \overline{\mathbb{R}}_+)$ be the function given by

$$u(x,y,t) = \zeta(r)\left(\frac{y}{r}\cos t - \frac{x}{r}\sin t\right),$$

where, as above, $r = \sqrt{x^2+y^2}$. It is easily checked that u is a classical solution of

$$\begin{cases} u_t(x,y,t) + H(x,y,u_x(x,y,t),u_y(x,y,t)) = 0 & \text{in } B_4 \times \mathbb{R}_+, \\ \nu(x,y) \cdot (u_x(x,y,t),u_y(x,y,t)) = 0 & \text{on } \partial B_4 \times \mathbb{R}_+, \end{cases}$$

where $\nu(x,y)$ denotes the outer unit normal at $(x,y) \in \partial B_4$. Note here that if we introduce the polar coordinate system

$$x = r\cos\theta, \quad y = r\sin\theta$$

and the new function

$$v(r,\theta,t) = u(r\cos\theta, r\sin\theta, t) \quad \text{for } (r,\theta,t) \in \mathbb{R}_+ \times \mathbb{R} \times \overline{\mathbb{R}}_+,$$

then the above Hamilton–Jacobi equation reads

$$v_t + \widetilde{H}(r,\theta,v_r,v_\theta) = 0,$$

where

$$\widetilde{H}(r,\theta,p_r,p_\theta) = |p_\theta+\zeta(r)|-\zeta(r)+\eta(r)\sqrt{p_r^2+\left(\frac{p_\theta}{r}\right)^2}+(1-\eta(r))\,(|p_r|-M)_+\,,$$

while the definition of u reads

$$v(r,\theta,t)=\zeta(r)\sin(\theta-t).$$

Note also that any constant function w on $\overline{B}_4$ is a classical solution of

$$\begin{cases} H(x,y,w_x(x,y,t),w_y(x,y,t))=0 & \text{in } B_4,\\ \nu(x,y)\cdot(w_x(x,y,t),w_y(x,y,t))=0 & \text{on } \partial B_4, \end{cases}$$

which implies that the eigenvalue $c^\#$ is zero.

It is clear that u does not have the asymptotic behavior (115). As is easily seen, the Hamiltonian H satisfies (A5)–(A7), but neither of (A9)$_\pm$.

6.1 Preliminaries to Asymptotic Solutions

According to Theorem 3.3 and Corollary 3.1, we know that $u\in \mathrm{BUC}(\overline{Q})$. We set

$$u_\infty(x)=\liminf_{t\to\infty} u(x,t) \quad \text{for all } x\in\overline{\Omega}.$$

Lemma 6.1. *The function u_∞ is a viscosity solution of* (SNP) *and* $u_\infty \in \mathrm{Lip}(\overline{\Omega})$.

Proof. Note that

$$u_\infty(x)=\lim_{t\to\infty}\inf\{u(x,t+r)\,:\,r>0\} \quad \text{for all } x\in\overline{\Omega}. \tag{116}$$

By Lemma 5.11, if we set

$$v(x,t)=\inf\{u(x,t+r)\,:\,r>0\} \quad \text{for } (x,t)\in\overline{Q},$$

then $v\in \mathrm{BUC}(\overline{Q})$ and it is a viscosity solution of (ENP). For each $x\in\overline{\Omega}$, the function $v(x,\cdot)$ is nondecreasing in $\overline{\mathbb{R}}_+$. Hence, by the Ascoli–Arzela theorem or Dini's lemma, we see that the convergence in (116) is uniform in $\overline{\Omega}$. By Proposition 1.9, we see that the function $u_\infty(x)$, as a function of (x,t), is a viscosity solution of (ENP), which means that u_∞ is a viscosity solution of (SNP). Finally, Proposition 1.14 guarantees that $u_\infty\in \mathrm{Lip}(\overline{\Omega})$. □

We introduce the following notation:

$$S = \{(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \ : \ \xi \in D_p^- H(x,p) \ \text{ for some } \ (x,p) \in Z\},$$
$$P(x,\xi) = \{p \in \mathbb{R}^n \ : \ \xi \in D_p^- H(x,p)\} \quad \text{for } \ (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Lemma 6.2. (i) $Z,\ S \subset \overline{\Omega} \times B_{R_0}$ *for some* $R_0 > 0$.

(ii) *Assume that* (A9)$_+$ *holds. Then there exist constants* $\delta > 0$ *and* $R_1 > 0$ *such that for any* $(x,\xi) \in S$ *and any* $\varepsilon \in (0,\delta)$, *we have* $P(x,(1+\varepsilon)\xi) \neq \emptyset$ *and* $P(x,(1+\varepsilon)\xi) \subset B_{R_1}$.

(iii) *Assume that* (A9)$_-$ *holds. Then there exist constants* $\delta > 0$ *and* $R_1 > 0$ *such that for any* $(x,\xi) \in S$ *and any* $\varepsilon \in (0,\delta)$, *we have* $P(x,(1-\varepsilon)\xi) \neq \emptyset$ *and* $P(x,(1-\varepsilon)\xi) \subset B_{R_1}$.

Proof. (i) It follows from coercivity (A6) that there exists a constant $R_1 > 0$ such that $Z \subset \mathbb{R}^n \times B_{R_1}$. Next, fix any $(x,\xi) \in S$. Then, by the definition of S, we may choose a point $p \in P(x,\xi)$ such that $(x,p) \in Z$. Note that $|p| < R_1$. By convexity (A7), we have

$$H(x,p') \geq H(x,p) + \xi \cdot (p'-p) \qquad \text{for all } p' \in \mathbb{R}^n.$$

Assuming that $\xi \neq 0$ and setting $p' = p + \xi/|\xi|$ in the above, we get

$$|\xi| = \xi \cdot (p'-p) \leq H(x,p') - H(x,p) < \sup_{\overline{\Omega} \times B_{R_1+1}} H - \inf_{\overline{\Omega} \times B_{R_1}} H.$$

We may choose a constant $R_2 > 0$ so that the right-hand side is less than R_2, and therefore $\xi \in B_{R_2}$. Setting $R_0 = \max\{R_1, R_2\}$, we conclude that $Z,\ S \subset \mathbb{R}^n \times B_{R_0}$.

(ii) By (i), there is a constant $R_0 > 0$ such that $Z, S \subset \overline{\Omega} \times B_{R_0}$. We set $\delta = \omega_0(1)$, where ω_0 is from (A9)$_+$. In view of coercivity (A6), replacing $R_0 > 0$ by a larger constant if necessary, we may assume that $H(x,p) \geq 1 + \omega_0(1)$ for all $(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_{R_0})$.

Fix any $(x,\xi) \in S$, $p \in P(x,\xi)$ and $\varepsilon \in (0,\delta)$. Note that $\xi, p \in B_{R_0}$. By (A9)$_+$, for all $x \in \mathbb{R}^n$ we have

$$H(x,q) \geq \xi \cdot (q-p) + \omega_0\left((\xi \cdot (q-p))_+\right).$$

We set $V := \{q \in \overline{B}_{2R_0}(p) \ : \ |\xi \cdot (q-p)| \leq 1\}$. Let $q \in V$ and observe the following: if $q \in \partial B_{2R_0}(p)$, which implies that $|q| \geq R_0$, then $H(x,q) \geq 1 + \omega_0(1) > 1 + \varepsilon \geq (1+\varepsilon)\xi \cdot (q-p)$. If $\xi \cdot (q-p) = 1$, then $H(x,q) \geq 1 + \omega_0(1) > 1 + \varepsilon = (1+\varepsilon)\xi \cdot (q-p)$. Also, if $\xi \cdot (q-p) = -1$, then $H(x,q) \geq \xi \cdot (q-p) > (1+\varepsilon)\xi \cdot (q-p)$. Accordingly, the function $G(q) := H(x,q) - (1+\varepsilon)\xi \cdot (q-p)$ on $\mathbb{R}^n$ is positive on ∂V while it vanishes at $q = p \in V$, and hence it attains a minimum over the set V at an interior point of V. Thus, $P(x,(1+\varepsilon)\xi) \neq \emptyset$. By the convexity of G, we see easily that $G(q) > 0$ for all $q \in \mathbb{R}^n \setminus V$ and conclude that $P(x,(1+\varepsilon)\xi) \subset B_{2R_0}$.

(iii) Let ω_0 be the function from $(\mathrm{A9})_-$. As before, we choose $R_0 > 0$ so that $Z, S \subset \overline{\Omega} \times B_{R_0}$ and $H(x,p) \geq 1 + \omega_0(1)$ for all $(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_{R_0})$, and set $\delta = \omega_0(1) \wedge 1$. Note that for all $x \in \mathbb{R}^n$,

$$H(x,q) \geq \xi \cdot (q-p) + \omega_0\left((\xi \cdot (q-p))_-\right).$$

Fix any $(x,\xi) \in S$, $p \in P(x,\xi)$ and $\varepsilon \in (0,\delta)$. Set $V := \{q \in \overline{B}_{2R_0}(p) : |\xi \cdot (q-p)| \leq 1\}$. Let $q \in V$ and observe the following: if $q \in \partial B_{2R_0}(p)$, then $H(x,q) \geq 1 + \omega_0(1) > 1 + \varepsilon \geq (1-\varepsilon)\xi \cdot (q-p)$. If $\xi \cdot (q-p) = -1$, then $H(x,q) \geq -1 + \omega_0(1) > -1 + \varepsilon = (1-\varepsilon)\xi \cdot (q-p)$. If $\xi \cdot (q-p) = 1$, then $H(x,q) \geq \xi \cdot (q-p) > (1-\varepsilon)\xi \cdot (q-p)$. As before, the function $G(q) := H(x,q) - (1-\varepsilon)\xi \cdot (q-p)$ attains a minimum over V at an interior point of V. Consequently, $P(x,(1-\varepsilon)\xi) \neq \emptyset$. Moreover, we get $P(x,(1-\varepsilon)\xi) \subset B_{2R_0}$. □

Lemma 6.3. *Assume that* $(\mathrm{A9})_+$ *(resp.,* $(\mathrm{A9})_-$*) holds. Then there exist a constant* $\delta_1 > 0$ *and a modulus* ω_1 *such that for any* $\varepsilon \in [0,\delta_1]$ *and* $(x,\xi) \in S$*,*

$$L(x,(1+\varepsilon)\,\xi) \leq (1+\varepsilon)L(x,\xi) + \varepsilon\,\omega_1(\varepsilon) \tag{117}$$

(resp.,

$$L(x,(1-\varepsilon)\,\xi) \leq (1-\varepsilon)L(x,\xi) + \varepsilon\,\omega_1(\varepsilon)\,). \tag{118}$$

Before going into the proof, we make the following observation: under the assumption that H, L are smooth, for any $(x,\xi) \in S$, if we set $p := D_\xi L(x,\xi)$, then

$$H(x,p) = 0,$$
$$p \cdot \xi = H(x,p) + L(x,\xi) = L(x,\xi),$$

and, as $\varepsilon \to 0$,

$$\begin{aligned} L(x,(1+\varepsilon)\xi) &= L(x,\xi) + \varepsilon p \cdot \xi + o(\varepsilon) \\ &= L(x,\xi) + \varepsilon L(x,\xi) + o(\varepsilon) = (1+\varepsilon)L(x,\xi) + o(\varepsilon). \end{aligned}$$

Proof. Assume that $(\mathrm{A9})_+$ holds. Let $R_0 > 0$, $R_1 > 0$ and $\delta > 0$ be the constants from Lemma 6.2. Fix any $(x,\xi) \in S$ and $\varepsilon \in [0,\delta)$. In view of Lemma 6.2, we may choose a $p_\varepsilon \in P(x,(1+\varepsilon)\xi)$. Then we have $|p_\varepsilon - p_0| < 2R_1$, $|\xi| < R_0$ and $|\xi \cdot (p_\varepsilon - p_0)| < 2R_0R_1$.

Note by $(\mathrm{A9})_+$ that

$$H(x,p_\varepsilon) \geq \xi \cdot (p_\varepsilon - p_0) + \omega_0\left((\xi \cdot (p_\varepsilon - p_0))_+\right).$$

Hence, we obtain

$$\begin{aligned} L(x,(1+\varepsilon)\,\xi) &= (1+\varepsilon)\,\xi\cdot p_\varepsilon - H(x,p_\varepsilon) \le (1+\varepsilon)\,\xi\cdot p_\varepsilon \\ &\quad - \xi\cdot(p_\varepsilon - p_0) - \omega_0\left((\xi\cdot(p_\varepsilon-p_0))_+\right) \\ &\le (1+\varepsilon)[\xi\cdot p_0 - H(x,p_0)] \\ &\quad + \varepsilon\,\xi\cdot(p_\varepsilon-p_0) - \omega_0\left((\xi\cdot(p_\varepsilon-p_0))_+\right) \\ &\le (1+\varepsilon)L(x,\xi) + \varepsilon \max_{0\le r\le 2R_0R_1}\left(r - \frac{1}{\varepsilon}\omega_0(r)\right). \end{aligned}$$

We define the function ω_1 on $[0,\infty)$ by setting $\omega_1(s) = \max_{0\le r\le 2R_0R_1}(r-\omega_0(r)/s)$ for $s>0$ and $\omega_1(0)=0$ and observe that $\omega_1\in C([0,\infty))$. We have also $L(x,(1+\varepsilon)\xi)\le(1+\varepsilon)L(x,\xi)+\varepsilon\omega_1(\varepsilon)$ for all $\varepsilon\in(0,\delta)$. Thus (117) holds with $\delta_1 := \delta/2$.

Next, assume that (A9)_ holds. Let $R_0>0$, $R_1>0$ and $\delta>0$ be the constants from Lemma 6.2. Fix any $(x,\xi)\in S$ and $\varepsilon\in[0,\delta)$.

As before, we may choose a $p_\varepsilon\in P(x,(1-\varepsilon)\xi)$, and observe that $|p_\varepsilon-p_0|<2R_1$, $|\xi|<R_0$ and $|\xi\cdot(p_\varepsilon-p_0)|<2R_0R_1$. Noting that

$$H(x,p_\varepsilon)\ge \xi\cdot(p_\varepsilon-p_0)+\omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right),$$

we obtain

$$\begin{aligned} L(x,(1-\varepsilon)\,\xi) &= (1-\varepsilon)\,\xi\cdot p_\varepsilon - H(x,p_\varepsilon) \le (1-\varepsilon)\,\xi\cdot p_\varepsilon \\ &\quad - \xi\cdot(p_\varepsilon - p_0) - \omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right) \\ &\le (1-\varepsilon)[\xi\cdot p_0 - H(x,p_0)] \\ &\quad - \varepsilon\,\xi\cdot(p_\varepsilon-p_0) - \omega_0\left((\xi\cdot(p_\varepsilon-p_0))_-\right) \\ &\le (1+\varepsilon)L(x,\xi) + \varepsilon \max_{0\le r\le 2R_0R_1}\left(r - \frac{1}{\varepsilon}\omega_0(r)\right). \end{aligned}$$

Setting $\omega_1(s) = \max_{0\le r\le 2R_0R_1}(r-\omega_0(r)/s)$ for $s>0$ and $\omega_1(0)=0$, we find a function $\omega_1\in C([0,\infty))$ vanishing at the origin for which $L(x,(1-\varepsilon)\xi)\le(1-\varepsilon)L(x,\xi)+\varepsilon\omega_1(\varepsilon)$ for all $\varepsilon\in(0,\delta)$. Thus (118) holds with $\delta_1:=\delta/2$. □

Theorem 6.2. *Let $u\in \mathrm{Lip}(\overline{\Omega})$ be a subsolution of* (SNP)*. Let $\eta\in \mathrm{AC}(\mathbb{R}_+,\mathbb{R}^n)$ be such that $\eta(t)\in\overline{\Omega}$ for all $t\in\mathbb{R}_+$. Set $\mathbb{R}_{+,b}=\{t\in\mathbb{R}_+ : \eta(t)\in\partial\Omega\}$. Then there exists a function $p\in L^\infty(\mathbb{R}_+,\mathbb{R}^n)$ such that*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u\circ\eta(t) = p(t)\cdot\dot\eta(t) & \text{for a.e. } t\in\mathbb{R}_+, \\ H(\eta(t),\ p(t))\le 0 & \text{for a.e. } t\in\mathbb{R}_+, \\ \gamma(\eta(t))\cdot p(t)\le g(\eta(t)) & \text{for a.e. } t\in\mathbb{R}_{+,b}. \end{cases}$$

Proof. According to Theorem 4.2, there is a collection $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ such that

$$\begin{cases} H(x, Du_\varepsilon(x)) \le \varepsilon & \text{for all } x \in \overline{\Omega}, \\ \dfrac{\partial u_\varepsilon}{\partial \gamma}(x) \le g(x) & \text{for all } x \in \partial\Omega, \\ \|u_\varepsilon - u\|_{\infty,\Omega} < \varepsilon, \\ \sup_{0<\varepsilon<1} \|Du_\varepsilon\|_{L^\infty(\Omega)} < \infty. \end{cases}$$

If we set $p_\varepsilon(t) = Du_\varepsilon \circ \eta(t)$ for all $t \in \overline{\mathbb{R}}_+$, then we have

$$\begin{cases} u_\varepsilon \circ \eta(t) - u_\varepsilon \circ \eta(0) = \displaystyle\int_0^t p_\varepsilon(s) \cdot \dot{\eta}(s)\mathrm{d}s & \text{for all } t \in \mathbb{R}_+, \\ H(\eta(t),\ p_\varepsilon(t)) \le \varepsilon \quad \text{for all } t \in \mathbb{R}_+, \\ \gamma(\eta(t)) \cdot p_\varepsilon(t) \le g(\eta(t)) \quad \text{for all } t \in \mathbb{R}_{+,\mathrm{b}}. \end{cases} \tag{119}$$

Since $\{p_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\mathbb{R}_+)$, there is a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converging to zero such that, as $j \to \infty$, the sequence $\{p_{\varepsilon_j}\}$ converges weakly-star in $L^\infty(\mathbb{R}_+)$ to some function $p \in L^\infty(\mathbb{R}_+)$. It is clear from (119) that

$$\begin{cases} u \circ \eta(t) - u \circ \eta(0) = \displaystyle\int_0^t p(s) \cdot \dot{\eta}(s)\mathrm{d}s \quad \text{for all } t \in \mathbb{R}_+, \\ \gamma(\eta(t)) \cdot p(t) \le g(\eta(t)) \quad \text{for a.e. } t \in \mathbb{R}_{+,\mathrm{b}}. \end{cases}$$

Now, we fix an $i \in \mathbb{N}$ so that $i > \|p\|_{L^\infty(\mathbb{R}_+)}$ and any $0 < T < \infty$, and set $J = [0,\ T]$. Using Lemma 5.6, for each $m \in \mathbb{N}$, we find a function $v_m \in L^\infty(J, \mathbb{R}^n)$ so that

$$H(\eta(s), p(s)) + L_i(\eta(s), -v_m(s)) < -v_m(s) \cdot p(s) + 1/m \quad \text{for a.e. } s \in J. \tag{120}$$

By the convex duality, we have

$$H(x, q) = \sup_{\xi\in\mathbb{R}^n} (\xi \cdot q - L_i(x, \xi)) \quad \text{for all } (x, q) \in \overline{\Omega} \times B_i.$$

(Note that $L_i(x, \cdot)$ is the convex conjugate of the function $H(x, \cdot) + \delta_{\overline{B}_i}$, where $\delta_{\overline{B}_i}(p) = 0$ if $p \in \overline{B}_i$ and $= \infty$ otherwise.) Hence, for any nonnegative function $\psi \in L^\infty(J, \mathbb{R})$ and any $(j, m) \in \mathbb{N}^2$, by (119) we get

$$\begin{aligned} \varepsilon_j \int_J \psi(s)\mathrm{d}s &\ge \int_J \psi(s) H(\eta(s), p_{\varepsilon_j}(s))\mathrm{d}s \\ &\ge \int_J \psi(s)[-v_m(s) \cdot p_{\varepsilon_j}(s) - L_i(\eta(s), -v_m(s))]\mathrm{d}s. \end{aligned}$$

Combining this observation with (120), after sending $j \to \infty$, we obtain

$$0 \geq \int_J \psi(s)(H(\eta(s), p(s)) - 1/m)\mathrm{d}s,$$

which implies that $H(\eta(s), p(s)) \leq 0$ for a.e. $s \in [0, T]$. Since $T > 0$ is arbitrary, we see that

$$H(\eta(s), p(s)) \leq 0 \quad \text{for a.e. } s \in \mathbb{R}_+.$$

The proof is complete. □

6.2 Proof of Convergence

This subsection is devoted to the proof of Theorem 6.1.

Proof (Theorem 6.1). It is enough to show that

$$\limsup_{t\to\infty} u(x,t) \leq u_\infty(x) \quad \text{for all } x \in \overline{\Omega}. \tag{121}$$

Indeed, once this is proved, it is obvious that $\lim_{t\to\infty} u(x,t) = u_\infty(x)$ for all $x \in \overline{\Omega}$, and moreover, since $u \in \mathrm{BUC}(Q)$, by the Ascoli–Arzela theorem, it follows that the convergence, $\lim_{t\to\infty} u(x,t) = u_\infty(x)$, is uniform in $\overline{\Omega}$.

Fix any $z \in \overline{\Omega}$. According to Lemma 6.1 and Corollary 5.2, we may choose a $(\eta, v, l) \in \mathrm{SP}(z)$ such that for all $t > 0$,

$$u_\infty(z) - u_\infty(\eta(t)) = \mathscr{L}(t, \eta, v, l). \tag{122}$$

Due to Theorem 6.2, there exists a function $q \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} u_\infty(\eta(s)) = q(s) \cdot \dot{\eta}(s) & \text{for a.e. } s \in \mathbb{R}_+, \\ H(\eta(s), q(s)) \leq 0 & \text{for a.e. } s \in \mathbb{R}_+, \\ \gamma(\eta(s)) \cdot q(s) \leq g(\eta(s)) & \text{for a.e. } s \in \mathbb{R}_{+,\mathrm{b}}, \end{cases} \tag{123}$$

where $\mathbb{R}_{+,\mathrm{b}} := \{s \in \mathbb{R}_+ \,:\, \eta(s) \in \partial\Omega\}$.

We now show that

$$\begin{cases} H(\eta(s), q(s)) = 0 & \text{for a.e. s } \in \mathbb{R}_+, \\ l(s)\gamma(\eta(s)) \cdot q(s) = l(s)g(\eta(s)) & \text{for a.e. } s \in \mathbb{R}_{+,\mathrm{b}}, \\ -q(s) \cdot v(s) = H(\eta(s), q(s)) + L(\eta(s), -v(s)) & \text{for a.e. s } \in \mathbb{R}_+. \end{cases} \tag{124}$$

We remark here that the last equality in (124) is equivalent to saying that

$$-v(s) \in D_p^- H(\eta(s),\, q(s)) \quad \text{for a.e. } s \in \mathbb{R}_+,$$

(or

$$q(s) \in D_\xi^- L(\eta(s),\, -v(s)) \quad \text{for a.e. } s \in \mathbb{R}_+.)$$

By differentiating (122), we get

$$-\frac{\mathrm{d}}{\mathrm{d}s} u_\infty(\eta(s)) = L(\eta(s), -v(s)) + l(s)g(\eta(s)) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

Combining this with (123), we calculate

$$\begin{aligned} 0 &= q(s)\cdot \dot\eta(s) + L(\eta(s), -v(s)) + l(s)g(\eta(s)) \\ &= q(s)\cdot (v(s) - l(s)\gamma(\eta(s))) + L(\eta(s), -v(s)) + l(s)g(\eta(s)) \\ &\geq -H(\eta(s), q(s)) - l(s)(q(s)\cdot\gamma(\eta(s)) - g(\eta(s))) \geq 0 \end{aligned}$$

for a.e. $s \in \mathbb{R}_+$, which guarantees that (124) holds.

Fix any $\varepsilon > 0$. We prove that there is a constant $\tau > 0$ and for each $x \in \overline{\Omega}$ a number $\sigma(x) \in [0,\, \tau]$ for which

$$u_\infty(x) + \varepsilon > u(x, \sigma(x)). \tag{125}$$

In view of the definition of u_∞, for each $x \in \overline{\Omega}$ there is a constant $t(x) > 0$ such that

$$u_\infty(x) + \varepsilon > u(x,\, t(x)).$$

By continuity, for each fixed $x \in \overline{\Omega}$, we can choose a constant $r(x) > 0$ so that

$$u_\infty(y) + \varepsilon > u(y,\, t(x)) \quad \text{for } y \in \overline{\Omega} \cap B_{r(x)}(x),$$

where $B_\rho(x) := \{y \in \mathbb{R}^n \,:\, |y - x| < \rho\}$. By the compactness of $\overline{\Omega}$, there is a finite sequence x_i, $i = 1, 2, \ldots, N$, such that

$$\overline{\Omega} \subset \bigcup_{1\leq i\leq N} B_{r(x_i)}(x_i),$$

That is, for any $y \in \overline{\Omega}$ there exists x_i, with $1 \leq i \leq N$, such that $y \in B_{r(x_i)}(x_i)$, which implies

$$u_\infty(y) + \varepsilon > u(y,\, t(x_i)).$$

Thus, setting

$$\tau = \max_{1\leq i\leq N} t(x_i),$$

we find that for each $x \in \overline{\Omega}$ there is a constant $\sigma(x) \in [0, \tau]$ such that (125) holds.

In what follows we fix $\tau > 0$ and $\sigma(x) \in [0, \tau]$ as above. Also, we choose a constant $\delta_1 > 0$ and a modulus ω_1 as in Lemma 6.3.

We divide our argument into two cases according to which hypothesis is valid, $(A9)_+$ or $(A9)_-$. We first argue under hypothesis $(A9)_+$. Choose a constant $T > \tau$ so that $\tau/(T - \tau) \le \delta_1$. Fix any $t \ge T$, and set $\theta = \sigma(\eta(t)) \in [0, \tau]$. We set $\delta = \theta/(t - \theta)$ and note that $\delta \le \tau/(t - \tau) \le \delta_1$. We define functions $\eta_\delta, v_\delta, l_\delta$ on $\overline{\mathbb{R}}_+$ by

$$\begin{aligned}\eta_\delta(s) &= \eta((1 + \delta)s),\\ v_\delta(s) &= (1 + \delta)v((1 + \delta)s),\\ l_\delta(s) &= (1 + \delta)l((1 + \delta)s),\end{aligned}$$

and note that $(\eta_\delta, v_\delta, l_\delta) \in \mathrm{SP}(z)$.

By (124) together with the remark after (124), we know that $H(\eta(s), q(s)) = 0$ and $-v(s) \in D_p^- H(\eta(s), q(s))$ for a.e. $s \in \mathbb{R}_+$. That is, $(\eta(s), -v(s)) \in S$ for a.e. $s \in \mathbb{R}_+$. Therefore, by (117), we get for a.e. $s \in \mathbb{R}_+$,

$$L(\eta_\delta(s), -v_\delta(s)) \le (1 + \delta)L\big(\eta((1 + \delta)s), -v((1 + \delta)s)\big) + \delta\omega_1(\delta).$$

Integrating this over $(0, t - \theta)$, making a change of variables in the integral and noting that $(1 + \delta)(t - \theta) = t$, we get

$$\begin{aligned}\int_0^{t-\theta} L(\eta_\delta(s), -v_\delta(s))\mathrm{d}s &\le \int_0^t L(\eta(s), -v(s))\mathrm{d}s + (t - \theta)\delta\omega_1(\delta)\\ &= \int_0^t L(\eta(s), -v(s))\mathrm{d}s + \theta\omega_1(\delta),\end{aligned}$$

as well as

$$\int_0^{t-\theta} l_\delta(s)g(\eta_\delta(s))\mathrm{d}s = \int_0^t l(s)g(\eta(s))\mathrm{d}s.$$

Moreover,

$$\begin{aligned}u(z, t) &\le \mathscr{L}(t - \theta, \eta_\delta, v_\delta, l_\delta) + u(\eta_\delta(t - \theta), \theta)\\ &\le \int_0^t \big(L(\eta(s), -v(s)) + l(s)g(\eta(s))\big)\mathrm{d}s + \theta\omega_1(\delta) + u\big(\eta(t), \sigma(\eta(t))\big)\\ &< u_\infty(z) - u_\infty(\eta(t)) + \tau\omega_1(\delta) + u_\infty(\eta(t)) + \varepsilon\\ &= u_\infty(z) + \tau\omega_1(\delta) + \varepsilon.\end{aligned}$$

Thus, recalling that $\delta \le \tau/(t - \tau)$, we get

$$u(z, t) \le u_\infty(z) + \tau\omega_1\Big(\frac{\tau}{t - \tau}\Big) + \varepsilon. \tag{126}$$

Next, we assume that (A9)$_-$ holds. We choose $T > \tau$ as before, and fix $t \geq T$. Set $\theta = \sigma(\eta(t-\tau)) \in [0, \tau]$ and $\delta = (\tau-\theta)/(t-\theta)$. Observe that $(1-\delta)(t-\theta) = t - \tau$ and $\delta \leq \tau/(t-\tau) \leq \delta_1$.

We set $\eta_\delta(s) = \eta((1-\delta)s)$, $v_\delta(s) = (1-\delta)v((1-\delta)s)$ and $l_\delta(s) = (1-\delta)l((1-\delta)s)$ for $s \in \overline{\mathbb{R}}_+$ and observe that $(\eta_\delta, v_\delta, l_\delta) \in \mathrm{SP}(z)$. As before, thanks to (118), we have

$$L(\eta_\delta(s), -v_\delta(s)) \leq (1-\delta)L(\eta((1-\delta)s), -v(1-\delta)s)) + \delta\omega_1(\delta) \quad \text{for a.e. } s \in \mathbb{R}_+.$$

Hence, we get

$$\begin{aligned}\int_0^{t-\theta} L(\eta_\delta(s), -v_\delta(s))\mathrm{d}s &\leq \int_0^{t-\tau} L(\eta(s), -v(s))\mathrm{d}s + (t-\theta)\delta\omega_1(\delta) \\ &= \int_0^{t-\tau} L(\eta(s), -v(s))\mathrm{d}s + (\tau-\theta)\omega_1(\delta),\end{aligned}$$

and

$$\int_0^{t-\theta} l_\delta(s)g(\eta_\delta(s))\mathrm{d}s = \int_0^{t-\tau} l(s)g(\eta(s))\mathrm{d}s.$$

Furthermore, we calculate

$$\begin{aligned}u(z,t) &\leq \mathscr{L}(t-\theta, \eta_\delta, v_\delta, l_\delta) + u(\eta_\delta(t-\theta), \theta) \\ &\leq \mathscr{L}(t-\tau, \eta, v, l) + \tau\omega_1(\delta) + u(\eta(t-\tau), \sigma(\eta(t-\tau))) \\ &< u_\infty(z) + \tau\omega_1(\delta) + \varepsilon.\end{aligned}$$

Thus, we get

$$u(z,t) \leq u_\infty(z) + \tau\omega_1\Big(\frac{\tau}{t-\tau}\Big) + \varepsilon,$$

From the above inequality and (126) we see that (121) is valid. □

6.3 Representation of the Asymptotic Solution u_∞

According to Theorem 6.1, if either (A9)$_+$ or (A9)$_-$ holds, then the solution $u(x,t)$ of (ENP)–(ID) converges to the function $u_\infty(x)$ in $C(\overline{\Omega})$ as $t \to \infty$, where the function u_∞ is given by

$$u_\infty(x) = \liminf_{t\to\infty} u(x,t) \quad \text{for } x \in \overline{\Omega}.$$

In this subsection, we *do not* assume (A9)$_+$ or (A9)$_-$ and give two characterizations of the function u_∞.

Let $\mathscr{S}^-$ and $\mathscr{S}$ denote the sets of all viscosity subsolutions of (SNP) and of all viscosity solutions of (SNP), respectively.

Theorem 6.3. *Set*

$$\mathscr{F}_1 = \{v \in \mathscr{S}^- \,:\, v \le u_0 \text{ in } \overline{\Omega}\},$$

$$u_0^- = \sup \mathscr{F}_1,$$

$$\mathscr{F}_2 = \{w \in \mathscr{S} \,:\, w \ge u_0^- \text{ in } \overline{\Omega}\}.$$

Then $u_\infty = \inf \mathscr{F}_2$.

Proof. By Proposition 1.10, we have $u_0^- \in \mathscr{S}^-$. It is clear that $u_0^- \le u_0$ in $\overline{\Omega}$. Hence, by Theorem 3.1 applied to the functions u_0^- and u, we get $u_0^-(x) \le u(x,t)$ for all $(x,t) \in Q$, which implies that $u_0^- \le u_\infty$ in $\overline{\Omega}$. This together with Lemma 6.1 ensures that $u_\infty \in \mathscr{F}_2$, which shows that $\inf \mathscr{F}_2 \le u_\infty$ in $\overline{\Omega}$.

Next, we set

$$u^-(x,t) = \inf_{r>0} u(x,t+r) \quad \text{for all } (x,t) \in \overline{Q}.$$

By Lemma 5.11, the function u^- is a solution of (ENP) and the function $u^-(\cdot,0)$ is a viscosity subsolution of (SNP). Also, it is clear that $u^-(x,0) \le u_0(x)$ for all $x \in \overline{\Omega}$, which implies that $u^-(\cdot,0) \le u_0^- \le \inf \mathscr{F}_2$ in $\overline{\Omega}$. We apply Theorem 3.1 to the functions u^- and $\inf \mathscr{F}_2$, to obtain $u^-(x,t) \le \inf \mathscr{F}_2(x)$ for all $(x,t) \in Q$, from which we get $u_\infty \le \inf \mathscr{F}_2$ in $\overline{\Omega}$, and conclude the proof. □

Let $d \,:\, \overline{\Omega}^2 \to \mathbb{R}$ and $\mathscr{A}$ denote the distance-like function and the Aubry set, respectively, as in Sect. 4.

Theorem 6.4. *We have the formula:*

$$u_\infty(x) = \inf\{d(x,y) + d(y,z) + u_0(z) \,:\, z \in \overline{\Omega},\ y \in \mathscr{A}\} \quad \textit{for all } x \in \overline{\Omega}.$$

Proof. We first show that

$$u_0^-(x) = \inf\{u_0(y) + d(x,y) \,:\, y \in \overline{\Omega}\} \quad \text{for all } x \in \overline{\Omega},$$

where u_0^- is the function defined in Theorem 6.3.

Let u_d^- denote the function given by the right hand side of the above formula. Since $u_0^- \in \mathscr{S}^-$, we have

$$u_0^-(x) - u_0^-(y) \le d(x,y) \quad \text{for all } x,y \in \overline{\Omega},$$

which ensures that $u_0^- \le u_d^-$ in $\overline{\Omega}$.

By Theorem 4.4, we have $u_d^- \in \mathscr{S}^-$. Also, by the definition of u_d^-, we have $u_d^-(x) \le u_0(x) + d(x,x) = u_0(x)$ for all $x \in \overline{\Omega}$. Hence, by the definition of u_0^-, we find that $u_0^- \ge u_d^-$ in $\overline{\Omega}$. Thus, we have $u_0^- = u_d^-$ in $\overline{\Omega}$.

It is now enough to show that

$$u_\infty(x) = \inf_{y \in \mathscr{A}} (u_0^-(y) + d(x,y)).$$

Let ϕ denote the function defined by the right hand side of the above formula. The version of Proposition 1.10 for supersolutions ensures that $\phi \in \mathscr{S}^+$, while Theorem 4.4 guarantees that $\phi \in \mathscr{S}^-$. Hence, we have $\phi \in \mathscr{S}$. Observe also that

$$u_0^-(x) \le u_0^-(y) + d(x,y) \quad \text{for all } \ x, y \in \overline{\Omega},$$

which yields $u_0^- \le \phi$ in $\overline{\Omega}$. Thus, we see by Theorem 6.3 that $u_\infty \leq \phi$ in $\overline{\Omega}$.

Now, applying Theorem 4.1 to u_∞, we observe that for all $x \in \overline{\Omega}$,

$$\begin{aligned} u_\infty(x) &= \inf\{u_\infty(y) + d(x,y) \ : \ y \in \mathscr{A}\} \\ &\ge \inf\{u_0^-(y) + d(x,y) \ : \ y \in \mathscr{A}\} = \phi(x). \end{aligned}$$

Thus we find that $u_\infty = \phi$ in $\overline{\Omega}$. The proof is complete. □

Combining the above theorem and Proposition 5.4, we obtain another representation formula for u_∞.

Corollary 6.1. *The following formula holds:*

$$\begin{aligned} u_\infty(x) = \inf\big\{&\mathscr{L}(T,\eta,v,l) + u_0(\eta(T)) \ : \ T > 0, \ (\eta, v, l) \in \mathrm{SP}(x) \\ &\textit{such that } \ \eta(t) \in \mathscr{A} \ \textit{ for some } t \in (0, \ T)\big\}. \end{aligned}$$

Example 6.2. As in Example 3.1, let $n = 1$, $\Omega = (-1, \ 1)$ and $\gamma = \nu$ on $\partial\Omega$ (i.e., $\gamma(\pm 1) = \pm 1$). Let $H = H(p) = |p|^2$ and $g \ : \ \partial\Omega \to \mathbb{R}$ be the function given by $g(-1) = -1$ and $g(1) = 0$. As in Example 3.1, we see that $c^\# = 1$. We set $\tilde{H}(p) = H(p) - c^\# = |p|^2 - 1$. Note that $\tilde{H}$ satisfies both (A9)$_\pm$, and consider the Neumann problem

$$\tilde{H}(v'(x)) = 0 \ \text{ in } \ \Omega, \qquad \gamma(x)\cdot v'(x) = g(x) \ \text{ on } \ \partial\Omega. \tag{127}$$

It is easily seen that the distance-like function $d \ : \ \overline{\Omega}^2 \to \mathbb{R}$ for this problem is given by $d(x,y) = |x - y|$. Let $\mathscr{A}$ denote the Aubry set for problem (127). By examining the function d, we see that $\mathscr{A} = \{-1\}$. For instance, by observing that

$$D_x^- d(x,-1) = \begin{cases} \{1\} & \text{if } x \in \Omega, \\ (-\infty, \ 1] & \text{if } x = -1, \\ [1, \ \infty) & \text{if } x = 1, \end{cases}$$

we find that $-1 \in \mathscr{A}$. Let $u_0(x) = 0$. Consider the problem

$$\begin{cases} u_t(x,t) + H(u_x(x,t)) = 0 & \text{for } \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \gamma(x) u_x(x,t) = g(x) & \text{for } \ (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{for } x \in \overline{\Omega}. \end{cases}$$

If u is the viscosity solution of this problem and the function v is given by $v(x,t) = u(x,t) + c^{\#}t = u(x,t) + t$, then v solves in the viscosity sense

$$\begin{cases} v_t(x,t) + \tilde{H}(v_x(x,t)) = 0 & \text{for } (x,t) \in \Omega \times \mathbb{R}_+, \\ \gamma(x)v_x(x,t) = g(x) & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ v(x,0) = u_0(x) & \text{for } x \in \overline{\Omega}. \end{cases}$$

Setting

$$u_\infty(x) = \min\{d(x,y) + d(y,z) + u_0(z) \,:\, y \in \mathscr{A},\, z \in \overline{\Omega}\} \quad \text{for } x \in \overline{\Omega},$$

we note that $u_\infty(x) = |x+1|$ for all $x \in \overline{\Omega}$. Thanks to Theorems 6.1 and 6.4, we have

$$\lim_{t\to\infty} v(x,t) = u_\infty(x) \quad \text{uniformly on } \overline{\Omega},$$

which reads

$$\lim_{t\to\infty} (u(x,t) + t - |x+1|) = 0 \quad \text{uniformly on } \overline{\Omega}.$$

That is, we have $u(x,t) \approx -t + |x+1|$ as $t \to \infty$. If we replace $u_0(x) = 0$ by the function $u_0(x) = -3x$, then

$$u_\infty(x) = \min_{y\in\overline{\Omega}}\{|x+1| + |1+y| - 3y\} = |x+1| - 1 \quad \text{for all } x \in \overline{\Omega},$$

and $u(x,t) \approx -t + |x+1| - 1$ as $t \to \infty$.

In some cases the variational formula in Corollary 6.1 is useful to see the convergence assertion of Theorem 6.1.

Under the hypothesis that $c^{\#} = 0$, which is our case, we call a point $y \in \overline{\Omega}$ an *equilibrium point* if $L(y,0) = 0$. This condition, $L(y,0) = 0$, is equivalent to $\min_{p\in\mathbb{R}^n} H(y,p) = 0$.

Let $y \in \overline{\Omega}$ be an equilibrium point. If we define $(\eta, v, l) \in \mathrm{SP}(y)$ by setting $(\eta, v, l)(s) = (y,\, 0,\, 0)$, then $\mathscr{L}(t, \eta, v, l) = 0$ for all $t \in \mathbb{R}_+$, and Propositions 5.4 and 5.5 guarantee that $y \in \mathscr{A}$.

We now assume that $\mathscr{A}$ consists of only equilibrium points. Fix any $\varepsilon > 0$ and $x \in \overline{\Omega}$. According to Corollary 6.1, we can choose $\tau, \sigma \in \mathbb{R}_+$ and $(\eta, v, l) \in \mathrm{SP}(x)$ so that $\eta(\tau) \in \mathscr{A}$ and

$$\mathscr{L}(\tau + \sigma, \eta, v, l) + u_0(\eta(\tau + \sigma)) < u_\infty(x) + \varepsilon. \tag{128}$$

Fix any $t > \tau + \sigma$. We define $(\tilde{\eta}, \tilde{v}, \tilde{l}) \in \mathrm{SP}(x)$ by

$$(\tilde{\eta}, \tilde{v}, \tilde{l})(s) = \begin{cases} (\eta, v, l)(s) & \text{for } s \in [0, \tau), \\ (\eta(\tau), 0, 0) & \text{for } s \in [\tau, \tau + \theta), \\ (\eta, v, l)(s - \theta) & \text{for } s \in [\tau + \theta, \infty), \end{cases}$$

where $\theta = t - (\tau + \sigma)$. Using (128), we get

$$u_\infty(x) + \varepsilon > \mathscr{L}(t, \tilde{\eta}, \tilde{v}, \tilde{l}) + u_0(\tilde{\eta}(t)) \geq u(x, t).$$

Therefore, recalling that $\liminf_{t\to\infty} u(x, t) = u_\infty(x)$, we see that $\lim_{t\to\infty} u(x, t) = u_\infty(x)$ for all $x \in \overline{\Omega}$.

6.4 Localization of Conditions (A9)$_\pm$

In this subsection we explain briefly that the following versions of (A9)$_\pm$ localized to the Aubry set $\mathscr{A}$ may replace the role of (A9)$_\pm$ in Theorem 6.1.

(A10)$_\pm$ Let

$$Z_{\mathscr{A}} = \{(x, p) \in \mathscr{A} \times \mathbb{R}^n \ : \ H(x, p) = 0\}.$$

There exists a function $\omega_0 \in C([0, \infty))$ satisfying $\omega_0(r) > 0$ for all $r > 0$ such that if $(x, p) \in Z_{\mathscr{A}}$, $\xi \in D_p^- H(x, p)$ and $q \in \mathbb{R}^n$, then

$$H(x, p + q) \geq \xi \cdot q + \omega_0((\xi \cdot q)_\pm).$$

As before, assume that $c^\# = 0$ and let u be the solution of (ENP)–(ID) and $u_\infty(x) := \liminf_{t\to\infty} u(x, t)$.

Theorem 6.5. *Assume that either* (A10)$_+$ *or* (A10)$_-$ *holds. Then*

$$\lim_{t\to\infty} u(x, t) = u_\infty(x) \quad \textit{uniformly on } \overline{\Omega}. \tag{129}$$

If we set

$$u_\infty^+(x) = \limsup_{t\to\infty} u(x, t) \quad \text{for } x \in \overline{\Omega},$$

we see by Theorem 1.3 that the function $u_\infty^+(x)$ is a subsolution of (ENP), as a function of (x, t), and hence a subsolution of (SNP). That is, $u_\infty^+ \in \mathscr{S}^-$. Since $u_\infty \in \mathscr{S}^+$, once we have shown that $u_\infty^+ \leq u_\infty$ on $\mathscr{A}$, then, by Theorem 4.6, we get

$$u_\infty^+ \leq u_\infty \quad \text{in } \overline{\Omega},$$

which shows that the uniform convergence (129) is valid. Thus we only need to show that $u_\infty^+ \leq u_\infty$ on $\mathscr{A}$.

Following [21] (see also [39]), one can prove the following lemma.

Lemma 6.4. *For any $z \in \mathscr{A}$ there exists an $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ such that*

$$d(z, \eta(t)) = \mathscr{L}(t, \alpha) = -d(\eta(t), z) \quad \textit{for all} \;\; t > 0.$$

Proof. By Proposition 5.5, for each $k \in \mathbb{N}$ there are an $\alpha_k = (\eta_k, v_k, l_k) \in \mathrm{SP}(z)$ and $\tau_k \geq k$ such that

$$\mathscr{L}(\tau_k, \alpha_k) < \frac{1}{k} \quad \text{and} \quad \eta_k(\tau_k) = z.$$

Observe that for any $j, k \in \mathbb{N}$ with $j < k$,

$$\begin{aligned} \frac{1}{k} &> \mathscr{L}(j, \alpha_k) + \int_j^{\tau_k} [L(\eta_k(s), -v_k(s)) + l_k(s) g(\eta_k(s))] \mathrm{d}s \\ &\geq \mathscr{L}(j, \alpha_k) + d(\eta_k(j), \eta_k(\tau_k)), \end{aligned} \tag{130}$$

and hence

$$\sup_{k \in \mathbb{N}} \mathscr{L}(j, \alpha_k) < \infty \quad \text{for all} \;\; j \in \mathbb{N}.$$

We apply Theorem 5.4, with $T = j \in \mathbb{N}$, and use the diagonal argument, to conclude from (130) that there is an $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ such that for all $j \in \mathbb{N}$,

$$\mathscr{L}(j, \alpha) \leq \liminf_{k \to \infty} \mathscr{L}(j, \alpha_k) \leq -d(\eta(j), z).$$

Let $0 < t < \infty$, and choose a $j \in \mathbb{N}$ such that $t < j$. Using Propositions 5.4 and 4.1 (ii) (the triangle inequality for d), we compute that

$$\begin{aligned} d(z, \eta(t)) &\leq \mathscr{L}(t, \alpha) = \mathscr{L}(j, \alpha) - \int_t^j [L(\eta(s), -v(s)) + l(s) g(\eta(s))] \mathrm{d}s \\ &\leq \mathscr{L}(j, \alpha) - d(\eta(t), \eta(j)) \leq -d(\eta(j), z) - d(\eta(t), \eta(j)) \\ &\leq -d(\eta(t), z). \end{aligned}$$

Moreover, by the triangle inequality, we get

$$-d(\eta(t), z) \leq d(z, \eta(t)).$$

These together yield

$$d(z, \eta(t)) = \mathscr{L}(t, \alpha) = -d(\eta(t), z) \quad \text{for all } t > 0,$$

which completes the proof. □

The above assertion is somehow related to the idea of the quotient Aubry set (see [46, 47]). Indeed, if we introduce the equivalence relation $\equiv$ on $\mathscr{A}$ by

$$x \equiv y \quad \Longleftrightarrow \quad d(x,y) + d(y,x) = 0,$$

and consider the quotient space $\hat{\mathscr{A}}$ consisting of the equivalence classes

$$[x] = \{y \in \mathscr{A} \,:\, y \equiv x\}, \quad \text{with } \ x \in \mathscr{A},$$

then the space $\hat{\mathscr{A}}$ is a metric space with its distance given by

$$\hat{d}([x],[y]) = d(x,y) + d(y,x).$$

The property of the curve η in the above lemma that $d(z,\eta(t)) = -d(\eta(t),z)$ is now stated as: $\eta(t) \equiv \eta(0)$.

Lemma 6.5. *Let $\psi \in \mathscr{S}^-$ and $x, y \in \mathscr{A}$. If $x \equiv y$, then*

$$\psi(x) - \psi(y) = d(x,y).$$

Proof. By the definition of d, we have

$$\psi(x) - \psi(y) \le d(x,y) \quad \text{and} \quad \psi(y) - \psi(x) \le d(y,x).$$

Hence,

$$\psi(x) - \psi(y) \le d(x,y) = -d(y,x) \le \psi(x) - \psi(y),$$

which shows that $\psi(x) - \psi(y) = d(x,y) = -d(y,x)$. □

Proof (Theorem 6.5). As we have noted above, we need only to show that

$$u_\infty^+(x) \le u_\infty(x) \quad \text{for all } \ x \in \mathscr{A}.$$

To this end, we fix any $z \in \mathscr{A}$. Let $\alpha = (\eta, v, l) \in \mathrm{SP}(z)$ be as in Lemma 6.4. In view of Lemma 6.5, we have

$$u_\infty(z) - u_\infty(\eta(t)) = d(z,\eta(t)) = \mathscr{L}(t,\alpha) \quad \text{for all } \ t > 0.$$

It is obvious that the same assertion as Lemma 6.3 holds if we replace S by

$$S_{\mathscr{A}} := \{(x,\xi) \in \mathscr{A} \times \mathbb{R}^n \,:\, \xi \in D_p^- H(x,p) \quad \text{for some } (x,p) \in Z_{\mathscr{A}}\}.$$

We now just need to follow the arguments in Sect. 6.2, to conclude that

$$u_\infty^+(z) \le u_\infty(z).$$

The details are left to the interested reader. □

Appendix

A.1 Local maxima to global maxima

We recall a proposition from [56] which is about partition of unity.

Proposition A.1. *Let $\mathscr{O}$ be a collection of open subsets of $\mathbb{R}^n$. Set $W := \bigcup_{U\in\mathscr{O}} U$. Then there is a collection $\mathscr{F}$ of C^∞ functions in $\mathbb{R}^n$ having the following properties:*

(i) $0 \le f(x) \le 1$ *for all $x \in W$ and $f \in \mathscr{F}$.*
(ii) *For each $x \in W$ there is a neighborhood V of x such that all but finitely many $f \in \mathscr{F}$ vanish in V.*
(iii) $\sum_{f\in\mathscr{F}} f(x) = 1$ *for all $x \in W$.*
(iv) *For each $f \in \mathscr{F}$ there is a set $U \in \mathscr{O}$ such that* $\operatorname{supp} f \subset U$.

Proposition A.2. *Let Ω be any subset of $\mathbb{R}^n$, $u \in \mathrm{USC}(\Omega, \mathbb{R})$ and $\phi \in C^1(\Omega)$. Assume that $u - \phi$ attains a local maximum at $y \in \Omega$. Then there is a function $\psi \in C^1(\Omega)$ such that $u - \psi$ attains a global maximum at y and $\psi = \phi$ in a neighborhood of y.*

Proof. As usual it is enough to prove the above proposition in the case when $(u - \phi)(y) = 0$.

By the definition of the space $C^1(\Omega)$, there is an open neighborhood W_0 of Ω such that ϕ is defined in W_0 and $\phi \in C^1(W_0)$.

There is an open subset $U_y \subset W_0$ of $\mathbb{R}^n$ containing y such that $\max_{U_y\cap\Omega}(u-\phi) = (u - \phi)(y)$. Since $u \in \mathrm{USC}(\Omega, \mathbb{R})$, for each $x \in \Omega \setminus \{y\}$ we may choose an open subset U_x of $\mathbb{R}^n$ so that $x \in U_x$, $y \notin U_x$ and $\sup_{U_x\cap\Omega} u < \infty$. Set $a_x = \sup_{U_x\cap\Omega} u$ for every $x \in \Omega \setminus \{y\}$.

We set $\mathscr{O} = \{U_z : z \in \Omega\}$ and $W = \bigcup_{U\in\mathscr{O}} U$. Note that W is an open neighborhood of Ω. By Proposition A.1, there exists a collection $\mathscr{F}$ of functions $f \in C^\infty(\mathbb{R}^n)$ satisfying the conditions (i)–(iv) of the proposition. According to the condition (iv), for each $f \in \mathscr{F}$ there is a point $z \in \Omega$ such that $\operatorname{supp} f \subset U_z$. For each $f \in \mathscr{F}$ we fix such a point $z \in \Omega$ and define the mapping $p : \mathscr{F} \to \Omega$ by $p(f) = z$. We set

$$\psi(x) = \sum_{f\in\mathscr{F},\ p(f)\neq y} a_{p(f)} f(x) + \sum_{f\in\mathscr{F},\ p(f)=y} \phi(x) f(x) \quad \text{for } x \in W.$$

By the condition (ii), we see that $\psi \in C^1(W)$. Fix any $x \in \Omega$ and $f \in \mathscr{F}$, and observe that if $f(x) > 0$ and $p(f) \neq y$, then we have $x \in \operatorname{supp} f \subset U_{p(f)}$ and, therefore, $a_{p(f)} = \sup_{U_{p(f)}\cap\Omega} u \ge u(x)$. Observe also that if $f(x) > 0$ and $p(f) = y$, then we have $x \in U_y$ and $\phi(x) \ge u(x)$. Thus we see that for all $x \in \Omega$,

$$\psi(x) \ge \sum_{f\in\mathscr{F},\ p(f)\neq y} u(x) f(x) + \sum_{f\in\mathscr{F},\ p(f)=y} u(x) f(x) = u(x) \sum_{f\in\mathscr{F}} f(x) = u(x).$$

Thanks to the condition (ii), we may choose a neighborhood $V \subset W$ of y and a finite subset $\{f_j\}_{j=1}^N$ of $\mathscr{F}$ so that

$$\sum_{j=1}^N f_j(x) = 1 \quad \text{for all } x \in V.$$

If $p(f_j) \neq y$ for some $j = 1, \ldots, N$, then $U_{p(f_j)} \cap \{y\} = \emptyset$ and hence $y \notin \operatorname{supp} f_j$. Therefore, by replacing V by a smaller one we may assume that $p(f_j) = y$ for all $j = 1, \ldots, N$. Since $f = 0$ in V for all $f \in \mathscr{F} \setminus \{f_1, \ldots, f_N\}$, we see that

$$\psi(x) = \sum_{j=1}^N \phi(x) f_j(x) = \phi(x) \quad \text{for all } x \in V.$$

It is now easy to see that $u - \psi$ has a global maximum at y. □

A.2 A Quick Review of Convex Analysis

We discuss here basic properties of convex functions on $\mathbb{R}^n$.

By definition, a subset C of $\mathbb{R}^n$ is convex if and only if

$$(1-t)x + ty \in C \quad \text{for all } \ x, y \in C,\ 0 < t < 1.$$

For a given function $f : U \subset \mathbb{R}^n \to [-\infty, \infty]$, its epigraph $\operatorname{epi}(f)$ is defined as

$$\operatorname{epi}(f) = \{(x, y) \in U \times \mathbb{R} : y \geq f(x)\}.$$

A function $f : U \to [-\infty, \infty]$ is said to be convex if $\operatorname{epi}(f)$ is a convex subset of $\mathbb{R}^{n+1}$.

We are henceforth concerned with functions defined on $\mathbb{R}^n$. When we are given a function f on U with U being a proper subset of $\mathbb{R}^n$, we may think of f as a function defined on $\mathbb{R}^n$ having value ∞ on the set $\mathbb{R}^n \setminus U$.

It is easily checked that a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is convex if and only if for all $x, y \in \mathbb{R}^n$, $t, s \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)t + \lambda s \quad \text{if } \ t \geq f(x) \ \text{ and } \ s \geq f(y).$$

From this, we see that a function $f : \mathbb{R}^n \to (-\infty, \infty]$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y).$$

Here we use the convention for extended real numbers, i.e., for any $x \in \mathbb{R}$, $-\infty < x < \infty$, $x \pm \infty = \pm\infty$, $x \cdot (\pm\infty) = \pm\infty$ if $x > 0$, $0 \cdot (\pm\infty) = 0$, etc.

Any affine function $f(x) = a \cdot x + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a convex function on $\mathbb{R}^n$. Moreover, if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}$ are nonempty sets, then the function on $\mathbb{R}^n$ given by

$$f(x) = \sup\{a \cdot x + b \,:\, (a, b) \in A \times B\}$$

is a convex function. Note that this function f is lower semicontinuous on $\mathbb{R}^n$. We restrict our attention to those functions which take values only in $(-\infty, \infty]$.

Proposition B.1. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. Assume that $p \in D^- f(y)$ for some $y, p \in \mathbb{R}^n$. Then*

$$f(x) \geq f(y) + p \cdot (x - y) \quad \textit{for all } \; x \in \mathbb{R}^n.$$

Proof. By the definition of $D^- f(y)$, we have

$$f(x) \geq f(y) + p \cdot (x - y) + o(|x - y|) \quad \text{as } \; x \to y.$$

Hence, fixing $x \in \mathbb{R}^n$, we get

$$f(y) \leq f(tx + (1 - t)y) - tp \cdot (x - y) + o(t) \quad \text{as } \; t \to 0+.$$

Using the convexity of f, we rearrange the above inequality and divide by $t > 0$, to get

$$f(y) \leq f(x) - p \cdot (x - y) + o(1) \quad \text{as } \; t \to 0+.$$

Sending $t \to 0+$ yields

$$f(x) \geq f(y) + p \cdot (x - y) \quad \text{for all } \; x \in \mathbb{R}^n.$$

□

Proposition B.2. *Let $\mathscr{F}$ be a nonempty set of convex functions on $\mathbb{R}^n$ with values in $(-\infty, \infty]$. Then $\sup \mathscr{F}$ is a convex function on $\mathbb{R}^n$ having values in $(-\infty, \infty]$.*

Proof. It is clear that $(\sup \mathscr{F})(x) \in (-\infty, \infty]$ for all $x \in \mathbb{R}^n$. If $f \in \mathscr{F}$, $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, then we have

$$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) \leq (1 - t)(\sup \mathscr{F})(x) + t(\sup \mathscr{F})(y)$$

and hence

$$(\sup \mathscr{F})((1 - t)x + ty) \leq (1 - t)(\sup \mathscr{F})(x) + t(\sup \mathscr{F})(y),$$

which proves the convexity of $\sup \mathscr{F}$. □

We call a function $f : \mathbb{R}^n \to (-\infty, \infty]$ *proper convex* if the following three conditions hold:

(a) f is convex on $\mathbb{R}^n$.
(b) $f \in \mathrm{LSC}(\mathbb{R}^n)$.
(c) $f(x) \not\equiv \infty$.

Let $f : \mathbb{R}^n \to [-\infty, \infty]$. The conjugate convex function (or the Legendre–Fenchel transform) of f is the function $f^\star : \mathbb{R}^n \to [-\infty, \infty]$ given by

$$f^\star(x) = \sup_{y\in\mathbb{R}^n} (x \cdot y - f(y)).$$

Proposition B.3. *If f is a proper convex function, then so is $f^\star$.*

Lemma B.1. *If f is a proper convex function on $\mathbb{R}^n$, then $D^- f(y) \neq \emptyset$ for some $y \in \mathbb{R}^n$.*

Proof. We choose a point $x_0 \in \mathbb{R}^n$ so that $f(x_0) \in \mathbb{R}$. Let $k \in \mathbb{N}$, and define the function g_k on $\bar{B}_1(x_0)$ by the formula $g_k(x) = f(x) + k|x - x_0|^2$. Since $g_k \in \mathrm{LSC}(\overline{B}_1(x_0))$, and $g_k(x_0) = g(x_0) \in \mathbb{R}$, the function g_k has a finite minimum at a point $x_k \in \overline{B}_1(x_0)$. Note that if k is sufficiently large, then

$$\min_{\partial B_1(x_0)} g_k = \min_{\partial B_1(x_0)} f + k > f(x_0).$$

Fix such a large k, and observe that $x_k \in B_1(x_0)$ and, therefore, $-2k(x_k - x_0) \in D^- f(x_k)$. □

Proof (Proposition B.3). The function $x \mapsto x \cdot y - f(y)$ is an affine function for any $y \in \mathbb{R}^n$. By Proposition B.2, the function $f^\star$ is convex on $\mathbb{R}^n$. Also, since the function $x \mapsto x \cdot y - f(y)$ is continuous on $\mathbb{R}^n$ for any $y \in \mathbb{R}^n$, as stated in Proposition 1.5, the function $f^\star$ is lower semicontinuous on $\mathbb{R}^n$.

Since f is proper convex on $\mathbb{R}^n$, there is a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. Hence, we have

$$f^\star(y) \geq y \cdot x_0 - f(x_0) > -\infty \quad \text{for all } \ y \in \mathbb{R}^n.$$

By Lemma B.1, there exist points $y, p \in \mathbb{R}^n$ such that $p \in D^- f(y)$. By Proposition B.1, we have

$$f(x) \geq f(y) + p \cdot (x - y) \quad \text{for all } \ x \in \mathbb{R}^n.$$

That is,

$$p \cdot y - f(y) \geq p \cdot x - f(x) \quad \text{for all } \ x \in \mathbb{R}^n,$$

which implies that $f^\star(p) = p \cdot y - f(y) \in \mathbb{R}$. Thus, we conclude that $f^\star : \mathbb{R}^n \to (-\infty, \infty]$, $f^\star$ is convex on $\mathbb{R}^n$, $f^\star \in \mathrm{LSC}(\mathbb{R}^n)$ and $f^\star(x) \not\equiv \infty$. □

The following duality (called convex duality or Legendre–Fenchel duality) holds.

Theorem B.1. *Let* $f : \mathbb{R}^n \to (-\infty, \infty]$ *be a proper convex function. Then*

$$f^{\star\star} = f.$$

Proof. By the definition of $f^\star$, we have

$$f^\star(x) \geq x \cdot y - f(y) \quad \text{for all } \ x, y \in \mathbb{R}^n,$$

which reads

$$f(y) \geq y \cdot x - f^\star(x) \quad \text{for all } \ x, y \in \mathbb{R}^n.$$

Hence,

$$f(y) \geq f^{\star\star}(y) \quad \text{for all } \ y \in \mathbb{R}^n.$$

Next, we show that

$$f^{\star\star}(x) \geq f(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

We fix any $a \in \mathbb{R}^n$ and choose a point $y \in \mathbb{R}^n$ so that $f(y) \in \mathbb{R}$. We fix a number $R > 0$ so that $|y - a| < R$. Let $k \in \mathbb{N}$, and consider the function $g_k \in \mathrm{LSC}(\overline{B}_R(a))$ defined by $g_k(x) = f(x) + k|x - a|^2$. Let $x_k \in \overline{B}_R(a)$ be a minimum point of the function g_k. Noting that if k is sufficiently large, then

$$g_k(x_k) \leq f(y) + k|y - a|^2 < \min_{\partial B_R(a)} f + kR^2 = \min_{\partial B_R(a)} g_k,$$

we see that $x_k \in B_R(a)$ for k sufficiently large. We henceforth assume that k is large enough so that $x_k \in B_R(a)$. We have

$$D^- g_k(x_k) = D^- f(x_k) + 2k(x_k - a) \ni 0.$$

Accordingly, if we set $\xi_k = -2k(x_k - a)$, then we have $\xi_k \in D^- f(x_k)$. By Proposition B.1, we get

$$f(x) \geq f(x_k) + \xi_k \cdot (x - x_k) \quad \text{for all } \ x \in \mathbb{R}^n,$$

or, equivalently,

$$\xi_k \cdot x_k - f(x_k) \geq \xi_k \cdot x - f(x) \quad \text{for all } \ x \in \mathbb{R}^n.$$

Hence,

$$\xi_k \cdot x_k - f(x_k) = f^\star(\xi_k).$$

Using this, we compute that

$$\begin{aligned} f^{\star\star}(a) &\geq a \cdot \xi_k - f^\star(\xi_k) = \xi_k \cdot a - \xi_k \cdot x_k + f(x_k) \\ &= 2k|x_k - a|^2 + f(x_k). \end{aligned}$$

We divide our argument into the following cases, (a) and (b).

Case (a): $\lim_{k\to\infty} k|x_k - a|^2 = \infty$. In this case, if we set $m = \min_{\bar{B}_R(a)} f$, then we have

$$f^{\star\star}(a) \ge \liminf_{k\to\infty} 2k|x_k - a|^2 + m = \infty,$$

and, therefore, $f^{\star\star}(a) \ge f(a)$.

Case (b): $\liminf_{k\to\infty} k|x_k - a|^2 < \infty$. We may choose a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}$ so that $\lim_{j\to\infty} x_{k_j} = a$. Then we have

$$f^{\star\star}(a) \ge \liminf_{j\to\infty} \big(2k_j|x_{k_j} - a|^2 + f(x_{k_j})\big) \ge \liminf_{j\to\infty} f(x_{k_j}) \ge f(a).$$

Thus, in both cases we have $f^{\star\star}(a) \ge f(a)$, which completes the proof. □

Theorem B.2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be proper convex and $x, \xi \in \mathbb{R}^n$. Then the following three conditions are equivalent each other.*

(i) $\xi \in D^- f(x)$.
(ii) $x \in D^- f^\star(\xi)$.
(iii) $x \cdot \xi = f(x) + f^\star(\xi)$.

Proof. Assume first that (i) holds. By Proposition B.1, we have

$$f(y) \ge f(x) + \xi \cdot (y - x) \quad \text{for all} \ \ y \in \mathbb{R}^n,$$

which reads

$$\xi \cdot x - f(x) \ge \xi \cdot y - f(y) \quad \text{for all } y \in \mathbb{R}^n.$$

Hence,

$$\xi \cdot x - f(x) = \max_{y\in\mathbb{R}^n} (\xi \cdot y - f(y)) = f^\star(\xi).$$

Thus, (iii) is valid.

Next, we assume that (iii) holds. Then the function $y \mapsto \xi \cdot y - f(y)$ attains a maximum at x. Therefore, $\xi \in D^- f(x)$. That is, (i) is valid.

Now, by the convex duality (Theorem B.1), (iii) reads

$$x \cdot \xi = f^{\star\star}(x) + f^\star(\xi).$$

The equivalence between (i) and (iii), with f replaced by $f^\star$, is exactly the equivalence between (ii) and (iii). The proof is complete. □

Finally, we give a Lipschitz regularity estimate for convex functions.

Theorem B.3. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. Assume that there are constants $M > 0$ and $R > 0$ such that*

$$|f(x)| \le M \quad \textit{for all} \ \ x \in B_{3R}.$$

Then

$$|f(x) - f(y)| \leq \frac{M}{R}|x - y| \quad \textit{for all} \;\; x, y \in B_R.$$

Proof. Let $x, y \in B_R$ and note that $|x - y| < 2R$. We may assume that $x \neq y$. Setting $\xi = (x - y)/|x - y|$ and $z = y + 2R\xi$ and noting that $z \in B_{3R}$,

$$x - y = \frac{|x - y|}{2R}(z - y),$$

and

$$x = y + \frac{|x - y|}{2R}(z - y) = \frac{|x - y|}{2R}z + \left(1 - \frac{|x - y|}{2R}\right) y,$$

we obtain

$$f(x) \leq \frac{|x - y|}{2R} f(z) + \left(1 - \frac{|x - y|}{2R}\right) f(y),$$

and

$$f(x) - f(y) \leq \frac{|x - y|}{2R}(f(z) - f(y)) \leq \frac{|x - y|}{2R}(|f(z)| + |f(y)|) \leq \frac{M|x - y|}{R}.$$

In view of the symmetry in x and y, we see that

$$|f(x) - f(y)| \leq \frac{M}{R}|x - y| \quad \text{for all} \;\; x, y \in B_R. \qquad \square$$

A.3 Global Lipschitz Regularity

We give here a proof of Lemmas 2.1 and 2.2.

Proof (Lemma 2.1). We first show that there is a constant $C > 0$, for each $z \in \overline{\Omega}$ a ball $B_r(z)$ centered at z, and for each $x, y \in B_r(z) \cap \overline{\Omega}$, a curve $\eta \in \mathrm{AC}([0, T], \mathbb{R}^n)$, with $T \in \overline{\mathbb{R}}_+$, such that $\eta(s) \in \Omega$ for all $s \in (0, T)$, $|\dot{\eta}(s)| \leq 1$ for a.e. $s \in (0, T)$ and $T \leq C|x - y|$.

Let ρ be a defining function of Ω. We may assume that $\|D\rho\|_{\infty,\mathbb{R}^n} \leq 1$ and $|D\rho(x)| \geq \delta$ for all $x \in (\partial\Omega)^\delta := \{y \in \mathbb{R}^n \,:\, \mathrm{dist}(y, \partial\Omega) < \delta\}$ and some constant $\delta \in (0, 1)$.

Let $z \in \Omega$. We can choose $r > 0$ so that $B_r(z) \subset \Omega$. Then, for each $x, y \in B_r(z)$, with $x \neq y$, the line $\eta(s) = x + s(y - x)/|y - x|$, with $s \in [0, |x - y|]$, connects two points x and y and lies inside Ω. Note as well that $\dot{\eta}(s) = (y - x)/|y - x| \in \partial B_1$ for all $s \in [0, |x - y|]$.

Let $z \in \partial\Omega$. Since $|D\rho(z)|^2 \geq \delta^2$, by continuity, we may choose $r \in (0, \delta^3/4)$ so that $D\rho(x) \cdot D\rho(z) \geq \delta^2/2$ for all $x \in B_{4\delta^{-2}r}(z)$. Fix any $x, y \in B_r(z) \cap \overline{\Omega}$.

Consider the curve $\xi(t) = x + t(y-x) - t(1-t)6\delta^{-2}|x-y|D\rho(z)$, with $t \in [0, 1]$, which connects the points x and y. Note that

$$\begin{aligned}|\xi(t) - z| &\le (1-t)|x-z| + t|y-z| + 6t(1-t)\delta^{-2}|x-y||D\rho(z)| \\ &< (1 + 3\delta^{-2})r < 4\delta^{-2}r\end{aligned}$$

and $4\delta^{-2}r < \delta$. Hence, we have $\xi(t) \in B_{4\delta^{-2}r}(z) \cap (\partial\Omega)^{\delta}$ for all $t \in [0, 1]$. If $t \in (0, 1/2]$, then we have

$$\begin{aligned}\rho(\xi(t)) &\le \rho(x) + tD\rho(\theta\xi(t) + (1-\theta)x) \cdot (y - x - 6(1-t)\delta^{-2}|x-y|D\rho(z)) \\ &\le t|x-y|(1 - 3(1-t)) < 0\end{aligned}$$

for some $\theta \in (0, 1)$. Similarly, if $t \in [1/2, 1)$, we have

$$\rho(\xi(t)) \le \rho(y) + (1-t)|x-y|(1-3t) < 0.$$

Hence, $\xi(t) \in \Omega$ for all $t \in (0, 1)$. Note that

$$|\dot{\xi}(t)| \le |y-x|(1 + 6\delta^{-2}).$$

If $x = y$, then we just set $\eta(s) = x = y$ for $s = 0$ and the curve $\eta : [0, 0] \to \mathbb{R}^n$ has the required properties. Now let $x \ne y$. We set $t(x, y) = (1 + 6\delta^{-2})|x-y|$ and $\eta(s) = \xi(s/t(x, y))$ for $s \in [0, t(x, y)]$. Then the curve $\eta : [0, t(x, y)] \to \mathbb{R}^n$ has the required properties with $C = 1 + 6\delta^{-2}$.

Thus, by the compactness of $\overline{\Omega}$, we may choose a constant $C > 0$ and a finite covering $\{B^i\}_{i=1}^N$ of $\overline{\Omega}$ consisting of open balls with the properties: for each $x, y \in \hat{B}_i \cap \overline{\Omega}$, where $\hat{B}_i$ denotes the concentric open ball of B_i with radius twice that of B_i, there exists a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ such that $\eta(s) \in \Omega$ for all $s \in (0, t(x, y))$, $|\dot{\eta}(s)| \le 1$ for a.e. $s \in [0, t(x, y)]$ and $t(x, y) \le C|x-y|$.

Let r_i be the radius of the ball B_i and set $r = \min r_i$ and $R = \sum r_i$, where i ranges all over $i = 1, \ldots, N$.

Let $x, y \in \overline{\Omega}$. If $|x-y| < r$, then $x, y \in \hat{B}_i$ for some i and there is a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ such that $\eta(s) \in \Omega$ for all $s \in (0, t(x, y))$, $|\dot{\eta}(s)| \le 1$ for a.e. $s \in [0, t(x, y)]$ and $t(x, y) \le C|x-y|$. Next, we assume that $|x-y| \ge r$. By the connectedness of Ω, we infer that there is a sequence $\{B_{i_j} : j = 1, \ldots, J\} \subset \{B_i : i = 1, \ldots, N\}$ such that $x \in B_{i_1}$, $y \in B_{i_J}$, $B_{i_j} \cap B_{i_{j+1}} \cap \Omega \ne \emptyset$ for all $1 \le j < J$, and $B_{i_j} \ne B_{i_k}$ if $j \ne k$. It is clear that $J \le N$. If $J = 1$, then we may choose a curve η with the required properties as in the case where $|x-y| < r$. If $J > 1$, then we may choose a curve $\eta \in \mathrm{AC}([0, t(x, y)], \mathbb{R}^n)$ joining x and y as follows. First, we choose a sequence $\{x_j : j = 1, \ldots, J-1\}$ of points in Ω so that $x_j \in B_{i_j} \cap B_{i_{j+1}} \cap \Omega$ for all $1 \le j < J$. Next, setting $x_0 = x$, $x_J = y$ and $t_0 = 0$, since $x_{j-1}, x_{i_j} \in B_j \cap \overline{\Omega}$ for all $1 \le j \le J$, we may select $\eta_j \in \mathrm{AC}([t_{j-1}, t_j], \mathbb{R}^n)$, with $1 \le j \le J$, inductively so that $\eta_j(t_{j-1}) = x_{j-1}$,

$\eta_j(t_j) = x_j$, $\eta_j(s) \in \Omega$ for all $s \in (t_{j-1}, t_j)$ and $t_j \le t_{j-1} + C|x_j - x_{j-1}|$. Finally, we define $\eta \in \mathrm{AC}([0,\, t(x,y)], \mathbb{R}^n)$, with $t(x,y) = t_J$, by setting $\eta(s) = \eta_i(s)$ for $s \in [t_j - 1, t_j]$ and $1 \le j \le J$. Noting that

$$T \le C \sum_{j=1}^{J} |x_j - x_{j-1}| \le C \sum_{j=1}^{J} r_{i_j} \le CR \le CRr^{-1}|x-y|,$$

we see that the curve $\eta \in \mathrm{AC}([0,\, t(x,y)],\, \mathbb{R}^n)$ has all the required properties with C replaced by CRr^{-1}. □

Remark C.1. (i) A standard argument, different from the above one, to prove the local Lipschitz continuity near the boundary points is to flatten the boundary by a local change of variables. (ii) One can easily modify the above proof to prove the proposition same as Lemma 2.1, except that Ω is a Lipschitz domain.

Proof (Lemma 2.2). Let $C > 0$ be the constant from Lemma 2.1. We show that $|u(x) - u(y)| \le CM|x-y|$ for all $x, y \in \Omega$.

To show this, we fix any $x, y \in \Omega$ such that $x \ne y$. By Lemma 2.1, there is a curve $\eta \in \mathrm{AC}([0,\, t(x,y)],\, \mathbb{R}^n)$ such that $\eta(0) = x$, $\eta(t(x,y)) = y$, $t(x,y) \le C|x-y|$, $\eta(s) \in \Omega$ for all $s \in [0,\, t(x,y)]$ and $|\dot\eta(s)| \le 1$ for a.e. $s \in [0,\, t(x,y)]$.

By the compactness of the image $\eta([0,\, t(x,y)])$ of interval $[0,\, t(x,y)]$ by η, we may choose a finite sequence $\{B_i\}_{i=1}^N$ of open balls contained in Ω which covers $\eta([0,\, t(x,y)])$. We may assume by rearranging the label i if needed that $x \in B_1$, $y \in B_N$ and $B_i \cap B_{i+1} \ne \emptyset$ for all $1 \le i < N$. We may choose a sequence $0 = t_0 < t_1 < \cdots < t_N = t(x,y)$ of real numbers so that the line segment $[\eta(t_{i-1}),\, \eta(t_i)]$ joining $\eta(t_{i-1})$ and $\eta(t_i)$ lies in B_i for any $i = 1, \ldots, N$.

Thanks to Proposition 1.14, we have

$$|u(\eta(t_i)) - u(\eta(t_{i-1}))| \le M|\eta(t_i) - \eta(t_{i-1})| \quad \text{for all } i = 1, \ldots, N.$$

Using this, we compute that

$$\begin{aligned}|u(y) - u(x)| &= |u(\eta(t_N)) - u(\eta(t_0))| \le \sum_{i=1}^{N} |u(\eta(t_i)) - u(\eta(t_{i-1}))| \\ &\le M \sum_{i=1}^{N} |\eta(t_i) - \eta(t_{i-1})| \le M \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |\dot\eta(s)|\mathrm{d}s \\ &= M \int_{t_0}^{t_N} |\dot\eta(s)|\mathrm{d}s \le M(t_N - t_0) = Mt(x,y) \le CM|x-y|.\end{aligned}$$

This completes the proof. □

A.4 Localized Versions of Lemma 4.2

Theorem D.1. *Let U, V be open subsets of $\mathbb{R}^n$ with the properties: $\overline{V} \subset U$ and $V \cap \Omega \neq \emptyset$. Let $u \in C(U \cap \overline{\Omega})$ be a viscosity solution of*

$$\begin{cases} H(x, Du(x)) \leq 0 & \text{in } U \cap \Omega, \\ \dfrac{\partial u}{\partial \gamma}(x) \leq g(x) & \text{on } U \cap \partial\Omega. \end{cases} \tag{131}$$

Then, for each $\varepsilon \in (0,\ 1)$, there exists a function $u^\varepsilon \in C^1(V \cap \overline{\Omega})$ such that

$$\begin{cases} H(x, Du^\varepsilon(x)) \leq \varepsilon & \text{in } V \cap \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x) \leq g(x) & \text{on } V \cap \partial\Omega, \\ \|u^\varepsilon - u\|_{\infty, V\cap\Omega} \leq \varepsilon. \end{cases}$$

Proof. We choose functions $\zeta, \eta \in C^1(\mathbb{R}^n)$ so that $0 \leq \zeta(x) \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\zeta(x) = 1$ for all $x \in V$, $\eta(x) = 1$ for all $x \in \operatorname{supp}\zeta$ and $\operatorname{supp}\eta \subset U$.

We define the function $v \in C(\overline{\Omega})$ by setting $v(x) = \eta(x)u(x)$ for $x \in U \cap \overline{\Omega}$ and $v(x) = 0$ otherwise. By the coercivity of H, u is locally Lipschitz continuous in $U \cap \overline{\Omega}$, and hence, v is Lipschitz continuous in $\overline{\Omega}$. Let $L > 0$ be a Lipschitz bound of v in $\overline{\Omega}$. Then v is a viscosity solution of

$$\begin{cases} |Dv(x)| \leq L & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma}(x) \leq M & \text{in } \partial\Omega, \end{cases}$$

where $M := L\|\gamma\|_{\infty,\partial\Omega}$. In fact, we have a stronger assertion that for any $x \in \overline{\Omega}$ and any $p \in D^+v(x)$,

$$\begin{cases} |p| \leq L & \text{if } x \in \Omega, \\ \gamma(x) \cdot p \leq M & \text{if } x \in \partial\Omega. \end{cases} \tag{132}$$

To check this, let $\phi \in C^1(\overline{\Omega})$ and assume that $v - \phi$ attains a maximum at $x \in \overline{\Omega}$. Observe that if $x \in \Omega$, then $|D\phi(x)| \leq L$ and that if $x \in \partial\Omega$, then

$$\begin{aligned} 0 &\leq \liminf_{t\to 0+} \frac{(v-\phi)(x - t\gamma(x)) - (v-\phi)(x)}{-t} \\ &= \liminf_{t\to 0+} \frac{v(x - t\gamma(x)) - v(x)}{-t} - \frac{\partial \phi}{\partial \gamma}(x), \end{aligned}$$

which yields

$$\gamma(x) \cdot D\phi(x) \le L|\gamma(x)| \le M.$$

Thus, (132) is valid.

We set

$$h(x) = \zeta(x)g(x) + (1 - \zeta(x))M \quad \text{for} \;\; x \in \partial\Omega,$$
$$G(x, p) = \zeta(x)H(x, p) + (1 - \zeta(x))(|p| - L) \quad \text{for} \;\; (x, p) \in \overline{\Omega} \times \mathbb{R}^n.$$

It is clear that $h \in C(\partial\Omega)$ and G satisfies (A5)–(A7), with H replaced by G

In view of the coercivity of H, we may assume by reselecting L if necessary that for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$, if $|p| > L$, then $H(x, p) > 0$. We now show that v is a viscosity solution of

$$\begin{cases} G(x, Dv(x)) \le 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma}(x) \le h(x) & \text{on } \partial\Omega. \end{cases} \tag{133}$$

To do this, let $\hat{x} \in \overline{\Omega}$ and $\hat{p} \in D^+v(\hat{x})$. Consider the case where $\zeta(\hat{x}) > 0$, which implies that $\hat{x} \in U$. We have $\eta(x) = 1$ near the point $\hat{x}$, which implies that $\hat{p} \in D^+u(\hat{x})$. As u is a viscosity subsolution of (131), we have $H(\hat{x}, \hat{p}) \le 0$ if $\hat{x} \in \Omega$ and $\min\{H(\hat{x}, \hat{p}),\, \gamma(\hat{x}) \cdot \hat{p} - h(\hat{x})\} \le 0$ if $\hat{x} \in \partial\Omega$. Assume in addition that $\hat{x} \in \partial\Omega$. By (132), we have $\gamma(\hat{x}) \cdot \hat{p} \le M$. If $|\hat{p}| > L$, we have both

$$\gamma(\hat{x}) \cdot \hat{p} \le g(\hat{x}) \quad \text{and} \quad \gamma(\hat{x}) \cdot \hat{p} \le M.$$

Hence, if $|\hat{p}| > L$, then $\gamma(\hat{x}) \cdot \hat{p} \le h(\hat{x})$. On the other hand, if $|\hat{p}| \le L$, we have two cases: in one case we have $H(\hat{x}, \hat{p}) \le 0$ and hence, $G(\hat{x}, \hat{p}) \le 0$. In the other case, we have $\gamma(\hat{x}) \cdot \hat{p} \le g(\hat{x})$ and then $\gamma(\hat{x}) \cdot \hat{p} \le h(\hat{x})$. These observations together show that

$$\min\{G(\hat{x}, \hat{p}),\, \gamma(\hat{x}) \cdot \hat{p} - h(\hat{x})\} \le 0.$$

We next assume that $\hat{x} \in \Omega$. In this case, we easily see that $G(\hat{x}, \hat{p}) \le 0$.

Next, consider the case where $\zeta(\hat{x}) = 0$, which implies that $G(\hat{x}, \hat{p}) = |\hat{p}| - L$ and $h(\hat{x}) = M$. By (132), we immediately see that $G(\hat{x}, \hat{p}) \le 0$ if $\hat{x} \in \Omega$ and $\min\{G(\hat{x}, \hat{p}), \gamma(\hat{x}) \cdot \hat{p} - h(\hat{x})\} \le 0$ if $\hat{x} \in \partial\Omega$. We thus conclude that v is a viscosity solution of (133).

We may invoke Theorem 4.2, to find a collection $\{v^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(\overline{\Omega})$ such that

$$\begin{cases} G(x, Dv^\varepsilon(x)) \le \varepsilon & \text{for all } x \in \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le h(x) & \text{for all } x \in \partial\Omega, \\ \|v^\varepsilon - v\|_{\infty,\Omega} \le \varepsilon. \end{cases}$$

But, this yields

$$\begin{cases} H(x, v^\varepsilon(x)) \le \varepsilon & \text{for all } x \in V \cap \Omega, \\ \dfrac{\partial v^\varepsilon}{\partial \gamma}(x) \le g(x) & \text{for all } x \in V \cap \partial\Omega, \\ \|v^\varepsilon - u\|_{\infty, V\cap\Omega} \le \varepsilon. \end{cases}$$

The functions v^ε have all the required properties. □

The above theorem has a version for Hamilton–Jacobi equations of evolution type.

Theorem D.2. *Let U, V be bounded open subsets of $\mathbb{R}^n \times \mathbb{R}_+$ with the properties: $\overline{V} \subset U$, $\overline{U} \subset \mathbb{R}^n \times \mathbb{R}_+$ and $V \cap Q \neq \emptyset$. Let $u \in \mathrm{Lip}(U \cap Q)$ be a viscosity solution of*

$$\begin{cases} u_t(x,t) + H(x, D_x u(x,t)) \le 0 & \text{in } U \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u}{\partial \gamma}(x,t) \le g(x) & \text{on } U \cap (\partial\Omega \times \mathbb{R}_+). \end{cases}$$

Then, for each $\varepsilon \in (0, 1)$, there exists a function $u^\varepsilon \in C^1(V \cap Q)$ such that

$$\begin{cases} u_t^\varepsilon(x,t) + H(x, D_x u^\varepsilon(x,t)) \le \varepsilon & \text{in } V \cap (\Omega \times \mathbb{R}_+), \\ \dfrac{\partial u^\varepsilon}{\partial \gamma}(x,t) \le g(x) & \text{on } V \cap (\partial\Omega \times \mathbb{R}_+), \\ \|u^\varepsilon - u\|_{\infty, V\cap Q} \le \varepsilon. \end{cases} \tag{134}$$

Proof. Choose constants $a, b \in \mathbb{R}_+$ so that $U \subset \mathbb{R}^n \times (a, b)$ and let ρ be a defining function of Ω. We may assume that ρ is bounded in $\mathbb{R}^n$. We choose a function $\zeta \in C^1(\mathbb{R})$ so that $\zeta(t) = 0$ for all $t \in [a, b]$, $\zeta'(t) > 0$ for all $t > b$, $\zeta'(t) < 0$ for all $t < a$ and $\min\{\zeta(a/2), \zeta(2b)\} > \|\rho\|_{\infty,\Omega}$.

We set

$$\tilde{\rho}(x,t) = \rho(x) + \zeta(t) \quad \text{for } (x,t) \in \mathbb{R}^{n+1},$$
$$\tilde{\Omega} = \{(x,t) \in \mathbb{R}^{n+1} \, : \, \tilde{\rho}(x,t) < 0\}.$$

It is easily seen that

$$\tilde{\Omega} \subset \Omega \times (a/2, 2b) \quad \text{and} \quad \tilde{\Omega} \cap (\mathbb{R}^n \times [a, b]) = \Omega \times [a, b].$$

Let $(x,t) \in \mathbb{R}^{n+1}$ be such that $\tilde{\rho}(x,t) = 0$. It is obvious that $(x,t) \in \overline{\Omega} \times [a/2, 2b]$. If $a \le t \le b$, then $\rho(x) = 0$ and thus $D\rho(x) \neq 0$. If either $t > b$ or $t < a$, then $|\zeta'(t)| > 0$. Hence, we have $D\tilde{\rho}(x,t) \neq 0$. Thus, $\tilde{\rho}$ is a defining function of $\tilde{\Omega}$.

Let $M > 0$ and define $\tilde{\gamma} \in C(\partial\tilde{\Omega}, \mathbb{R}^{n+1})$ by

$$\tilde{\gamma}(x,t) = \big((1 + M\rho(x))_+\gamma(x),\ \zeta'(t)\big),$$

where we may assume that γ is defined and continuous in $\overline{\Omega}$. We note that for any $(x,t) \in \partial\tilde{\Omega}$,

$$\tilde{\gamma}(x,t) \cdot D\tilde{\rho}(x,t) = (1 + M\rho(x))_+\gamma(x) \cdot D\rho(x) + \zeta'(t)^2.$$

Note as well that $(1 + M\rho(x))_+ = 1$ for all $x \in \partial\Omega$ and

$$\lim_{M\to\infty}(1 + M\rho(x))_+ = 0 \quad \text{locally uniformly in } \Omega.$$

Thus we can fix $M > 0$ so that for all $(x,t) \in \partial\tilde{\Omega}$,

$$\tilde{\gamma}(x,t) \cdot D\tilde{\rho}(x,t) = (1 + M\rho(x))_+\gamma(x) \cdot D\rho(x) + \zeta'(t)^2 > 0.$$

Noting that for each $x \in \Omega$, the x-section $\{t \in \mathbb{R} : (x,t) \in \tilde{\Omega}\}$ of $\tilde{\Omega}$ is an open interval (or, line segment), we deduce that $\tilde{\Omega}$ is a connected set. We may assume that g is defined and continuous in $\overline{\Omega}$. We define $\tilde{g} \in C(\partial\tilde{\Omega})$ by $\tilde{g}(x,t) = g(x)$. Thus, assumptions (A1)–(A4) hold with $n+1$, $\tilde{\Omega}$, $\tilde{\gamma}$ and $\tilde{g}$ in place of n, Ω, γ and g.

Let $L > 0$ be a Lipschitz bound of the function u in $U \cap Q$. Set

$$\tilde{H}(x,t,p,q) = H(x,p) + q + 2(|q| - L)_+ \quad \text{for } (x,t,p,q) \in \overline{\tilde{\Omega}} \times \mathbb{R}^{n+1},$$

and note that $\tilde{H} \in C(\overline{\tilde{\Omega}} \times \mathbb{R}^{n+1})$ satisfies (A5)–(A7), with Ω replaced by $\tilde{\Omega}$.

We now claim that u is a viscosity solution of

$$\begin{cases} \tilde{H}(x,t,Du(x,t)) \le 0 & \text{in } U \cap \tilde{\Omega}, \\ \tilde{\gamma}(x,t) \cdot Du(x,t) \le \tilde{g}(x,t) & \text{on } U \cap \partial\tilde{\Omega}. \end{cases}$$

Indeed, since $U \cap \tilde{\Omega} = U \cap Q$ and $U \cap \partial\tilde{\Omega} = U \cap \partial Q$, if $(x,t) \in U \cap \overline{\tilde{\Omega}}$ and $(p,q) \in D^+u(x,t)$, then we get $|q| \le L$ by the cylindrical geometry of Q and, by the viscosity property of u,

$$\begin{cases} q + H(x,p) + 2(|q| - L)_+ \le 0 & \text{if } (x,t) \in \tilde{\Omega}, \\ \min\{q + H(x,p) + 2(|q| - L)_+, \gamma(x) \cdot p - g(x)\} \le 0 & \text{if } (x,t) \in \partial\tilde{\Omega}. \end{cases}$$

We apply Theorem D.1, to find a collection $\{u^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(V \cap \overline{\tilde{\Omega}})$ such that

$$\begin{cases} \tilde{H}(x,t,Du^\varepsilon(x,t)) \le \varepsilon & \text{in } V \cap \tilde{\Omega}, \\ \tilde{\gamma}(x,t)\cdot Du^\varepsilon(x,t) \le \tilde{g}(x,t) & \text{on } U \cap \tilde{\Omega}, \\ \|u^\varepsilon - u\|_{\infty,V\cap\tilde{\Omega}} \le \varepsilon. \end{cases}$$

It is straightforward to see that the collection $\{u^\varepsilon\}_{\varepsilon\in(0,1)} \subset C^1(V \cap Q)$ satisfies (134). □

A.5 A Proof of Lemma 5.4

This subsection is mostly devoted to the proof of Lemma 5.4, a version of the Dunford–Pettis theorem. We also give a proof of the weak-star compactness of bounded sequences in $L^\infty(J,\mathbb{R}^m)$, where $J = [a, b]$ is a finite interval in $\mathbb{R}$.

Proof (Lemma 5.4). We define the functions $F_j \in C(J,\mathbb{R}^m)$ by

$$F_j(x) = \int_a^x f_j(t)\mathrm{d}t.$$

By the uniform integrability of $\{f_j\}$, the sequence $\{F_j\}_{j\in\mathbb{N}}$ is uniformly bounded and equi-continuous in J. Hence, the Ascoli–Arzela theorem ensures that it has a subsequence converging to a function F uniformly in J. We fix such a subsequence and denote it again by the same symbol $\{F_j\}$. Because of the uniform integrability assumption, the sequence $\{F_j\}$ is equi-absolutely continuous in J. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le b, \qquad \sum_{i=1}^n (b_i - a_i) < \delta,$$

$$\Longrightarrow \quad \sum_{i=1}^n |f_j(b_i) - f_j(a_i)| < \varepsilon \quad \text{for all } \ j \in \mathbb{N}.$$

An immediate consequence of this is that $F \in \mathrm{AC}(J,\mathbb{R}^m)$. Hence, for some $f \in L^1(J,\mathbb{R}^m)$, we have

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{for all } \ x \in J.$$

Next, let $\phi \in C^1(J)$, and we show that

$$\lim_{j\to\infty} \int_a^b f_j(x)\phi(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x. \tag{135}$$

Integrating by parts, we observe that as $j \to \infty$,

$$\int_a^b f_j(x)\phi(x)\,\mathrm{d}x = \left[F_j\phi\right]_a^b - \int_a^b F_j(x)\phi'(x)\,\mathrm{d}x$$
$$\to \left[F\phi\right]_a^b - \int_a^b F(x)\phi'(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x.$$

Hence, (135) is valid.

Now, let $\phi \in L^\infty(J)$. We regard the functions f_j, f, ϕ as functions defined in $\mathbb{R}$ by setting $f_j(x) = f(x) = \phi(x) = 0$ for $x < a$ or $x > b$. Let $\{k_\varepsilon\}_{\varepsilon>0}$ be a collection of standard mollification kernels. We recall that

$$\lim_{\varepsilon\to 0} \|k_\varepsilon * \phi - \phi\|_{L^1(J)} = 0, \tag{136}$$

$$|k_\varepsilon * \phi(x)| \le \|\phi\|_{L^\infty(J)} \quad \text{for all } x \in J,\ \varepsilon > 0. \tag{137}$$

Fix any $\delta > 0$. By the uniform integrability assumption, we have

$$M := \sup_{j\in\mathbb{N}} \|f_j - f\|_{L^1(J)} < \infty.$$

Let $\alpha > 0$ and set

$$E_j := \{x \in J \,:\, |(f_j - f)(x)| > \alpha\}.$$

By the Chebychev inequality, we get

$$|E_j| \le \frac{M}{\alpha}.$$

By the uniform integrability assumption, if $\alpha > 0$ is sufficiently large, then

$$\int_{E_j} |(f_j - f)(x)|\,\mathrm{d}x < \delta. \tag{138}$$

In what follows we fix $\alpha > 0$ large enough so that (138) holds. We write $f_j - f = g_j + b_j$, where $g_j = (f_j - f)(1 - \mathbf{1}_{E_j})$ and $b_j = (f_j - f)\mathbf{1}_{E_j}$. Then,

$$|g_j(x)| \le \alpha \quad \text{for all } x \in J \qquad \text{and} \qquad \|b_j\|_{L^1(J)} < \delta.$$

Observe that

$$I_j := \int_J f_j(x)\phi(x)\,\mathrm{d}x - \int_J f(x)\phi(x)\,\mathrm{d}x$$
$$= \int_J (f_j - f)(x)\,k_\varepsilon * \phi(x)\,\mathrm{d}x + \int_J (f_j - f)(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x$$

and

$$\Big|\int_J (f_j - f)(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big|$$
$$\le \Big|\int_J g_j(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big| + \Big|\int_J b_j(x)(\phi - k_\varepsilon * \phi)(x)\,\mathrm{d}x\Big|$$
$$\le \alpha\|k_\varepsilon * \phi - \phi\|_{L^1(J)} + 2\delta\|\phi\|_{L^\infty(J)}.$$

Hence, in view of (135) and (136), we get $\limsup_{j\to\infty}|I_j| \le 2\delta\|\phi\|_{L^\infty(J)}$. As $\delta > 0$ is arbitrary, we get $\lim_{j\to\infty} I_j = 0$, which completes the proof. □

As a corollary of Lemma 5.4, we deduce that the weak-star compactness of bounded sequences in $L^\infty(J, \mathbb{R}^m)$:

Lemma E.1. *Let $J = [a, b]$, with $-\infty < a < b < \infty$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence of functions in $L^\infty(J, \mathbb{R}^m)$. Then $\{f_k\}$ has a subsequence which converges weakly-star in $L^\infty(J, \mathbb{R}^m)$.*

Proof. Set $M = \sup_{k\in\mathbb{N}} \|f_k\|_{L^\infty(J)}$. Let $E \subset J$ be a measurable set, and observe that

$$\int_E |f_k(t)|\mathrm{d}t \le M|E| \quad \text{for all } k \in \mathbb{N},$$

which shows that the sequence $\{f_k\}$ is uniformly integrable in J. Thanks to Lemma 5.4, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ of $\{f_k\}$ which converges to a function f weakly in $L^1(J, R^m)$.

Let $i \in \mathbb{N}$ and set $E_i = \{t \in J : |f(t)| > M + 1/i\}$ and $g_i(t) = \mathbf{1}_{E_i}(t)f(t)/|f(t)|$ for $t \in J$. Since $g_i \in L^\infty(J, \mathbb{R}^m)$, we get

$$\int_J f_{k_j}(t)\cdot g_i(t)\mathrm{d}t \to \int_J |f(t)|\mathbf{1}_{E_i}(t)\mathrm{d}t \quad \text{as } j \to \infty.$$

Hence, using the Chebychev inequality, we obtain

$$\Big(M + \frac{1}{i}\Big)|E_i| \le \int_J |f(t)|\mathbf{1}_{E_i}(t)\mathrm{d}t \le \int_J M\mathbf{1}_{E_i}(t)\mathrm{d}t = M|E_i|,$$

which ensures that $|E_i| = 0$. Thus, we find that $|f(t)| \le M$ a.e. in J.

Now, fix any $\phi \in L^1(J, \mathbb{R}^m)$. We select a sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset L^\infty(J, \mathbb{R}^m)$ so that, as $i \to \infty$, $\phi_i \to \phi$ in $L^1(J, \mathbb{R}^m)$. For each $i \in \mathbb{N}$, we have

$$\lim_{j\to\infty}\int_J f_{k_j}(t)\cdot\phi_i(t)\mathrm{d}t = \int_J f(t)\cdot\phi_i(t)\mathrm{d}t.$$

On the other hand, we have

$$\Big|\int_J f_{k_j}(t)\cdot\phi(t)\mathrm{d}t - \int_J f_{k_j}(t)\cdot\phi_i(t)\mathrm{d}t\Big| \le M\|\phi - \phi_i\|_{L^1(J)} \quad \text{for all } j \in \mathbb{N}$$

and

$$\Big|\int_J f(t)\cdot\phi(t)\mathrm{d}t - \int_J f(t)\cdot\phi_i(t)\mathrm{d}t\Big| \le M\|\phi-\phi_i\|_{L^1(J)}.$$

These together yield

$$\lim_{j\to\infty}\int_J f_{k_j}(t)\cdot\phi(t)\mathrm{d}t = \int_J f(t)\cdot\phi(t)\mathrm{d}t. \qquad \square$$

A.6 Rademacher's Theorem

We give here a proof of Rademacher's theorem.

Theorem F.1 (Rademacher). *Let* $B = B_1 \subset \mathbb{R}^n$ *and* $f \in \mathrm{Lip}(B)$. *Then* f *is differentiable almost everywhere in* B.

To prove the above theorem, we mainly follow the proof given in [1].

Proof. We first show that f has a distributional gradient $Df \in L^\infty(B)$.

Let $L > 0$ be a Lipschitz bound of the function f. Let $i \in \{1, 2, \dots, n\}$ and e_i denote the unit vector in $\mathbb{R}^n$ with unity as the i-th entry. Fix any $\phi \in C_0^1(B)$ and observe that

$$\begin{aligned}\int_B f(x)\phi_{x_i}(x)\mathrm{d}x &= \lim_{r\to 0+}\int_B f(x)\frac{\phi(x+re_i)-\phi(x)}{r}\mathrm{d}x\\ &= \lim_{r\to 0+}\int_B \frac{f(x-re_i)-f(x)}{r}\phi(x)\mathrm{d}x\end{aligned}$$

and

$$\Big|\int_B f(x)\phi_{x_i}(x)\mathrm{d}x\Big| \le L\int_B|\phi(x)|\mathrm{d}x \le L|B|^{1/2}\|\phi\|_{L^2(B)}.$$

Thus, the map

$$C_0^1(B)\ni\phi\mapsto -\int_B f(x)\phi_{x_i}(x)\mathrm{d}x \in \mathbb{R}$$

extends uniquely to a bounded linear functional G_i on $L^2(B)$. By the Riesz representation theorem, there is a function $g_i \in L^2(B)$ such that

$$G_i(\phi) = \int_B g_i(x)\phi(x)\mathrm{d}x \quad \text{for all } \phi\in L^2(B).$$

This shows that $g = (g_1, \dots, g_n)$ is the distributional gradient of f.

We plug the function $\phi \in L^2(B)$ given by $\phi(x) = (g_i(x)/|g_i(x)|)\mathbf{1}_{E_k}(x)$, where $k \in \mathbb{N}$ and $E_k = \{x \in B : |g_i(x)| > L + 1/k\}$, into the inequality $|G_i(\phi)| \le L\|\phi\|_{L^1(B)}$, to obtain

$$\int_B |g_i(x)|\mathbf{1}_{E_k}(x)\mathrm{d}x \le L\int_B \mathbf{1}_{E_k}(x)\mathrm{d}x = L|E_k|,$$

which yields

$$(L + 1/k)|E_k| \le L|E_k|.$$

Hence, we get $|E_k| = 0$ for all $k \in \mathbb{N}$ and $|\{x \in B : |g_i(x)| > L\}| = 0$. That is, $g_i \in L^\infty(B)$ and $|g_i(x)| \le L$ a.e. in B.

The Lebesgue differentiation theorem (see [57]) states that for a.e. $x \in B$, we have $g(x) \in \mathbb{R}^n$ and

$$\lim_{r\to 0+} \frac{1}{r^n} \int_{B_r} |g(x+y) - g(x)|\mathrm{d}y = 0. \tag{139}$$

Now, we fix such a point $x \in B$ and show that f is differentiable at x. Fix an $r > 0$ so that $B_r(x) \subset B$. For $\delta \in (0, r)$, consider the function $h_\delta \in C(\overline{B})$ given by

$$h_\delta(y) = \frac{f(x+\delta y) - f(x)}{\delta}.$$

We claim that

$$\lim_{\delta\to 0} h_\delta(y) = g(x)\cdot y \quad \text{uniformly for } \ y \in \overline{B}. \tag{140}$$

Note that $h_\delta(0) = 0$ and h_δ is Lipschitz continuous with Lipschitz bound L. By the Ascoli–Arzela theorem, for any sequence $\{\delta_j\} \subset (0, r)$ converging to zero, there exist a subsequence $\{\delta_{j_k}\}_{k\in\mathbb{N}}$ of $\{\delta_j\}$ and a function $h_0 \in C(\overline{B})$ such that

$$\lim_{k\to\infty} h_{\delta_{j_k}}(x) = h_0(y) \quad \text{uniformly for } \ y \in \overline{B}.$$

In order to prove (140), we need only to show that $h_0(y) = g(x)\cdot y$ for all $y \in B$.

Since $h_\delta(0) = 0$ for all $\delta \in (0, r)$, we have $h_0(0) = 0$. We observe from (139) that

$$\int_B |g(x+\delta y) - g(x)|\mathrm{d}y = \int_{B_\delta} |g(x+y) - g(x)|\delta^{-n}\mathrm{d}y \ \to \ 0 \quad \text{as } \ \delta \to 0.$$

Using this, we compute that for all $\phi \in C_0^1(B)$,

$$\begin{aligned}
\int_B h_0(y)\phi_{y_i}(y)\mathrm{d}y &= \lim_{k\to\infty} \int_B h_{\delta_{j_k}}(y)\phi_{y_i}(y)\mathrm{d}y \\
&= -\lim_{k\to\infty} \int_B g_i(x+\delta_{j_k}y)\phi(y)\mathrm{d}y \\
&= -\int_B g_i(x)\phi(y)\mathrm{d}y = \int_B g(x)\cdot y\phi_{y_i}(y)\mathrm{d}y.
\end{aligned}$$

This guarantees that $h_0(y) - g(x) \cdot y$ is constant for all $y \in B$ while $h_0(0) = 0$. Thus, we see that $h_0(y) = g(x) \cdot y$ for all $y \in B$, which proves (140).

Finally, we note that (140) yields

$$f(x+y) = f(x) + g(x) \cdot y + o(|y|) \quad \text{as} \quad y \to 0. \qquad \square$$

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Idempotent/Tropical Analysis, the Hamilton–Jacobi and Bellman Equations

Grigory L. Litvinov

In dear memory of my beloved wife Irina.

Abstract Tropical and idempotent analysis with their relations to the Hamilton–Jacobi and matrix Bellman equations are discussed. Some dequantization procedures are important in tropical and idempotent mathematics. In particular, the Hamilton–Jacobi–Bellman equation is treated as a result of the Maslov dequantization applied to the Schrödinger equation. This leads to a linearity of the Hamilton–Jacobi–Bellman equation over tropical algebras. The correspondence principle and the superposition principle of idempotent mathematics are formulated and examined. The matrix Bellman equation and its applications to optimization problems on graphs are discussed. Universal algorithms for numerical algorithms in idempotent mathematics are investigated. In particular, an idempotent version of interval analysis is briefly discussed.

1 Introduction

In these lecture notes we shall discuss some important problems of tropical and idempotent mathematics and especially those of idempotent and tropical analysis. Relations to the Hamilton–Jacobi and matrix Bellman equations will be examined. Applications of general principles of idempotent mathematics to numerical algorithms and their computer implementations will be discussed.

G.L. Litvinov (✉)
Institute for Information Transmission Problems, B. Karetnyi per. 19/1, Moscow, 127994 Russia

National Research University Higher School of Economics, Moscow, Russia
e-mail: glitvinov@gmail.com

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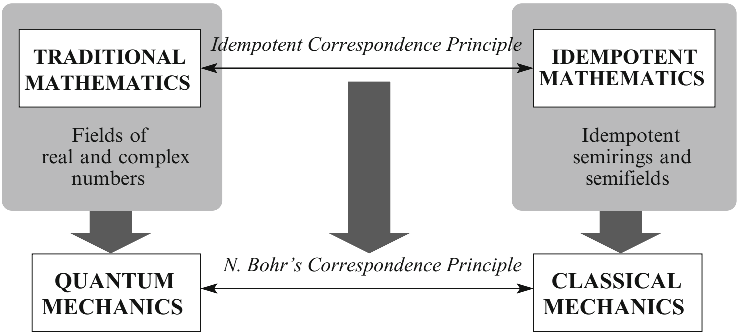

Fig. 1 Relations between idempotent and traditional mathematics

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in Sects. 2 and 3). For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$), see [39–42]. Some other dequantization procedures lead to interesting applications, e.g., to convex geometry, see below and [46, 55, 56].

Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be regarded as a result of the Maslov dequantization applied to the traditional algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [32, 72, 73, 94–96]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar constructions and results over idempotent semirings. A systematic application of this correspondence principle leads to a variety of theoretical and applied results [39–43], see Fig. 1.

The history of the subject is discussed, e.g., in [39], with extensive bibliography. See also [15, 17, 18, 20, 22, 40–42, 45].

Maslov's idempotent superposition principle means that many nonlinear problems related to extremal problems are linear over suitable idempotent semirings. The principle is very important for applications including numerical and parallel computations. See Maslov's original formulation in [63–65], as well as [6, 14, 15, 17, 18, 20, 22, 33, 39–43, 45], and below.

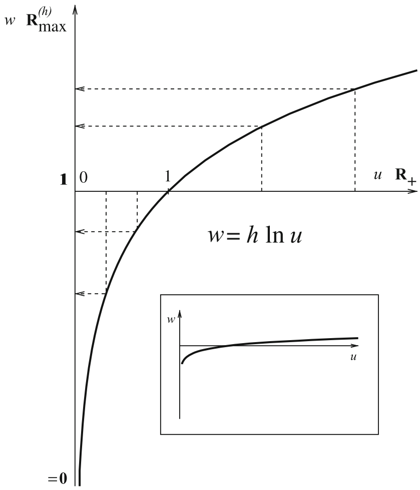

Fig. 2 Deformation of $\mathbf{R}_+$ to $\mathbf{R}^{(h)}$. *Inset*: the same for a small value of h

2 The Maslov Dequantization

Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real and complex numbers. The so-called max-plus algebra $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be seen as a result of the *Maslov dequantization* of the semifield $\mathbf{R}_+$ of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x,$$

where $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$, see Fig. 2. Let the addition and multiplication operations be mapped from $\mathbf{R}_+$ to $\mathbf{R} \cup \{-\infty\}$ by Φ_h, i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,$$
$$\mathbf{0} = -\infty = \Phi_h(0), \quad \mathbf{1} = 0 = \Phi_h(1).$$

It can be easily checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. This deformation of the algebraic structure borrowed from $\mathbf{R}_+$ brings us to the semifield $\mathbf{R}_{\max}$, known as the *max-plus algebra*, with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$.

The semifield $\mathbf{R}_{\max}$ is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element x of this semiring.

The semifield $\mathbf{R}_{\max}$ is also called a *tropical algebra*. The semifield $\mathbf{R}^{(h)} = \Phi_h(\mathbf{R}_+)$ with operations $\oplus_h$ and $\odot$ (i.e.$+$) is called a *subtropical algebra*.

The semifield $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with operations $\oplus = \min$ and $\odot = +$ ($\mathbf{0} = +\infty$, $\mathbf{1} = 0$) is isomorphic to $\mathbf{R}_{\max}$.

The analogy with quantization is obvious; the parameter h plays the role of the Planck constant. The map $x \mapsto |x|$ and the Maslov dequantization for $\mathbf{R}_+$ give us a natural transition from the field $\mathbf{C}$ (or $\mathbf{R}$) to the max-plus algebra $\mathbf{R}_{\max}$. *We will also call this transition the Maslov dequantization*. In fact the Maslov dequantization corresponds to the usual Schrödinger dequantization but for imaginary values of the Planck constant (see below). The transition from numerical fields to the max-plus algebra $\mathbf{R}_{\max}$ (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics*. The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is the transition from basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called *idempotent mathematics*, i.e. mathematics over idempotent semifields and semirings.

Remark. The term "tropical" appeared in [89] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand Maslov used this term in 1980s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, "tropical" means "over $\mathbf{R}_{\max}$ (or $\mathbf{R}_{\min}$)" and tropical algebras are $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. The terms "max-plus", "max-algebra" and "min-plus" are often used in the same sense.

3 Semirings and Semifields: The Idempotent Correspondence Principle

Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is a *semiring* if the following conditions are satisfied:

- The addition $\oplus$ and the multiplication $\odot$ are associative.
- The addition $\oplus$ is commutative.
- The multiplication $\odot$ is distributive with respect to the addition $\oplus$:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

and

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all $x, y, z \in S$.

A *unity* (we suppose that it exists) of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* (if it exists) of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral element $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible with respect to the multiplication. For the theory of semirings and semifields the reader is referred, e.g., to [26].

The analogy with quantum physics discussed in Sect. 2 and below leads to the following *idempotent correspondence principle*:

There is a (heuristic) correspondence between important, useful and interesting constructions and results over the field of complex (or real) numbers (or the semifield of nonnegative numbers) and similar constructions and results over idempotent semirings in the spirit of Bohr's correspondence principle in quantum theory [40–42].

This principle can be also applied to algorithms and their software and hardware implementations. Examples are discussed below; see also [39–42, 47–50, 53–57].

4 Idempotent Analysis

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by Maslov and his collaborators and then developed by many authors. The subject is presented in the book of Kolokoltsov and Maslov [33] (a version of this book in Russian was published in 1994).

Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, and unit $\mathbf{1}$. The set S is equipped with the *standard partial order* $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. If S contains a zero element $\mathbf{0}$, then all elements of S are nonnegative: $\mathbf{0} \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic "topological" concepts and results of idempotent analysis on the purely algebraic level; this line of reasoning was examined systematically in [18, 39–57].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $B(X, S)$ of all functions $X \to S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\,dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum\limits_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus\limits_i \varphi(x_i) \odot \sigma_i = \max\limits_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An *idempotent* (or *Maslov*) *measure* on X is defined by the formula $m_\psi(Y) = \sup_{x\in Y} \psi(x)$, where $\psi \in B(X,S)$ is a fixed function. The integral with respect to this measure is defined by the formula

$$I_\psi(\varphi) = \int_X^\oplus \varphi(x)\, dm_\psi = \int_X^\oplus \varphi(x) \odot \psi(x)\, dx = \sup_{x\in X}(\varphi(x) \odot \psi(x)). \quad (2)$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order is opposite to the conventional order $\leq$, so in this case (2) takes the form

$$\int_X^\oplus \varphi(x)\, dm_\psi = \int_X^\oplus \varphi(x) \odot \psi(x)\, dx = \inf_{x\in X}(\varphi(x) \odot \psi(x)),$$

where inf is understood in the sense of the conventional order $\leq$.

We shall see that in idempotent analysis measures and generalized functions (versions of distributions in the sense of L. Schwartz) are generated by usual functions. For example the δ-functional $\delta_y : \varphi(\cdot) \mapsto \varphi(y)$ is generated by the function

$$\delta_y(x) = \begin{cases} \mathbf{1}, & \text{if } x = y, \\ \mathbf{0}, & \text{if } x \neq y. \end{cases}$$

It is clear that

$$\varphi(y) = \int_X^\oplus \delta_y(x) \odot \varphi(x) dx = \sup_x(\delta_y(x) \odot \varphi(x)).$$

5 The Superposition Principle and Linear Equations

5.1 Heuristics

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings; this is Maslov's idempotent superposition principle, see [63–65]. More generally, the idempotent superposition principle means that although some important problems and equations (related to extremal problems, e.g., optimization problems, the Bellman equation and its instances, the Hamilton–Jacobi equation) are nonlinear in the usual sense, they can be treated as linear over appropriate idempotent semirings. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F, see below and [6, 14, 15, 20, 22, 28, 29]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [14, 15].

The linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$, which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),$$

where $x = (x_1, \ldots, x_N)$ are generalized coordinates, $p = (p_1, \ldots, p_N)$ are generalized momenta, m_i are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

$$S = \int_{t_0}^{t} L(x(t), \dot{x}(t), t)\, dt,$$

where the integration is performed along the actual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of t and t_0 and arbitrary trajectories $x(t)$, the action functional $S = S(x(t))$ can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as elements of $\mathbf{R}_{\min}$. In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of e^{-S}, i.e. idempotent integral $\int_{\{paths\}}^{\oplus} e^{-S(x(t))} D\{x(t)\}$ with respect to the max-plus algebra $\mathbf{R}_{\max}$. Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleĭnik solution formula for the Hamilton–Jacobi equation.

Since $\partial S/\partial x_i = p_i$, $\partial S/\partial t = -H(p,x)$, the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \tag{3}$$

Quantization leads to the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \hat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi, \tag{4}$$

where $\psi = \psi(x,t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbf{R}^N)$, and $\hat{H}$ is the energy operator obtained by substitution of the momentum operators $\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$ and the coordinate operators $\hat{x}_i : \psi \mapsto x_i\psi$ for the variables p_i and x_i in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x,t) = a(x,t)e^{iS(x,t)/\hbar}$, and to keep only the leading order as $\hbar \to 0$ (the "semiclassical" limit).

Instead of doing this, we switch to imaginary values of the Planck constant $\hbar$ by the substitution $h = i\hbar$, assuming $h > 0$. Then the Schrödinger equation (4) becomes similar to the heat equation:

$$h\frac{\partial u}{\partial t} = H\left(-h\frac{\partial}{\partial x_i}, \hat{x}_i\right)u, \tag{5}$$

where the real-valued function u corresponds to the wave function ψ. A similar idea (a switch to imaginary time) is used in the Euclidean quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of (5). Thus if u_1 and u_2 are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \tag{6}$$

Let $S = h\ln u$ or $u = e^{S/h}$ as in Sect. 2 above. It can easily be checked that (5) thus turns to

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N}\frac{1}{2m_i}\left(\frac{\partial S}{\partial x_i}\right)^2 + h\sum_{i=1}^{n}\frac{1}{2m_i}\frac{\partial^2 S}{\partial x_i^2}. \tag{7}$$

Thus we have a transition from (3) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{ReS/h}$, where ReS is the real part of S. Now let us consider S as a real variable. Equation (7) is nonlinear in the conventional sense. However, if S_1 and S_2 are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2 \tag{8}$$

obtained from (6) by means of the substitution $S = h \ln u$. Here the generalized multiplication $\odot$ coincides with the ordinary addition and the generalized addition $\oplus_h$ is the image of the conventional addition under the above change of variables. As $h \to 0$, we obtain the operations of the idempotent semiring $\mathbf{R}_{\max}$, i.e., $\oplus = \max$ and $\odot = +$, and (7) becomes the Hamilton–Jacobi equation (3), since the third term in the right-hand side of (7) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over $\mathbf{R}_{\max}$. This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see, e.g., [33, 43, 85]. Notice that if h is changed to $-h$, then we have that the resulting Hamilton–Jacobi equation is linear over $\mathbf{R}_{\min}$.

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. The idempotent linear functional analysis (see below) is a natural tool for investigation of those nonlinear infinite-dimensional problems that possess this property.

5.2 The Cauchy Problem for the Hamilton–Jacobi Equations

A rigorous "idempotent" approach to the investigation of the Hamilton–Jacobi equation was developed by Kolokoltsov and Maslov [33] (a Russian version of this book was published in 1994); see also [71, 85, 92, 93].

Let us consider, inspired by a long tradition, the well-known Cauchy problem for the Hamilton–Jacobi equation (3). Given the action function at time T

$$S(T,x) = S_T(x) = \varphi(x), \quad x \in \mathbf{R}^N, \tag{9}$$

the Cauchy problem asks to reconstruct $S(t,x)$ for $x \in \mathbf{R}^N$ during the time interval $0 \le t \le T$.

We shall discuss the min-plus linearity of this problem and denote by U_t the resolving operator, i.e. the map which assigns to each given $S_T(x)$ the solution $S(t,x)$ of the Cauchy problem in the interval $0 \le t \le T$. Then the map U_t, for each t, is a linear (over $\mathbf{R}_{\min}$) operator in the space LSC$(\mathbf{R}^n, \mathbf{R}_{\min})$ of lower semicontinuous functions taking their values in $\mathbf{R}_{\min}$. Moreover U_t is an integral operator (in the sense of idempotent mathematics) of the form:

$$(U_t\varphi)(x) = \int^{\oplus} \varphi(y)K_t(x,y)dy = \inf_y\{\varphi(y) + K_t(x,y)\}, \qquad (10)$$

where $K_t(x, y)$, as a function of $y \in \mathbf{R}^n$, is bounded from below and lower semicontinuous. See [33, 85] for details.

The operator U_t (as well as other integral operators, see Sect. 7 below) has the following property:

$$U_t(\bigoplus_\nu \varphi_\nu) = \bigoplus_\nu (U_t\varphi_\nu), \qquad (11)$$

where $\{\varphi_\nu\}$ is a bounded set of elements in $\mathrm{LSC}(\mathbf{R}^n, \mathbf{R}_{\min})$. So if we have such a family of functions $S_\nu(T, x)$ and $S(T, x) = \int^{\oplus} S_\nu(T, x)d\nu = \inf_\nu(S_\nu(T, x))$, then the solution of the Cauchy problem is expressed as $S(t, x) = \inf_\nu(S_\nu(t, x))$.

Relations between the "idempotent approach", viscosity solutions and minimax solutions in the sense of Subbotin [92, 93] are examined, e.g., in [85] in details; see also McEneaney [71]. To this end, let us mention that more general Hamiltonians of the form $H = H(t, x, p)$ (satisfying some additional conditions) and different kinds of solution spaces are also considered in the literature.

The situation is similar for the Cauchy problem for the homogeneous Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial x}) = 0, \quad S_{t=0} = S_0(x),$$

where $H : \mathbf{R}^n \mapsto \mathbf{R}$ is a convex (not strictly) first order homogeneous function

$$H(p) = \sup_{(f,g)\in V} (f \cdot p + g), \quad f \in \mathbf{R}^n, \quad g \in \mathbf{R},$$

and V is a compact set in $\mathbf{R}^{n+1}$. See [33].

To develop a rigorous "idempotent" approach to differential equations and other problems, one needs an idempotent version of analysis and, especially, functional analysis. See Sect. 7 below.

6 Convolution and the Fourier–Legendre Transform

Let G be a group. Then the space $\mathscr{B}(G, \mathbf{R}_{\max})$ of all bounded functions $G \to \mathbf{R}_{\max}$ (see above) is an idempotent semiring with respect to the following analog $\circledast$ of the usual convolution:

$$(\varphi(x) \circledast \psi)(g) == \int_G^{\oplus} \varphi(x) \odot \psi(x^{-1} \cdot g)\, dx = \sup_{x\in G}(\varphi(x) + \psi(x^{-1} \cdot g)).$$

Of course, it is possible to consider other "function spaces" (and other basic semirings instead of $\mathbf{R}_{\max}$).

Let $G = \mathbf{R}^n$, where $\mathbf{R}^n$ is considered as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x)\, dx \tag{12}$$

where $e^{i\xi \cdot x}$ is a character of the group G, i.e., a solution of the following functional equation:

$$f(x + y) = f(x) f(y).$$

The idempotent analog of this equation is

$$f(x + y) = f(x) \odot f(y) = f(x) + f(y),$$

so "continuous idempotent characters" are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (12) assumes the form

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^{\oplus} \xi \cdot x \odot \varphi(x)\, dx = \sup_{x \in G}(\xi \cdot x + \varphi(x)). \tag{13}$$

The transform in (13) is the *Legendre transform* (up to some change of notation) [65]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics. The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [61].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution $\circledast$ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform.

The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory, see, e.g., [50]. In particular, "idempotent" representations of groups can be examined as representations of the corresponding convolution semirings (i.e. idempotent group semirings) in semimodules.

7 Idempotent Functional Analysis

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity take $S = \mathbf{R}_{\max}$ and let X be an arbitrary set. The idempotent integration can be defined by the formula (1), see above. The functional $I(\varphi)$ is linear over S and its values correspond to limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions φ and ψ is defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$

So it is natural to construct idempotent analogs of integral operators in the form

$$\varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x, y) \odot \varphi(y)\, dy = \sup_{y \in Y} \{K(x, y) + \varphi(y)\}, \tag{14}$$

where $\varphi(y)$ is an element of a space of functions defined on a set Y, and $K(x, y)$ is an S-valued function on $X \times Y$. Of course, expressions of this type are standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings which are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces it was proved that every "good" linear operator (in the idempotent sense) can be presented in the form (14); this is an idempotent version of the kernel theorem of Schwartz; results of this type were proved by Kolokoltsov, Dudnikov and Samborskiĭ, Singer, Shubin and others. So every 'good' linear functional can be presented in the form $\varphi \mapsto \langle \varphi, \psi \rangle$, where $\langle , \rangle$ is an idempotent scalar product.

In the framework of idempotent functional analysis results of this type can be proved in a very general situation. In [47–50, 54, 57] an algebraic version of the idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms (see below). The treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known theorems of Hahn–Banach, Riesz, and Riesz–Fisher) to idempotent analogs of Grothendieck's concepts and results on topological tensor products, nuclear spaces and operators. Abstract idempotent versions of the kernel theorem are formulated. Note that the transition from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many non-isomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) were obtained by Cohen, Gaubert, and Quadrat. Idempotent functional analysis has received much attention in the last years, see, e.g., [3, 18, 28–30, 33–57, 68, 88] and works cited in [39]. All the results presented in this section are proved in [49] (Sects. 7.1–7.4) and in [57] (Sects. 7.5–7.10)

7.1 Idempotent Semimodules and Idempotent Linear Spaces

An additive semigroup S with commutative addition $\oplus$ is called an *idempotent semigroup* if the relation $x \oplus x = x$ is fulfilled for all elements $x \in S$. If S contains a neutral element, this element is denoted by the symbol $\mathbf{0}$. Any idempotent semigroup is a partially ordered set with respect to the following standard order: $x \preceq y$ if and only if $x \oplus y = y$. It is obvious that this order is well defined and $x \oplus y = \sup\{x, y\}$. Thus, any idempotent semigroup is an upper semilattice; moreover, the concepts of idempotent semigroup and upper semilattice coincide, see [10]. An idempotent semigroup S is called *a-complete* (or *algebraically complete*) if it is complete as an ordered set, i.e., if any subset X in S has the least upper bound $\sup(X)$ denoted by $\oplus X$ and the greatest lower bound $\inf(X)$ denoted by $\wedge X$. This semigroup is called *b-complete* (or *boundedly complete*), if any bounded above subset X of this semigroup (including the empty subset) has the least upper bound $\oplus X$ (in this case, any nonempty subset Y in S has the greatest lower bound $\wedge Y$ and S in a lattice). Note that any a-complete or b-complete idempotent semiring has the zero element $\mathbf{0}$ that coincides with $\oplus\emptyset$, where $\emptyset$ is the empty set. Certainly, a-completeness implies the b-completeness. Completion by means of cuts [10] yields an embedding $S \to \hat{S}$ of an arbitrary idempotent semigroup S into an a-complete idempotent semigroup $\hat{S}$ (which is called a *normal completion* of S); in addition, $\hat{\hat{S}} = S$. The b-completion procedure $S \to \hat{S}_b$ is defined similarly: if $S \ni \infty = \sup S$, then $\hat{S}_b = \hat{S}$; otherwise, $\hat{S} = \hat{S}_b \cup \{\infty\}$. An arbitrary b-complete idempotent semigroup S also may differ from $\hat{S}$ only by the element $\infty = \sup S$.

Let S and T be b-complete idempotent semigroups. Then, a homomorphism $f : S \to T$ is said to be a *b-homomorphism* if $f(\oplus X) = \oplus f(X)$ for any bounded subset X in S. If the b-homomorphism f is extended to a homomorphism $\hat{S} \to \hat{T}$ of the corresponding normal completions and $f(\oplus X) = \oplus f(X)$ for all $X \subset S$, then f is said to be an *a-homomorphism*. An idempotent semigroup S equipped with a topology such that the set $\{s \in S | s \preceq b\}$ is closed in this topology for any $b \in S$ is called a *topological idempotent semigroup* S.

Proposition 7.1. *Let S be an a-complete topological idempotent semigroup and T be a b-complete topological idempotent semigroup such that, for any nonempty subsemigroup X in T, the element $\oplus X$ is contained in the topological closure of X in T. Then, a homomorphism $f : T \to S$ that maps zero into zero is an a-homomorphism if and only if the mapping f is lower semicontinuous in the sense that the set $\{t \in T | f(t) \preceq s\}$ is closed in T for any $s \in S$.*

An idempotent semiring K is called *a-complete* (respectively *b-complete*) if K is an a-complete (respectively b-complete) idempotent semigroup and, for any subset (respectively, for any bounded subset) X in K and any $k \in K$, the generalized distributive laws $k \odot (\oplus X) = \oplus(k \odot X)$ and $(\oplus X) \odot k = \oplus(X \odot k)$ are fulfilled. Generalized distributivity implies that any a-complete or b-complete idempotent semiring has a zero element that coincides with $\oplus\emptyset$, where $\emptyset$ is the empty set.

The set $\mathbf{R}(\max, +)$ of real numbers equipped with the idempotent addition $\oplus = \max$ and multiplication $\odot = +$ is an idempotent semiring; in this case, $\mathbf{1} = 0$. Adding the element $\mathbf{0} = -\infty$ to this semiring, we obtain a b-complete semiring $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with the same operations and the zero element. Adding the element $+\infty$ to $\mathbf{R}_{\max}$ and assuming that $\mathbf{0} \odot (+\infty) = \mathbf{0}$ and $x \odot (+\infty) = +\infty$ for $x \neq \mathbf{0}$ and $x \oplus (+\infty) = +\infty$ for any x, we obtain the a-complete idempotent semiring $\hat{\mathbf{R}}_{\max} = \mathbf{R}_{\max} \cup \{+\infty\}$. The standard order on $\mathbf{R}(\max, +)$, $\mathbf{R}_{\max}$ and $\hat{\mathbf{R}}_{\max}$ coincides with the ordinary order. The semirings $\mathbf{R}(\max, +)$ and $\mathbf{R}_{\max}$ are semifields. On the contrary, an a-complete semiring that does not coincide with $\{\mathbf{0}, \mathbf{1}\}$ cannot be a semifield. An important class of examples is related to (topological) vector lattices (see, for example, [10] and [86, Chap. 5]). Defining the sum $x \oplus y$ as $\sup\{x, y\}$ and the multiplication $\odot$ as the addition of vectors, we can interpret the vector lattices as idempotent semifields. Adding the zero element $\mathbf{0}$ to a complete vector lattice (in the sense of [10, 86]), we obtain a b-complete semifield. If, in addition, we add the infinite element, we obtain an a-complete idempotent semiring (which, as an ordered set, coincides with the normal completion of the original lattice).

Important definitions. Let V be an idempotent semigroup and K be an idempotent semiring. Suppose that a multiplication $k, x \mapsto k \odot x$ of all elements from K by the elements from V is defined; moreover, this multiplication is associative and distributive with respect to the addition in V and $\mathbf{1} \odot x = x$, $\mathbf{0} \odot x = \mathbf{0}$ for all $x \in V$. In this case, the semigroup V is called an *idempotent semimodule* (or simply, a *semimodule*) over K. The element $\mathbf{0}_V \in V$ is called the *zero* of the semimodule V if $k \odot \mathbf{0}_V = \mathbf{0}_V$ and $\mathbf{0}_V \oplus x = x$ for any $k \in K$ and $x \in V$. Let V be a semimodule over a b-complete idempotent semiring K. This semimodule is called *b-complete* if it is b-complete as an idempotent semiring and, for any bounded subsets Q in K and X in V, the generalized distributive laws $(\oplus Q) \odot x = \oplus(Q \odot x)$ and $k \odot (\oplus X) = \oplus(k \odot X)$ are fulfilled for all $k \in K$ and $x \in X$. This semimodule is called *a-complete* if it is b-complete and contains the element $\infty = \sup V$.

A semimodule V over a b-complete semifield K is said to be an *idempotent a-space* (*b-space*) if this semimodule is a-complete (respectively, b-complete) and the equality $(\wedge Q) \odot x = \wedge(Q \odot x)$ holds for any nonempty subset Q in K and any $x \in V$, $x \neq \infty = \sup V$. The normal completion $\hat{V}$ of a b-space V (as an idempotent semigroup) has the structure of an idempotent a-space (and may differ from V only by the element $\infty = \sup V$).

Let V and W be idempotent semimodules over an idempotent semiring K. A mapping $p : V \to W$ is said to be *linear* (over K) if

$$p(x \oplus y) = p(x) \oplus p(y) \text{ and } p(k \odot x) = k \odot p(x)$$

for any $x, y \in V$ and $k \in K$. Let the semimodules V and W be b-complete. A linear mapping $p : V \to W$ is said to be *b-linear* if it is a b-homomorphism of the idempotent semigroup; this mapping is said to be *a-linear* if it can be extended to an a-homomorphism of the normal completions $\hat{V}$ and $\hat{W}$. Proposition 7.1 (see above) shows that a-linearity simulates (semi)continuity for linear mappings. The normal completion $\hat{K}$ of the semifield K is a semimodule over K. If $W = \hat{K}$, then the linear mapping p is called a *linear functional*.

Linear, a-linear and b-linear mappings are also called *linear, a-linear* and *b-linear operators* respectively.

Examples of idempotent semimodules and spaces that are the most important for analysis are either subsemimodules of topological vector lattices [86] (or coincide with them) or are dual to them, i.e., consist of linear functionals subject to some regularity condition, for example, consist of a-linear functionals. Concrete examples of idempotent semimodules and spaces of functions (including spaces of bounded, continuous, semicontinuous, convex, concave and Lipschitz functions) see in [33, 48, 49, 57] and below.

7.2 Basic Results

Let V be an idempotent b-space over a b-complete semifield K, $x \in \hat{V}$. Denote by x^* the functional $V \to \hat{K}$ defined by the formula $x^*(y) = \wedge\{k \in K | y \preceq k \odot x\}$, where y is an arbitrary fixed element from V.

Theorem 7.1. *For any $x \in \hat{V}$ the functional x^* is a-linear. Any nonzero a-linear functional f on V is given by $f = x^*$ for a unique suitable element $x \in V$. If $K \neq \{\mathbf{0}, \mathbf{1}\}$, then $x = \oplus\{y \in V | f(y) \preceq \mathbf{1}\}$.*

Note that results of this type obtained earlier concerning the structure of linear functionals cannot be carried over to subspaces and subsemimodules.

A subsemigroup W in V closed with respect to the multiplication by an arbitrary element from K is called a *b-subspace* in V if the imbedding $W \to V$ can be extended to a b-linear mapping. The following result is obtained from Theorem 7.1 and is the idempotent version of the Hahn–Banach theorem.

Theorem 7.2. *Any a-linear functional defined on a b-subspace W in V can be extended to an a-linear functional on V. If $x, y \in V$ and $x \neq y$, then there exists an a-linear functional f on V that separates the elements x and y, i.e., $f(x) \neq f(y)$.*

The following statements are easily derived from the definitions and can be regarded as the analogs of the well-known results of the traditional functional analysis (the Banach–Steinhaus and the closed-graph theorems).

Proposition 7.2. *Suppose that P is a family of a-linear mappings of an a-space V into an a-space W and the mapping $p : V \to W$ is the pointwise sum of the mappings of this family, i.e., $p(x) = \sup\{p_\alpha(x) | p_\alpha \in P\}$. Then the mapping p is a-linear.*

Proposition 7.3. *Let V and W be a-spaces. A linear mapping $p : V \to W$ is a-linear if and only if its graph Γ in $V \times W$ is closed with respect to passing to sums (i.e., to least upper bounds) of its arbitrary subsets.*

In [18] the basic results were generalized for the case of semimodules over the so-called reflexive b-complete semirings.

7.3 Idempotent b-semialgebras

Let K be a b-complete semifield and A be an idempotent b-space over K equipped with the structure of a semiring compatible with the multiplication $K \times A \to A$ so that the associativity of the multiplication is preserved. In this case, A is called an *idempotent b-semialgebra* over K.

Proposition 7.4. *For any invertible element $x \in A$ from the b-semialgebra A and any element $y \in A$, the equality $x^*(y) = \mathbf{1}^*(y \odot x^{-1})$ holds, where $\mathbf{1} \in A$.*

The mapping $A \times A \to \hat{K}$ defined by the formula $(x, y) \mapsto \langle x, y \rangle = \mathbf{1}^*(x \odot y)$ is called the *canonical scalar product* (or simply *scalar product*). The basic properties of the scalar product are easily derived from Proposition 7.4 (in particular, the scalar product is commutative if the b-semialgebra A is commutative). The following theorem is an idempotent version of the Riesz–Fisher theorem.

Theorem 7.3. *Let a b-semialgebra A be a semifield. Then any nonzero a-linear functional f on A can be represented as $f(y) = \langle y, x \rangle$, where $x \in A$, $x \neq \mathbf{0}$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product on A.*

Remark 7.1. Using the completion procedures, one can extend all the results obtained to the case of incomplete semirings, spaces, and semimodules, see [49].

Example 7.1. Let $\mathscr{B}(X)$ be a set of all bounded functions with values belonging to $\mathbf{R}(\max, +)$ on an arbitrary set X and let $\hat{\mathscr{B}}(X) = \mathscr{B}(X) \cup \{\mathbf{0}\}$. The pointwise idempotent addition of functions $(\varphi_1 \oplus \varphi_2)(x) = \varphi_1(x) \oplus \varphi_2(x)$ and the multiplication $(\varphi_1 \odot \varphi_2)(x) = (\varphi_1(x)) \odot (\varphi_2(x))$ define on $\hat{\mathscr{B}}(X)$ the structure of a b-semialgebra over the b-complete semifield $\mathbf{R}_{\max}$. In this case, $\mathbf{1}^*(\varphi) = \sup_{x \in X} \varphi(x)$ and the scalar product is expressed in terms of idempotent integration: $\langle \varphi_1, \varphi_2 \rangle = \sup_{x \in X}(\varphi_1(x) \odot \varphi_2(x)) = \sup_{x \in X}(\varphi_1(x) + \varphi_2(x)) = \int_X^{\oplus} (\varphi_1(x) \odot \varphi_2(x))\ dx$. Scalar products of this type were systematically used in idempotent analysis. Using Theorems 7.1 and 7.3, one can easily describe a-linear functionals on idempotent spaces in terms of idempotent measures and integrals.

Example 7.2. Let X be a linear space in the traditional sense. The idempotent semiring (and linear space over $\mathbf{R}(\max, +)$) of convex functions $\mathrm{Conv}(X, \mathbf{R})$ is b-complete but it is not a b-semialgebra over the semifield $K = \mathbf{R}(\max, +)$.

Any nonzero a-linear functional f on $\mathrm{Conv}(X, \mathbf{R})$ has the form

$$\varphi \mapsto f(\varphi) = \sup_x \{\varphi(x) + \psi(x)\} = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx,$$

where ψ is a concave function, i.e., an element of the idempotent space $\mathrm{Conc}(X, \mathbf{R}) = -\mathrm{Conv}(X, \mathbf{R})$.

7.4 Linear Operator, b-semimodules and Subsemimodules

In what follows, we suppose that all semigroups, semirings, semifields, semimodules, and spaces are idempotent unless otherwise specified. We fix a basic semiring K and examine semimodules and subsemimodules over K. We suppose that every linear functional takes it values in the basic semiring.

Let V and W be b-complete semimodules over a b-complete semiring K. Denote by $L_b(V, W)$ the set of all b-linear mappings from V to W. It is easy to check that $L_b(V, W)$ is an idempotent semigroup with respect to the pointwise addition of operators; the composition (product) of b-linear operators is also a b-linear operator, and therefore the set $L_b(V, V)$ is an idempotent semiring with respect to these operations, see, e.g., [49]. The following proposition can be treated as a version of the Banach–Steinhaus theorem in idempotent analysis (as well as Proposition 7.2 above).

Proposition 7.5. *Assume that S is a subset in $L_b(V, W)$ and the set $\{g(v) \mid g \in S\}$ is bounded in W for every element $v \in V$; thus the element $f(v) = \sup_{g\in S} g(v)$ exists, because the semimodule W is b-complete. Then the mapping $v \mapsto f(v)$ is a b-linear operator, i.e., an element of $L_b(V, W)$. The subset S is bounded; moreover,* $\sup S = f$.

Corollary 7.1. *The set $L_b(V, W)$ is a b-complete idempotent semigroup with respect to the (idempotent) pointwise addition of operators. If $V = W$, then $L_b(V, V)$ is a b-complete idempotent semiring with respect to the operations of pointwise addition and composition of operators.*

Corollary 7.2. *A subset S is bounded in $L_b(V, W)$ if and only if the set $\{g(v) \mid g \in S\}$ is bounded in the semimodule W for every element $v \in V$.*

A subset of an idempotent semimodule is called a *subsemimodule* if it is closed under addition and multiplication by scalar coefficients. A subsemimodule V of a b-complete semimodule W is *b-closed* if V is closed under sums of any subsets of V that are bounded in W. A subsemimodule of a b-complete semimodule is called a *b-subsemimodule* if the corresponding embedding is a b-homomorphism. It is easy to see that each b-closed subsemimodule is a b-subsemimodule, but the converse is not true. The main feature of b-subsemimodules is that restrictions of b-linear operators and functionals to these semimodules are b-linear.

The following definitions are very important for our purposes. Assume that W is an idempotent b-complete semimodule over a b-complete idempotent semiring K and V is a subset of W such that V is closed under multiplication by scalar coefficients and is an upper semilattice with respect to the order induced from W. Let us define an addition operation in V by the formula $x \oplus y = \sup\{x, y\}$, where sup means the least upper bound in V. If K is a semifield, then V is a semimodule over K with respect to this addition.

For an arbitrary b-complete semiring K, we will say that V is a *quasisubsemimodule* of W if V is a semimodule with respect to this addition (this means that the corresponding distribution laws hold).

Recall that the symbol $\wedge$ means the greatest lower bound (see Sect. 7.1 above). A quasisubsemimodule V of an idempotent b-complete semimodule W is called a $\wedge$-*subsemimodule* if it contains $\mathbf{0}$ and is closed under the operations of taking infima (greatest lower bounds) in W. It is easy to check that *each* $\wedge$-*subsemimodule is a* b-*complete semimodule*.

Note that quasisubsemimodules and $\wedge$-subsemimodules may fail to be subsemimodules, because only the order is induced and not the corresponding addition (see Example 7.6 below).

Recall that idempotent semimodules over semifields are *idempotent spaces*. In idempotent mathematics, such spaces are analogs of traditional linear (vector) spaces over fields. In a similar way we use the corresponding terms like *b-spaces, b-subspaces, b-closed subspaces*, $\wedge$-*subspaces*, etc.

Some examples are presented below.

7.5 Functional Semimodules

Let X be an arbitrary nonempty set and K be an idempotent semiring. By $K(X)$ denote the semimodule of all mappings (functions) $X \to K$ endowed with the pointwise operations. By $K_b(X)$ denote the subsemimodule of $K(X)$ consisting of all bounded mappings. If K is a b-complete semiring, then $K(X)$ and $K_b(X)$ are b-complete semimodules. Note that $K_b(X)$ is a b-subsemimodule but not a b-closed subsemimodule of $K(X)$. Given a point $x \in X$, by δ_x denote the functional on $K(X)$ that maps f to $f(x)$. It can easily be checked that the functional δ_x is b-linear on $K(X)$.

Recall that the functional δ_x is generated by the usual function

$$\delta_x(y) = \begin{cases} \mathbf{1}, & \text{if } x = y, \\ \mathbf{0}, & \text{if } x \neq y, \end{cases}$$

so $\varphi(x) = \int^{\oplus} \delta_x(y)\varphi(y)dy = \sup_y(\delta_x(y) \odot \varphi(y))$. Note that δ-functions form a natural (continuous in general) basis in any typical functional semimodule.

We say that a quasisubsemimodule of $K(X)$ is an (idempotent) *functional semimodule* on the set X. An idempotent functional semimodule in $K(X)$ is called *b-complete* if it is a b-complete semimodule.

A functional semimodule $V \subset K(X)$ is called a *functional b-semimodule* if it is a b-subsemimodule of $K(X)$; a functional semimodule $V \subset K(X)$ is called a *functional* $\wedge$-*semimodule* if it is a $\wedge$-subsemimodule of $K(X)$.

In general, a functional of the form δ_x on a functional semimodule is not even linear, much less b-linear (see Example 7.6 below). However, the following proposition holds, which is a direct consequence of our definitions.

Proposition 7.6. *An arbitrary b-complete functional semimodule W on a set X is a b-subsemimodule of $K(X)$ if and only if each functional of the form δ_x (where $x \in X$) is b-linear on W.*

Example 7.3. The semimodule $K_b(X)$ (consisting of all bounded mappings from an arbitrary set X to a b-complete idempotent semiring K) is a functional $\wedge$-semimodule. Hence it is a b-complete semimodule over K. Moreover, $K_b(X)$ is a b-subsemimodule of the semimodule $K(X)$ consisting of all mappings $X \to K$.

Example 7.4. If X is a finite set consisting of n elements ($n > 0$), then $K_b(X) = K(X)$ is an "n-dimensional" semimodule over K; it is denoted by K^n. In particular, $\mathbf{R}^n_{max}$ is an idempotent space over the semifield $\mathbf{R}_{max}$, and $\hat{\mathbf{R}}^n_{\max}$ is a semimodule over the semiring $\hat{\mathbf{R}}_{\max}$. Note that $\hat{\mathbf{R}}^n_{\max}$ can be treated as a space over the semifield $\mathbf{R}_{max}$. For example, the semiring $\hat{\mathbf{R}}_{\max}$ can be treated as a space (semimodule) over $\mathbf{R}_{\max}$.

Example 7.5. Let X be a topological space. Denote by $USC(X)$ the set of all upper semicontinuous functions with values in $\mathbf{R}_{\max}$. By definition, a function $f(x)$ is upper semicontinuous if the set $X_s = \{x \in X \mid f(x) \geq s\}$ is closed in X for every element $s \in \mathbf{R}_{\max}$ (see, e.g., [49, Sect. 2.8]). If a family $\{f_\alpha\}$ consists of upper semicontinuous (e.g., continuous) functions and $f(x) = \inf_\alpha f_\alpha(x)$, then $f(x) \in USC(X)$. It is easy to check that $USC(X)$ has a natural structure of an idempotent space over $\mathbf{R}_{\max}$. Moreover, $USC(X)$ is a functional $\wedge$-space on X and a b-space. The subspace $USC(X) \cap K_b(X)$ of $USC(X)$ consisting of bounded (from above) functions has the same properties.

Example 7.6. Note that an idempotent functional semimodule (and even a functional $\wedge$-semimodule) on a set X is not necessarily a subsemimodule of $K(X)$. The simplest example is the functional space (over $K = \mathbf{R}_{\max}$) Conc($\mathbf{R}$) consisting of all concave functions on $\mathbf{R}$ with values in $\mathbf{R}_{\max}$. Recall that a function f belongs to Conc($\mathbf{R}$) if and only if the subgraph of this function is convex, i.e., the formula $f(ax + (1-a)y) \geq af(x) + (1-a)f(y)$ is valid for $0 \leq a \leq 1$. The basic operations with $\mathbf{0} \in \mathbf{R}_{\max}$ can be defined in an obvious way. If $f, g \in$Conc($\mathbf{R}$), then denote by $f \oplus g$ the sum of these functions in Conc($\mathbf{R}$). The subgraph of $f \oplus g$ is the convex hull of the subgraphs of f and g. Thus $f \oplus g$ does not coincide with the pointwise sum (i.e., $\max\{f(x), g(x)\}$).

Example 7.7. Let X be a nonempty metric space with a fixed metric r. Denote by Lip(X) the set of all functions defined on X with values in $\mathbf{R}_{\max}$ satisfying the following *Lipschitz condition*:

$$| f(x) \odot (f(y))^{-1} | = | f(x) - f(y) | \leq r(x, y),$$

where x, y are arbitrary elements of X. The set Lip(X) consists of continuous real-valued functions (but not all of them!) and (by definition) the function equal to $-\infty = \mathbf{0}$ at every point $x \in X$. The set Lip(X) has the structure of an idempotent space over the semifield $\mathbf{R}_{\max}$. Spaces of the form Lip(X) are said to be *Lipschitz spaces*. These spaces are b-subsemimodules in $K(X)$.

7.6 Integral Representations of Linear Operators in Functional Semimodules

Let W be an idempotent b-complete semimodule over a b-complete semiring K and $V \subset K(X)$ be a b-complete functional semimodule on X. A mapping $A : V \to W$ is called an *integral operator* or an operator with an *integral representation* if there exists a mapping $k : X \to W$, called the *integral kernel* (or *kernel*) *of the operator* A, such that

$$Af = \sup_{x \in X} (f(x) \odot k(x)). \tag{15}$$

In idempotent analysis, the right-hand side of formula (11) is often written as $\int_X^{\oplus} f(x) \odot k(x) dx$. Regarding the kernel k, it is assumed that the set $\{f(x) \odot k(x) | x \in X\}$ is bounded in W for all $f \in V$ and $x \in X$. We denote the set of all functions with this property by $\mathrm{kern}_{V,W}(X)$. In particular, if $W = K$ and A is a functional, then this functional is called *integral*. Thus each integral functional can be presented in the form of a "scalar product" $f \mapsto \int_X^{\oplus} f(x) \odot k(x)\ dx$, where $k(x) \in K(X)$; in idempotent analysis, this situation is standard.

Note that a functional of the form δ_y (where $y \in X$) is a typical integral functional; in this case, $k(x) = \mathbf{1}$ if $x = y$ and $k(x) = \mathbf{0}$ otherwise.

We call a functional semimodule $V \subset K(X)$ *nondegenerate* if for every point $x \in X$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$, and *admissible* if for every function $f \in V$ and every point $x \in X$ such that $f(x) \neq \mathbf{0}$ there exists a function $g \in V$ such that $g(x) = \mathbf{1}$ and $f(x) \odot g \preceq f$.

Note that all idempotent functional semimodules over semifields are admissible (it is sufficient to set $g = f(x)^{-1} \odot f$).

Proposition 7.7. *Denote by X_V the subset of X defined by the formula $X_V = \{x \in X \mid \exists f \in V : f(x) = \mathbf{1}\}$. If the semimodule V is admissible, then the restriction to X_V defines an embedding $i : V \to K(X_V)$ and its image $i(V)$ is admissible and nondegenerate.*

If a mapping $k : X \to W$ is a kernel of a mapping $A : V \to W$, then the mapping $k_V : X \to W$ that is equal to k on X_V and equal to $\mathbf{0}$ on $X \setminus X_V$ is also a kernel of A.

A mapping $A : V \to W$ is integral if and only if the mapping $i_{-1}A : i(A) \to W$ is integral.

In what follows, K always denotes a fixed b-complete idempotent (basic) semiring. If an operator has an integral representation, this representation may not be unique. However, if the semimodule V is nondegenerate, then the set of all kernels of a fixed integral operator is bounded with respect to the natural order in the set of all kernels and is closed under the supremum operation applied to its arbitrary subsets. In particular, *any integral operator defined on a nondegenerate functional semimodule has a unique maximal kernel.*

An important point is that an integral operator is not necessarily b-linear and even linear except when V is a b-subsemimodule of $K(X)$ (see Proposition 7.8 below).

If W is a functional semimodule on a nonempty set Y, then an integral kernel k of an operator A can be naturally identified with the function on $X \times Y$ defined by the formula $k(x, y) = (k(x))(y)$. This function will also be called an *integral kernel* (or *kernel*) of the operator A. As a result, the set $\mathrm{kern}_{V,W}(X)$ is identified with the set $\mathrm{kern}_{V,W}(X, Y)$ of all mappings $k : X \times Y \to K$ such that for every point $x \in X$ the mapping $k_x : y \mapsto k(x, y)$ lies in W and for every $v \in V$ the set $\{v(x) \odot k_x | x \in X\}$ is bounded in W. Accordingly, the set of all integral kernels of b-linear operators can be embedded into $\mathrm{kern}_{V,W}(X, Y)$.

If V and W are functional b-semimodules on X and Y, respectively, then the set of all kernels of b-linear operators can be identified with $\mathrm{kern}_{V,W}(X, Y)$ and the following formula holds:

$$Af(y) = \sup_{x \in X} (f(x) \odot k(x, y)) = \int_X^{\oplus} f(x) \odot k(x, y) dx. \tag{16}$$

This formula coincides with the usual definition of an integral representation of an operator. Note that formula (15) can be rewritten in the form

$$Af = \sup_{x \in X} (\delta_x(f) \odot k(x)). \tag{17}$$

Proposition 7.8. *An arbitrary b-complete functional semimodule V on a nonempty set X is a functional b-semimodule on X (i.e., a b-subsemimodule of $K(X)$) if and only if all integral operators defined on V are b-linear.*

The following notion (definition) is especially important for our purposes. Let $V \subset K(X)$ be a b-complete functional semimodule over a b-complete idempotent semiring K. We say that the *kernel theorem* holds for the semimodule V if every b-linear mapping from V into an arbitrary b-complete semimodule over K has an integral representation.

Theorem 7.4. *Assume that a b-complete semimodule W over a b-complete semiring K and an admissible functional $\wedge$-semimodule $V \subset K(X)$ are given. Then every b-linear operator $A : V \to W$ has an integral representation of the form* (15). *In particular, if W is a functional b-semimodule on a set Y, then the operator A has an integral representation of the form* (16). *Thus for the semimodule V the kernel theorem holds.*

Remark 7.2. Examples of admissible functional $\wedge$-semimodules (and $\wedge$-spaces) appearing in Theorem 7.4 are presented above, see, e.g., Examples 7.3–7.5. Thus for these functional semimodules and spaces V over K, the kernel theorem holds and every b-linear mapping V into an arbitrary b-complete semimodule W over K has an integral representation (16). Recall that every functional space over a b-complete semifield is admissible, see above.

7.7 Nuclear Operators and Their Integral Representations

Let us introduce some important definitions. Assume that V and W are b-complete semimodules. A mapping $g : V \to W$ is called *one-dimensional* (or a *mapping of rank* 1) if it is of the form $v \mapsto \phi(v) \odot w$, where ϕ is a b-linear functional on V and $w \in W$. A mapping g is called *b-nuclear* if it is the sum (i.e., supremum) of a bounded set of one-dimensional mappings. Since every one-dimensional mapping is b-linear (because the functional ϕ is b-linear), *every b-nuclear operator is b-linear* (see Corollary 7.1 above). Of course, b-nuclear mappings are closely related to tensor products of idempotent semimodules, see [48].

By $\phi \odot w$ we denote the one-dimensional operator $v \mapsto \phi(v) \odot w$. In fact, this is an element of the corresponding tensor product.

Proposition 7.9. *The composition (product) of a b-nuclear and a b-linear mapping or of a b-linear and a b-nuclear mapping is a b-nuclear operator.*

Theorem 7.5. *Assume that W is a b-complete semimodule over a b-complete semiring K and $V \subset K(X)$ is a functional b-semimodule. If every b-linear functional on V is integral, then a b-linear operator $A : V \to W$ has an integral representation if and only if it is b-nuclear.*

7.8 The b-approximation Property and b-nuclear Semimodules and Spaces

We say that a b-complete semimodule V has the *b-approximation property* if the identity operator id: $V \to V$ is b-nuclear (for a treatment of the approximation property for locally convex spaces in the traditional functional analysis, see [86]).

Let V be an arbitrary b-complete semimodule over a b-complete idempotent semiring K. We call this semimodule a *b-nuclear semimodule* if any b-linear mapping of V to an arbitrary b-complete semimodule W over K is a b-nuclear operator. Recall that, in the traditional functional analysis, a locally convex space is nuclear if and only if all continuous linear mappings of this space to any Banach space are nuclear operators, see [86].

Proposition 7.10. *Let V be an arbitrary b-complete semimodule over a b-complete semiring K. The following statements are equivalent:*

1. *The semimodule V has the b-approximation property.*
2. *Every b-linear mapping from V to an arbitrary b-complete semimodule W over K is b-nuclear.*
3. *Every b-linear mapping from an arbitrary b-complete semimodule W over K to the semimodule V is b-nuclear.*

Corollary 7.3. *An arbitrary b-complete semimodule over a b-complete semiring K is b-nuclear if and only if this semimodule has the b-approximation property.*

Recall that, in the traditional functional analysis, any nuclear space has the approximation property but the converse is not true.

Concrete examples of b-nuclear spaces and semimodules are described in Examples 7.3, 7.4 and 7.7 (see above). Important b-nuclear spaces and semimodules (e.g., the so-called Lipschitz spaces and semi-Lipschitz semimodules) are described in [57]. In this paper there is a description of all functional b-semimodules for which the kernel theorem holds (as semi-Lipschitz semimodules); this result is due to Shpiz.

It is easy to show that the idempotent spaces $USC(X)$ and Conc($\mathbf{R}$) (see Examples 7.5 and 7.6) are not b-nuclear (however, for these spaces the kernel theorem is true). The reason is that these spaces are not functional b-spaces and the corresponding δ-functionals are not b-linear (and even linear).

7.9 Kernel Theorems for Functional b-Semimodules

Let $V \subset K(X)$ be a b-complete functional semimodule over a b-complete semiring K. Recall that for V the *kernel theorem* holds if every b-linear mapping of this semimodule to an arbitrary b-complete semimodule over K has an integral representation.

Theorem 7.6. *Assume that a b-complete semiring K and a nonempty set X are given. The kernel theorem holds for any functional b-semimodule $V \subset K(X)$ if and only if every b-linear functional on V is integral and the semimodule V is b-nuclear, i.e., has the b-approximation property.*

Corollary 7.4. *If for a functional b-semimodule the kernel theorem holds, then this semimodule is b-nuclear.*

Note that the possibility to obtain an integral representation of a functional means that one can decompose it into a sum of functionals of the form δ_x.

Corollary 7.5. *Assume that a b-complete semiring K and a nonempty set X are given. The kernel theorem holds for a functional b-semimodule $V \subset K(X)$ if and only if the identity operator id: $V \to V$ is integral.*

7.10 Integral Representations of Operators in Abstract Idempotent Semimodules

In this subsection, we examine the following problem: when a b-complete idempotent semimodule V over a b-complete semiring is isomorphic to a functional b-semimodule W such that the kernel theorem holds for W.

Assume that V is a b-complete idempotent semimodule over a b-complete semiring K and ϕ is a b-linear functional defined on V. We call this functional a *δ-functional* if there exists an element $v \in V$ such that

$$\phi(w) \odot v \preceq w$$

for every element $w \in V$. It is easy to see that every functional of the form δ_x is a δ-functional in this sense (but the converse is not true in general).

Denote by $\Delta(V)$ the set of all δ-functionals on V. Denote by i_Δ the natural mapping $V \to K(\Delta(V))$ defined by the formula

$$(i_\Delta(v))(\phi) = \phi(v)$$

for all $\phi \in \Delta(V)$. We say that an element $v \in V$ is *pointlike* if there exists a b-linear functional ϕ such that $\phi(w) \odot v \preceq w$ for all $w \in V$. The set of all pointlike elements of V will be denoted by $P(V)$. Recall that by $\phi \odot v$ we denote the one-dimensional operator $w \mapsto \phi(w) \odot v$.

The following assertion is an obvious consequence of our definitions (including the definition of the standard order) and the idempotency of our addition.

Remark 7.3. If a one-dimensional operator $\phi \odot v$ appears in the decomposition of the identity operator on V into a sum of one-dimensional operators, then $\phi \in \Delta(V)$ and $v \in P(V)$.

Denote by *id* and *Id* the identity operators on V and $i_\Delta(V)$, respectively.

Proposition 7.11. *If the operator id is b-nuclear, then i_Δ is an embedding and the operator Id is integral.*

If the operator i_Δ is an embedding and the operator Id is integral, then the operator id is b-nuclear.

Theorem 7.7. *A b-complete idempotent semimodule V over a b-complete idempotent semiring K is isomorphic to a functional b-semimodule for which the kernel theorem holds if and only if the identity mapping on V is a b-nuclear operator, i.e., V is a b-nuclear semimodule.*

The following proposition shows that, in a certain sense, the embedding i_Δ is a universal representation of a b-nuclear semimodule in the form of a functional b-semimodule for which the kernel theorem holds.

Proposition 7.12. *Let K be a b-complete idempotent semiring, X be a nonempty set, and $V \subset K(X)$ be a functional b-semimodule on X for which the kernel theorem holds. Then there exists a natural mapping $i : X \to \Delta(V)$ such that the corresponding mapping $i_* : K(\Delta(V)) \to K(X)$ is an isomorphism of $i_\Delta(V)$ onto V.*

8 The Dequantization Transform, Convex Geometry and the Newton Polytopes

Let X be a topological space. For functions $f(x)$ defined on X we shall say that a certain property is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X. Suppose X is $\mathbf{C}^n$ or $\mathbf{R}^n$; denote by $\mathbf{R}^n_+$ the set $x = \{ (x_1, \dots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \dots, n$. For $x = (x_1, \dots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \dots, \exp(x_n))$; so if $x \in \mathbf{R}^n$, then $\exp(x) \in \mathbf{R}^n_+$.

Denote by $\mathscr{F}(\mathbf{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbf{C}^n$ such that $U \supset \mathbf{R}^n_+$. It is clear that $\mathscr{F}(\mathbf{C}^n)$ is a ring (and an algebra over $\mathbf{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathscr{F}(\mathbf{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|, \tag{18}$$

where h is a (small) real positive parameter and $x \in \mathbf{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to +0} \hat{f}_h(x), \tag{19}$$

if the right-hand side of (19) exists almost everywhere.

We shall say that the function $\hat{f}(x)$ is a *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbf{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to $\mathbf{R}^n_+$ only; so in fact the dequantization transform is constructed for functions defined on $\mathbf{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$.

Of course, similar definitions can be given for functions defined on $\mathbf{R}^n$ and $\mathbf{R}^n_+$. If $s = 1/h$, then we have the following version of (18) and (19):

$$\hat{f}(x) = \lim_{s \to \infty} (1/s) \log |f(e^{sx})|. \tag{20}$$

Denote by $\partial \hat{f}$ the subdifferential of the function $\hat{f}$ at the origin.

If f is a polynomial we have

$$\partial \hat{f} = \{ v \in \mathbf{R}^n \mid (v, x) \leq \hat{f}(x) \ \forall x \in \mathbf{R}^n \}.$$

It is well known that all the convex compact subsets in $\mathbf{R}^n$ form an idempotent semiring $\mathscr{S}$ with respect to the Minkowski operations: for $\alpha, \beta \in \mathscr{S}$ the sum $\alpha \oplus \beta$ is the convex hull of the union $\alpha \cup \beta$; the product $\alpha \odot \beta$ is defined in the following way: $\alpha \odot \beta = \{ x \mid x = a + b$, where $a \in \alpha, b \in \beta$, see Fig. 3. In fact $\mathscr{S}$ is an idempotent linear space over $\mathbf{R}_{\max}$.

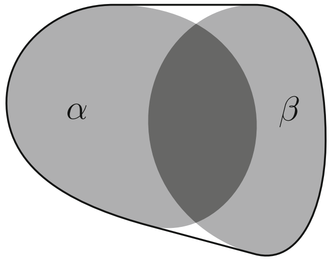

Fig. 3 Algebra of convex subsets

Of course, the Newton polytopes of polynomials in n variables form a subsemiring $\mathscr{N}$ in $\mathscr{S}$. If f, g are polynomials, then $\partial(\widehat{fg}) = \partial\hat{f} \odot \partial\hat{g}$; moreover, if f and g are "in general position", then $\partial(\widehat{f+g}) = \partial\hat{f} \oplus \partial\hat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this "traditional" semiring to the idempotent semiring $\mathscr{N}$.

Theorem 8.1. *If f is a polynomial, then the subdifferential $\partial\hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of f. For the semiring of polynomials with nonnegative coefficients, the transform $f \mapsto \partial\hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the Minkowski operations (see above).*

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see below and [55].

8.1 Dequantization Transform: Algebraic Properties

Denote by V the set $\mathbf{R}^n$ treated as a linear Euclidean space (with the scalar product $(x, y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$) and set $V_+ = \mathbf{R}^n_+$. We shall say that a function $f \in \mathscr{F}(\mathbf{C}^n)$ is *dequantizable* whenever its dequantization $\hat{f}(x)$ exists (and is defined on an open dense subset of V). By $\mathscr{D}(\mathbf{C}^n)$ denote the set of all dequantizable functions and by $\hat{\mathscr{D}}(V)$ denote the set $\{\, \hat{f} \mid f \in \mathscr{D}(\mathbf{C}^n)\,\}$. Recall that functions from $\mathscr{D}(\mathbf{C}^n)$ (and $\hat{\mathscr{D}}(V)$) are defined almost everywhere and $f = g$ means that $f(x) = g(x)$ a.e., i.e., for x ranging over an open dense subset of $\mathbf{C}^n$ (resp., of V). Denote by $\mathscr{D}_+(\mathbf{C}^n)$ the set of all functions $f \in \mathscr{D}(\mathbf{C}^n)$ such that $f(x_1, \ldots, x_n) \geq 0$ if $x_i \geq 0$ for $i = 1, \ldots, n$; so $f \in \mathscr{D}_+(\mathbf{C}^n)$ if the restriction of f to $V_+ = \mathbf{R}^n_+$ is a nonnegative function. By $\hat{\mathscr{D}}_+(V)$ denote the image of $\mathscr{D}_+(\mathbf{C}^n)$ under the dequantization transform. We shall say that functions $f, g \in \mathscr{D}(\mathbf{C}^n)$ are in *general position* whenever $\hat{f}(x) \neq \hat{g}(x)$ for x running an open dense subset of V.

Theorem 8.2. *For functions $f, g \in \mathscr{D}(\mathbf{C}^n)$ and any nonzero constant c, the following equations are valid:*

(1) $\widehat{fg} = \hat{f} + \hat{g}$
(2) $|\hat{f}| = \hat{f}$; $\widehat{cf} = f$; $\hat{c} = 0$

(3) $\widehat{(f+g)}(x) = \max\{\hat{f}(x), \hat{g}(x)\}$ *a.e. if f and g are nonnegative on V_+ (i.e., $f, g \in \mathscr{D}_+(\mathbf{C}^n)$) or f and g are in general position.*

Left-hand sides of these equations are well-defined automatically.

Corollary 8.1. *The set $\mathscr{D}_+(\mathbf{C}^n)$ has a natural structure of a semiring with respect to the usual addition and multiplication of functions taking their values in $\mathbf{C}$. The set $\hat{\mathscr{D}}_+(V)$ has a natural structure of an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$; elements of $\hat{\mathscr{D}}_+(V)$ can be naturally treated as functions taking their values in $\mathbf{R}_{\max}$. The dequantization transform generates a homomorphism from $\mathscr{D}_+(\mathbf{C}^n)$ to $\hat{\mathscr{D}}_+(V)$.*

8.2 Generalized Polynomials and Simple Functions

For any nonzero number $a \in \mathbf{C}$ and any vector $d = (d_1, \dots, d_n) \in V = \mathbf{R}^n$ we set $m_{a,d}(x) = a \prod_{i=1}^{n} x_i^{d_i}$; functions of this kind we shall call *generalized monomials*. Generalized monomials are defined a.e. on $\mathbf{C}^n$ and on V_+, but not on V unless the numbers d_i take integer or suitable rational values. We shall say that a function f is a *generalized polynomial* whenever it is a finite sum of linearly independent generalized monomials. For instance, Laurent polynomials and Puiseax polynomials are examples of generalized polynomials.

As usual, for $x, y \in V$ we set $(x, y) = x_1 y_1 + \cdots + x_n y_n$. The following proposition is a result of a trivial calculation.

Proposition 8.1. *For any nonzero number $a \in V = \mathbf{C}$ and any vector $d \in V = \mathbf{R}^n$ we have $(\widehat{m_{a,d}})_h(x) = (d, x) + h \log |a|$.*

Corollary 8.2. *If f is a generalized monomial, then $\hat{f}$ is a linear function.*

Recall that a real function p defined on $V = \mathbf{R}^n$ is *sublinear* if $p = \sup_\alpha p_\alpha$, where $\{p_\alpha\}$ is a collection of linear functions. Sublinear functions defined everywhere on $V = \mathbf{R}^n$ are convex; thus these functions are continuous, see [61]. We discuss sublinear functions of this kind only. Suppose p is a continuous function defined on V, then p is sublinear whenever

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.
2. $p(cx) = cp(x)$ for all $x \in V$, $c \in \mathbf{R}_+$.

So if p_1, p_2 are sublinear functions, then $p_1 + p_2$ is a sublinear function.

We shall say that a function $f \in \mathscr{F}(\mathbf{C}^n)$ is *simple*, if its dequantization $\hat{f}$ exists and a.e. coincides with a sublinear function; by misuse of language, we shall denote this (uniquely defined everywhere on V) sublinear function by the same symbol $\hat{f}$.

Recall that simple functions f and g are *in general position* if $\hat{f}(x) \neq \hat{g}(x)$ for all x belonging to an open dense subset of V. In particular, generalized monomials are in general position whenever they are linearly independent.

Denote by $Sim(\mathbf{C}^n)$ the set of all simple functions defined on V and denote by $Sim_+(\mathbf{C}^n)$ the set $Sim(\mathbf{C}^n) \cap \mathscr{D}_+(\mathbf{C}^n)$. By $Sbl(V)$ denote the set of all (continuous) sublinear functions defined on $V = \mathbf{R}^n$ and by $Sbl_+(V)$ denote the image $\widehat{Sim_+}(\mathbf{C}^n)$ of $Sim_+(\mathbf{C}^n)$ under the dequantization transform.

The following statements can be easily deduced from Theorem 8.2 and definitions.

Corollary 8.3. *The set $Sim_+(\mathbf{C}^n)$ is a subsemiring of $\mathscr{D}_+(\mathbf{C}^n)$ and $Sbl_+(V)$ is an idempotent subsemiring of $\widehat{\mathscr{D}_+}(V)$. The dequantization transform generates an epimorphism of $Sim_+(\mathbf{C}^n)$ onto $Sbl_+(V)$. The set $Sbl(V)$ is an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$.*

Corollary 8.4. *Polynomials and generalized polynomials are simple functions.*

We shall say that functions $f, g \in \mathscr{D}(V)$ are *asymptotically equivalent* whenever $\hat{f} = \hat{g}$; any simple function f is an *asymptotic monomial* whenever $\hat{f}$ is a linear function. A simple function f will be called an *asymptotic polynomial* whenever $\hat{f}$ is a sum of a finite collection of nonequivalent asymptotic monomials.

Corollary 8.5. *Every asymptotic polynomial is a simple function.*

Example 8.1. Generalized polynomials, logarithmic functions of (generalized) polynomials, and products of polynomials and logarithmic functions are asymptotic polynomials. This follows from our definitions and formula (19).

8.3 Subdifferentials of Sublinear Functions

We shall use some elementary results from convex analysis. These results can be found, e.g., in [61, Chap. 1, Sect. 1].

For any function $p \in Sbl(V)$ we set

$$\partial p = \{ v \in V \mid (v, x) \leq p(x)\ \forall x \in V \}. \tag{21}$$

It is well known from convex analysis that for any sublinear function p the set ∂p is exactly the *subdifferential* of p at the origin. The following propositions are also known in convex analysis.

Proposition 8.2. *Suppose $p_1, p_2 \in Sbl(V)$, then*

(1) $\partial(p_1 + p_2) = \partial p_1 \odot \partial p_2 = \{ v \in V \mid v = v_1 + v_2,$ *where* $v_1 \in \partial p_1, v_2 \in \partial p_2 \}$.
(2) $\partial(\max\{p_1(x), p_2(x)\}) = \partial p_1 \oplus \partial p_2$.

Recall that $\partial p_1 \oplus \partial p_2$ is a convex hull of the set $\partial p_1 \cup \partial p_2$.

Proposition 8.3. *Suppose $p \in Sbl(V)$. Then ∂p is a nonempty convex compact subset of V.*

Corollary 8.6. *The map $p \mapsto \partial p$ is a homomorphism of the idempotent semiring $Sbl(V)$ (see Corollary 8.1) to the idempotent semiring $\mathscr{S}$ of all convex compact subsets of V (see Sect. 8.1 above).*

8.4 Newton Sets for Simple Functions

For any simple function $f \in Sim(\mathbf{C}^n)$ let us denote by $N(f)$ the set $\partial(\hat{f})$. We shall call $N(f)$ the *Newton set* of the function f.

Proposition 8.4. *For any simple function f, its Newton set $N(f)$ is a nonempty convex compact subset of V.*

This proposition follows from Proposition 8.3 and definitions.

Theorem 8.3. *Suppose that f and g are simple functions. Then*

(1) $N(fg) = N(f) \odot N(g) = \{ v \in V \mid v = v_1 + v_2$ *with* $v_1 \in N(f), v_2 \in N(g) \}$.
(2) $N(f + g) = N(f) \oplus N(g)$, *if f_1 and f_2 are in general position or $f_1, f_2 \in Sim_+(\mathbf{C}^n)$ (recall that $N(f) \oplus N(g)$ is the convex hull of $N(f) \cup N(g)$).*

This theorem follows from Theorem 8.2, Proposition 8.2 and definitions.

Corollary 8.7. *The map $f \mapsto N(f)$ generates a homomorphism from $Sim_+(\mathbf{C}^n)$ to $\mathscr{S}$.*

Proposition 8.5. *Let $f = m_{a,d}(x) = a \prod_{i=1}^{n} x_i^{d_i}$ be a monomial; here $d = (d_1, \dots, d_n) \in V = \mathbf{R}^n$ and a is a nonzero complex number. Then $N(f) = \{d\}$.*

This follows from Proposition 8.1, Corollary 8.2 and definitions.

Corollary 8.8. *Let $f = \sum_{d \in D} m_{a_d, d}$ be a polynomial. Then $N(f)$ is the polytope $\oplus_{d \in D}\{d\}$, i.e. the convex hull of the finite set D.*

This statement follows from Theorem 8.3 and Proposition 8.5. Thus in this case $N(f)$ is the well-known classical Newton polytope of the polynomial f.
Now the following corollary is obvious.

Corollary 8.9. *Let f be a generalized or asymptotic polynomial. Then its Newton set $N(f)$ is a convex polytope.*

Example 8.2. Consider the one dimensional case, i.e., $V = \mathbf{R}$ and suppose $f_1 = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $f_2 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$, where $a_n \neq 0, b_m \neq 0, a_0 \neq 0, b_0 \neq 0$. Then $N(f_1)$ is the segment $[0, n]$ and $N(f_2)$ is the segment $[0, m]$. So the map $f \mapsto N(f)$ corresponds to the map $f \mapsto \deg(f)$, where $\deg(f)$ is a degree of the polynomial f. In this case Theorem 2 means that $\deg(fg) = \deg f + \deg g$ and $\deg(f + g) = \max\{\deg f, \deg g\} = \max\{n, m\}$ if $a_i \geq 0, b_i \geq 0$ or f and g are in general position.

9 Dequantization of Set Functions and Measures on Metric Spaces

The following results are presented in [56].

Example 9.1. Let M be a metric space, S its arbitrary subset with a compact closure. It is well-known that a Euclidean d-dimensional ball B_ρ of radius ρ has volume

$$\mathrm{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1+d/2)}\rho^d,$$

where d is a natural parameter. By means of this formula it is possible to define a volume of B_ρ for any *real* d. Cover S by a finite number of balls of radii ρ_m. Set

$$v_d(S) := \lim_{\rho \to 0} \inf_{\rho_m < \rho} \sum_m \mathrm{vol}_d(B_{\rho_m}).$$

Then there exists a number D such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number D is called the *Hausdorff–Besicovich dimension* (or *HB-dimension*) of S, see, e.g., [67]. Note that a set of non-integral HB-dimension is called a fractal in the sense of Mandelbrot.

Theorem 9.1. *Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius ρ covering S. Then*

$$D(S) = \underline{\lim}_{\rho \to +0} \log_\rho(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of S. Set $\rho = e^{-s}$, then

$$D(S) = \underline{\lim}_{s \to +\infty} (1/s) \cdot \log \mathcal{N}_{exp(-s)}(S).$$

So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $\mathcal{N}_\rho(S)$.

Example 9.2. Let μ be a set function on M (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball B_ρ. Let $B_{x,\rho}$ be a ball of radius ρ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let $\rho = e^{-s}$ and

$$D_{x,\mu} := \underline{\lim}_{s \to +\infty} -(1/s) \cdot \log(|\mu_x(e^{-s})|).$$

This number could be treated as a dimension of M at the point x with respect to the set function μ. So this dimension is a result of a dequantization of the function

$\mu_x(\rho)$, where x is fixed. There are many dequantization procedures of this type in different mathematical areas. In particular, Maslov's negative dimension (see [67]) can be treated similarly.

10 Dequantization of Geometry

An idempotent version of real algebraic geometry was discovered in the report of Viro for the Barcelona Congress [94]. Starting from the idempotent correspondence principle Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problem for constructing real algebraic varieties with prescribed properties and parameters) and relations to complex algebraic geometry and amoebas in the sense of Gelfand et al., see [25,95]. Then complex algebraic geometry was dequantized by Mikhalkin and the result turned out to be the same; this new "idempotent" (or asymptotic) geometry is now often called the *tropical algebraic geometry*, see, e.g., [32, 43, 46, 53, 72, 73].

There is a natural relation between the Maslov dequantization and amoebas.

Suppose $(\mathbf{C}^*)^n$ is a complex torus, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ is the group of nonzero complex numbers under multiplication. For $z = (z_1, \dots, z_n) \in (\mathbf{C}^*)^n$ and a positive real number h denote by $\mathrm{Log}_h(z) = h\log(|z|)$ the element

$$(h\log|z_1|, h\log|z_2|, \dots, h\log|z_n|) \in \mathbf{R}^n.$$

Suppose $V \subset (\mathbf{C}^*)^n$ is a complex algebraic variety; denote by $\mathscr{A}_h(V)$ the set $\mathrm{Log}_h(V)$. If $h = 1$, then the set $\mathscr{A}(V) = \mathscr{A}_1(V)$ is called the *amoeba* of V; the amoeba $\mathscr{A}(V)$ is a closed subset of $\mathbf{R}^n$ with a non-empty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity suppose V is a hypersurface in $(\mathbf{C}^*)^n$ defined by a polynomial f; then there is a deformation $h \mapsto f_h$ of this polynomial generated by the Maslov dequantization and $f_h = f$ for $h = 1$. Let $V_h \subset (\mathbf{C}^*)^n$ be the zero set of f_h and set $\mathscr{A}_h(V_h) = \mathrm{Log}_h(V_h)$. Then there exists a tropical variety $Tro(V)$ such that the subsets $\mathscr{A}_h(V_h) \subset \mathbf{R}^n$ tend to $Tro(V)$ in the Hausdorff metric as $h \to 0$. The tropical variety $Tro(V)$ is a result of a deformation of the amoeba $\mathscr{A}(V)$ and the Maslov dequantization of the variety V. The set $Tro(V)$ is called the *skeleton* of $\mathscr{A}(V)$.

Example 10.1. For the line $V = \{\,(x, y) \in (\mathbf{C}^*)^2 \mid x + y + 1 = 0\,\}$ the piecewise-linear graph $Tro(V)$ is a tropical line, see Fig. 4a. The amoeba $\mathscr{A}(V)$ is represented in Fig. 4b, while Fig. 4c demonstrates the corresponding deformation of the amoeba.

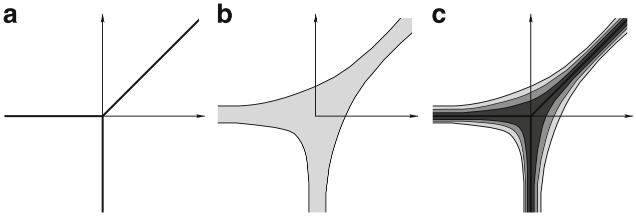

Fig. 4 Tropical line and deformations of an amoeba

11 Some Semiring Constructions and the Matrix Bellman Equation

11.1 Complete Idempotent Semirings and Examples

Recall that a partially ordered set S is *complete* if for every subset $T \subset S$ there exist elements $\sup T \in S$ and $\inf T \in S$. We say that an idempotent semiring S is *complete* if it is complete as an ordered set with respect to the standard order. Of course, any a-complete semiring (see Sect. 7.1) is complete. The most well-known and important examples are "numerical semirings" consisting of (a subset of) real numbers and ordered by the usual linear order $\leq$.

Example 11.1. Consider the semiring $\hat{\mathbf{R}}_{\max} = \mathbf{R}_{\max} \cup \{\infty\}$ with standard operations $\oplus = \max$, $\odot = +$ and neutral elements $\mathbf{0} = -\infty$, $\mathbf{1} = 0$, $x \leq \infty$, $x \oplus \infty = \infty$ for all x, $x \odot \infty = \infty \odot x = \infty$ if $x \neq \mathbf{0}$, and $\mathbf{0} \odot \infty = \infty \odot \mathbf{0}$. The semiring $\hat{\mathbf{R}}_{\max}$ is complete and a-complete. The semiring $\hat{\mathbf{R}}_{\min} = \mathbf{R}_{\min} \cup \{-\infty\}$ with obvious operations is also complete; $\hat{\mathbf{R}}_{\min}$ and $\hat{\mathbf{R}}_{\max}$ are isomorphic.

Example 11.2. Consider the semiring $S^{[a,b]}_{\max,\min}$ defined on the real interval $[a,b]$ with operations $\oplus = \max$, $\odot = \min$ and neutral elements $\mathbf{0} = a$ and $\mathbf{1} = b$. The semiring is complete and a-complete. Set $S_{\max,\min} = S^{[a,b]}_{\max,\min}$ with $a = -\infty$ and $b = +\infty$. If $-\infty \leq a < b \leq +\infty$ then $S^{[a,b]}_{\max,\min}$ and $S_{\max,\min}$ are isomorphic.

Example 11.3. The Boolean algebra $B = \{\mathbf{0}, \mathbf{1}\}$ is a complete and a-complete semifield consisting of two elements.

11.2 Closure Operations

Let a semiring S be endowed with a partial unary *closure (or Kleene) operation* $*$ such that $x \preceq y$ implies $x^* \preceq y^*$ and $x^* = \mathbf{1} \oplus (x^* \odot x) = \mathbf{1} \oplus (x \odot x^*)$ on its domain of definition. In particular, $\mathbf{0}^* = \mathbf{1}$ by definition. These axioms imply that $x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \cdots \oplus (x^* \odot x^n)$ if $n \geqslant 1$. Thus x^* can be considered as a 'regularized sum' of the series $x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \dots$; in an idempotent semiring, by definition, $x^* = \sup\{\mathbf{1}, x, x^2, \dots\}$ if this supremum exists. So if S is complete, then the closure operation is well-defined for every element $x \in S$.

In numerical semirings the operation $*$ is defined as follows: $x^* = (1-x)^{-1}$ if $x \prec 1$ in $\mathbf{R}_+$, or $\hat{\mathbf{R}}_+$ and $x^* = \infty$ if $x \succcurlyeq 1$ in $\hat{\mathbf{R}}_+$; $x^* = \mathbf{1}$ if $x \preceq \mathbf{1}$ in $\mathbf{R}_{\max}$ and $\hat{\mathbf{R}}_{\max}$, $x^* = \infty$ if $x \succ \mathbf{1}$ in $\hat{\mathbf{R}}_{\max}$, $x^* = \mathbf{1}$ for all x in $S^{[a,b]}_{\max,\min}$. In all other cases x^* is undefined. Note that the closure operation is very easy to implement.

11.3 Matrices Over Semirings

Denote by $\mathrm{Mat}_{mn}(S)$ a set of all matrices $A = (a_{ij})$ with m rows and n columns whose coefficients belong to a semiring S. The sum $A \oplus B$ of matrices $A, B \in \mathrm{Mat}_{mn}(S)$ and the product AB of matrices $A \in \mathrm{Mat}_{lm}(S)$ and $B \in \mathrm{Mat}_{mn}(S)$ are defined according to the usual rules of linear algebra: $A \oplus B = (a_{ij} \oplus b_{ij}) \in \mathrm{Mat}_{mn}(S)$ and

$$AB = \left(\bigoplus_{k=1}^{m} a_{ij} \odot b_{kj} \right) \in \mathrm{Mat}_{ln}(S),$$

where $A \in \mathrm{Mat}_{lm}(S)$ and $B \in \mathrm{Mat}_{mn}(S)$. Note that we write AB instead of $A \odot B$.

If the semiring S is ordered, then the set $\mathrm{Mat}_{mn}(S)$ is ordered by the relation $A = (a_{ij}) \preceq B = (b_{ij})$ iff $a_{ij} \preceq b_{ij}$ in S for all $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

The matrix multiplication is consistent with the order $\preceq$ in the following sense: if $A, A' \in \mathrm{Mat}_{lm}(S)$, $B, B' \in \mathrm{Mat}_{mn}(S)$ and $A \preceq A'$, $B \preceq B'$, then $AB \preceq A'B'$ in $\mathrm{Mat}_{ln}(S)$. The set $\mathrm{Mat}_{nn}(S)$ of square $(n \times n)$ matrices over an idempotent semiring S forms a idempotent semiring with a zero element $O = (o_{ij})$, where $o_{ij} = \mathbf{0}$, $1 \leqslant i, j \leqslant n$, and a unit element $I = (\delta_{ij})$, where $\delta_{ij} = \mathbf{1}$ if $i = j$ and $\delta_{ij} = \mathbf{0}$ otherwise.

The set Mat_{nn} is an example of a noncommutative semiring if $n > 1$.

The closure operation in matrix semirings over an idempotent semiring S can be defined inductively (another way to do that see in [26] and below): $A^* = (a_{11})^* = (a_{11}^*)$ in $\mathrm{Mat}_{11}(S)$ and for any integer $n > 1$ and any matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathrm{Mat}_{kk}(S)$, $A_{12} \in \mathrm{Mat}_{kn-k}(S)$, $A_{21} \in \mathrm{Mat}_{n-kk}(S)$, $A_{22} \in \mathrm{Mat}_{n-kn-k}(S)$, $1 \leqslant k \leqslant n$, by definition,

$$A^* = \begin{pmatrix} A_{11}^* \oplus A_{11}^* A_{12} D^* A_{21} A_{11}^* & A_{11}^* A_{12} D^* \\ D^* A_{21} A_{11}^* & D^* \end{pmatrix}, \tag{22}$$

where $D = A_{22} \oplus A_{21}A_{11}^*A_{12}$. It can be proved that this definition of A^* implies that the equality $A^* = A^*A \oplus I$ is satisfied and thus A^* is a 'regularized sum' of the series $I \oplus A \oplus A^2 \oplus \dots$.

Note that this recurrence relation coincides with the formulas of escalator method of matrix inversion in the traditional linear algebra over the field of real or complex numbers, up to the algebraic operations used. Hence this algorithm of matrix closure requires a polynomial number of operations in n.

11.4 Discrete Stationary Bellman Equations

Let S be a semiring. The *discrete stationary Bellman equation* has the form

$$X = AX \oplus B, \tag{23}$$

where $A \in \mathrm{Mat}_{nn}(S)$, $X, B \in \mathrm{Mat}_{ns}(S)$, and the matrix X is unknown. Let A^* be the closure of the matrix A. It follows from the identity $A^* = A^*A \oplus I$ that the matrix A^*B satisfies this equation; moreover, it can be proved that for idempotent semirings this solution is the least in the set of solutions to equation (23) with respect to the partial order in $\mathrm{Mat}_{ns}(S)$.

Equation (23) over max-plus semiring arises in connection with Bellman optimality principle and discretization of Hamilton–Jacobi equations, see e.g., [71]. It is also intimately related with optimization problems on graphs to be discussed below.

11.5 Weighted Directed Graphs and Matrices Over Semirings

Suppose that S is a semiring with zero $\mathbf{0}$ and unity $\mathbf{1}$. It is well-known that any square matrix $A = (a_{ij}) \in \mathrm{Mat}_{nn}(S)$ specifies a *weighted directed graph*. This geometrical construction includes three kinds of objects: the set X of n elements $x_1, \dots, x_n$ called *nodes*, the set Γ of all ordered pairs (x_i, x_j) such that $a_{ij} \neq \mathbf{0}$ called *arcs*, and the mapping $A\colon \Gamma \to S$ such that $A(x_i, x_j) = a_{ij}$. The elements a_{ij} of the semiring S are called *weights* of the arcs, see Fig. 5.

Conversely, any given weighted directed graph with n nodes specifies a unique matrix $A \in \mathrm{Mat}_{nn}(S)$.

This definition allows for some pairs of nodes to be disconnected if the corresponding element of the matrix A is $\mathbf{0}$ and for some channels to be "loops" with coincident ends if the matrix A has nonzero diagonal elements. This concept is convenient for analysis of parallel and distributed computations and design of computing media and networks (see, e.g., [5, 45, 69, 97]).

Recall that a sequence of nodes of the form

$$p = (y_0, y_1, \dots, y_k)$$

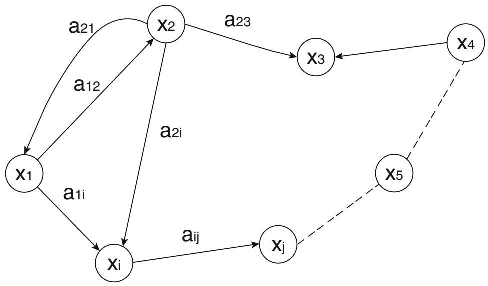

Fig. 5 A weighted directed graph

with $k \geqslant 0$ and $(y_i, y_{i+1}) \in \Gamma$, $i = 0, \ldots, k-1$, is called a *path* of length k connecting y_0 with y_k. Denote the set of all such paths by $P_k(y_0, y_k)$. The weight $A(p)$ of a path $p \in P_k(y_0, y_k)$ is defined to be the product of weights of arcs connecting consecutive nodes of the path:

$$A(p) = A(y_0, y_1) \odot \cdots \odot A(y_{k-1}, y_k).$$

By definition, for a "path" $p \in P_0(x_i, x_j)$ of length $k = 0$ the weight is $\mathbf{1}$ if $i = j$ and $\mathbf{0}$ otherwise.

For each matrix $A \in \mathrm{Mat}_{nn}(S)$ define $A^0 = I = (\delta_{ij})$ (where $\delta_{ij} = \mathbf{1}$ if $i = j$ and $\delta_{ij} = \mathbf{0}$ otherwise) and $A^k = AA^{k-1}$, $k \geqslant 1$. Let $a_{ij}^{(k)}$ be the (i, j)th element of the matrix A^k. It is easily checked that

$$a_{ij}^{(k)} = \bigoplus_{\substack{i_0=i,\, i_k=j \\ 1\leqslant i_1,\ldots,i_{k-1}\leqslant n}} a_{i_0 i_1} \odot \cdots \odot a_{i_{k-1} i_k}.$$

Thus $a_{ij}^{(k)}$ is the supremum of the set of weights corresponding to all paths of length k connecting the node $x_{i_0} = x_i$ with $x_{i_k} = x_j$.

Denote the elements of the matrix A^* by $a_{ij}^{(*)}$, $i, j = 1, \ldots, n$; then

$$a_{ij}^{(*)} = \bigoplus_{0\leqslant k<\infty} \bigoplus_{p\in P_k(x_i,x_j)} A(p).$$

The closure matrix A^* solves the well-known *algebraic path problem*, which is formulated as follows: for each pair (x_i, x_j) calculate the supremum of weights of all paths (of arbitrary length) connecting node x_i with node x_j. The closure operation in matrix semirings has been studied extensively (see, e.g., [1, 2, 6–8, 14, 15, 20–22, 26–30, 33, 34, 59] and references therein).

Example 11.4 (The shortest path problem.). Let $S = \mathbf{R}_{\min}$, so the weights are real numbers. In this case

$$A(p) = A(y_0, y_1) + A(y_1, y_2) + \cdots + A(y_{k-1}, y_k).$$

If the element a_{ij} specifies the length of the arc (x_i, x_j) in some metric, then $a_{ij}^{(*)}$ is the length of the shortest path connecting x_i with x_j.

Example 11.5 (The maximal path width problem.). Let $S = \mathbf{R} \cup \{\mathbf{0}, \mathbf{1}\}$ with $\oplus =$ max, $\odot =$ min. Then

$$a_{ij}^{(*)} = \max_{p \in \bigcup_{k \geqslant 1} P_k(x_i, x_j)} A(p), \quad A(p) = \min(A(y_0, y_1), \ldots, A(y_{k-1}, y_k)).$$

If the element a_{ij} specifies the "width" of the arc (x_i, x_j), then the width of a path p is defined as the minimal width of its constituting arcs and the element $a_{ij}^{(*)}$ gives the supremum of possible widths of all paths connecting x_i with x_j.

Example 11.6 (A simple dynamic programming problem.). Let $S = \mathbf{R}_{\max}$ and suppose a_{ij} gives the *profit* corresponding to the transition from x_i to x_j. Define the vector $B = (b_i) \in \mathrm{Mat}_{n1}(\mathbf{R}_{\max})$ whose element b_i gives the *terminal profit* corresponding to exiting from the graph through the node x_i. Of course, negative profits (or, rather, losses) are allowed. Let m be the total profit corresponding to a path $p \in P_k(x_i, x_j)$, i.e.

$$m = A(p) + b_j.$$

Then it is easy to check that the supremum of profits that can be achieved on paths of length k beginning at the node x_i is equal to $(A^k B)_i$ and the supremum of profits achievable without a restriction on the length of a path equals $(A^* B)_i$.

Example 11.7 (The matrix inversion problem.). Note that in the formulas of this section we are using distributivity of the multiplication $\odot$ with respect to the addition $\oplus$ but do not use the idempotency axiom. Thus the algebraic path problem can be posed for a nonidempotent semiring S as well (see, e.g., [84]). For instance, if $S = \mathbf{R}$, then

$$A^* = I + A + A^2 + \cdots = (I - A)^{-1}.$$

If $\|A\| > 1$ but the matrix $I - A$ is invertible, then this expression defines a regularized sum of the divergent matrix power series $\sum_{i \geqslant 0} A^i$.

There are many other important examples of problems (in different areas) related to algorithms of linear algebra over semirings (transitive closures of relations, accessible sets, critical paths, paths of greatest capacities, the most reliable paths, interval and other problems), see [1, 2, 5–7, 12, 14–17, 20–24, 26–31, 33, 34, 58, 59, 69, 75, 76, 81–84, 87, 89, 98–101].

We emphasize that this connection between the matrix closure operation and solution to the Bellman equation gives rise to a number of different algorithms for numerical calculation of the closure matrix. All these algorithms are adaptations of the well-known algorithms of the traditional computational linear algebra, such as the Gauss–Jordan elimination, various iterative and escalator schemes, etc. This is a special case of the idempotent superposition principle.

In fact, the theory of the discrete stationary Bellman equation can be developed using the identity $A^* = AA^* \oplus I$ as an additional axiom without any substantial interpretation (the so-called *closed semirings*, see, e.g., [7, 26, 38, 84]).

12 Universal Algorithms

Computational algorithms are constructed on the basis of certain primitive operations. These operations manipulate data that describe "numbers." These "numbers" are elements of a "numerical domain," i.e., a mathematical object such as the field of real numbers, the ring of integers, or an idempotent semiring of numbers.

In practice elements of the numerical domains are replaced by their computer representations, i.e., by elements of certain finite models of these domains. Examples of models that can be conveniently used for computer representation of real numbers are provided by various modifications of floating point arithmetics, approximate arithmetics of rational numbers [52], and interval arithmetics. The difference between mathematical objects ("ideal" numbers) and their finite models (computer representations) results in computational (e.g., rounding) errors.

An algorithm is called *universal* if it is independent of a particular numerical domain and/or its computer representation. A typical example of a universal algorithm is the computation of the scalar product (x, y) of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ by the formula $(x, y) = x_1 y_1 + \cdots + x_n y_n$. This algorithm (formula) is independent of a particular domain and its computer implementation, since the formula is well-defined for any semiring. It is clear that one algorithm can be more universal than another. For example, the simplest Newton–Cotes formula, the rectangular rule, provides the most universal algorithm for numerical integration; indeed, this formula is valid even for idempotent integration (over any idempotent semiring, see above and [5, 33, 39, 40, 42–44, 51, 62–65]. Other quadrature formulas (e.g., combined trapezoid rule or the Simpson formula) are independent of computer arithmetics and can be used (e.g., in an iterative form) for computations with arbitrary accuracy. In contrast, algorithms based on Gauss–Jacobi formulas are designed for fixed accuracy computations: they include constants (coefficients and nodes of these formulas) defined with fixed accuracy. Certainly, algorithms of this type can be made more universal by including procedures for computing the constants; however, this results in an unjustified complication of the algorithms.

Computer algebra algorithms used in such systems as Mathematica, Maple, REDUCE, and others are highly universal. Most of the standard algorithms used in linear algebra can be rewritten in such a way that they will be valid over any field and complete idempotent semiring (including semirings of intervals; see below and [58, 59, 90], where an interval version of the idempotent linear algebra and the corresponding universal algorithms are discussed).

As a rule, iterative algorithms (beginning with the successive approximation method) for solving differential equations (e.g., methods of Euler, Euler–Cauchy, Runge–Kutta, Adams, a number of important versions of the difference approximation method, and the like), methods for calculating elementary and some special functions based on the expansion in Taylor's series and continuous fractions (Padé approximations) and others are independent of the computer representation of numbers.

Calculations on computers usually are based on a floating-point arithmetic with a mantissa of a fixed length; i.e., computations are performed with fixed accuracy. Broadly speaking, with this approach only the relative rounding error is fixed, which can lead to a drastic loss of accuracy and invalid results (e.g., when summing series and subtracting close numbers). On the other hand, this approach provides rather high speed of computations. Many important numerical algorithms are designed to use floating-point arithmetic (with fixed accuracy) and ensure the maximum computation speed. However, these algorithms are not universal. The above mentioned Gauss–Jacobi quadrature formulas, computation of elementary and special functions on the basis of the best polynomial or rational approximations or Padé–Chebyshev approximations, and some others belong to this type. Such algorithms use nontrivial constants specified with fixed accuracy.

Recently, problems of accuracy, reliability, and authenticity of computations (including the effect of rounding errors) have gained much attention; in part, this fact is related to the ever-increasing performance of computer hardware. When errors in initial data and rounding errors strongly affect the computation results, such as in ill-posed problems, analysis of stability of solutions, etc., it is often useful to perform computations with improved and variable accuracy. In particular, the rational arithmetic, in which the rounding error is specified by the user [52], can be used for this purpose. This arithmetic is a useful complement to the interval analysis [70]. The corresponding computational algorithms must be universal (in the sense that they must be independent of the computer representation of numbers).

13 Universal Algorithms of Linear Algebra Over Semirings

The most important linear algebra problem is to solve the system of linear equations

$$AX = B, \tag{24}$$

where A is a matrix with elements from the basic field and X and B are vectors (or matrices) with elements from the same field. It is required to find X if A and B are given. If A in (24) is not the identity matrix I, then system (24) can be written in form (23), i.e.,

$$X = AX + B. \tag{25}$$

It is well known that the form (25) is convenient for using the successive approximation method. Applying this method with the initial approximation $X_0 = 0$, we obtain the solution

$$X = A^* B, \tag{26}$$

where

$$A^* = I + A + A^2 + \cdots + A^n + \cdots \tag{27}$$

On the other hand, it is clear that

$$A^* = (I - A)^{-1}, \tag{28}$$

if the matrix $I - A$ is invertible. The inverse matrix $(I - A)^{-1}$ can be considered as a regularized sum of the formal series (27).

The above considerations can be extended to a broad class of semirings.

The closure operation for matrix semirings $\mathrm{Mat}_n(S)$ can be defined and computed in terms of the closure operation for S (see Sect. 11.3 above); some methods are described in [1, 2, 7, 14, 15, 26–29, 33, 37, 51, 59, 83, 84, 87]. One such method is described below (*LDM*-factorization), see [45].

If S is a field, then, by definition, $x^* = (1 - x)^{-1}$ for any $x \neq 1$. If S is an idempotent semiring, then, by definition,

$$x^* = \mathbf{1} \oplus x \oplus x^2 \oplus \cdots = \sup\{\mathbf{1}, x, x^2, \ldots\}, \tag{29}$$

if this supremum exists. Recall that it exists if S is complete, see Sect. 11.2.

Consider a nontrivial universal algorithm applicable to matrices over semirings with the closure operation defined.

Example 13.1 (Semiring LDM-Factorization). Factorization of a matrix into the product $A = LDM$, where L and M are lower and upper triangular matrices with a unit diagonal, respectively, and D is a diagonal matrix, is used for solving matrix equations $AX = B$. We construct a similar decomposition for the Bellman equation $X = AX \oplus B$.

For the case $AX = B$, the decomposition $A = LDM$ induces the following decomposition of the initial equation:

$$LZ = B, \qquad DY = Z, \qquad MX = Y. \tag{30}$$

Hence, we have

$$A^{-1} = M^{-1} D^{-1} L^{-1}, \tag{31}$$

if A is invertible. In essence, it is sufficient to find the matrices L, D and M, since the linear system (30) is easily solved by a combination of the forward substitution for Z, the trivial inversion of a diagonal matrix for Y, and the back substitution for X.

Using (30) as a pattern, we can write

$$Z = LZ \oplus B, \qquad Y = DY \oplus Z, \qquad X = MX \oplus Y. \tag{32}$$

Then

$$A^* = M^* D^* L^*. \tag{33}$$

A triple (L, D, M) consisting of a lower triangular, diagonal, and upper triangular matrices is called an *LDM-factorization* of a matrix A if relations (32) and (33) are satisfied. We note that in this case, the principal diagonals of L and M are zero.

The modification of the notion of *LDM*-factorization used in matrix analysis for the equation $AX = B$ is constructed in analogy with a construction suggested by Carré in [14, 15] for LU-factorization.

We stress that the algorithm described below can be applied to matrix computations over any semiring under the condition that the unary operation $a \mapsto a^*$ is applicable every time it is encountered in the computational process. Indeed, when constructing the algorithm, we use only the basic semiring operations of addition $\oplus$ and multiplication $\odot$ and the properties of associativity, commutativity of addition, and distributivity of multiplication over addition.

If A is a symmetric matrix over a semiring with a commutative multiplication, the amount of computations can be halved, since M and L are mapped into each other under transposition.

We begin with the case of a triangular matrix $A = L$ (or $A = M$). Then, finding X is reduced to the forward (or back) substitution.

Forward substitution

We are given:

- $L = \|l^i_j\|^n_{i,j=1}$, where $l^i_j = \mathbf{0}$ for $i \leq j$ (a lower triangular matrix with a zero diagonal).
- $B = \|b^i\|^n_{i=1}$.

It is required to find the solution $X = \|x^i\|^n_{i=1}$ to the equation $X = LX \oplus B$. The program fragment solving this problem is as follows:

```
for i = 1 to n do
{     x^i := b^i;
      for j = 1 to i − 1 do
            x^i := x^i ⊕ (l^i_j ⊙ x^j); }
```

Back substitution

We are given:

- $M = \|m^i_j\|^n_{i,j=1}$, where $m^i_j = \mathbf{0}$ for $i \geq j$ (an upper triangular matrix with a zero diagonal).
- $B = \|b^i\|^n_{i=1}$.

It is required to find the solution $X = \|x^i\|^n_{i=1}$ to the equation $X = MX \oplus B$. The program fragment solving this problem is as follows:

```
for i = n to 1 step −1 do
{     x^i := b^i;
      for j = n to i + 1 step −1 do
            x^i := x^i ⊕ (m^i_j ⊙ x^i); }
```

Both algorithms require $(n^2 - n)/2$ operations $\oplus$ and $\odot$.

Closure of a diagonal matrix

We are given:

- $D = \operatorname{diag}(d_1, \dots, d_n)$.
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = DX \oplus B$. The program fragment solving this problem is as follows:

```
for i = 1 to n do
    x^i := (d_i)* ⊙ b^i;
```

This algorithm requires n operations $*$ and n multiplications $\odot$.

General case

We are given:

- $L = \|l^i_j\|_{i,j=1}^n$, where $l^i_j = \mathbf{0}$ if $i \le j$.
- $D = \operatorname{diag}(d_1, \dots, d_n)$.
- $M = \|m^i_j\|_{i,j=1}^n$, where $m^i_j = \mathbf{0}$ if $i \ge j$.
- $B = \|b^i\|_{i=1}^n$.

It is required to find the solution $X = \|x^i\|_{i=1}^n$ to the equation $X = AX \oplus B$, where L, D, and M form the *LDM*-factorization of A. The program fragment solving this problem is as follows:

```
FORWARD SUBSTITUTION
for i = 1 to n do
{ x^i := b^i;
 for j = 1 to i − 1 do
  x^i := x^i ⊕ (l^i_j ⊙ x^j); }
CLOSURE OF A DIAGONAL MATRIX
for i = 1 to n do
 x^i := (d_i)* ⊙ b^i;
BACK SUBSTITUTION
for i = n to 1 step −1 do
{ for j = n to i + 1 step −1 do
  x^i := x^i ⊕ (m^i_j ⊙ x^j); }
```

Note that x^i is not initialized in the course of the back substitution. The algorithm requires $n^2 - n$ operations $\oplus$, n^2 operations $\odot$, and n operations $*$.

LDM-factorization

We are given:

- $A = \|a^i_j\|_{i,j=1}^n$.

It is required to find the *LDM*-factorization of A: $L = \|l^i_j\|_{i,j=1}^n$, $D = \operatorname{diag}(d_1, \dots, d_n)$, and $M = \|m^i_j\|_{i,j=1}^n$, where $l^i_j = \mathbf{0}$ if $i \le j$, and $m^i_j = \mathbf{0}$ if $i \ge j$.

The program uses the following internal variables:

- $C = \|c^i_j\|^n_{i,j=1}$
- $V = \|v^i\|^n_{i=1}$
- d.

INITIALISATION
for $i = 1$ to n do
    for $j = 1$ to n do
        $c^i_j = a^i_j$;
MAIN LOOP
for $j = 1$ to n do
{   for $i = 1$ to j do
        $v^i := a^i_j$;
    for $k = 1$ to $j - 1$ do
        for $i = k + 1$ to j do
            $v^i := v^i \oplus (a^i_k \odot v^k)$;
    for $i = 1$ to $j - 1$ do
        $a^i_j := (a^i_i)^* \odot v^i$;
    $a^j_j := v^j$;
    for $k = 1$ to $j - 1$ do
        for $i = j + 1$ to n do
            $a^i_j := a^i_j \oplus (a^i_k \odot v^k)$;
    $d = (v^j)^*$;
    for $i = j + 1$ to n do
        $a^i_j := a^i_j \odot d$; }

This algorithm requires $(2n^3 - 3n^2 + n)/6$ operations $\oplus$, $(2n^3 + 3n^2 - 5n)/6$ operations $\odot$, and $n(n+1)/2$ operations $*$. After its completion, the matrices L, D, and M are contained, respectively, in the lower triangle, on the diagonal, and in the upper triangle of the matrix C. In the case when A is symmetric about the principal diagonal and the semiring over which the matrix is defined is commutative, the algorithm can be modified in such a way that the number of operations is reduced approximately by a factor of two.

Other examples can be found in [14, 15, 26–29, 37, 38, 84, 87].

Note that to compute the matrices A^* and A^*B it is convenient to solve the Bellman equation (25).

Some other interesting and important problems of linear algebra over semirings are examined, e.g., in [9, 12, 13, 16, 23, 24, 26–29, 31, 75–77, 79, 98–101].

Remark 13.1. It is well known that linear problems and equations are especially convenient for parallelization, see, e.g., [97]. Standard methods (including the so-called block methods) constructed in the framework of the traditional mathematics can be extended to universal algorithms over semirings (the correspondence principle!). For example, formula (22) discussed in Sect. 11.3 leads to a simple block method for parallelization of the closure operations. Other standard methods of linear algebra [97] can be used in a similar way.

14 The Correspondence Principle for Computations

Of course, the idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [40, 42, 44, 51]. Thus:

If we have an important and interesting numerical algorithm, then there is a good chance that its semiring analogs are important and interesting as well.

In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. Note that numerical algorithms for standard infinite-dimensional linear problems over idempotent semirings (i.e., for problems related to idempotent integration, integral operators and transformations, the Hamilton–Jacobi and generalized Bellman equations) deal with the corresponding finite-dimensional (or finite) "linear approximations". Nonlinear algorithms often can be approximated by linear ones. Thus the idempotent linear algebra is a basis for the idempotent numerical analysis.

Moreover, it is well-known that linear algebra algorithms easily lend themselves to parallel computation; their idempotent analogs admit parallelization as well. Thus we obtain a systematic way of applying parallel computing to optimization problems.

Basic algorithms of linear algebra (such as inner product of two vectors, matrix addition and multiplication, etc.) often do not depend on concrete semirings, as well as on the nature of domains containing the elements of vectors and matrices. Algorithms to construct the closure $A^* = I \oplus A \oplus A^2 \oplus \cdots \oplus A^n \oplus \cdots = \bigoplus_{n=1}^{\infty} A^n$ of an idempotent matrix A can be derived from standard methods for calculating $(I - A)^{-1}$. For the Gauss–Jordan elimination method (via LU-decomposition) this trick was used in [84], and the corresponding algorithm is universal and can be applied both to the Bellman equation and to computing the inverse of a real (or complex) matrix $(I - A)$. Computation of A^{-1} can be derived from this universal algorithm with some obvious cosmetic transformations.

Thus it seems reasonable to develop universal algorithms that can deal equally well with initial data of different domains sharing the same basic structure [40, 42, 44].

15 The Correspondence Principle for Hardware Design

A systematic application of the correspondence principle to computer calculations leads to a unifying approach to software and hardware design.

The most important and standard numerical algorithms have many hardware realizations in the form of technical devices or special processors. *These devices* *often can be used as prototypes for new hardware units generated by substitution of the usual arithmetic operations for its semiring analogs and by addition tools for performing neutral elements* **0** *and* **1** (the latter usually is not difficult). Of course, the case of numerical semirings consisting of real numbers (maybe except neutral elements) and semirings of numerical intervals is the most simple and natural [39, 40, 42–44, 51, 58, 59, 90]. Note that for semifields (including $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$) the operation of division is also defined.

Good and efficient technical ideas and decisions can be transferred from prototypes to new hardware units. Thus the correspondence principle generated a regular heuristic method for hardware design. Note that to get a patent it is necessary to present the so-called "invention formula", that is to indicate a prototype for the suggested device and the difference between these devices.

Consider (as a typical example) the most popular and important algorithm of computing the scalar product of two vectors:

$$(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \tag{34}$$

The universal version of (34) for any semiring A is obvious:

$$(x, y) = (x_1 \odot y_1) \oplus (x_2 \odot y_2) \oplus \cdots \oplus (x_n \odot y_n). \tag{35}$$

In the case $A = \mathbf{R}_{\max}$ this formula turns into the following one:

$$(x, y) = \max\{x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n\}. \tag{36}$$

This calculation is standard for many optimization algorithms, so it is useful to construct a hardware unit for computing (36). There are many different devices (and patents) for computing (34) and every such device can be used as a prototype to construct a new device for computing (36) and even (35). Many processors for matrix multiplication and for other algorithms of linear algebra are based on computing scalar products and on the corresponding "elementary" devices respectively, etc.

There are some methods to make these new devices more universal than their prototypes. There is a modest collection of possible operations for standard numerical semirings: max, min, and the usual arithmetic operations. So, it is easy to construct programmable hardware processors with variable basic operations. Using modern technologies it is possible to construct cheap special-purpose multi-processor chips implementing examined algorithms. The so-called systolic processors are especially convenient for this purpose. A systolic array is a "homogeneous" computing medium consisting of elementary processors, where the general scheme and processor connections are simple and regular. Every elementary processor pumps data in and out performing elementary operations in a such way that the corresponding data flow is kept up in the computing medium; there is an analogy with the blood circulation and this is a reason for the term "systolic", see e.g., [40, 42, 44, 45, 66, 83, 84, 87].

Some systolic processors for the general algebraic path problem are presented in [83, 84, 87]. In particular, there is a systolic array of $n(n+1)$ elementary processors which performs computations of the Gauss–Jordan elimination algorithm and can solve the algebraic path problem within $5n-2$ time steps. Of course, hardware implementations for important and popular basic algorithms increase the speed of data processing.

The so-called GPGPU (General-Purpose computing on Graphics Processing Units) technique is another important field for applications of the correspondence principle. The matter is that graphic processing units (hidden in modern laptop and desktop computers) are potentially powerful processors for solving numerical problems. The recent tremendous progress in graphical processing hardware and software resulted in new "open" programmable parallel computational devices (special processors), see, e.g., [11, 78, 102]. These devices are going to be standard for coming PC (personal computers) generations. Initially used for graphical processing only (at that time they were called GPU), today they are used for various fields, including audio and video processing, computer simulation, and encryption. But this list can be considerably enlarged following the correspondence principle: the basic operations would be used as parameters. Using the technique described in this paper (see also our references), standard linear algebra algorithms can be used for solving different problems in different areas. In fact, the hardware supports all operations needed for the most important idempotent semirings: plus, times, min, max. The most popular linear algebra packages [ATLAS (Automatically Tuned Linear Algebra Software), LAPACK, PLASMA (Parallel Linear Algebra for Scalable Multicore Architectures)] can already use GPGPU, see [103–105]. We propose to make these tools more powerful by using parameterized algorithms.

Linear algebra over the most important numerical semirings generates solutions for many concrete problems in different areas, see above.

Note that to be consistent with operations we have to redefine zero (0) and unit (1) elements (see above); comparison operations must be also redefined as it is described above. Once the operations are redefined, then the most of basic linear algebra algorithms, including back and forward substitution, Gauss elimination method, Jordan elimination method and others could be rewritten for new domains and data structures. Combined with the power of the new parallel hardware this approach could change PC from entertainment devices to power full instruments.

16 The Correspondence Principle for Software Design

Software implementations for universal semiring algorithms are not as efficient as hardware ones (with respect to the computation speed) but they are much more flexible. Program modules can deal with abstract (and variable) operations and data types. These operations and data types can be defined by the corresponding input data. In this case they can be generated by means of additional program modules. For programs written in this manner it is convenient to use special techniques of the so-called object oriented (and functional) design, see, e.g., [60, 80, 91]. Fortunately, powerful tools supporting the object-oriented software design have recently appeared including compilers for real and convenient programming languages (e.g. C^{++} and Java) and modern computer algebra systems.

Recently, this type of programming technique has been dubbed generic programming (see, e.g., [8, 80]). To help automate the generic programming, the so-called Standard Template Library (STL) was developed in the framework of C^{++} [80,91]. However, high-level tools, such as STL, possess both obvious advantages and some disadvantages and must be used with caution.

It seems that it is natural to obtain an implementation of the correspondence principle approach to scientific calculations in the form of a powerful software system based on a collection of universal algorithms. This approach ensures a working time reduction for programmers and users because of the software unification. The arbitrary necessary accuracy and safety of numeric calculations can be ensured as well.

This software system may be especially useful for designers of algorithms, software engineers, students and mathematicians.

Note that there are some software systems oriented to calculations with idempotent semirings like $\mathbf{R}_{\max}$; see, e.g., [82]. However these systems do not support universal algorithms.

17 Interval Analysis in Idempotent Mathematics

Traditional interval analysis is a nontrivial and popular mathematical area, see, e.g., [4, 24, 35, 70, 74, 77]. An "idempotent" version of interval analysis (and moreover interval analysis over positive semirings) appeared in [58,59,90]. Later the idempotent interval analysis has attracted many experts in tropical linear algebra and applications, see, e.g., [16, 24, 31, 75, 76, 101]. We also mention the closely related interval analysis over the positive semiring $\mathbf{R}_+$ discussed in [9].

Let a set S be partially ordered by a relation $\preceq$. A *closed interval* in S is a subset of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{\, x \in S \mid \underline{\mathbf{x}} \preceq x \preceq \overline{\mathbf{x}}\,\}$, where the elements $\underline{\mathbf{x}} \preceq \overline{\mathbf{x}}$ are called *lower* and *upper bounds* of the interval $\mathbf{x}$. The order $\preceq$ induces a partial ordering on the set of all closed intervals in S: $\mathbf{x} \preceq \mathbf{y}$ iff $\underline{\mathbf{x}} \preceq \underline{\mathbf{y}}$ and $\overline{\mathbf{x}} \preceq \overline{\mathbf{y}}$.

A *weak interval extension* $I(S)$ of an ordered semiring S is the set of all closed intervals in S endowed with operations $\oplus$ and $\odot$ defined as $\mathbf{x} \oplus \mathbf{y} = [\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}, \overline{\mathbf{x}} \oplus \overline{\mathbf{y}}]$, $\mathbf{x} \odot \mathbf{y} = [\underline{\mathbf{x}} \odot \underline{\mathbf{y}}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}]$ and a partial order induced by the order in S. The closure operation in $I(S)$ is defined by $\mathbf{x}^* = [\underline{\mathbf{x}}^*, \overline{\mathbf{x}}^*]$. There are some other interval extensions (including the so-called strong interval extension [59]) but the weak extension is more convenient.

The extension $I(S)$ is idempotent if S is an idempotent semiring. A universal algorithm over S can be applied to $I(S)$ and we shall get an interval version of the initial algorithm. Usually both the versions have the same complexity. For the discrete stationary Bellman equation and the corresponding optimization problems on graphs, interval analysis was examined in [58, 59] in details. Other problems of idempotent linear algebra were examined in [16, 24, 31, 75, 76].

Idempotent mathematics appears to be remarkably simpler than its traditional analog. For example, in traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, whereas in idempotent interval arithmetic this distributivity is preserved. Moreover, in traditional interval analysis the set of all square interval matrices of a given order does not form even a semigroup with respect to matrix multiplication: this operation is not associative since distributivity is lost in the traditional interval arithmetic. On the contrary, in the idempotent (and positive) case associativity is preserved. Finally, in traditional interval analysis some problems of linear algebra, such as solution of a linear system of interval equations, can be very difficult (more precisely, they are NP-hard, see [19, 24, 35, 36] and references therein). It was noticed in [58, 59] that in the idempotent case solving an interval linear system requires a polynomial number of operations (similarly to the usual Gauss elimination algorithm). The remarkable simplicity of idempotent interval arithmetic is due to the following properties: the monotonicity of arithmetic operations and the positivity of all elements of an idempotent semiring.

Interval estimates in idempotent mathematics are usually exact. In the traditional theory such estimates tend to be overly pessimistic.

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